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MULTIFRACTAL ANALYSIS OF SUPERPROCESSES WITH STABLE BRANCHING IN DIMENSION ONE

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We show that density functions of a $(\alpha, 1, \beta)$-superprocesses are almost sure multifractal for $\alpha > \beta + 1$, $\beta \in (0, 1)$ and calculate the corresponding spectrum of singularities.

1. Introduction, main results and discussion. For $0 < \alpha \le 2$ and $1 + \beta \in (1, 2]$, the so-called (α, d, β)-*superprocess* $X = \{X_t : t \ge 0\}$ in R^d is a finite measure-valued process related to the log-Laplace equation

$$\frac{d}{dt}u = \Delta_\alpha u + au - bu^{1+\beta}, \tag{1.1}$$

where $a \in \mathsf{R}$ and $b > 0$ are any fixed constants. Its underlying motion is described by the fractional Laplacian $\Delta_\alpha := -(-\Delta)^{\alpha/2}$ determining a symmetric α-stable motion in R^d of index $\alpha \in (0, 2]$ (Brownian motion corresponds to $\alpha = 2$), whereas its continuous-state branching mechanism

$$v \mapsto -av + bv^{1+\beta}, \qquad v \ge 0, \tag{1.2}$$

belongs to the domain of attraction of a stable law of index $1 + \beta \in (1, 2]$ (the branching mechanism is critical if $a = 0$).

Let $d < \frac{\alpha}{\beta}$. Then, for any fixed time t, $X_t(dx)$ is a.s. absolutely continuous with respect to the Lebesgue measure (cf. Fleischmann [3] for $a = 0$).

In the case of $\beta = 1$, there is a continuous version of the density of $X_t(dx)$ for all $\alpha > 1$; see Konno and Shiga [12]. A careful examination of their arguments shows that this density is Hölder continuous with any exponent smaller than $(\alpha - 1)/2$.

Now we consider the case $\beta < 1$. As shown in Fleischmann, Mytnik and Wachtel ([4], Theorem 1.2(a), (c)), there is a *dichotomy* for the density function of the measure (in what follows, we just say the "density of the measure"): there is a continuous version of the density of $X_t(dx)$ if $d = 1$ and $\alpha > 1 + \beta$, but otherwise the density is unbounded on open sets of positive $X_t(dx)$-measure. Note that the case

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of $\alpha = 2$ had been studied earlier in Mytnik and Perkins [15]. In the case of continuity, Hölder regularity properties of the density had been studied in [4, 5].

From now on, we always assume that $d = 1$, $\beta < 1$ and $\alpha > 1 + \beta$, that is, there is a continuous version of the density at fixed time t. This density, with a slight abuse of notation, will be also denoted by $X_t(x)$, $x \in \mathsf{R}$.

Let us first recall the notion of pointwise regularity (see, e.g., Jaffard [7]). We say that a function f has regularity of index $\eta > 0$ at a point $x \in \mathsf{R}$, if there exists an open neighborhood $U(x)$ of x, a constant $C > 0$ and a polynomial P_x of degree at most $\lfloor \eta \rfloor$ such that

$$|f(y) - P_x(y)| \le C|y - x|^\eta \qquad \text{for all } y \in U(x). \tag{1.3}$$

For $\eta \in (0, 1)$, the above definition coincides with the definition of Hölder continuity with index η at a point. Note that sometimes the class of functions satisfying (1.3) is denoted by $C^\eta(x)$. Now, given f one would like to find the supremum over all η such that (1.3) holds for some constant C and polynomial P_x. This leads to the definition of so-called *optimal* Hölder exponent (or index) of f at x:

$$H_f(x) := \sup\{\eta > 0 : f \in C^\eta(x)\}, \tag{1.4}$$

and we set it to 0 if $f \notin C^\eta(x)$ for all $\eta > 0$. To simplify the exposition, we will sometimes call $H_f(x)$ the Hölder exponent of f at x.

Let us fix $t > 0$ and return to the continuous density X_t of the $(\alpha, 1, \beta)$-superprocess. In what follows, $H_X(x)$ will denote the optimal Hölder exponent of X_t at $x \in \mathsf{R}$. In Theorem 1.2(a), (b) of [4], the so-called *optimal index* for *local* Hölder continuity of X_t had been determined as

$$\eta_{\mathrm{c}} := \frac{\alpha}{1+\beta} - 1 \in (0, 1). \tag{1.5}$$

This means that $\inf_{x \in K} H_X(x) \ge \eta_{\mathrm{c}}$ for any compact K and, moreover, in any nonempty open set $U \subset \mathsf{R}$ with $X_t(U) > 0$ one can find (random) points x such that $H_X(x) = \eta_{\mathrm{c}}$. Moreover, it was proved in [5] that for any fixed point $x \in \mathsf{R}$ we have

$$H_X(x) = \bar{\eta}_{\mathrm{c}} := \frac{1+\alpha}{1+\beta} - 1 \qquad \text{a.s. on } \{X_t(x) > 0\}$$

in the case of $\beta > (\alpha - 1)/2$, and

$$H_X(x) \ge 1 \qquad \text{a.s. on } \{X_t(x) > 0\}$$

if $\beta \le (\alpha - 1)/2$.

REMARK 1.1. In [5], the classical definition of Hölder exponent was used, which can take only values between 0 and 1. Hence, the result in [5] states that the optimal index of Hölder continuity (in classical sense) equals $\min\{\frac{1+\alpha}{1+\beta} - 1, 1\}$, for any $\beta \in (0, \alpha - 1)$.

The purpose of this paper includes proving that on any open set of positive X_t measure, and for any $\eta \in (\eta_c, \bar{\eta}_c) \setminus \{1\}$ there are, with probability one, (random) points $x \in \mathsf{R}$ such that the optimal Hölder index $H_X(x)$ of X_t at x is exactly η. Moreover, for an open set $U \subset \mathsf{R}$, we are going to determine the *Hausdorff dimension*, say $D_U(\eta)$, of the (random) set

$$\mathcal{E}_{U,X,\eta} := \{x \in U : H_X(x) = \eta\}.$$

We will show that the function $\eta \mapsto D_U(\eta)$ is independent of U; it reveals the so-called *multifractal spectrum* related to the optimal Hölder index at points.

To formulate our main result, we need also the following notation. Let $\mathcal{M}_f$ denote the set of finite measures on R, and for $\mu \in \mathcal{M}_f$, $|\mu|$ will denote the total mass $\mu(\mathsf{R})$. Our main result is as follows.

