Local asymptotics for the area of random walk excursions

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Abstract

We prove a local limit theorem for the area of the positive excursion of random walks with zero mean and finite variance. Our main result complements previous work of Caravenna and Chaumont; Sohier, as well as Kim and Pittel.

1. *Introduction and statement of results*

Let $\{S_n\}$ be an integer-valued centred random walk with finite second moments, and let τ denote the first time when the random walk is negative, that is, $\tau := \min\{n \geqslant 1 : S_n \leqslant 0\}$. The path $\{S_1, S_2, \dots, S_{\tau-1}\}$ we shall call the positive excursion of $\{S_n\}$. It follows easily from recent results of Caravenna and Chaumont [4] and Sohier [18] that the rescaled excursion of the random walk conditioned on $\tau = n + 1$ converges weakly to the standard Brownian excursion which we shall denote by $e(t), t \in [0, 1]$. This implies that an appropriately rescaled area converges towards the corresponding functional of the Brownian excursion. More precisely,

$$\mathbf{P}(n^{-3/2}A_n \leqslant x \,|\, \tau = n+1) \longrightarrow \mathbf{P}\left(\int_0^1 e(t)\,dt \leqslant x\right), \quad x > 0, \tag{1.1}$$

where

$$A_n := \sum_{k=1}^{n} S_k.$$

For simple random walks, this convergence was proved by Takacs [20], who also identified the limiting distribution, the so-called Airy distribution. (We give below an exact expression for its density.) His motivation was partially rooted in combinatorics. More precisely, he was interested in the investigation of the asymptotic number of random trees on n vertices with given total height, see Takacs [20–22] and Spencer [19]. Using the well-known one-to-one correspondence between random trees and random walk excursions, this problem is equivalent to a problem concerning the area under random walk path. It is worth mentioning that areas of random walk excursions appear also in other combinatorial problems such as:

(1) analysis of linear probing hashing, Flajolet, Poblete and Viola [11];
(2) enumeration of paths below a line of rational slope, Banderier and Gittenberger [1];
(3) Winston–Kleitman problem on tournament scores, Winston and Kleitman [26] and Takacs [20].

Assertion (1.1) allows one to find the asymptotic number of random trees on n vertices with the total height bounded by $xn^{3/2}$. But in order to find the number of trees with fixed total height, one needs a local version of (1.1). Moreover, such a result allows one to confirm the Kleitman–Winston conjecture mentioned above, see Takacs [20, p. 565]. This conjecture was proved by Kim and Pittel [14] by deriving a uniform upper bound for probabilities $\mathbf{P}(A_n = a \,|\, \tau = n+1)$ in the case of a simple random walk.

The main purpose of the present paper is to extend the result of Kim and Pittel to a local limit theorem for the excursion area of all random walks with finite variance.

We say that X is (d,ρ)-lattice if its distribution is lattice with span d and shift $\rho \in [0,d)$, that is, d is the maximal number such that $\mathbf{P}(X \in \{\rho + dk,\ k \in \mathbb{Z}\}) = 1$. Furthermore, we define $N_x := \{n \geqslant 1 : \mathbf{P}(S_n = x) > 0\}$, $x \geqslant 0$.

Theorem 1.1. *Assume that* $\mathbf{E}X = 0$, $\mathbf{E}X^2 := \sigma^2 \in (0,\infty)$ *and* X *is* (d,ρ)*-lattice. Then*

$$\sup_{a \in n(n+1)\rho/2 + d\mathbb{Z}} \left| n^{3/2}\mathbf{P}(A_n = a \,|\, \tau = n+1) - \frac{d}{\sigma} w_{\mathrm{ex}}\left(\frac{a}{\sigma n^{3/2}}\right)\right| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty \tag{1.2}$$

and, for every fixed $x \geqslant 0$,

$$\sup_{a \in n(n+1)\rho/2 + d\mathbb{Z}} \left| n^{3/2}\mathbf{P}(A_n = a \,|\, \tau = n+1, S_n = x) - \frac{d}{\sigma} w_{\mathrm{ex}}\left(\frac{a}{\sigma n^{3/2}}\right)\right| \longrightarrow 0 \tag{1.3}$$

as $N_x \ni n \to \infty$. *Here* w_{ex} *denotes the density of* $\int_0^1 e(t)\,dt$.

Takacs [20, Theorem 5] has obtained an exact expression for w_{ex}:

$$w_{\mathrm{ex}}(x) = \frac{2^{13/6}}{3^{3/2}x^{10/3}} \sum_{k=1}^{\infty} a_k^2 \exp\left\{-\frac{2a_k^3}{27x^2}\right\} U\left(-\frac{5}{6}, \frac{4}{3}, \frac{2a_k^3}{27x^2}\right),$$

where $U(a,b,z)$ is a confluent hypergeometric function and $\{-a_k\}$ is a sequence of zeros of the Airy function

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^{\infty} \cos(t^3/3 + tx)\,dt$$

arranged so that $a_k < a_{k+1}$ for all k. (For further properties of the Airy function, we refer to Janson [13, Section 12].)

Using the asymptotics $w_{\mathrm{ex}}(x) \to 0$ as $x \to 0$ or $x \to \infty$ from [13, Section 15], we conclude $\sup_{x \geqslant 0} w_{\mathrm{ex}}(x) < \infty$. From this fact and (1.2), we infer that

$$\sup_{a \geqslant 1} \mathbf{P}(A_n = a \,|\, \tau = n+1) \leqslant \frac{C}{n^{3/2}}, \quad n \geqslant 1,$$

reproducing the main result of Kim and Pittel [14].

Example 1. To demonstrate the relevance of our theorem in a combinatorial context, we apply it to the following problem of enumeration of Dyck paths below a line of rational slope. Following Banderier and Gittenberger [1], we look at walks on $\mathbb{N}^2$ with steps $(1,0)$ and $(0,1)$ constrained to stay below a line $y = (\alpha/\beta)x$ with $\alpha,\beta \in \mathbb{N}$. We are interested in the asymptotic number of such walks of length n which start at $(0,0)$ and end on the line, and have a fixed area between the line and the path. According to [1, Theorem 8], this number is equal, up to the factor $\alpha+\beta$, to the number of random walk excursions of length n with the endpoint 0 and the same area. The set of jumps of this random walk is $\{\alpha, -\beta\}$. Let $N(n,a)$ denote the number of excursions with area a. Then

$$N(n,a) = 2^n\mathbf{P}(A_n = a, \tau = n, S_n = 0),$$

where S_n is a random walk with $\mathbf{P}(X = \alpha) = \mathbf{P}(X = -\beta) = \frac{1}{2}$. This walk is obviously $(\alpha, \alpha+\beta)$-lattice. Since $\mathbf{E}X = \alpha - \beta$ is not necessarily zero, we cannot apply Theorem 1.1 directly. To obtain a driftless random walk, we perform an exponential change of measure. Set

$h_0 = (\alpha+\beta)^{-1}\log(\beta/\alpha)$ and define a new measure $\widehat{\mathbf{P}}$ by the equality

$$\widehat{\mathbf{P}}(X=x) = \frac{e^{h_0 x}}{\varphi(h_0)}\mathbf{P}(X=x), \quad x\in\{\alpha,-\beta\},$$

where $\varphi(h) = \mathbf{E}\, e^{hX} = (e^{\alpha h} - e^{-\beta h})/2$. Then

$$\mathbf{P}(A_n = a, \tau = n, S_n = 0) = (\varphi(h_0))^n \widehat{\mathbf{P}}(A_n = a, \tau = n, S_n = 0).$$

Combining now (1.3) with [**25**, Theorem 6], we obtain

$$\widehat{\mathbf{P}}(A_n = a, \tau = n, S_n = 0) = C(\alpha,\beta) w_{\mathrm{ex}}\left(\frac{a}{\sigma n^{3/2}}\right) n^{-3} + o(n^{-3}),$$

where $\sigma^2 = \sigma^2(\alpha,\beta) = \widehat{\mathbf{E}} X^2$. (We cannot give an analytical expression for the constant $C(\alpha,\beta)$, due to the fact that we do not know exact the form of the renewal function of ascending ladder epochs.) As a result, we have

$$N(n,a) \approx C(\alpha,\beta) w_{\mathrm{ex}}\left(\frac{a}{\sigma n^{3/2}}\right) n^{-3} \left(\left(\frac{\beta}{\alpha}\right)^{\alpha/(\alpha+\beta)} + \left(\frac{\alpha}{\beta}\right)^{\beta/(\alpha+\beta)}\right)^n.$$

The proof of (1.2) and (1.3) is based on a representation of the positive excursion of $\{S_n\}$ as a concatenation of two meanders of random walks. Then the area of the excursion is the sum of areas of two independent meanders with equal endpoints, see formula (3.8) below. A similar approach has been used in [**6**] to derive local asymptotics for random walks in cones.

To derive Theorem 1.1 from this representation, we need to prove the following local limit theorem for the joint distribution of a discrete meander and its area.

