AC-driven Brownian motors: A Fokker-Planck treatment

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I. INTRODUCTION

To pan for gold, sand containing gold is placed into a pan and is agitated in a circular motion, back and forth, left and right. The sand eventually goes away with the water and gold grains remain in the bottom of the pan. However, if we simply incline the pan, then all the content will go away.

This example illustrates the basic mechanism behind the ratchet idea: in a driven system without a preferable direction of motion, transport velocities depend on the characteristics of movable objects such as charge, spin, mass, and size.[1–4] The intensive development of this idea during the last decade has brought new tools for the smart control of transport in systems ranging from mechanical engines[5] to quantum devices.[6]

The ratchet setup requires three ingredients—nonlinearity, asymmetry (spatial and/or temporal), and a fluctuating input force of zero mean. Nonlinearity is necessary because, otherwise, the system will produce a zero-mean output from a zero-mean input. Asymmetry is needed to violate the left/right symmetry of the response because transport in a fully symmetrical system is unbiased. A zero-mean fluctuating force should break thermodynamic equilibrium, which forbids appearance of any directed transport due to the second law of thermodynamics.[7,8]

The basic model of dynamical ratchets is a classical particle in a one-dimensional periodic potential exposed to an AC field.[9–12] Although the symmetry analysis of microscopic equations of motion[9] enables us to formulate the conditions necessary for the appearance of directed transport,[13] both the sign of the current and its strength depend on dynamical mechanisms. In the deterministic limit for which noise is absent, the evolution of a damped particle is governed by attractors, regular (limit cycles) or chaotic ones.[14] The transport properties are encoded in the characteristics of the attractors, so that if there is only one attractor in phase space, the DC current is equal to the mean velocity of the attractor.[12]

The situation changes drastically when noise contributes to the dynamics. During the evolution, a particle may jump out of an attractor, evolve outside of its vicinity, and return to the attractor. In other words, the particle explores the whole phase space, and the dynamics of the particle cannot be described only in terms of the properties of the attractor. The situation becomes even more complicated when several attractors coexist in a system's phase space.[16]

The standard approach to the problem (at the undergraduate level) is based on direct simulation of the corresponding equations of motion[17] and requires a very long time to overcome transient effects and to produce sufficient self-averaging over phase space. The goal of this paper is to show that these problems can be tackled by using the time-dependent Fokker-Planck equation.[18–20] Our approach allows us to reduce the stochastic problem to a system of ordinary linear algebraic equations, which can be solved by standard numerical methods. The solution of these equations provides full information on the transport properties of the system.

The outline of this article is as follows. In Sec. II we introduce the model and set up the problem within the Fokker-Planck framework. Then we formulate the symmetries which need to be violated to obtain a nonzero DC current. In Sec. III we discuss the method of solution of AC-driven Fokker-Planck equations. In Sec. IV we illustrate our approach with the tilting ratchets.[4] Section V contains some conclusions and perspectives.

II. THE MODEL

The one-dimensional classical dynamics of a particle (for example, a cold atom[13] in an optical lattice or a colloidal microsphere in a magnetic bubble lattice[3]) of mass m and friction coefficient γ, exposed to an AC-driven periodic potential $U(x,t)$ and coupled to a heat bath, is described by the equation:

$$m\ddot{x} + \gamma\dot{x} = g(x,t) + \xi(t), \tag{1}$$

where the force $g(x,t) = -\partial U(x,t)/\partial x$ is time and space periodic,

$$g(x+L,t) = g(x,t+T) = g(x,t). \tag{2}$$

The noise is modeled by a δ-correlated Gaussian white noise, $\langle \xi(t) \rangle = 0$, $\langle \xi(t)\xi(t') \rangle = 2\gamma D \delta(t-t')$, with noise intensity $D = kT$.