THEOREM 1.2 (Multifractal spectrum). *Fix $t > 0$, and $X_0 = \mu \in \mathcal{M}_f$. Let $d = 1$ and $\alpha > 1 + \beta$. Then, for any $\eta \in [\eta_c, \bar{\eta}_c) \setminus \{1\}$, with probability one,*

$$D_U(\eta) = (\beta + 1)(\eta - \eta_c) \qquad \textit{for any open set } U \subset \mathsf{R},$$

whenever $X_t(U) > 0$.

REMARK 1.3. Let us consider the case $\eta = \bar{\eta}_c$. First note that if $\bar{\eta}_c < 1$ then, for every fixed x, $H_X(x) = \bar{\eta}_c$ almost surely on the event $\{X_t(x) > 0\}$, see Theorem 1.1 from [5]. We conjecture that this statement is also valid whenever $\bar{\eta}_c \geq 1$. This would imply that $D_U(\bar{\eta}_c) = 1$ almost surely on $\{X_t(U) > 0\}$. Indeed, for B being an arbitrary interval in (0, 1) define

$$\lambda(\mathcal{E}_{B,X,\bar{\eta}_c}) = \int_B 1_{\{H_X(x)=\bar{\eta}_c\}}\, dx,$$

where λ is the Lebesgue measure on R. Then, by the Fubini theorem, we have

$$\begin{aligned}
\mathbf{E}\Big[\lambda(\mathcal{E}_{B,X,\bar{\eta}_c}) \Big| \inf_{y\in B} X_t(y) > 0\Big] &= \mathbf{E}\Big[\int_B 1_{\{H_X(x)=\bar{\eta}_c\}}\, dx \Big| \inf_{y\in B} X_t(y) > 0\Big] \\
&= \int_B \mathbf{P}\Big(H_X(x) = \bar{\eta}_c \Big| \inf_{y\in B} X_t(y) > 0\Big)\, dx \\
&= \lambda(B),
\end{aligned}$$

in the last step we used our conjecture that $H_X(x) = \bar{\eta}_c$ with probability one for every fixed point x. That is, given $\{\inf_{y\in B} X_t(y) > 0\}$, we get that, with probability one, $D_B(\bar{\eta}_c) = 1$. We may fix ω outside a $\mathbf{P}$-null set so that this holds for any rational ball B, that is, for any ball with a rational radius and center. Let U be an arbitrary open set such that $\{\inf_{y\in U} X_t(y) > 0\}$. Then there is always a rational ball B in U such that $\{\inf_{y\in B} X_t(y) > 0\}$, and so the result follows immediately from what we derived for the fixed ball.

REMARK 1.4. The fact that our proof fails in the case $\eta = 1$ is even more disappointing. Formally, it happens for some technical reasons, but one has also to note, that this point is critical: it is the borderline between differentiable and nondifferentiable functions. However, we still believe that the function $D_U(\cdot)$ can be continuously extended to $\eta = 1$, that is, $D_U(1) = (\beta + 1)(1 - \eta_c)$ almost surely on $\{X_t(U) > 0\}$.

REMARK 1.5. The condition $\alpha > 1 + \beta$ excludes the case of the quadratic super-Brownian motion, that is, $\alpha = 2$, $\beta = 1$. But it is a known "folklore" result that the super-Brownian motion $X_t(\cdot)$ is almost surely monofractal on any open set of strictly positive density. That is, $\mathbf{P}$-a.s., for any x with $X_t(x) > 0$ we have $H_X(x) = 1/2$. For the fact that $H_X(x) \geq 1/2$, for any x, see Konno and Shiga [12] and Walsh [17]. To get that $H_X(x) \leq 1/2$ on the event $\{X_t(x) > 0\}$, one can show that

$$\limsup_{\delta \to 0} \frac{|X_t(x+\delta) - X_t(x)|}{\delta^\eta} = \infty \qquad \text{for all } x \text{ such that } X_t(x) > 0, \mathbf{P}\text{-a.s.},$$

for every $\eta > 1/2$. This result follows from the fact that for $\beta = 1$ the noise driving the corresponding stochastic equation for X_t is Gaussian (see (0.4) in [12]) in contrast to the case of $\beta < 1$ considered here, where we have driving discontinuous noise with Lévy type intensity of jumps.

The multifractal spectrum of random functions and measures has attracted attention for many years and has been studied, for example, in Dembo et al. [1], Durand [2], Hu and Taylor [6], Klenke and Mörters [11], Le Gall and Perkins [13], Mörters and Shieh [14] and Perkins and Taylor [16]. The multifractal spectrum of singularities that describes the Hausdorff dimension of sets of different Hölder exponents of functions was investigated for deterministic and random functions in Jaffard [7–9] and Jaffard and Meyer [10].

We now turn to the description of our approach. We would like to verify the spectrum of singularities of $X_t(\cdot)$ on any open (random) set U whenever $X_t(U) > 0$. Based on the ideas of the proof of Theorem 1.1(b) in [15], it is enough to verify the spectrum of singularities of $X_t(\cdot)$ on any *fixed* open ball U in R. In what follows, we fix, without loss of generality, $U = (0, 1)$. The extension of our argument to general open U is trivial.

Next, we derive a representation for the density $X_t(\cdot)$, which will be used in the proof. Let p^α denote the continuous α-stable transition kernel related to the fractional Laplacian $\Delta_\alpha = -(-\Delta)^{\alpha/2}$ in R, and $(S_t^\alpha, t \geq 0)$ the related semigroup, that is,

$$S_t^\alpha f(x) = \int_{\mathsf{R}} p^\alpha(x-y) f(y)\, dy \qquad \text{for any bounded function } f$$

and

$$S_t^\alpha \nu(x) = \int_{\mathsf{R}} p^\alpha(x-y) \nu(dy) \qquad \text{for any finite measure } \nu.$$