THEOREM 1.2. *Assume that the conditions of Theorem 1.1 are satisfied and let $(M_t)_{t\geqslant 0}$ denote the Brownian meander. Then, for every $z \geqslant 0$,*

$$\sup_{\substack{a\in n(n+1)\rho/2+d\mathbb{Z},\\ x\in n\rho+d\mathbb{Z}}} \left| n^2 \mathbf{P}_z(A_n = a, S_n = x \,|\, \tau > n) - \frac{d^2}{\sigma^2} h\left(\frac{a}{\sigma n^{3/2}}, \frac{x}{\sigma n^{1/2}}\right)\right| \longrightarrow 0,$$

where $h(u,v)$ is the density function of the vector $(\int_0^1 M_t\,dt, M_1)$ and P_z is the distribution of the walk starting at z.

This theorem is deduced by combining weak convergence towards the Brownian meander with a local limit result for the unconditioned pair (A_n, S_n). Technically, the hardest part of the proof of this theorem consists in showing that the distribution of the vector $(\int_0^1 M_t\,dt, M_1)$ has a continuous density $h(u,v)$. This is done in the next section: we give two independent proofs of this fact. It is worth mentioning that our approach gives, as a byproduct, an integral representation of $w_{\mathrm{ex}}(x)$ in terms of $h(u,v)$.

We conclude the introduction by stating the following simple consequence of Theorem 1.2.

COROLLARY 1.3. *As $n\to\infty$,*

$$\sup_{a\in n(n+1)\rho/2+d\mathbb{Z}} \left| n^2 \mathbf{P}(A_n = a \,|\, \tau > n) - \frac{d}{\sigma} w_{me}\left(\frac{a}{\sigma n^{3/2}}\right)\right| \longrightarrow 0,$$

where w_{me} is the density of $\int_0^1 M_t\,dt$.

This result is a local counterpart of Theorem 4 in Takacs [**23**].

2. *Brownian meander and its area*

Set $I_t = \int_0^t B_s\,ds$. Let

$$p_t(x,y;u,v) = \frac{3}{\sqrt{\pi}t^2}\exp\left\{-\frac{6(u-x-ty)^2}{t^3} + \frac{6(u-x-ty)(v-y)}{t^2} - \frac{2(v-y)^2}{t}\right\}$$

be the transition function of the process $(I_t, B_t)_{t\geqslant 0}$.

We next recall the definition and some basic properties of the Brownian meander $\{M_t, t \in [0,1]\}$. Let $T = \sup\{t \in [0,1] : B_t = 0\}$ and $\Delta = 1 - T$. Then one sets

$$M_t = \Delta^{-1/2} B_{T+t\Delta}, \quad t \in [0,1].$$

This is a time inhomogeneous Markov process with continuous paths. For our purposes, it is important to know that the meander can be seen as a weak limit of conditioned Brownian motion. More precisely, Durrett, Iglehart and Miller [9] have shown that

$$\mathcal{L}\left\{B_t, t \in [0,1] \,\middle|\, \min_{\leqslant 1} B_t > -\varepsilon\right\} \Rightarrow \mathcal{L}\{M_t, t \in [0,1]\} \quad \text{as } \varepsilon \longrightarrow 0$$

on the space of continuous functions endowed with the topology of the uniform convergence. In particular, the conditional distribution of the vector (I_1, B_1) conditioned on $\{\min_{t\leqslant 1} B_t > -\varepsilon\}$ converges weakly towards the distribution of the vector $(\int_0^1 M_t\,dt, M_1)$. Using this convergence, we prove the following proposition, which is needed for the derivation of our main result.

PROPOSITION 2.1. *The joint distribution of $\int_0^1 M_t\,dt$ and M_1 is absolutely continuous with a continuous density $h(u,v)$. Furthermore, there exists a measure ν such that*

$$\begin{aligned} h(u,v) &= \sqrt{\frac{\pi}{2}}\left(6u - 2v + \sqrt{\frac{2}{\pi}}\right) p_1(0,0;u,v) \\ &\quad + \sqrt{\frac{\pi}{2}} \int_0^1 \int_0^\infty \nu(ds,dz)[p_1(0,0;u,v) - p_{1-s}(0,0;u-z,v)]. \end{aligned} \tag{2.1}$$

In the proof of Proposition 2.1, we first derive the representation given by (2.1). The continuity of h is then immediate from the uniform continuity of the integrand in this formula. It is worth mentioning that (2.1) can be used for checking further smoothness properties of the density function h. To this end, one needs appropriate dominated convergence bounds for the integrand $p_1(0,0;u,v) - p_{1-s}(0,0,u-z,v)$.

Proof of Proposition 2.1. Define

$$\overline{p}_t(x,y;u,v) = \frac{\mathbf{P}_{(x,y)}(I_t \in du,\ B_t \in dv,\ \tau > t)}{du\,dv},$$

where $\tau := \inf\{t > 0 : B_t = 0\}$.

Using the strong Markov property, it can be easily seen that

$$\begin{aligned} \overline{p}_1(0,\varepsilon;u,v) &= p_1(0,\varepsilon;u,v) - \int_0^1 \int_0^\infty \mathbf{P}_{(0,\varepsilon)}(\tau \in ds, I_s \in dz) p_{1-s}(z,0,u,v) \\ &= p_1(0,\varepsilon;u,v) - p_1(0,0;u,v)\mathbf{P}_{(0,\varepsilon)}(\tau \leqslant 1) \\ &\quad + \int_0^1 \int_0^\infty \mathbf{P}_{(0,\varepsilon)}(\tau \in ds, I_s \in dz)[p_1(0,0;u,v) - p_{1-s}(z,0,u,v)]. \end{aligned} \tag{2.2}$$

Since

$$\mathbf{P}_{(0,\varepsilon)}(\tau > 1) = 2\Phi(\varepsilon) - 1 \sim \sqrt{\frac{2}{\pi}}\varepsilon, \quad \varepsilon \longrightarrow 0, \tag{2.3}$$

we have

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}p_1(0,0;u,v)\mathbf{P}_{(0,\varepsilon)}(\tau>1)=\sqrt{\frac{2}{\pi}}p_1(0,0;u,v).$$

Furthermore, by Taylor's formula,

$$\begin{aligned}&p_1(0,\varepsilon;u,v)-p_1(0,0;u,v)\\&\quad=\frac{3}{\sqrt{\pi}}(\exp\{-6(u-\varepsilon)^2+6(u-\varepsilon)(v-\varepsilon)-2(v-\varepsilon)^2\}-\exp\{-6u^2+6uv-2v^2\})\\&\quad=\frac{3}{\sqrt{\pi}}\exp\{-6u^2+6uv-2v^2\}(12u-6u-6v+4v)\varepsilon+O(\varepsilon^2),\end{aligned}$$

which implies that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(p_1(0,\varepsilon;u,v)-p_1(0,0;u,v))=p_1(0,0;u,v)(6u-2v).$$

As a result,

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(p_1(0,\varepsilon;u,v)-p_1(0,0;u,v)\mathbf{P}_{(0,\varepsilon)}(\tau\leqslant 1))=p_1(0,0;u,v)\left(6u-2v+\sqrt{\frac{2}{\pi}}\right).\tag{2.4}$$

In order to deal with the integral term in (2.2), we write

$$\begin{aligned}&\int_0^1\int_0^\infty \mathbf{P}_{(0,\varepsilon)}(\tau\in ds,I_s\in dz)[p_1(0,0;u,v)-p_{1-s}(z,0;u,v)]\\&\quad=\int_0^1\int_0^\infty \mathbf{P}_{(0,\varepsilon)}(\tau\in ds,I_s\in dz)[p_1(0,0;u,v)-p_{1-s}(0,0;u,v)]\\&\qquad+\int_0^1\int_0^\infty \mathbf{P}_{(0,\varepsilon)}(\tau\in ds,I_s\in dz)[p_{1-s}(0,0;u,v)-p_{1-s}(z,0;u,v)]\\&\quad=\int_0^{1/\varepsilon^2}\mathbf{P}_{(0,1)}(\tau\in ds)[p_1(0,0;u,v)-p_{1-\varepsilon^2 s}(0,0;u,v)]\\&\qquad+\int_0^{1/\varepsilon^2}\int_0^\infty \mathbf{P}_{(0,1)}(\tau\in ds,I_s\in dz)[p_{1-\varepsilon^2 s}(0,0;u,v)-p_{1-\varepsilon^2 s}(0,0;u-\varepsilon^3 z,v)],\end{aligned}$$

where the last equality follows from the Brownian scaling. Fix some $r\in(0,1/2)$ and write

$$\begin{aligned}&\int_0^{1/\varepsilon^2}\mathbf{P}_{(0,1)}(\tau\in ds)[p_1(0,0;u,v)-p_{1-\varepsilon^2 s}(0,0;u,v)]\\&\quad=\int_0^{r/\varepsilon^2}\mathbf{P}_{(0,1)}(\tau\in ds)[p_1(0,0;u,v)-p_{1-\varepsilon^2 s}(0,0;u,v)]\\&\qquad+\int_{r/\varepsilon^2}^{1/\varepsilon^2}\mathbf{P}_{(0,1)}(\tau\in ds)[p_1(0,0;u,v)-p_{1-\varepsilon^2 s}(0,0;u,v)].\end{aligned}$$