The state of the system can be represented as a point in three-dimensional phase space, (x,v,t), for $m>0$ (underdamped regime), and in two-dimensional phase space, (x,t), when $m=0$ (overdamped limit).[14] The stationary asymptotic current is equal to

$$J = \lim_{t\to\infty} J(t) = \lim_{t\to\infty} \frac{x(t)}{t}. \tag{3}$$

In the deterministic limit, $D=0$, Eqs. (1) and (2) can be studied numerically by using the fourth-order Runge-Kutta method.[15]

The simplest strategy to study the system with noise is to perform the numerical integration of the stochastic differential equation (1) by using, for example, the Euler-Maruyama method[17]

$$x_{n+1} = x_n + v_n h, \tag{4}$$

$$v_{n+1} = v_n + \frac{1}{m} g(x_n, nh)h + \sqrt{\frac{2\gamma D}{m}} h\tilde{\xi}(nh), \tag{5}$$

where $v=\dot{x}$, h is the time step, $t=nh$, and $\tilde{\xi}$ is a δ-correlated Gaussian white noise, $\langle \xi(t)\rangle=0$, $\langle \xi(t)\xi(t')\rangle=\delta(t-t')$. The simplest approach is not always optimal: it requires a huge number of individual stochastic trajectories in order to obtain reliable average of the observable with acceptable accuracy. The stochastic equations (4) and (5) which govern the stochastic variables x and v can be "absorbed" into the Fokker-Planck equation. This partial differential equation describes the time evolution of the probability distribution in the phase space,[18–20] rather than a single stochastic trajectory. It reads

$$\left\{ \partial_t + \frac{\partial}{\partial x} v - \frac{1}{m}\frac{\partial}{\partial v}[\gamma v - g(x,t)] - \frac{\gamma D}{m^2}\frac{\partial^2}{\partial v^2} \right\} P(x,v,t) = 0. \tag{6}$$

The Fokker-Planck equation for the overdamped limit, when inertia is negligible, $m=0$, is[18–20]

$$\gamma \dot{P}(x,t) = -\left[\frac{\partial}{\partial x} g(x,t) - D\frac{\partial^2}{\partial x^2} \right] P(x,t). \tag{7}$$

The overdamped limit is the appropriate description for microdynamics at low Reynolds number when a particle moves in an extremely viscous media.[21]

Equations (6) and (7) are linear, dissipative, and preserve the norm, $\int P(x,v,t)\, dx\, dv$ for Eq. (6) and $\int P(x,t)\, dx$ for Eq. (7).[20] In addition, these equations possess discrete time and space translation symmetries, so that the operations $x \to x + L$ and $t \to t+T$ leave the equations invariant. For a given boundary condition and a fixed norm, any initial distribution, $P(\ldots,0)$, will converge to a single time-periodic attractor solution, $\tilde{P}(\ldots,t)=\tilde{P}(\ldots,t+T)$. What are the proper spatial boundary conditions for the ratchet problem given by Eqs. (1)–(3)? If we want to calculate the asymptotic current Eq. (3), we should use periodic boundary conditions, $\tilde{P}(x,\ldots)=\tilde{P}(x+L,\ldots)$.[4]

The DC-components of the directed current Eq. (3) in terms of the spatially periodic attractor solution $\tilde{P}$ are given by[4]

$$J = \left\langle \int_{-\infty}^{\infty} v\hat{P}(x,v,t)dv \right\rangle_{T,L} \quad (m>0), \tag{8}$$

$$J = \gamma^{-1}\langle g(x,t)\hat{P}(x,t)\rangle_{T,L} \quad (m=0), \tag{9}$$

where $\langle\cdots\rangle_T = \int_0^T \cdots\, dt$ and $\langle\cdots\rangle_L = \int_0^L \cdots\, dx$. Without loss of generality we set $L=2\pi$ and $m=1$ (for the underdamped limit).

Let us assume that Eq. (6) [or Eq. (7)] is invariant under some transformation of the variables x and t, which does not affect the boundary conditions. Then the unique solution $\tilde{P}$ is also invariant under the transformation. The strategy is to identify symmetry operations that invert the sign of the current J in Eq. (8) [or Eq. (9)] and, at the same time, leave the corresponding Fokker-Planck equation invariant. If such a transformation exists, the DC-current J will vanish.[22] Sign changes of the current can be obtained by either inverting the spatial coordinate x or time t (plus simultaneously inverting the velocity, $v \to -v$, for the underdamped case).