Fix $X_0 = \mu \in \mathcal{M}_{\mathrm{f}} \setminus \{0\}$. First, we want to recall the *martingale decomposition* of the $(\alpha, 1, \beta)$-superprocess X (valid for any $\alpha \in (0, 2]$, $\beta \in (0, 1)$; see, e.g., [4], Lemma 1.6): for all sufficiently smooth bounded nonnegative functions φ on R and $t \geq 0$,

$$\langle X_t, \varphi\rangle = \langle \mu, \varphi\rangle + \int_0^t ds \langle X_s, \Delta_\alpha \varphi\rangle + M_t(\varphi) + a I_t(\varphi) \tag{1.6}$$

with discontinuous martingale

$$t \mapsto M_t(\varphi) := \int_{(0,t]\times\mathsf{R}\times\mathsf{R}_+} \tilde{\mathcal{N}}\big(d(s, x, r)\big) r\varphi(x) \tag{1.7}$$

and increasing process

$$t \mapsto I_t(\varphi) := \int_0^t ds \langle X_s, \varphi\rangle. \tag{1.8}$$

Here, $\tilde{\mathcal{N}} := \mathcal{N} - \widehat{\mathcal{N}}$, where $\mathcal{N}(d(s, x, r))$ is a random measure on $(0, \infty) \times \mathsf{R} \times (0, \infty)$ describing all the jumps $r\delta_x$ of X at times s at sites x of size r (which are the only discontinuities of the process X). Moreover,

$$\widehat{\mathcal{N}}\big(d(s, x, r)\big) = \varrho\, ds X_s(dx) r^{-2-\beta}\, dr \tag{1.9}$$

is the compensator of $\mathcal{N}$, where $\varrho := b(1 + \beta)\beta/\Gamma(1 - \beta)$ with Γ denoting the Gamma function.

Under our assumptions, the random measure $X_t(dx)$ is a.s. absolutely continuous for every fixed $t > 0$. From the Green function representation related to (1.6) (see, e.g., [4], (1.9)), we obtain the following representation of a version of the density function of $X_t(dx)$ (see, e.g., [4], (1.12)):

$$\begin{aligned} X_t(x) &= \mu * p_t^\alpha(x) + \int_{(0,t]\times\mathsf{R}} M\big(d(s, y)\big) p_{t-s}^\alpha(y - x) \\ &\quad + a \int_{(0,t]\times\mathsf{R}} I\big(d(s, y)\big) p_{t-s}^\alpha(y - x) \\ &=: Z^1(x) + Z^2(x) + Z^3(x), \qquad x \in \mathsf{R} \end{aligned} \tag{1.10}$$

(with notation in the obvious correspondence). Note that although $Z^i, i = 1, 2, 3$, depend on t, we omit the corresponding subscript since t is fixed throughout the paper. $M(d(s, y))$ in (1.10) is the martingale measure related to (1.7) and $I(d(s, y))$ the random measure related to (1.8). Note that by Lemma 1.7 of [4] the class of "legitimate" integrands with respect to the martingale measure $M(d(s, y))$ includes the set of functions ψ such that for some $p \in (1 + \beta, 2)$,

$$\int_0^T ds \int_{\mathsf{R}} dx S_s^\alpha \mu(x) \big|\psi(s, x)\big|^p < \infty \qquad \forall T > 0. \tag{1.11}$$

We let $\mathcal{L}_{\mathrm{loc}}^p$ denote the space of equivalence classes of measurable functions satisfying (1.11). For $\alpha > 1 + \beta$, it is easy to check that, for any $t > 0, z \in \mathsf{R}$,

$(s,x) \mapsto p^{\alpha}_{t-s}(z-x)1_{\{s<t\}}$ is in $\mathcal{L}^{p}_{\mathrm{loc}}$ for any $p \in (1+\beta, 2)$, and hence the stochastic integral in the representation (1.10) is well defined.

Z^1 is obviously twice differentiable. Moreover, it turns out (see Corollary 2.7) that Z^3 is Hölder continuous of index $\alpha 1_{\{\bar\eta_c>1\}} + 1_{\{\bar\eta_c\le 1\}}$. Noting that this index is not smaller that $\overline{\eta}_c$, we conclude that the multifractal structure of X_t coincides with that of Z^2. Recalling the definitions of Z^2 and $M(ds,dy)$, we see that there is a "competition" between branching and motion: jumps of the martingale measure M try to destroy smoothness of $X_t(\cdot)$ and p^{α} tries to make $X_t(\cdot)$ smoother. Thus, it is natural to expect that $\{x : H_{Z^2}(x) = \eta\}$ can be described by jumps of a certain order depending on η.

In Section 3, we construct a random set S_η such that $\dim(S_\eta) \le (1+\beta)(\eta - \eta_c)$ and $\{H_{Z^2}(x) = \eta\} \subseteq S_{\eta+\varepsilon}$. This, after some simple manipulations, allows us to obtain the bound $\dim(\{H_{Z^2}(x) = \eta\}) \le (1+\beta)(\eta - \eta_c)$.

It turns out that S_η is not very convenient for the derivation of the lower bound for $\dim(\{H_{Z^2}(x) = \eta\})$. For this reason, in Section 4, we introduce an alternative random set $\tilde J_{\eta,1}$ with $\dim(\tilde J_{\eta,1}) \ge (\beta+1)(\eta - \eta_c)$, on which we show existence of jumps which should lead to $H_{Z^2}(x) \le \eta$ for $x \in \tilde J_{\eta,1}$. However, if several jumps occur in a proximity of a point x then they can compensate each other. To show that this possible scenario does not affect the Hausdorff dimension of the set, $\{H_{Z^2}(x) = \eta\}$ is the most difficult part of our proof. This is not unexpected: in our previous papers [4, 5], the proofs of the optimality of Hölder indices were much harder than the derivation of the Hölder continuity. More precisely, we prove that such a compensation is possible only on a set of the Hausdorff dimension strictly smaller than $(\beta+1)(\eta - \eta_c)$, and hence this does not influence the dimension result.

2. Preliminaries.

2.1. *Estimates for the transition kernel of the α-stable motion.* The symbol C will always denote a generic positive constant, which might change from line to line. On the other hand, $C_{(\#)}$ denotes a constant appearing in formula line (or array) (#).

Throughout the paper, we will need the following bound; see [4], Lemma 2.1.

LEMMA 2.1. *For every* $\delta \in [0,1]$, *there exists a constant* $C > 0$ *such that*

$$|p^{\alpha}_t(x) - p^{\alpha}_t(y)| \le C \frac{|x-y|^{\delta}}{t^{\delta/\alpha}}\bigl(p^{\alpha}_t(x/2) + p^{\alpha}_t(y/2)\bigr), \qquad t > 0, x, y \in \mathsf{R}. \tag{2.1}$$

By methods very similar to those used for the proof of the previous lemma, one can get the following result.