It is easily seen that

$$\begin{aligned}\frac{\partial}{\partial t}p_t(0,0;u,v)=&-\frac{6}{\sqrt{\pi}t^3}\exp\left\{-\frac{6u^2}{t^3}+\frac{6uv}{t^2}-\frac{2v^2}{t}\right\}\\&+\frac{2}{\sqrt{\pi}t^2}\exp\left\{-\frac{6u^2}{t^3}+\frac{6uv}{t^2}-\frac{2v^2}{t}\right\}\left(\frac{18u^2}{t^4}-\frac{12uv}{t^3}+\frac{2v^2}{t^2}\right).\end{aligned}$$

Noting that this derivative is globally bounded, we infer from Taylor's formula that

$$\sup_{s\leqslant 1/2}s^{-1}|p_1(0,0;u,v)-p_{1-s}(0,0;u,v)|\leqslant C.\tag{2.5}$$

Combining this with the equality

$$\frac{\mathbf{P}_{(0,1)}(\tau \in ds)}{ds} = \frac{1}{\sqrt{2\pi}} s^{-3/2} e^{-1/2s}, \tag{2.6}$$

we obtain

$$\left| \int_0^{r/\varepsilon^2} \mathbf{P}_{(0,1)}(\tau \in ds)[p_1(0,0;u,v) - p_{1-\varepsilon^2 s}(0,0;u,v)] \right| \leqslant C \int_0^{r/\varepsilon^2} s^{-3/2}\varepsilon^2 s\, ds = 2C\sqrt{r}\varepsilon. \tag{2.7}$$

Noting that (2.6) implies

$$\frac{\mathbf{P}_{(0,1)}(\tau \in ds)}{ds} \sim (2\pi)^{-1/2} s^{-3/2} \quad \text{as } s \longrightarrow \infty,$$

we obtain, as $\varepsilon \to 0$,

$$\begin{aligned} &\int_{r/\varepsilon^2}^{1/\varepsilon^2} \mathbf{P}_{(0,1)}(\tau \in ds)[p_1(0,0;u,v) - p_{1-\varepsilon^2 s}(0,0;u,v)] \\ &\quad = \frac{1+o(1)}{\sqrt{2\pi}} \int_{r/\varepsilon^2}^{1/\varepsilon^2} s^{-3/2}[p_1(0,0;u,v) - p_{1-\varepsilon^2 s}(0,0;u,v)]\, ds \\ &\quad = \frac{1+o(1)}{\sqrt{2\pi}} \varepsilon \int_r^1 s^{-3/2}[p_1(0,0;u,v) - p_{1-s}(0,0;u,v)]\, ds. \end{aligned}$$

Using (2.5) once again, we conclude that

$$\begin{aligned} &\lim_{r\to 0} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{r/\varepsilon^2}^{1/\varepsilon^2} \mathbf{P}_{(0,1)}(\tau \in ds)[p_1(0,0;u,v) - p_{1-\varepsilon^2 s}(0,0;u,v)] \\ &\quad = \frac{1}{\sqrt{2\pi}} \int_0^1 s^{-3/2}[p_1(0,0;u,v) - p_{1-s}(0,0;u,v)]\, ds. \end{aligned}$$

Combining this relation with (2.7), we finally obtain

$$\begin{aligned} &\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^{1/\varepsilon^2} \mathbf{P}_{(0,1)}(\tau \in ds)[p_1(0,0;u,v) - p_{1-\varepsilon^2 s}(0,0;u,v)] \\ &\quad = \frac{1}{\sqrt{2\pi}} \int_0^1 s^{-3/2}[p_1(0,0;u,v) - p_{1-s}(0,0;u,v)]\, ds. \end{aligned} \tag{2.8}$$

We now turn to the integral

$$\int_0^{1/\varepsilon^2} \int_0^\infty \mathbf{P}_{(0,1)}(\tau \in ds, I_s \in dz)[p_{1-\varepsilon^2 s}(0,0;u,v) - p_{1-\varepsilon^2 s}(0,0;u-\varepsilon^3 z,v)].$$

Since the derivative

$$\frac{\partial}{\partial u} p_t(0,0;u,v) = \frac{3}{\sqrt{\pi} t^2} \left\{ -\frac{6u^2}{t^3} + \frac{6uv}{t^2} - \frac{2v^2}{t} \right\} \left(\frac{6v}{t^2} - \frac{12u}{t^3} \right)$$

is uniformly bounded in t,

$$|p_{1-\varepsilon^2 s}(0,0;u,v) - p_{1-\varepsilon^2 s}(0,0;u-\varepsilon^3 z,v)| \leqslant C(u,v)\varepsilon^3 z.$$

Therefore,

$$\begin{aligned} &\left| \int_0^{1/\varepsilon^2} \int_0^{r/\varepsilon^3} \mathbf{P}_{(0,1)}(\tau \in ds, I_s \in dz)[p_{1-\varepsilon^2 s}(0,0;u,v) - p_{1-\varepsilon^2 s}(0,0;u-\varepsilon^3 z,v)] \right| \\ &\quad \leqslant C\varepsilon^3 \int_0^{r/\varepsilon^3} z\mathbf{P}_{(0,1)}(I_\tau \in dz). \end{aligned}$$

According to formula (2.10) in Isozaki and Watanabe [12]

$$\frac{\mathbf{P}_{(0,1)}(I_\tau \in dz)}{dz} = \frac{2^{1/3}}{3^{2/3}\Gamma(1/3)} z^{-4/3} \exp\{-2/9z\}, \quad z > 0.$$

This implies that

$$\int_0^{r/\varepsilon^3} z\mathbf{P}_{(0,1)}(I_\tau \in dz) \leqslant \frac{2^{1/3}}{3^{2/3}\Gamma(1/3)} \int_0^{r/\varepsilon^3} z^{-1/3}\, dz = \frac{3^{1/3}}{2^{2/3}\Gamma(1/3)} r^{2/3}\varepsilon^{-2}$$

and, consequently,

$$\begin{aligned} &\left| \int_0^{1/\varepsilon^2} \int_0^{r/\varepsilon^3} \mathbf{P}_{(0,1)}(\tau \in ds, I_s \in dz)[p_{1-\varepsilon^2 s}(0,0;u,v) - p_{1-\varepsilon^2 s}(0,0;u-\varepsilon^3 z, v)] \right| \\ &\quad \leqslant Cr^{2/3}\varepsilon. \end{aligned} \tag{2.9}$$

Since p_t is uniformly bounded in all variables,

$$\begin{aligned} &\left| \int_0^{r/\varepsilon^2} \int_{r/\varepsilon^3}^{\infty} \mathbf{P}_{(0,1)}(\tau \in ds, I_s \in dz)[p_{1-\varepsilon^2 s}(0,0;u,v) - p_{1-\varepsilon^2 s}(0,0;u-\varepsilon^3 z, v)] \right| \\ &\quad \leqslant C\mathbf{P}_{(0,1)}(\tau \leqslant r\varepsilon^{-2}, I_\tau \geqslant r\varepsilon^{-3}) \leqslant C\mathbf{P}_{(0,1)}\left(\tau \leqslant r\varepsilon^{-2}, \max_{t\leqslant\tau} B_t \geqslant \varepsilon^{-1}\right), \end{aligned}$$

where in the last step we used the bound $I_\tau \leqslant \tau \max_{t\leqslant\tau} B_t$. Applying now a good-$\lambda$-inequality (see Durrett [8, p.153]) and Doob's inequality, we have

$$\mathbf{P}_{(0,1)}\left(\tau \leqslant r\varepsilon^{-2}, \max_{t\leqslant\tau} B_t \geqslant \varepsilon^{-1}\right) \leqslant 4r\mathbf{P}_{(0,1)}\left(\max_{t\leqslant\tau} B_t \geqslant \varepsilon^{-1}\right) \leqslant 8r\varepsilon.$$

Therefore,

$$\left| \int_0^{r/\varepsilon^2} \int_{r/\varepsilon^3}^{\infty} \mathbf{P}_{(0,1)}(\tau \in ds, I_s \in dz)[p_{1-\varepsilon^2 s}(0,0;u,v) - p_{1-\varepsilon^2 s}(0,0;u-\varepsilon^3 z, v)] \right| \leqslant Cr\varepsilon. \tag{2.10}$$

It remains to consider the integral

$$\int_{r/\varepsilon^2}^{1/\varepsilon^2} \int_{r/\varepsilon^3}^{\infty} \mathbf{P}_{(0,1)}(\tau \in ds, I_s \in dz)[p_{1-\varepsilon^2 s}(0,0;u,v) - p_{1-\varepsilon^2 s}(0,0;u-\varepsilon^3 z, v)].$$