All such transformations together with the requirements for the force $g(x,t)$ are[9,22]

$$\hat{S}_1 : x \to -x + x', t \to t + t', \quad \hat{S}_1(g) \to -g, \tag{10}$$

$$\hat{S}_2 : x \to x + x', t \to -t + t' \quad \hat{S}_2(g) \to g \quad (\text{if } \gamma = 0), \tag{11}$$

where x' and t' depend on the particular shape of $g(x,t)$.

There is an additional symmetry for the overdamped limit,[23]

$$\hat{S}_3 : x \to x + x', t \to -t + t', \quad \hat{S}_3(g) \to -g \quad (m=0), \tag{12}$$

which does not follow from Eq. (7). This symmetry is not a symmetry of Eq. (7) due to the last term on the right-hand side of Eq. (7), which is called the diffusive term.[18] It has been shown, nevertheless, that the symmetry $\hat{S}_3$ corresponds to a certain property of the asymptotic attractor solution $\tilde{P}(x,t)$ such that the current Eq. (9) vanishes when $\hat{S}_3$ holds.[22]

By a proper choice of $g(x,t)$ all relevant symmetries can be broken, and we can expect the appearance of a nonzero DC current J [Eqs. (8) and (9)].

III. METHOD OF SOLUTION

It is impossible to solve the Fokker-Planck equation in Eqs. (6) and (7) analytically. We will treat them by using a Fourier expansion. For the overdamped case (7) we have

$$P(x,t) = \frac{1}{\sqrt{2\pi T}} \sum_{n,k=-N,-K}^{N,K} P_{nk} e^{i(nx+k\omega t)}. \tag{13}$$

For the underdamped regime Eq. (6) we use the matrix continued fraction technique[18]

$$P(x,v,t) = \frac{\psi_0(v)}{\sqrt{2\pi T}} \sum_{n,k=-N,-K}^{N,K} \sum_{s=0}^{S} P_{nks} e^{i(nx+k\omega t)} \psi_s(v), \tag{14}$$

where $\psi_s(v)$ is the Hermite function of order s.[24] The expansions in Eqs. (13) and (14) are both truncated, so we can control the convergence and the precision of the solution $\tilde{P}$ by varying the parameters N, K, and S.

By substituting the expansions in Eqs. (13) and (14) into the corresponding Eqs. (7) and (6), respectively, we obtain the following systems of linear algebraic equations,

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$$(ik\omega + Dn^2)P_{lk} + in\sum_{l,q} g_{lq}P_{(n-l)(k-q)} = 0, \tag{15}$$

for the overdamped limit, and

$$(ik\omega + s\gamma D)P_{nks} + i\sqrt{D}n(\sqrt{s}P_{nk(s-1)} + \sqrt{s+1}P_{nk(s+1)}) - \sqrt{\frac{s}{D}}\sum_{l,q} g_{lq}P_{(n-l)(k-q)(s-1)} = 0, \tag{16}$$

for the underdamped regime. Here g_{nk} is the Fourier coefficient of the force function, $g(x,t)=\Sigma_{n,k}g_{nk}e^{i(nx+k\omega t)}$. Note that the equation for the DC-element, P_{00} in Eq. (7) [P_{000} in Eq. (6)], should be replaced by $P_{00}=1$ ($P_{000}=1$). These equations correspond to the normalization condition, $\int_0^{2\pi}P(x,t)\,dx=1$ ($\int_0^{2\pi}\int_{-\infty}^{\infty}P(x,v,t)\,dx\,dv=1$).

The expression for the current takes the form

$$J = \sqrt{\frac{D}{LT}}P_{001}, \tag{17}$$

for the underdamped case in Eq. (6), and

$$J = \frac{1}{\sqrt{LT}}\sum_{n,k} g_{nk}P_{-n-k}, \tag{18}$$

for the overdamped limit in Eq. (7).