LEMMA 2.2. (a) *For every* $\delta \in [0, 1]$, *there exists a constant* $C > 0$ *such that, for all* $t > 0$ *and* $x, y \in \mathsf{R}$,

$$\left|\frac{\partial p_t^\alpha(x)}{\partial x} - \frac{\partial p_t^\alpha(y)}{\partial y}\right| \le C\frac{|x-y|^\delta}{t^{(1+\delta)/\alpha}}\big(p_t^\alpha(x/2) + p_t^\alpha(y/2)\big). \tag{2.2}$$

(b) *There exists a constant* $C > 0$ *such that*

$$\left|\frac{\partial p_t^\alpha(x)}{\partial x}\right| \le Ct^{-1/\alpha}p_t^\alpha(x/2), \qquad t > 0, x \in \mathsf{R}. \tag{2.3}$$

An immediate corollary from the above lemma is as follows.

COROLLARY 2.3. *For every* $\delta \in [1, 2]$,

$$\begin{aligned}&\left|p_t^\alpha(x) - p_t^\alpha(y) - (x-y)\frac{\partial p_t^\alpha(y)}{\partial y}\right| \\ &\quad\le C\frac{|x-y|^\delta}{t^{\delta/\alpha}}\big(p_t^\alpha(x/2) + p_t^\alpha(y/2)\big), \qquad t > 0, x, y \in \mathsf{R}.\end{aligned} \tag{2.4}$$

With Lemma 2.2(b) at hand, it is easy to check that if $\beta < (\alpha - 1)/2$, then for any $t > 0, z \in \mathsf{R}$, $(s, x) \mapsto \frac{\partial p_{t-s}^\alpha(z-x)}{\partial x}1_{\{s<t\}}$ is in $\mathcal{L}_{\mathrm{loc}}^p$ for any $p \in (1 + \beta, \frac{1+\alpha}{2})$. Then, using again condition (1.11), it is easy to show the following result.

LEMMA 2.4. *Let* $\beta < (\alpha - 1)/2$. *Then for any fixed* $t > 0, x \in \mathsf{R}$, *the stochastic integral*

$$\int_{(0,t]\times\mathsf{R}} M\big(d(s, y)\big)\frac{\partial p_{t-s}^\alpha(y-x)}{\partial x}$$

is well defined.

In what follows, we let $\frac{\partial Z^2(x)}{\partial x}$ denote the integral $\int_{(0,t]\times\mathsf{R}} M(d(s, y))\frac{\partial p_{t-s}^\alpha(y-x)}{\partial x}$.

2.2. *Bound for stable processes*. Let $L = \{L_t : t \ge 0\}$ denote a spectrally positive stable process of index $\kappa \in (1, 2)$. That is, L is an R-valued time-homogeneous process with independent increments and with Laplace transform given by

$$\mathbf{E}e^{-\lambda L_t} = e^{t\lambda^\kappa}, \qquad \lambda, t \ge 0. \tag{2.5}$$

Let $\mathbf{\Delta}L_s := L_s - L_{s-} > 0$ denote the (positive) jumps of L. The next technical result gives an exponential upper bound for the tail of $\sup_{0\le u\le t}|L_u|$ under the condition that all the jumps of L are not too large.

LEMMA 2.5. *There exists a constant $C_{(2.6)} = C_{(2.6)}(\kappa)$ such that*

$$\begin{aligned}
&\mathbf{P}\Big(\sup_{0\le u\le t} |L_u| 1_{\{\sup_{0\le v\le u} \Delta L_v \le y\}} \ge x\Big) \\
&\qquad \le \left(\frac{C_{(2.6)} t}{x y^{\kappa-1}}\right)^{x/y} + \exp\left\{-\frac{x^{\kappa/(\kappa-1)}}{C_{(2.6)} t^{1/(\kappa-1)}}\right\}
\end{aligned} \tag{2.6}$$

and

$$\mathbf{P}\Big(\sup_{0\le u\le t} L_u 1_{\{\sup_{0\le v\le u} \Delta L_v \le y\}} \ge x\Big) \le \left(\frac{C_{(2.6)} t}{x y^{\kappa-1}}\right)^{x/y} \tag{2.7}$$

for all $t, x, y > 0$.

This bound (2.7) has been proven in [4] (see Lemma 2.3 there). To prove the inequality for $\sup_{0\le u\le t} |L_u|$, one has to combine (2.7) with Lemma 2.4 from [4].

2.3. *Analysis of Z^1 and Z^3.* Consider Z^1, Z^3 on the right-hand side of (1.10). Clearly, Z^1 is twice differentiable. Noting that $\bar{\eta}_c < 2$ for all α, β, we see that Z^1 does not affect the optimal Hölder exponent of X_t. As for Z^3, we have the following result.

LEMMA 2.6. *Let $\beta < (\alpha - 1)/2$. Then* $\mathbf{P}$*-a.s., $Z^3(x)$ is differentiable for any $x \in (0, 1)$, and the mapping*

$$x \mapsto \frac{d}{dx} Z^3(x), \qquad x \in (0, 1),$$

is, $\mathbf{P}$*-a.s., Hölder continuous with any exponent $\eta < \alpha - 1$.*

PROOF. Using Lemma 2.12 in [4] with $\theta = \delta = 1$, we see that $Z^3(x)$ is differentiable and, furthermore,

$$\frac{d}{dx} Z^3(x) = a \int_0^t ds \int_{\mathsf{R}} X_s(dy) \frac{\partial}{\partial x} p_{t-s}^{\alpha}(x - y).$$

Therefore, for any $x_1, x_2 \in (0, 1)$,

$$\begin{aligned}
&\left|\frac{d}{dx} Z^3(x_1) - \frac{d}{dx} Z^3(x_2)\right| \\
&\qquad \le |a| \int_0^t ds \int_{\mathsf{R}} X_s(dy) \left|\frac{\partial}{\partial x_1} p_{t-s}^{\alpha}(x_1 - y) - \frac{\partial}{\partial x_2} p_{t-s}^{\alpha}(x_2 - y)\right|.
\end{aligned}$$