We start with the Laplace transform of the function $\mathbf{P}_{(0,1)}(\tau > t, I_\tau > z)$. It is easy to verify that

$$\begin{aligned} F(\lambda,\mu) &:= \lambda\mu \int_0^\infty \int_0^\infty e^{-\lambda t - \mu z}\mathbf{P}_{(0,1)}(\tau > t, I_\tau > z)\, dt\, dz \\ &= 1 - \mathbf{E}_{(0,1)}[e^{-\lambda\tau}] - \mathbf{E}_{(0,1)}[e^{-\mu I_\tau}] + \mathbf{E}_{(0,1)}[e^{-\lambda\tau-\mu I_\tau}]. \end{aligned}$$

It is well known that

$$\mathbf{E}_{(0,1)}[e^{-\lambda\tau}] = e^{-\sqrt{2\lambda}}.$$

Furthermore, for all positive μ, one has (see [16, Theorem 1])

$$\mathbf{E}_{(0,1)}[e^{-\lambda\tau-\mu I_\tau}] = \frac{\mathrm{Ai}((2\mu)^{1/3} + 2\lambda/((2\mu)^{2/3}))}{\mathrm{Ai}(2\lambda/((2\mu)^{2/3}))}.$$

From these equalities, we conclude that

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} F(\varepsilon^2\lambda, \varepsilon^3\mu) = \sqrt{2\lambda} - \frac{\mathrm{Ai}'(0)}{\mathrm{Ai}(0)}(2\mu)^{1/3} + \frac{\mathrm{Ai}'(2\lambda/((2\mu)^{2/3}))}{\mathrm{Ai}(2\lambda/((2\mu)^{2/3}))}(2\mu)^{1/3}. \tag{2.11}$$

According to Theorem 2.1(i) in Omey and Willekens [17], the latter convergence implies that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbf{P}_{(0,1)}(\tau>t\varepsilon^{-2},I_\tau>z\varepsilon^{-3})=G(t,z), \tag{2.12}$$

where the function G is determined by the right-hand side in (2.11).

By the fundamental theorem of calculus, we have

$$\begin{aligned}&p_{1-\varepsilon^2 s}(0,0;u,v)-p_{1-\varepsilon^2 s}(0,0;u-\varepsilon^3 z,v)\\&\quad=-\int_0^{\varepsilon^3 z}\int_0^{\varepsilon^2 s}\frac{\partial^2}{\partial q\partial w}p_{1-q}(0,0;u-w,v)\,dq\,dw-\int_0^{\varepsilon^3 z}\frac{\partial}{\partial w}p_1(0,0;u-w,v).\end{aligned} \tag{2.13}$$

Using this representation and exchanging the integrals, we obtain

$$\begin{aligned}&\int_{r/\varepsilon^2}^{1/\varepsilon^2}\int_{r/\varepsilon^3}^{\infty}\mathbf{P}_{(0,1)}(\tau\in ds,I_\tau\in dz)[p_{1-\varepsilon^2 s}(0,0;u,v)-p_{1-\varepsilon^2 s}(0,0;u-\varepsilon^3 z,v)]\\&\quad=-\int_0^1\int_0^\infty\frac{\partial^2}{\partial q\partial w}p_{1-q}(0,0;u-w,v)\int_{(q\vee r)/\varepsilon^2}^{1/\varepsilon^2}\int_{(w\vee r)/\varepsilon^3}^{\infty}\mathbf{P}_{(0,1)}(\tau\in ds,I_\tau\in dz)\,dw\,dq\\&\qquad-\int_0^\infty\frac{\partial}{\partial w}p_1(0,0;u-w,v)\int_{(w\vee r)/\varepsilon^3}^{\infty}\mathbf{P}_{(0,1)}(\tau\in(r/\varepsilon^2,1/\varepsilon^2),I_\tau\in dz)\,dw\\&\quad=-\int_0^1\int_0^\infty\frac{\partial^2}{\partial q\partial w}p_{1-q}(0,0;u-w,v)\mathbf{P}_{(0,1)}\left(\tau\in\left(\frac{q\vee r}{\varepsilon^2},\frac{1}{\varepsilon^2}\right),I_\tau>\frac{w\vee r}{\varepsilon^3}\right)dw\,dq\\&\qquad-\int_0^\infty\frac{\partial}{\partial w}p_1(0,0;u-w,v)\mathbf{P}_{(0,1)}\left(\tau\in\left(\frac{r}{\varepsilon^2},\frac{1}{\varepsilon^2}\right),I_\tau>\frac{w\vee r}{\varepsilon^3}\right)dw.\end{aligned}$$

Applying now (2.12), we obtain

$$\begin{aligned}&\frac{1}{\varepsilon}\int_{r/\varepsilon^2}^{1/\varepsilon^2}\int_{r/\varepsilon^3}^{\infty}\mathbf{P}_{(0,1)}(\tau\in ds,I_\tau\in dz)[p_{1-\varepsilon^2 s}(0,0;u,v)-p_{1-\varepsilon^2 s}(0,0;u-\varepsilon^3 z,v)]\\&\quad\longrightarrow-\int_0^1\int_0^\infty\frac{\partial^2}{\partial q\partial w}p_{1-q}(0,0;u-w,v)(G(q\vee r,w\vee r)-G(1,w\vee r))\,dw\,dq\\&\qquad-\int_0^\infty\frac{\partial}{\partial w}p_1(0,0;u-w,v)(G(r,w\vee r)-G(1,w\vee r))\,dw\quad\text{as }\varepsilon\longrightarrow 0.\end{aligned} \tag{2.14}$$

The use of the Lebesgue's dominated convergence theorem is justified by the fact that $\varepsilon^{-1}\mathbf{P}_{(0,1)}(\tau\in(r\vee q/\varepsilon^2,1/\varepsilon^2),I_\tau>w\vee r/\varepsilon^3)$ is uniformly bounded in q,w and derivatives $(\partial^2/\partial q\partial w)p_{1-q}(0,0;u-w,v)$, $(\partial/\partial w)p_1(0,0;u-w,v)$ are integrable.

Let ν denote the measure which corresponds to G, that is,

$$G(t,z)=\int_t^\infty\int_z^\infty\nu(ds,dy).$$

Using this representation and (2.13) with $\varepsilon=1$, we can rewrite the limit in (2.14) in the following way

$$\int_r^1\int_r^\infty\nu(ds,dz)[p_{1-s}(0,0;u,v)-p_{1-s}(0,0;u-z,v)].$$

Letting here $r\to 0$ and taking into account (2.9) and (2.10), we conclude that

$$\begin{aligned}&\frac{1}{\varepsilon}\int_0^{1/\varepsilon^2}\int_0^\infty\mathbf{P}_{(0,1)}(\tau\in ds,I_\tau\in dz)[p_{1-\varepsilon^2 s}(0,0;u,v)-p_{1-\varepsilon^2 s}(0,0;u-\varepsilon^3 z,v)]\\&\quad\longrightarrow\int_0^1\int_0^\infty\nu(ds,dz)[p_{1-s}(0,0;u,v)-p_{1-s}(0,0;u-z,v)]\quad\text{as }\varepsilon\longrightarrow 0.\end{aligned} \tag{2.15}$$

Substituting (2.4), (2.8) and (2.15) into (2.2), we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\overline{p}_1(0,\varepsilon;u,v) &= \left(6u-2v+\sqrt{\frac{2}{\pi}}\right)p_1(0,0;u,v)\\ &\quad+\frac{1}{\sqrt{2\pi}}\int_0^1 s^{-3/2}[p_1(0,0;u,v)-p_{1-s}(0,0;u,v)]\\ &\quad+\int_0^1\int_0^\infty \nu(ds,dz)[p_{1-s}(0,0;u,v)-p_{1-s}(0,0;u-z,v)]. \qquad (2.16)\end{aligned}$$

From the definition of the measure ν and (2.6), we obtain

$$\begin{aligned}\frac{\int_{z=0}^\infty \nu(ds,dz)}{ds} &= \frac{d}{ds}\lim_{\varepsilon\to 0}\frac{\mathbf{P}_{(0,1)}(\tau>s\varepsilon^{-2})}{\varepsilon}\\ &= \frac{d}{ds}\lim_{\varepsilon\to 0}\frac{\int_{s\varepsilon^{-2}}^\infty \sqrt{(2/\pi)}u^{-1/2}\,e^{-1/2u}\,du}{\varepsilon}\\ &= \frac{d}{ds}\left(\sqrt{\frac{2}{\pi}}s^{-1/2}\right)=\frac{1}{\sqrt{2\pi}}s^{-3/2}.\end{aligned}$$

This equality implies that the expression on the right-hand side of (2.16) is equal to

$$\left(6u-2v+\sqrt{\frac{2}{\pi}}\right)p_1(0,0;u,v)+\int_0^1\int_0^\infty \nu(ds,dz)[p_1(0,0;u,v)-p_{1-s}(0,0;u-z,v)].$$