To use standard numerical methods for the solution of Eqs. (15) and (16) we have to transform the original variables P_{nk} (overdamped limit) and P_{nks} (underdamped regime) to the single-index variable $\mathcal{P}_z$, $z=1,\ldots,Z$. There is the one-to-one index transformation,

$$\{n,k\} \to z = 1 + (n+N)(2K+1) + k + K, \tag{19}$$

$$Z = (2N+1)(2K+1), \tag{20}$$

for the overdamped limit, and

$$\{n,k,s\} \to z = 1 + s(2N+1)(2K+1) + (n+N)(2K+1) + k + K, \tag{21}$$

$$Z = (2N+1)(2K+1)(S+1), \tag{22}$$

for the underdamped regime, which transforms the original two- and three-dimensional matrices, P_{nk} and P_{nks}, to the vector-column $\mathcal{P}_z$. The corresponding system of equations for $\mathcal{P}_z$ can be solved by using standard numerical routines.[25]

IV. EXAMPLE: TILTING RATCHETS

There are many choices of the driving potential $U(x,t)$.[4] In the following we consider a particle placed in a stationary periodic potential and driven by an alternating tilting force,[9,11,12,22]

$$g(x,t) = -V'(x) + E(t). \tag{23}$$

For the simple potential $V(x)=V_0(1-\cos x)$, the two-frequency driving force,

$$E(t) = E_1\sin(\omega t) + E_2\sin(2\omega t + \theta), \tag{24}$$

ensures that for E_1, $E_2\neq 0$ the symmetry $\hat{S}_1$ is always violated. The symmetry $\hat{S}_2$ is broken for $\theta\neq 0, \pm\pi$. In addition, $\hat{S}_3$ does not hold at $\theta\neq \pm\pi/2$.[22]

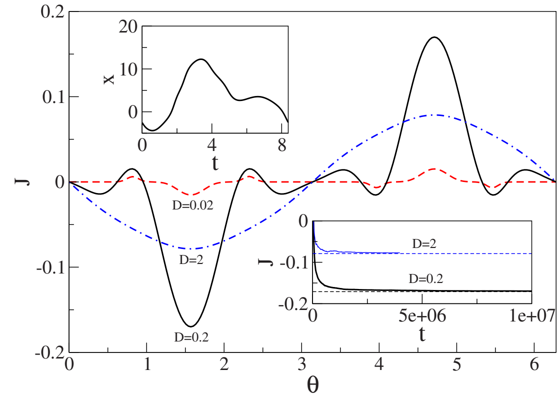

Fig. 1. (Color online) The dependence of the current J in Eq. (18) on θ for the system given by Eq. (15) with $N=K=60$ at different temperatures, $D=kT$. The parameters are $\gamma=1$, $V_0=2$, $E_1=4.6$, $E_2=-6$, and $\omega=0.75$. Insets: (top left) the attractor of the corresponding deterministic system from Eq. (1) for $\theta=-\pi/2$; (bottom right) the dependence of the running current $J(t)=x(t)/t$ on t at $\theta=-\pi/2$ for $D=0.2$ and $D=2$. Dashed lines correspond to the current values J from Eq. (18).

We start from the overdamped limit, $m=0$ and $\gamma=1$. For the parameters $V_0=2$, $E_1=4.6$, $E_2=-6$, and $\omega=0.75$ there is only one limit cycle in the phase space of the system (see inset in Fig. 1). The limit cycle is nontransporting for the entire range $-\pi\leq\theta\leq\pi$: the particle returns to its initial position after one period T of the external driving force. There is no DC transport in the deterministic limit $D=0$ for this set of parameters.