Applying now Lemma 2.2 with $\delta < \alpha - 1$, we obtain

$$\begin{aligned}
&\left|\frac{d}{dx}Z^3(x_1) - \frac{d}{dx}Z^3(x_2)\right| \\
&\quad \le C|a||x_1 - x_2|^{\delta} \int_0^t ds (t-s)^{-(1+\delta)/\alpha} \\
&\qquad \times \int_{\mathsf{R}} X_s(dy)\left(p^{\alpha}_{t-s}\left(\frac{x_1 - y}{2}\right) + p^{\alpha}_{t-s}\left(\frac{x_2 - y}{2}\right)\right) \\
&\quad = C|a||x_1 - x_2|^{\delta} \int_0^t ds (t-s)^{-(1+\delta)/\alpha} \\
&\qquad \times \left(S^{\alpha}_{2^{\alpha}(t-s)} X_s(x_1) + S^{\alpha}_{2^{\alpha}(t-s)} X_s(x_2)\right) \\
&\quad \le C|a||x_1 - x_2|^{\delta} \sup_{s \le t, x \in (0,1)} S^{\alpha}_{2^{\alpha}(t-s)} X_s(x) \int_0^t ds s^{-(1+\delta)/\alpha} \\
&\quad = C\alpha(\alpha - 1 - \delta)^{-1}|a||x_1 - x_2|^{\delta} \sup_{s \le t, x \in (0,1)} S^{\alpha}_{2^{\alpha}(t-s)} X_s(x).
\end{aligned}$$

Taking into account Lemma 2.11 from [4], which states that

$$V := \sup_{s \le t, x \in (0,1)} S^{\alpha}_{2^{\alpha}(t-s)} X_s(x) < \infty, \qquad \mathbf{P}\text{-a.s.,} \tag{2.8}$$

we see that $x \mapsto \frac{d}{dx}Z^3(x)$ is Hölder continuous with the exponent δ. $\square$

Combining this lemma with [4], Remark 2.13, we obtain

COROLLARY 2.7. *$\mathbb{P}$-a.s., for any $x \in (0, 1)$ we have*

$$H_{Z^3}(x) \ge \alpha \mathsf{1}_{\{\bar{\eta}_c > 1\}} + \mathsf{1}_{\{\bar{\eta}_c \le 1\}}. \tag{2.9}$$

From this corollary and the fact that the right-hand side in (2.9) is not smaller than $\bar{\eta}_c$, we conclude that Z^3 does not affect the multifractal structure of X_t either. More precisely, the spectrum of singularities of X_t coincides with that of Z^2. Consequently, to prove Theorem 1.2, we have to determine Hausdorff dimensions of the sets

$$\begin{aligned}
\mathcal{E}_{Z^2,\eta} &:= \{x \in (0, 1) : H_{Z^2}(x) = \eta\}, \\
\tilde{\mathcal{E}}_{Z^2,\eta} &:= \{x \in (0, 1) : H_{Z^2}(x) \le \eta\},
\end{aligned}$$

and this is done in the next two sections.

3. Upper bound for the Hausdorff dimension. The aim of this section is to prove the following proposition.

PROPOSITION 3.1. *For every* $\eta \in [\eta_c, \overline{\eta}_c)$,

$$\dim(\mathcal{E}_{Z^2,\eta}) \le \dim(\tilde{\mathcal{E}}_{Z^2,\eta}) \le (1+\beta)(\eta - \eta_c), \qquad \mathbf{P}\text{-}a.s.$$

We need to introduce an additional notation. In what follows, for any $\eta \in (\eta_c, \bar{\eta}_c) \setminus \{1\}$, we fix an arbitrary small $\gamma = \gamma(\eta) \in (0, \frac{10^{-2}\eta_c}{\alpha})$ such that

$$\gamma < \begin{cases} \dfrac{10^{-2}}{\alpha} \min\{1-\eta, \eta\}, & \text{if } \eta < 1, \\ \dfrac{10^{-2}}{\alpha} \min\{\eta - 1, 2-\eta\}, & \text{if } \eta > 1, \end{cases}$$

and define

$$S_\eta := \bigl\{x \in (0,1) : \text{there exists a sequence } (s_n, y_n) \to (t, x) \\ \text{with } \mathbf{\Delta} X_{s_n}(\{y_n\}) \ge (t - s_n)^{1/(1+\beta) - \gamma} |x - y_n|^{\eta - \eta_c}\bigr\}.$$

To prove the above proposition we have to verify the following two lemmas.

LEMMA 3.2. *For every* $\eta \in (\eta_c, \overline{\eta}_c) \setminus \{1\}$, *we have*

$$\mathbf{P}\bigl(H_{Z^2}(x) \ge \eta - 2\alpha\gamma \textit{ for all } x \in (0,1) \setminus S_\eta\bigr) = 1.$$

LEMMA 3.3. *For every* $\eta \in (\eta_c, \overline{\eta}_c) \setminus \{1\}$, *we have*

$$\dim(S_\eta) \le (1+\beta)(\eta - \eta_c), \qquad \mathbf{P}\text{-}a.s.$$

With Lemmas 3.2 and 3.3 in hand, we immediately get the following.

PROOF OF PROPOSITION 3.1. It follows easily from Lemma 3.2 that $\tilde{\mathcal{E}}_{Z^2,\eta} \subset S_{\eta + 2\alpha\gamma + \varepsilon}$ for every $\varepsilon > 0$ and every $\eta \in (\eta_c, \overline{\eta}_c) \setminus \{1\}$. Therefore,

$$\dim(\tilde{\mathcal{E}}_{Z^2,\eta}) \le \lim_{\varepsilon \to 0} \dim(S_{\eta + 2\alpha\gamma + \varepsilon}).$$

Using Lemma 3.3, we then get

$$\dim(\tilde{\mathcal{E}}_{Z^2,\eta}) \le (1+\beta)(\eta + 2\alpha\gamma - \eta_c), \qquad \mathbf{P}\text{-a.s.}$$

Since γ can be chosen arbitrary small, the result for $\eta \ne 1$ follows immediately. The inequality for $\eta = 1$ follows from the monotonicity in η of the sets $\tilde{\mathcal{E}}_{Z^2,\eta}$. □

Let $\varepsilon \in (0, \eta_c/2)$ be arbitrarily small. We introduce a "good" event A^ε which will be frequently used throughout the proofs. On this event, with high probability, V from (2.8) is bounded by a constant, and there is a bound on the sizes of jumps. By Lemma 2.14 of [4], there exists a constant $C_{(3.1)} = C_{(3.1)}(\varepsilon, \gamma)$ such that

$$\mathbf{P}\bigl(|\mathbf{\Delta} X_s| > C_{(3.1)}(t-s)^{(1+\beta)^{-1} - \gamma} \text{ for some } s < t\bigr) \le \varepsilon/3. \tag{3.1}$$