Recalling (2.3), we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{\overline{p}_1(0,\varepsilon;u,v)}{\mathbf{P}_{(0,\varepsilon)}(\tau>1)} &= \sqrt{\frac{\pi}{2}}\left(6u-2v+\sqrt{\frac{2}{\pi}}\right)p_1(0,0;u,v)\\ &\quad+\sqrt{\frac{\pi}{2}}\int_0^1\int_0^\infty \nu(ds,dz)[p_1(0,0;u,v)-p_{1-s}(0,0;u-z,v)]. \qquad (2.17)\end{aligned}$$

Since this limit is locally uniform in u and v

$$\begin{aligned}\lim_{\varepsilon\to 0}\mathbf{P}(I_1\in[u_1,u_2],B_1\in[v_1,v_2]\mid \min_{t\leqslant 1}B_t>-\varepsilon) &= \lim_{\varepsilon\to 0}\int_{u_1}^{u_2}\int_{v_1}^{v_2}\frac{\overline{p}_1(0,\varepsilon;u,v)}{\mathbf{P}_{(0,\varepsilon)}(\tau>1)}\,du\,dv\\ &= \int_{u_1}^{u_2}\int_{v_1}^{v_2}\lim_{\varepsilon\to 0}\frac{\overline{p}_1(0,\varepsilon;u,v)}{\mathbf{P}_{(0,\varepsilon)}(\tau>1)}\,du\,dv\end{aligned}$$

for all $0<u_1<u2$ and all $0<v_1<v_2$. According to the weak convergence towards the Brownian meander, the limit in the first line of the previous display is equal to $\mathbf{P}(\int_0^1 M_t\,dt\in[u_1,u_2],M_1\in[v_1,v_2])$. Therefore, the distribution of $(\int_0^1 M_t\,dt,M_1)$ is absolutely continuous and its density is given by the right-hand side in (2.17). Therefore, (2.1) is proved. The continuity of the density again follows from the fact that all the limits are locally uniform in u and v. □

The existence of the density $h(u,v)$ can be seen as follows. Denote

$$\psi^s_{me}(x,y,t)=\mathbf{E}_x[e^{-sI_t},B_t\in dy,\tau>t]/dy.$$

By the Feynmann–Kac formula, we conclude that the generator $\mathcal{A}_s$ of the semigroup corresponding to $\psi^s_{me}(x,y,t)$ is given by the differential operator

$$\mathcal{A}_s=\frac{1}{2}\frac{\partial^2}{\partial y^2}-sy$$

with the Dirichlet boundary condition. First, observe that the operator $\mathcal{A}_s$ with Dirichlet boundary conditions is essentially self-adjoint (compare, for example, [15, Lemma 3.1]) and thus gives rise to a unique self-adjoint extension. It is well known (see, for example, [2, Theorem 3.1]) that the spectrum of this operator is purely discrete, and its eigenvalues $-\lambda_n$ can be found by the solving the equation

$$f''(y) - 2syf(y) = -\lambda_n f(y)$$

with the boundary condition $f(0) = 0$. The general solution is given by $\mathrm{Ai}((2s)^{1/3}y - 2^{1/3}\lambda/s^{2/3})$. In order to satisfy the boundary condition, we need to require $\lambda_n = a_n s^{2/3}/2^{1/3}$, where $-a_n$ are zeros of the Airy function.

The sequence $(2s)^{1/6}\mathrm{Ai}(y(2s)^{1/3} - a_n)/\mathrm{Ai}'(-a_n)$ is orthonormal, see [24, Section 4.4] for more details. Therefore, by diagonalization of the self-adjoint operator $\mathcal{A}_s$,

$$\psi^s_{me}(x,y,t) = -\sum_{n=1}^{\infty} e^{-2^{-1/3}s^{2/3}a_n t}(2s)^{1/3}\frac{\mathrm{Ai}(y(2s)^{1/3} - a_n)\mathrm{Ai}(x(2s)^{1/3} - a_n)}{(\mathrm{Ai}'(-a_n))^2}.$$

In view of (2.3), as $x \to 0$,

$$\frac{\psi^s_{me}(x,y,1)}{\mathbf{P}_x(\tau > 1)} \longrightarrow \sqrt{\frac{\pi}{2}}(2s)^{2/3}\sum_{n=1}^{\infty} e^{-2^{-1/3}s^{2/3}a_n}\frac{\mathrm{Ai}(y(2s)^{1/3} - a_n)}{\mathrm{Ai}'(-a_n)}.$$

Consequently,

$$\mathbf{E}[e^{-s\int_0^1 M_t\,dt}, M_1 \in dy]/dy = \sqrt{\frac{\pi}{2}}(2s)^{2/3}\sum_{n=1}^{\infty} e^{-2^{-1/3}s^{2/3}a_n}\frac{\mathrm{Ai}(y(2s)^{1/3} - a_n)}{\mathrm{Ai}'(-a_n)}. \tag{2.18}$$

Integrating over y, we obtain the formula for the Laplace transform of the area of the standard meander, see also formula (209) in Janson [13],

$$\mathbf{E}[e^{-s\int_0^1 M_t\,dt}] = \sqrt{\frac{\pi}{2}}(2s)^{1/3}\sum_{n=1}^{\infty} r_n\, e^{-2^{-1/3}s^{2/3}a_n},$$

where

$$r_n := \frac{1}{\mathrm{Ai}'(-a_n)}\int_{-a_n}^{\infty}\mathrm{Ai}(z)\,dz, \quad n \geqslant 1.$$

Setting $s = -ir = e^{-i\pi/2}r$ in (2.18) and noting that the real part of $(e^{-i\pi/2})^{2/3}$ is always positive, we conclude that

$$\mathbf{E}[e^{ir\int_0^1 M_t\,dt}, M_1 \in dy]/dy = \sqrt{\frac{\pi}{2}}(-ir)^{2/3}\sum_{n=1}^{\infty} e^{-2^{-1/3}(-ir)^{2/3}a_n}\frac{\mathrm{Ai}(y(-2ir)^{1/3} - a_n)}{\mathrm{Ai}'(-a_n)}$$

is decreasing exponentially. In particular, this Fourier transform is integrable. Therefore, the corresponding measure is absolutely continuous with respect to the Lebesgue measure. Moreover, this Fourier transform multiplied by any power of r is still integrable. Thus, for every fixed y, the joint density $h(x,y)$ has derivatives of all orders in x. To obtain the continuity in y, it suffices to note that the Fourier transform is continuous in y and that this continuity carries over under the inverse transformation.

3. *Local asymptotics for discrete meanders: Proof of Theorem* 1.2

First, we state some known limit theorems for random walks and discrete meanders.

PROPOSITION 3.1. *If the variance of S_1 is one, then, for every $B \in \mathcal{B}(\mathbb{R}_+^2)$ such that the Lebesgue measure of ∂B equals zero and every starting point $z \geqslant 0$,*

$$\lim_{n\to\infty} \mathbf{P}_z \left(\left(\frac{A_n}{n^{3/2}}, \frac{S_n}{n^{1/2}} \right) \in B \middle| \tau > n \right) = \mathbf{P} \left(\left(\int_0^1 M_t \, dt, M_1 \right) \in B \right),$$

where M_t is the Brownian meander.

This convergence is immediate from the functional limit theorem for random walks conditioned to stay positive, which was proved by Bolthausen [3].

Another crucial ingredient of the proof of Theorem 1.2 is the following result.

PROPOSITION 3.2. *Under the conditions of Theorem 1.1,*

$$\sup_{a\in n(n+1)\rho/2+d\mathbb{Z}, x\in n\rho+d\mathbb{Z}} \left| n^2 \mathbf{P}(A_n = a, S_n = x) - \frac{d^2}{\sigma^2} g\left(\frac{a}{\sigma n^{3/2}}, \frac{x}{\sigma n^{1/2}} \right) \right| \longrightarrow 0,$$

where $g(u, v) = p_1(0, 0; u, v)$ is the density of the vector $(\int_0^1 B_t \, dt, B_1)$.

A version of this convergence for absolutely continuous distributions has been proved by Caravenna and Deuschel [5]. Since the case of discrete random walks needs only some obvious changes, we omit the proof of this result.

Proposition 3.2 and the boundedness of g imply the following result.

COROLLARY 3.3. *There exists a constant C such that*

$$\sup_{a,x\in\mathbb{Z}} \mathbf{P}(A_n = a, S_n = x) \leqslant Cn^{-2}, \quad n \geqslant 1. \tag{3.1}$$

To simplify notation, we give a proof of Theorem 1.2 for $z = 0$ only. Moreover, we assume, for the same reason, that $d = 1$ and $\rho = 0$.