Noise changes the dynamics, and the particle can leave the attractor, which leads to a finite current (see Fig. 1). Thermal fluctuations play a constructive role by providing a way for the manifestation of the asymmetry hidden in the AC-driving. Because the current depends non-monotonically on the noise intensity D, there is a kind of stochastic resonance effect.[26,27] For weak noise the dynamics is still localized near the attractor [Fig. 2(a)], so the current is low (Fig. 1, dashed line). In the high-temperature limit where the noise starts to dominate the dynamics, the particle dynamics is "smeared" over the phase space [Fig. 2(b)]. The relevant space-temporal correlations are suppressed by the noise and the current tends to decrease (Fig. 1, dashed-dotted line). There is a resonance temperature near $D\approx 0.27$ (see Fig. 3), where the DC-current can be resonantly enhanced. In this case the directed flow of particles is analogous to the flow of information for stochastic resonance.[26,27]

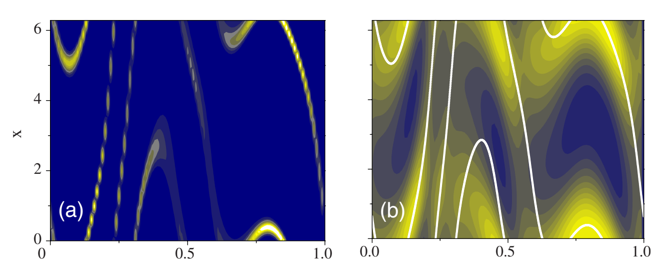

Fig. 2. (Color online) The attractor solution $\widetilde{P}(x,t)$ of the Fokker-Planck equation in Eq. (7) for (a) $D=0.02$ and (b) $D=2$. The white line in (b) marks the corresponding deterministic attractor. Here $\theta=-\pi/2$; the other parameters are the same as in Fig. 1.

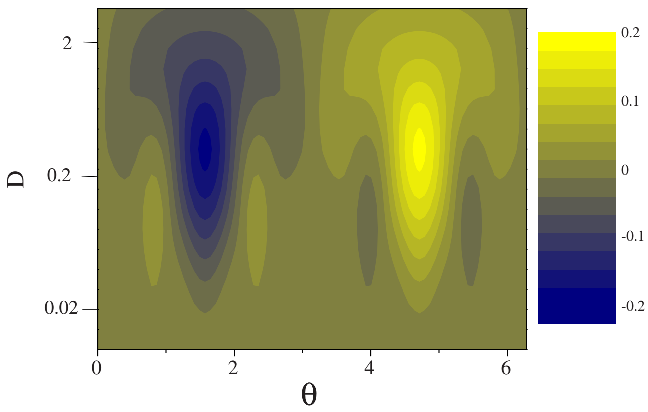

Fig. 3. (Color online) The current J from Eq. (18) as a function of θ and D.

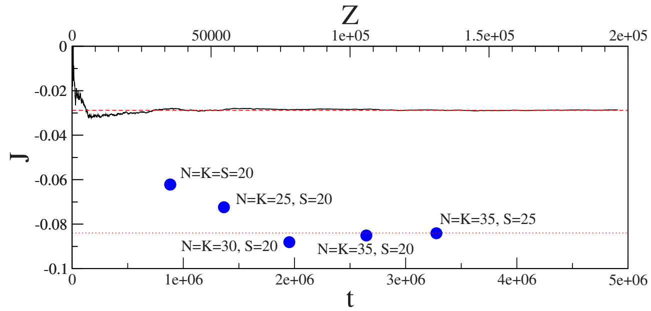

Fig. 5. (Color online) The current $J(t)=x(t)/t$ versus t for the deterministic system in Eq. (1) and the dependence of the current J from Eq. (17) on the number of basis states Z. The other parameters are the same as in Fig. 4.

The noise variation cannot violate the basic symmetries: current is always absent at $\theta=0$ and $\theta=\pm\pi$. We have corroborated our analysis by a simulation of Eqs. (4) and (5) and found perfect agreement with the numerical solutions of the corresponding Fokker-Planck equation (inset on Fig. 1).