Then we fix another constant $C_{(3.2)} = C_{(3.2)}(\varepsilon, \gamma)$ such that

$$\mathbf{P}(V \le C_{(3.2)}) \ge 1 - \varepsilon/3. \tag{3.2}$$

Recall that, by Theorem 1.2 in [4], $x \mapsto X_t(x)$ is $\mathbf{P}$-a.s. Hölder continuous with any exponent less than η_c. Hence, we can define a constant $C_{(3.3)} = C_{(3.3)}(\varepsilon)$ such that

$$\mathbf{P}\biggl(\sup_{x_1, x_2 \in (0,1), x_1 \neq x_2} \frac{|X_t(x_1) - X_t(x_2)|}{|x_1 - x_2|^{\eta_c - \varepsilon}} \le C_{(3.3)}\biggr) \ge 1 - \varepsilon/3. \tag{3.3}$$

Now we are ready to define

$$\begin{aligned} A^\varepsilon := {} & \bigl\{|\mathbf{\Delta} X_s| \le C_{(3.1)}(t-s)^{(1+\beta)^{-1} - \gamma} \text{ for all } s < t\bigr\} \\ & \cap \{V \le C_{(3.2)}\} \cap \biggl\{\sup_{x_1, x_2 \in (0,1), x_1 \neq x_2} \frac{|X_t(x_1) - X_t(x_2)|}{|x_1 - x_2|^{\eta_c - \varepsilon}} \le C_{(3.3)}\biggr\}. \end{aligned} \tag{3.4}$$

Clearly, by (3.1), (3.2) and (3.3), $\mathbf{P}(A^\varepsilon) \ge 1 - \varepsilon$. See (3.4) in [4] for the analogous definition.

The proof of Lemma 3.3 is rather short, so we will give it first.

PROOF OF LEMMA 3.3. To every jump (s, y, r) of the measure $\mathcal{N}$ (in what follows in the paper we will usually call them simply "jumps") with

$$(s, y, r) \in D_{j,n} := \bigl[t - 2^{-j}, t - 2^{-j-1}\bigr) \times (0,1) \times \bigl[2^{-n-1}, 2^{-n}\bigr)$$

we assign the ball

$$B^{(s,y,r)} := B\biggl(y, \biggl(\frac{2^{-n}}{(2^{-j-1})^{1/(1+\beta) - \gamma}}\biggr)^{1/(\eta - \eta_c)}\biggr). \tag{3.5}$$

We used here the obvious notation $B(y, \delta)$ for the ball in R with the center at y and radius δ. Define $n_0(j) := j[\frac{1}{1+\beta} - \frac{\gamma}{4}]$. It follows from (3.1) and (3.4) that, on A^ε, there are no jumps bigger than $2^{-n_0(j)}$ in the time interval $[t - 2^{-j}, t - 2^{-j-1})$.

It is easy to see that every point from S_η is contained in infinitely many balls $B^{(s,y,r)}$. Therefore, for every $J \ge 1$, the set

$$\bigcup_{j \ge J, n \ge 1} \bigcup_{(s,y,r) \in D_{j,n}} B^{(s,y,r)}$$

covers S_η. From (3.1) and (3.4), we conclude that, on A^ε, there are no jumps bigger than $C_{(3.1)} 2^{-(j+1)(1/(1+\beta) - \gamma)}$ in the time interval $s \in [t - 2^{-j}, t - 2^{-j-1})$ for any $j \ge 1$. Define $n_0(j) := j[\frac{1}{1+\beta} - \frac{\gamma}{4}]$. Clearly, there exists J_0 such that for all $j \ge J_0$ there are no jumps bigger than $2^{-n_0(j)}$ in the time interval $[t - 2^{-j}, t - 2^{-j-1})$. Hence, for every $J \ge J_0$, the set

$$S_\eta(J) := \bigcup_{j \ge J, n \ge n_0(j)} \bigcup_{(s,y,r) \in D_{j,n}} B^{(s,y,r)}$$

covers S_η for every $\omega \in A^\varepsilon$.

It follows from the formula for the compensator that, on the event $\{\sup_{s\le t} X_s((0,1)) \le N\}$, the intensity of jumps with $(s,y,r) \in D_{j,n}$ is bounded by

$$N2^{-j-1}\int_{2^{-n-1}}^{2^{-n}} \varrho r^{-2-\beta}\,dr = \frac{N\varrho(2^{1+\beta}-1)}{2(1+\beta)}2^{n(1+\beta)-j} =: \lambda_{j,n}.$$

Therefore, the intensity of jumps with $(s,y,r) \in \bigcup_{n=n_0(j)}^{n_1(j)} D_{j,n} =: \tilde{D}_j$, where $n_1(j) = j[\frac{1}{1+\beta} + \frac{\gamma}{4}]$, is bounded by

$$\sum_{n=n_0(j)}^{n_1(j)} \lambda_{j,n} \le \frac{N\varrho 2^\beta}{(\beta+1)} 2^{j(1+\beta)\gamma/4} =: \Lambda_j.$$

The number of such jumps does not exceed $2\Lambda_j$ with the probability $1 - e^{-(1-2\log 2)\Lambda_j}$. This is immediate from the exponential Chebyshev inequality applied to Poisson distributed random variables. Analogously, the number of jumps with $(s,y,r) \in D_{j,n}$ does not exceed $2\lambda_{j,n}$ with the probability at least $1 - e^{-(1-2\log 2)\lambda_{j,n}}$. Since

$$\sum_j \left(e^{-(1-2\log 2)\Lambda_j} + \sum_{n=n_1(j)}^{\infty} e^{-(1-2\log 2)\lambda_{j,n}} \right) < \infty,$$

we conclude, applying the Borel–Cantelli lemma that, for almost every ω from the set $A^\varepsilon \cap \{\sup_{s\le t} X_s((0,1)) \le N\}$, there exists $J(\omega)$ such that for all $j \ge J(\omega)$ and $n \ge n_1(j)$, the numbers of jumps in $\tilde{D}_j$ and in $D_{j,n}$ are bounded by $2\Lambda_j$ and $2\lambda_{j,n}$, respectively.