We start by considering various 'boundary' values of a and x. Splitting the trajectory of S_n at $n - m$, we obtain

$$\begin{aligned} &\mathbf{P}(A_n = a, S_n = x, \tau > n) \\ &\quad = \sum_{y,b\geqslant 1} \mathbf{P}(A_{n-m} = b, S_{n-m} = y, \tau > n-m) \mathbf{P}_y(A_m = a - b, S_m = x, \tau > m). \end{aligned} \tag{3.2}$$

Applying now (3.1) to probabilities $\mathbf{P}_y(A_m = a - b, S_m = x, \tau > m)$ and using the following well-known relation (see, for example, formula (8.10) on p. 419 in Feller's book [10])

$$\mathbf{P}(\tau > n) \sim \theta n^{-1/2} \quad \text{with some positive } \theta, \tag{3.3}$$

we obtain, choosing $m = [n/2]$,

$$\sup_{a,x\in\mathbb{Z}} \mathbf{P}(A_n = a, S_n = x, \tau > n) \leqslant \frac{C}{m^2} \mathbf{P}(\tau > n - m) \leqslant \frac{C}{n^{5/2}}. \tag{3.4}$$

If $a \leqslant \delta n^{3/2}$, then we infer from (3.2) with $m = [n/2]$ that

$$\begin{aligned} \mathbf{P}(A_n = a, S_n = x, \tau > n) &\leqslant \mathbf{P}(A_{n-m} \leqslant a, \tau > n - m) \frac{C}{m^2} \\ &\leqslant \mathbf{P}(A_{n-m} \leqslant \delta n^{3/2} \,|\, \tau > n - m) \frac{C}{n^{5/2}}. \end{aligned}$$

In view of Proposition 3.1,

$$\mathbf{P}(A_{n-m} \leqslant \delta n^{3/2} \,|\, \tau > n - m) \longrightarrow \mathbf{P} \left(\int_0^1 M_t \, dt \leqslant 2^{3/2} \delta \right).$$

According to formula (212) in Janson [13],

$$\mathbf{P}\left(\int_0^1 M_t\,dt \leqslant u\right) \sim c_1\, e^{-c_2/u^2} \quad \text{as } u \longrightarrow 0.$$

Consequently,

$$n^{5/2} \sup_{a\leqslant \delta n^{3/2},\, x\geqslant 1} \mathbf{P}(A_n = a, S_n = x, \tau > n) \leqslant C\, e^{-1/\delta}. \tag{3.5}$$

For $a \geqslant 2Rn^{3/2}$, we have

$$\begin{aligned}
\mathbf{P}(A_n = a, S_n = x, \tau > n) &= \mathbf{P}\left(A_n = a, S_n = x, \max_{k\leqslant n} S_k \geqslant 2R\sqrt{n}, \tau > n\right)\\
&\leqslant \mathbf{P}\left(A_n = a, S_n = x, \max_{k\leqslant m} S_k \geqslant R\sqrt{n}, \tau > n\right)\\
&\quad + \mathbf{P}\left(A_n = a, S_n = x, \max_{k\leqslant n-m} (S_{m+k} - S_m) \geqslant R\sqrt{n}, \tau > n\right).
\end{aligned}$$

Using the Markov property and (3.1), we obtain for $m = [n/2]$

$$\begin{aligned}
&\mathbf{P}\left(A_n = a, S_n = x, \max_{k\leqslant m} S_k \geqslant R\sqrt{n}, \tau > n\right)\\
&\quad\leqslant \mathbf{P}\left(\max_{k\leqslant m} S_k \geqslant R\sqrt{n}, \tau > m\right) \sup_{y,b\in\mathbb{Z}} \mathbf{P}(A_{n-m} = b, S_{n-m} = y)\\
&\quad\leqslant \frac{C}{n^{5/2}}\mathbf{P}\left(\max_{k\leqslant m} S_k \geqslant R\sqrt{n}\,\middle|\, \tau > m\right) \leqslant \frac{C}{n^{5/2}}\mathbf{P}\left(\sup_{t\leqslant 1} M_t \geqslant R\sqrt{2}\right).
\end{aligned}$$

In the last step, we used functional limit theorem for random walks conditioned to stay positive. Furthermore, using (3.4), we obtain

$$\begin{aligned}
&\mathbf{P}\left(A_n = a, S_n = x, \max_{k\leqslant n-m} (S_{m+k} - S_m) \geqslant R\sqrt{n}, \tau > n\right)\\
&\quad\leqslant \sup_{y,b\in\mathbb{Z}} \mathbf{P}(A_m = b, S_m = y, \tau > m)\mathbf{P}\left(\max_{k\leqslant n-m} (S_{m+k} - S_m) \geqslant R\sqrt{n}\right)\\
&\quad\leqslant \frac{C}{n^{5/2}}\mathbf{P}\left(\max_{t\leqslant 1} B_t > R\sqrt{2}\right).
\end{aligned}$$

As a result, we have

$$n^{5/2} \sup_{a\geqslant 2Rn^{3/2},\, x\geqslant 1} \mathbf{P}(A_n = a, S_n = x, \tau > n) \leqslant \Delta(R), \tag{3.6}$$

where $\Delta(R) \to 0$ as $R \to \infty$. Since for $x \geqslant 2R\sqrt{n}$ the equality

$$\mathbf{P}(A_n = a, S_n = x, \tau > n) = \mathbf{P}\left(A_n = a, S_n = x, \max_{k\leqslant n} S_k \geqslant 2R\sqrt{n}, \tau > n\right)$$

holds, we have

$$n^{5/2} \sup_{a\geqslant 1,\ x\geqslant 2R\sqrt{n}} \mathbf{P}(A_n = a, S_n = x, \tau > n) \leqslant \Delta(R). \tag{3.7}$$

For $x \leqslant 2\varepsilon\sqrt{n}$, we use an alternative representation for $\mathbf{P}(A_n = a, S_n = x, \tau > n)$. Set $X_i' := -X_{m+1-i}$, $i \in \{1, 2, \dots, m\}$ and $S_k' = S_0' + \sum_{i=1}^k X_i'$, $A_k' = \sum_{i=1}^k S_i'$. Then it is easy to see that if $S_0 = S'_0 = 0$, then

$$\left\{ym + \sum_{i=1}^{m} S_i = a - b,\ y + S_m = x,\ \min_{i\leqslant m}(y + S_i) > 0\right\}$$
$$= \left\{xm + \sum_{i=1}^{m} S'_i = a - b + y - x,\ x + S'_m = y,\ \min_{i\leqslant m}(x + S'_i) > 0\right\}.$$

Consequently,

$$\mathbf{P}_y(A_m = a - b, S_m = x, \tau > m) = \mathbf{P}_x(A'_m = a - b + y - x, S'_m = y, \tau' > m)$$

and

$$\mathbf{P}(A_n = a, S_n = x, \tau > n) = \sum_{y,b\geqslant 1} \mathbf{P}(A_{n-m} = b, S_{n-m} = y, \tau > n - m) \times \mathbf{P}_x(A'_m = a - b + y - x, S'_m = y, \tau' > m). \tag{3.8}$$

From this representation and (3.4), we conclude that

$$\mathbf{P}(A_n = a, S_n = x, \tau > n) \leqslant \frac{C}{(n-m)^{5/2}} \mathbf{P}_x(\tau' > m) \leqslant \frac{C}{n^{5/2}} \mathbf{P}_{2\varepsilon\sqrt{n}}(\tau' > m).$$

It is immediate from the functional Central Limit Theorem (CLT) that $\mathbf{P}_{2\varepsilon\sqrt{n}}(\tau' > [n/2]) \leqslant C\varepsilon$. Therefore,

$$n^{5/2} \sup_{a\geqslant 1,\ x\leqslant 2\varepsilon\sqrt{n}} \mathbf{P}(A_n = a, S_n = x, \tau > n) \leqslant C\varepsilon. \tag{3.9}$$

We now turn to 'normal' values for the vector (A_n, S_n), that is,

$$\delta n^{3/2} \leqslant a \leqslant 2Rn^{3/2} \quad \text{and} \quad 2\varepsilon\sqrt{n} \leqslant x \leqslant 2R\sqrt{n}.$$

For every x define

$$B_1 = B_1(x) := \{y \geqslant 1 : |y - x| \leqslant \varepsilon\sqrt{n}\} \quad \text{and} \quad B_2 = B_2(x) := \mathbb{Z}_+ \setminus B_1(x).$$

For every $m \geqslant 1$, we have

$$\mathbf{P}(A_n = a, S_n = x, \tau > n) = \mathbf{P}(A_n = a, S_n = x, S_{n-m} \in B_1, \tau > n) + \mathbf{P}(A_n = a, S_n = x, S_{n-m} \in B_2, \tau > n). \tag{3.10}$$

Set $m = [\varepsilon^3 n]$. Then, applying (3.4), we obtain, uniformly in $a, x \geqslant 1$,

$$\begin{aligned}
&\mathbf{P}(A_n = a, S_n = x, S_{n-m} \in B_2, \tau > n)\\
&\quad = \sum_{y\in B_2, b\geqslant 1} \mathbf{P}(A_{n-m} = b, S_{n-m} = y, \tau > n - m)\mathbf{P}_y(A_m = a - b, S_m = x, \tau > m)\\
&\quad \leqslant \frac{C}{n^{5/2}} \sum_{y\in B_2} \mathbf{P}_y(S_m = x) \leqslant \frac{C}{n^{5/2}} \mathbf{P}(|S_m| > \varepsilon\sqrt{n}) \leqslant \frac{C}{n^{5/2}} \overline{\Phi}(\varepsilon^{-1/2}),
\end{aligned} \tag{3.11}$$

where $\overline{\Phi}(x) = \int_x^\infty (1/\sqrt{2\pi})\, e^{-u^2/2}\, du$.