The overdamped limit is not suitable for the description of all realistic situations. For example, for the modeling of Josephson ratchets it is necessary to take into account the inertial term, $m\ddot{x}$.[28] The underdamped regime without periodic driving has been much studied. To our knowledge there is only one paper where the Fokker-Planck equation for an underdamped system with AC-driving has been solved numerically.[29]

The results for $m=1$, $V_0=1$, $E_1=E_2=1.2$, $\omega=1$, $\gamma=0.1$, and $\theta=0$ are shown in Fig. 4. There is a single chaotic attractor for the deterministic limit, $D=0$. In Fig. 4(a) the Poincaré section[14] of the attractor (the position of the particle in the phase space, $[x(t),v(t)]$ at times $t_n=nT$) is depicted. Because the relevant symmetry, S_1, is violated by the presence of the second harmonic E_2, the attractor generates a finite DC-current, $J\approx-0.0288$ (see Fig. 5).

Although the attractor solution of the corresponding Fokker-Planck equation for $D=0.1$ clearly resembles its deterministic counterpart (Fig. 4), it produces a much stronger current, $J\approx-0.083$, than in the deterministic limit. Thermal fluctuations again play a constructive role in the process of current rectification.

The convergence of the quantity J in Eq. (17) with the total number of states Z is slow (Fig. 5). For weak noise the solution $P(x,v,t)$ is concentrated around the deterministic attractor. In this case the distribution exhibits a fine structure which becomes richer with decreasing noise. We have to take into account 71 spatial and 71 temporal harmonics to estimate accurately the asymptotic current for $D=0.1$. This slow convergence implies that the fast temporal modes (chaotic dynamics on a short time scale, $t_c \ll T$) and short spatial modes $(x_c \ll L)$ contribute substantially to the overall ratchet effect.

In the strong noise limit, $D\sim 1$, we expect smeared-out smooth distributions with a minimum structure and also fast convergence. The bad news is that the ratchet current will be weak in this limit: noise will prevail over the system's dynamics.

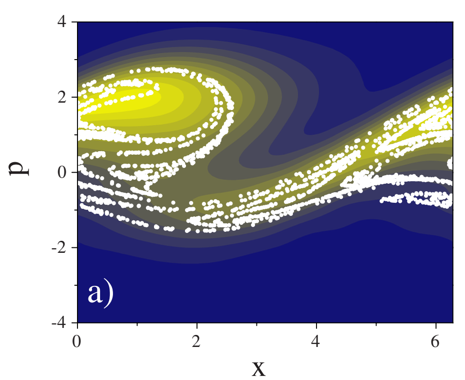

Fig. 4. (Color online) The Poincaré representation of the attractor solution $\tilde{P}(x,v,0)$ [Eq. (14)] of Eq. (16) for $D=0.1$, $N=K=35$, and $S=25$, with the Poincaré section of the corresponding deterministic attractor, $D=0$, from Eq. (1) (white dots). The parameters are $m=1$, $V_0=1$, $E_1=E_2=1.2$, $\omega=1$, $\gamma=0.1$, and $\theta=0$.

V. CONCLUSIONS

We have shown that AC-driven ratchets in a thermal environment can be understood by using the Fokker-Planck equation. Our approach maps the original problem onto a set of ordinary algebraic linear equations which can be solved by standard numerical routines.

The approach opens new possibilities for further developments. For example, it allows a generalization to the case of colored noise with an exponential correlation function.[18] The corresponding one-dimensional "nonthermal" process can be embedded into a two-dimensional "thermal" dynamics by introducing the additional dynamical variable $\varepsilon(t)$, which replaces the noise term, $\xi(t)$, in Eq. (1).[30] Another possibility is the study of the quantum limit of underdamped AC-driving ratchets. The master equation for the corresponding Wigner function can be represented as a Fokker-Planck equation [Eq. (16)] with an additional coupling between elements.[31] The absolute negative mobility in AC-driven inertial systems[32,33] is another possible target for our method.

The driven particle in a periodic potential is a corner stone problem of nonlinear science, and is frequently used as an introductory and illustrative example of chaotic dynamics. Therefore, our work provides another answer to the question: "What convincing applications of the Fokker-Planck equation (beside simple Brownian motion) exist that fit into a undergraduate's system of concepts?"[34]

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