The radius of every ball corresponding to the jump in $\tilde{D}_j$ is bounded by $r_j := C2^{-(3\gamma)/(4(\eta-\eta_c))j}$. Thus, one can easily see that

$$\sum_{j=1}^{\infty}\left(2\Lambda_j r_j^\theta + \sum_{n=n_1(j)}^{\infty} 2\lambda_{j,n}\left(\frac{2^{-n}}{(2^{-j-1})^{1/(1+\beta)-\gamma}}\right)^{\theta/(\eta-\eta_c)}\right) < \infty$$

for every $\theta > (1+\beta)(\eta-\eta_c)$. This yields the desired bound for the Hausdorff dimension for almost every $\omega \in A^\varepsilon \cap \{\sup_{s\le t} X_s((0,1)) \le N\}$. Letting $N \to \infty$ and $\varepsilon \to 0$ completes the proof. □

The remaining part of this section will be devoted to the proof of Lemma 3.2. Since $S_\eta = \bigcap_{J\ge1} S_\eta(J)$,

$$\begin{aligned}\{H_{Z^2}(x) \ge \eta - 2\alpha\gamma, \forall x \in (0,1)\setminus S_\eta\} \\ = \bigcap_{J\ge1}\{H_{Z^2}(x) \ge \eta - 2\alpha\gamma, \forall x \in (0,1)\setminus S_\eta(J)\}.\end{aligned}$$

Thus, it suffices to show that

$$\mathbf{P}\big(H_{Z^2}(x) \ge \eta - 2\alpha\gamma, \forall x \in (0,1)\setminus S_\eta(J)\big) = 1 \tag{3.6}$$

for every $J \ge 1$.

Before we start proving (3.6), let us introduce some further notation. For any $x_1, x_2 \in \mathsf{R}$, $\eta \in (\eta_c, \bar{\eta}_c)$, define

$$\tilde{p}_s^{\alpha,\eta}(x, y) := \begin{cases} p_s^\alpha(x) - p_s^\alpha(y), & \text{if } \eta \le 1, s > 0, \\ p_s^\alpha(x) - p_s^\alpha(y) - (x - y)\dfrac{\partial p_s^\alpha(y)}{\partial y}, & \text{if } \eta \in (1, \bar{\eta}_c), s > 0, \end{cases}$$

$$\tilde{p}_s^{\alpha,\eta,\prime}(x, y) := \frac{\partial p_s^\alpha(x)}{\partial x} - \frac{\partial p_s^\alpha(y)}{\partial y}, \qquad \eta \in (1, \bar{\eta}_c),$$

$$\tilde{Z}_s^{2,\eta}(x_1, x_2) := \int_0^s \int_{\mathsf{R}} M\big(d(u, y)\big)\tilde{p}_{t-u}^{\alpha,\eta}(x_1 - y, x_2 - y), \qquad s \in [0, t],$$

$$\boldsymbol{\Delta}\tilde{Z}_s^{2,\eta}(x_1, x_2) := \tilde{Z}_s^{2,\eta}(x_1, x_2) - \tilde{Z}_{s-}^{2,\eta}(x_1, x_2), \qquad s \in (0, t],$$

$$\tilde{Z}_s^{2,\eta,\prime}(x_1, x_2) := \int_0^s \int_{\mathsf{R}} M\big(d(u, y)\big)\tilde{p}_{t-u}^{\alpha,\eta,\prime}(x_1 - y, x_2 - y), \qquad s \in [0, t],$$

$$\boldsymbol{\Delta}\tilde{Z}_s^{2,\eta,\prime}(x_1, x_2) := \tilde{Z}_s^{2,\eta,\prime}(x_1, x_2) - \tilde{Z}_{s-}^{2,\eta,\prime}(x_1, x_2), \qquad \eta \in (1, \bar{\eta}_c), s \in (0, t].$$

Also for any $N, J \ge 1$, let

$$\tilde{S}_\eta(N, J) := \big\{(x_1, x_2) \in \mathsf{R}^2 : \exists x_0 \in (0, 1) \setminus S_\eta(J) \text{ such that } x_1, x_2 \in B\big(x_0, 2^{-N}\big)\big\}$$

and

$$\tilde{S}'_\eta(J) = \big\{(x_1, x_2) \in \mathsf{R}^2 : \exists x_0 \in (0, 1) \setminus S_\eta(J) \text{ such that } x_1, x_2 \in B\big(x_0, 4|x_1 - x_2|\big)\big\}.$$

We split the proof of (3.6) into several steps.

LEMMA 3.4. *Fix arbitrary* (*deterministic*) $x_1, x_2 \in \mathsf{R}$, *and* $\eta \in (\eta_c, \bar{\eta}_c)$. *Then for any* $N, J \ge 1$, *there exists a constant* $C_{(3.7)}(J) \ge 1$ *such that*

$$\begin{aligned} &\big|\boldsymbol{\Delta}\tilde{Z}_s^{2,\eta}(x_1, x_2)\big| 1_{A^\varepsilon} 1_{\{(x_1,x_2)\in\tilde{S}_\eta(N,J)\}} \\ &\quad \le C_{(3.7)}(J)|x_1 - x_2|^{\eta_c - \alpha\gamma} 2^{-N(\eta-\eta_c)} \qquad \forall s \le t. \end{aligned} \tag{3.7}$$

PROOF. Let (y, s, r) be the point of an arbitrary jump of the measure $\mathcal{N}$ with $s \le t$. Then for the corresponding jump of $\tilde{Z}_s^{2,\eta}(x_1, x_2)$ we get the following bound:

$$\big|\boldsymbol{\Delta}\tilde{Z}_s^{2,\eta}(x_1, x_2)\big| \le r\big|\tilde{p}_{t-s}^{\alpha,\eta}(x_1 - y, x_2 - y)\big|. \tag{3.8}$$

Now on the event $\{(x_1, x_2) \in \tilde{S}_\eta(N, J)\}$, there exists a point $x_0 \in (0, 1) \setminus S_\eta(J)$ such that $x_1, x_2 \in B(x_0, 2^{-N})$, and for $s \ge t - 2^{-J}$ we have

$$r \le (t - s)^{1/(1+\beta)-\gamma}|y - x_0|^{\eta-\eta_c}.$$

This and (3.8) imply that for $s \ge t - 2^{-J}$

$$I = I(s, y, x_1, x_2) := \left|\mathbf{\Delta}\tilde{Z}_s^{2,\eta}(x_1, x_2)\right| 1_{A^\varepsilon} 1_{\{(x_1,x_2)\in\tilde{S}_\eta(N,J)\}} \le (t-s)^{1/(1+\beta)-\gamma}|y - x_0|^{\eta-\eta_c}\left|\tilde{p}_{t-s}^{\alpha,\eta}(x_1 - y, x_2 - y)\right|. \tag{3.9}$$