Further,

$$\begin{aligned}
&\mathbf{P}(A_n = a, S_n = x, S_{n-m} \in B_1, \tau > n)\\
&\quad = \sum_{y\in B_1, b\geqslant 1} \mathbf{P}(A_{n-m} = b, S_{n-m} = y, \tau > n - m)\mathbf{P}_y(A_m = a - b, S_m = x, \tau > m)\\
&\quad = \sum_{y\in B_1, b\geqslant 1} \mathbf{P}(A_{n-m} = b, S_{n-m} = y, \tau > n - m)\mathbf{P}_y(A_m = a - b, S_m = x)\\
&\qquad - \sum_{y\in B_1, b\geqslant 1} \mathbf{P}(A_{n-m} = b, S_{n-m} = y, \tau > n - m)\mathbf{P}_y(A_m = a - b, S_m = x, \tau \leqslant m).
\end{aligned}$$

Applying (3.4) to the probabilities in the second sum, we obtain

$$\sum_{y\in B_1, b\geqslant 1} \mathbf{P}(A_{n-m}=b, S_{n-m}=y, \tau>n-m)\mathbf{P}_y(A_m=a-b, S_m=x, \tau\leqslant m)$$
$$\leqslant \frac{C}{n^{5/2}}\sum_{y\in B_1}\mathbf{P}_y(S_m=x,\tau\leqslant m)=\frac{C}{n^{5/2}}\mathbf{P}_x(S'_m\in B_1,\tau'\leqslant m).$$

For $x\geqslant 2\varepsilon\sqrt{n}$, we have

$$\mathbf{P}_x(S'_m\in B_1,\tau'\leqslant m)\leqslant \mathbf{P}\left(\max_{k\leqslant m} S_k>2\varepsilon\sqrt{n}\right)\leqslant C\overline{\Phi}(\varepsilon^{-1/2}).$$

Therefore,

$$\sum_{y\in B_1, b\geqslant 1} \mathbf{P}(A_{n-m}=b, S_{n-m}=y, \tau>n-m)\mathbf{P}_y(A_m=a-b, S_m=x, \tau\leqslant m)$$
$$\leqslant \frac{C}{n^{5/2}}\overline{\Phi}(\varepsilon^{-1/2}). \tag{3.12}$$

It follows from Proposition 3.2 that

$$\sum_{y\in B_1, b\geqslant 1} \mathbf{P}(A_{n-m}=b, S_{n-m}=y, \tau>n-m)\mathbf{P}_y(A_m=a-b, S_m=x)$$
$$=\sum_{y\in B_1, b\geqslant 1} \mathbf{P}(A_{n-m}=b, S_{n-m}=y, \tau>n-m)m^{-2}g\left(\frac{a-b-my}{m^{3/2}},\frac{x-y}{m^{1/2}}\right)$$
$$+o(m^{-2}\mathbf{P}(\tau>n-m)) \tag{3.13}$$

uniformly in $a,x\geqslant 1$. Recalling that $m=[\varepsilon^3 n]$ and using (3.3), we obtain

$$\sum_{y\in B_1, b\geqslant 1} \mathbf{P}(A_{n-m}=b, S_{n-m}=y, \tau>n-m)m^{-2}g\left(\frac{a-b-my}{m^{3/2}},\frac{x-y}{m^{1/2}}\right)$$
$$=\frac{1-\varepsilon^3}{\varepsilon^6}\frac{\theta}{n^{5/2}}\mathbf{E}\left[g\left(\frac{a-A_{n-m}-\varepsilon^3 nS_{n-m}}{\varepsilon^{9/2}n^{3/2}},\frac{x-S_{n-m}}{\varepsilon^{3/2}n^{1/2}}\right)1\{S_{n-m}\in B_1\}\middle|\tau>n-m\right].$$

Since $g(u,v)\to 0$ as $v\to\infty$ uniformly in u,

$$\sup_{a,x}\mathbf{E}\left[g\left(\frac{a-A_{n-m}-\varepsilon^3 nS_{n-m}}{\varepsilon^{9/2}n^{3/2}},\frac{x-S_{n-m}}{\varepsilon^{3/2}n^{1/2}}\right)1\{S_{n-m}\in B_2\}\middle|\tau>n-m\right]\leqslant r_1(\varepsilon), \tag{3.14}$$

where $r_1(\varepsilon)\to 0$ as $\varepsilon\to 0$.

It is easy to see that the family $\{f_{a,x}(u,v):=g(a-u,x-v),\ a,x>0\}$ is equicontinuous. Then, applying Proposition 3.1, we obtain

$$\mathbf{E}\left[g\left(\frac{a-A_{n-m}-\varepsilon^3 nS_{n-m}}{\varepsilon^{9/2}n^{3/2}},\frac{x-S_{n-m}}{\varepsilon^{3/2}n^{1/2}}\right)\middle|\tau>n-m\right]$$
$$=o(1)+\int_{\mathbb{R}_+^2} g\left(\frac{an^{-3/2}-(1-\varepsilon^3)^{3/2}u-\varepsilon^3(1-\varepsilon^3)^{1/2}v}{\varepsilon^{9/2}},\frac{xn^{-1/2}-(1-\varepsilon^3)^{1/2}v}{\varepsilon^{3/2}}\right)h(u,v)\,du\,dv$$

uniformly in a and x.

Furthermore, as $\varepsilon\to 0$,

$$\varepsilon^{-6}\int_{\mathbb{R}_+^2} g\left(\frac{b-(1-\varepsilon^3)^{3/2}u-\varepsilon^3(1-\varepsilon^3)^{1/2}v}{\varepsilon^{9/2}},\frac{y-(1-\varepsilon^3)^{1/2}v}{\varepsilon^{3/2}}\right)h(u,v)\,du\,dv\longrightarrow h(b,y)$$

locally uniformly in b, y. From this convergence and (3.14), we conclude that, uniformly in $a \in [\delta n^{3/2}, 2Rn^{3/2}]$ and $x \in [2\varepsilon\sqrt{n}, 2R\sqrt{n}]$,

$$\limsup_{n\to\infty} \Bigg| \mathbf{E}\left[g\left(\frac{a - A_{n-m} - \varepsilon^3 ny}{\varepsilon^{9/2} n^{3/2}}, \frac{x - S_{n-m}}{\varepsilon^{3/2} n^{1/2}} \right) 1\{S_{n-m} \in B_1\} \Bigg| \tau > n-m \right] \\ -h\left(\frac{a}{n^{3/2}}, \frac{x}{n^{1/2}} \right) \Bigg| \leqslant r_2(\varepsilon). \tag{3.15}$$

Combining (3.10)–(3.13) and (3.15), we conclude that

$$\limsup_{n\to\infty} \left| n^{5/2} \mathbf{P}(A_n = a, S_n = x, \tau > n) - \theta h\left(\frac{a}{n^{3/2}}, \frac{x}{n^{1/2}} \right) \right| \leqslant r_3(\varepsilon)$$

uniformly in $\delta n^{3/2} \leqslant a \leqslant 2Rn^{3/2}$ and $2\varepsilon\sqrt{n} \leqslant x \leqslant 2R\sqrt{n}$. Taking into account (3.9)–(3.7), we arrive at the desired local asymptotic.

4. *Proof of Theorem* 1.1

We are going to split the path of the excursion and to inverse the time in the second half of the path. For this reason, we need information on the position of our random walk immediately before τ occurs. Let $H(x)$ be the renewal function corresponding to strict ascending ladder epochs.

LEMMA 4.1. *For every fixed* $x \in \mathbb{Z}_+$,

$$\mathbf{P}(S_n = x, \tau = n+1) \sim \frac{H(x)}{\sqrt{2\pi}} n^{-3/2} P(X < -x).$$

Furthermore,

$$\mathbf{P}(\tau = n+1) \sim \left(\sum_{x>0} H(x) P(X < -x) \right) (2\pi)^{-1/2} n^{-3/2}.$$

Proof. First, we note that

$$\mathbf{P}(S_n = x, \tau = n+1) = \mathbf{P}(S_n = x, \tau > n) \mathbf{P}(X_{n+1} < -x).$$

Further, according to [**25**, Theorem 6],

$$\mathbf{P}(S_n = x, \tau > n) \sim \frac{H(x)}{\sqrt{2\pi}} n^{-3/2}. \tag{4.1}$$

Thus, the first statement is proved.

Obviously,

$$\mathbf{P}(\tau = n+1) = \sum_{x>0} \mathbf{P}(S_n = x, \tau > n) \mathbf{P}(X_{n+1} < -x).$$

Since $\sup_x \mathbf{P}(S_n = x, \tau > n) \leqslant Cn^{-1}$, see [**25**, Lemma 19],

$$\begin{aligned} \sum_{x \geqslant N} \mathbf{P}(S_n = x, \tau > n) \mathbf{P}(X_{n+1} < -x) &\leqslant \frac{C}{n} \sum_{x \geqslant N} \mathbf{P}(X_{n+1} < -x) \\ &\leqslant \frac{C}{n} \mathbf{E}[|X|, |X| > N], \quad N \geqslant 0. \end{aligned}$$

From the finiteness of the second moment, we infer that there exist δ_n such that $\delta_n \to 0$, $\delta_n n^{1/2} \to \infty$ and $\mathbf{E}[|X|, |X| > \delta_n n^{1/2}] = o(n^{-1/2})$. Indeed, the finiteness of the second moment implies the existence of an increasing function f such that $\lim_{x\to\infty} f(x) = \infty$ and

$\mathbf{E}X^2 f(|X|) < \infty$. Then, by the Markov inequality,

$$\mathbf{E}[|X|; |X| > \delta_n n^{1/2}] \leqslant \frac{\mathbf{E}[X^2 f(|X|); |X| > \delta_n n^{1/2}]}{\delta_n n^{1/2} f(\delta_n n^{1/2})}$$

for every sequence δ_n. Now it is easy to see that the sequence

$$\delta_n := \max\{n^{-1/4}, 1/f(n^{1/4})\} \tag{4.2}$$

possesses all needed properties.

Consequently,

$$\sum_{x \geqslant \delta_n n^{1/2}} \mathbf{P}(S_n = x, \tau > n)\mathbf{P}(X_{n+1} < -x) = o(n^{-3/2}). \tag{4.3}$$

Using [**25**, Theorem 6] once again, we obtain

$$\begin{aligned}\sum_{x < \delta_n n^{1/2}} &\mathbf{P}(S_n = x, \tau > n)\mathbf{P}(X_{n+1} < -x) \\ &= \frac{1}{\sqrt{2\pi}} n^{-3/2}(1 + o(1)) \left(\sum_{x < \delta_n n^{1/2}} H(x)\mathbf{P}(X < -x) \right).\end{aligned} \tag{4.4}$$

Combining (4.3), (4.4) and noting that $\sum_{x>0} H(x)\mathbf{P}(X < -x)$ is finite, we complete the proof. □

We are now in a position to prove Theorem 1.1. We start with the representation

$$\begin{aligned}\mathbf{P}(A_n = a, \tau = n+1) &= \sum_{x=1}^{\infty} \mathbf{P}(A_n = a, S_n = x, \tau = n+1) \\ &= \sum_{x=1}^{\infty} \mathbf{P}(A_n = a, S_n = x, \tau > n)\mathbf{P}(X \leqslant -x).\end{aligned} \tag{4.5}$$

Using (3.4), we conclude that

$$\begin{aligned}\sum_{x \geqslant \delta_n n^{1/2}}^{\infty} &\mathbf{P}(A_n = a, S_n = x, \tau > n)\mathbf{P}(X \leqslant -x) \\ &\leqslant C n^{-5/2}\mathbf{E}[-X, X \leqslant -\delta_n n^{1/2}] = o(n^{-3})\end{aligned} \tag{4.6}$$

for δ_n defined in (4.2).

Combining (3.2) and (3.4), we obtain

$$\begin{aligned}\mathbf{P}(A_n = a, S_n = x, \tau > n) &\leqslant C n^{-5/2} \sum_{y,b \geqslant 1} \mathbf{P}_x(A'_{n/2} = a - b + y - x, S'_{n/2} = y, \tau' > n/2) \\ &\leqslant C n^{-5/2}\mathbf{P}_x(\tau' > n/2).\end{aligned}$$

According to Corollary 3 from [**7**], $\mathbf{P}_x(\tau' > n/2) \leqslant C x n^{-1/2}$ uniformly in $x \leqslant \delta_n n^{1/2}$. Therefore, uniformly in a,

$$\sum_{N \leqslant x \leqslant \delta_n n^{1/2}} \mathbf{P}(A_n = a, S_n = x, \tau > n)\mathbf{P}(X \leqslant -x) \leqslant C n^{-3}\mathbf{E}[X^2, X \leqslant -N]. \tag{4.7}$$

It remains to consider fixed values of x. From (3.8), we obtain

$$\mathbf{P}(A_n = a, S_n = x, \tau > n) = \mathbf{P}(\tau > n - m)\mathbf{P}_x(\tau' > m)\Sigma(a, x), \tag{4.8}$$

where

$$\Sigma(a,x) := \sum_{y,b\geqslant 1} \mathbf{P}(A_{n-m}=b, S_{n-m}=y \,|\, \tau>n-m) \\ \times \mathbf{P}_x(A'_m = a-b+y-x, S'_m=y \,|\, \tau'>m).$$

Applying Theorem 1.2 to probabilities in the previous display, we obtain, uniformly in a and for every fixed x,

$$\begin{aligned}\Sigma(a,x) &= \frac{2^4+o(1)}{n^4}\sum_{y,b\geqslant 1} h\left(\frac{2^{3/2}b}{n^{3/2}},\frac{2^{1/2}y}{n^{1/2}}\right) h\left(\frac{2^{3/2}(a-b+y-x)}{n^{3/2}},\frac{2^{1/2}y}{n^{1/2}}\right)\\ &= \frac{2^4+o(1)}{n^2}\int_0^\infty\int_0^{a/n^{3/2}} h(2^{3/2}u,2^{1/2}v)h(2^{3/2}(a/n^{3/2}-u),2^{1/2}v)\,du\,dv\\ &=: \frac{2^4+o(1)}{n^2}q(a/n^{3/2}).\end{aligned} \tag{4.9}$$

Summing (4.8) over a, we have

$$\mathbf{P}(S_n=x,\tau>n) = \mathbf{P}(\tau>n-m)\mathbf{P}_x(\tau'>m) \\ \times\sum_{b\geqslant 1}\mathbf{P}(S_{n-m}=y\,|\,\tau>n-m)\mathbf{P}_x(S'_m=y\,|\,\tau'>m).$$

By the local limit theorem for random walks conditioned to stay positive, see [**25**, Theorem 5],

$$\sup_{y\geqslant 1}\left|\sqrt{n}\mathbf{P}(S_n=y\,|\,\tau>n)-\frac{y}{\sqrt{n}}e^{-y^2/2n}\right|\longrightarrow 0.$$

Combining this with the fact that S'_n conditioned on $\tau'>n$ converges, for every fixed x, towards M_1, we obtain

$$\begin{aligned}\sum_{b\geqslant 1}\mathbf{P}(S_{n-m}=y\,|\,\tau>n-m)\mathbf{P}_x(S'_m=y\,|\,\tau'>m) &\sim \sqrt{\frac{2}{n}}\int_0^\infty z\,e^{-z^2/2}\mathbf{P}(M_1\in dz)\\ &\sim\sqrt{\frac{2}{n}}\int_0^\infty z^2\,e^{-z^2}\,dz = \frac{\Gamma(3/2)}{\sqrt{2n}}.\end{aligned}$$

As a result,

$$\mathbf{P}(S_n=x,\tau>n)\sim\frac{\Gamma(3/2)}{\sqrt{2n}}\mathbf{P}(\tau>n-m)\mathbf{P}_x(\tau'>m).$$

Combining this with (4.1), we obtain

$$\mathbf{P}(\tau>n-m)\mathbf{P}_x(\tau'>m)\sim\frac{H(x)}{\sqrt{\pi}\Gamma(3/2)}\frac{1}{n}. \tag{4.10}$$

Combining (4.8)–(4.10), we obtain, uniformly in a,

$$n^3\mathbf{P}(A_n=a,S_n=x,\tau>n)\longrightarrow\frac{2^4H(x)}{\sqrt{\pi}\Gamma(3/2)}q(a/n^{3/2}).$$

Summing over x from 1 to N, we obtain

$$n^3\sum_{x=1}^N\mathbf{P}(A_n=a,S_n=x,\tau>n)\mathbf{P}(X\leqslant -x) \\ = \frac{2^4}{\sqrt{\pi}\Gamma(3/2)}\left(\sum_{x=1}^N H(x)\mathbf{P}(X\leqslant -x)\right)q(a/n^{3/2})+o(n^{-3}).$$

Combining this with (4.5)–(4.7), we conclude that

$$n^3\mathbf{P}(A_n = a, \tau = n+1) = \frac{2^4 q(a/n^{3/2})}{\sqrt{\pi}\Gamma(3/2)} \sum_{x=1}^{\infty} H(x)\mathbf{P}(X \leqslant -x) + o(n^{-3})$$

uniformly in a. Hence, in view of Lemma 4.1,

$$n^{3/2}\mathbf{P}(A_n = a \,|\, \tau = n+1) = \frac{2^{9/2}}{\Gamma(3/2)} q(a/n^{3/2}) + o(n^{-3/2}). \tag{4.11}$$

Uniqueness of the limit implies that $(2^{9/2}/\Gamma(3/2))q(x) = w_{\mathrm{ex}}(x)$. This completes the proof of the theorem.

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