Applying Lemma 2.1 (if $\eta \le 1$) or Corollary 2.3 (if $\eta > 1$) with $\delta = \eta - \alpha\gamma$ to $\tilde{p}_{t-s}^{\alpha,\eta}(x_1 - y, x_2 - y)$, we conclude from (3.9) that, for $s \ge t - 2^{-J}$,

$$\begin{aligned} I &\le (t-s)^{-(\eta-\eta_c)/\alpha}|y - x_0|^{\eta-\eta_c}|x_1 - x_2|^{\eta-\alpha\gamma} \\ &\quad \times \left(p_1^\alpha\left(\frac{x_1 - y}{2(t-s)^{1/\alpha}}\right) + p_1^\alpha\left(\frac{x_2 - y}{2(t-s)^{1/\alpha}}\right)\right) \\ &\le C_{(3.10)}(t-s)^{-(\eta-\eta_c)/\alpha}|y - x_0|^{\eta-\eta_c}|x_1 - x_2|^{\eta-\alpha\gamma} \\ &\quad \times \left(\frac{|x_1 - y| + |x_2 - y|}{(t-s)^{1/\alpha}} \vee 1\right)^{-\alpha-1}, \end{aligned} \tag{3.10}$$

where the last inequality follows from the standard bound

$$p_1^\alpha(z) \le C_{(3.11)}\big(|z| \vee 1\big)^{-\alpha-1}, \qquad z \in \mathsf{R}. \tag{3.11}$$

One can easily check by separating the cases $|x_1 - y| + |x_2 - y| < (t-s)^{1/\alpha}$ and $|x_1 - y| + |x_2 - y| \ge (t-s)^{1/\alpha}$ that

$$|x_1 - x_2|^{\eta-\eta_c}\left(\frac{|x_1 - y| + |x_2 - y|}{(t-s)^{1/\alpha}} \vee 1\right)^{-\alpha-1} \le (t-s)^{(\eta-\eta_c)/\alpha},$$

and hence

$$I \le C_{(3.10)}|x_1 - x_2|^{\eta_c-\alpha\gamma}|y - x_0|^{\eta-\eta_c}, \qquad s \ge t - 2^{-J}. \tag{3.12}$$

If $|y - x_0| \le 2^{-N+1}$, then we obtain the bound

$$I \le 2C_{(3.10)}|x_1 - x_2|^{\eta_c-\alpha\gamma}2^{-N(\eta-\eta_c)}, \qquad s \ge t - 2^{-J}. \tag{3.13}$$

Now consider the case $|y - x_0| > 2^{-N+1}$. Here, we treat separately two subcases: $|y - x_0| \le (t-s)^{1/\alpha}$ and $|y - x_0| > (t-s)^{1/\alpha}$. First, if $|y - x_0| \le (t-s)^{1/\alpha}$, then it follows from (3.10) that

$$\begin{aligned} I &\le C_{(3.10)}(t-s)^{-(\eta-\eta_c)/\alpha}|y - x_0|^{\eta-\eta_c}|x_1 - x_2|^{\eta-\alpha\gamma} \\ &\le C_{(3.10)}|x_1 - x_2|^{\eta-\alpha\gamma} \\ &\le C_{(3.10)}|x_1 - x_2|^{\eta_c-\alpha\gamma}2^{-(N-1)(\eta-\eta_c)}, \qquad s \ge t - 2^{-J}. \end{aligned} \tag{3.14}$$

Second, if $|y - x_0| > (t-s)^{1/\alpha}$, then we recall that $|y - x_0| > (t-s)^{1/\alpha} \vee 2^{-N+1}$ and $|x_i - x_0| \le 2^{-N}$, $i = 1, 2$, to get that

$$|x_i - y| \ge |x_0 - y|/2, \qquad i = 1, 2. \tag{3.15}$$

Combining this with (3.10), we obtain

$$
\begin{aligned}
I &\le C_{(3.10)}(t-s)^{-(\eta-\eta_c)/\alpha}|y-x_0|^{\eta-\eta_c}|x_1-x_2|^{\eta-\alpha\gamma}\left(\frac{|x_0-y|}{(t-s)^{1/\alpha}}\right)^{-\alpha-1} \\
&\le C_{(3.10)}|x_1-x_2|^{\eta-\alpha\gamma}\left(\frac{|x_0-y|}{(t-s)^{1/\alpha}}\right)^{\eta-\eta_c-\alpha-1} \\
&\le C_{(3.10)}|x_1-x_2|^{\eta-\alpha\gamma} \\
&\le C_{(3.10)}|x_1-x_2|^{\eta_c-\alpha\gamma}2^{-(N-1)(\eta-\eta_c)}, \qquad s \ge t-2^{-J}.
\end{aligned}
\tag{3.16}
$$

Finally, we consider the jumps (y,s,r) with $s < t-2^{-J}$. On the event A^{ε},

$$
r \le (t-s)^{1/(1+\beta)-\gamma}.
$$

Using Lemma 2.1 (or Corollary 2.3) with $\delta = \eta-\alpha\gamma$ once again, we see from (3.8) that

$$
\begin{aligned}
I &\le C_{(3.17)}|x_1-x_2|^{\eta-\alpha\gamma}(t-s)^{-(\eta-\eta_c)/\alpha} \\
&\le C_{(3.17)}2^{J(\eta-\eta_c)/\alpha}|x_1-x_2|^{\eta-\alpha\gamma} \\
&\le C_{(3.17)}2^{J(\eta-\eta_c)/\alpha}|x_1-x_2|^{\eta_c-\alpha\gamma}2^{-(N-1)(\eta-\eta_c)}, \qquad s < t-2^{-N}.
\end{aligned}
\tag{3.17}
$$

Combining (3.13)–(3.17), we get the desired result. □

By a similar argument, we can get the following result.

LEMMA 3.5. *Let* $\bar{\eta}_c > 1$. *Fix arbitrary (deterministic)* $x_1, x_2 \in \mathsf{R}$, *and* $\eta \in (1, \bar{\eta}_{\mathrm{c}})$. *Then for any* $J \ge 1$, *there exists a constant* $C_{(3.18)}(J)$ *such that*

$$
\begin{aligned}
&\left|\mathbf{\Delta}\tilde{Z}_s^{\eta,2,\prime}(x_1,x_2)\right|\mathbf{1}_{A^{\varepsilon}}\mathbf{1}_{\{(x_1,x_2)\in\tilde{S}'_\eta(J)\}} \\
&\qquad \le C_{(3.18)}(J)|x_1-x_2|^{\eta-1-\alpha\gamma} \qquad \forall s \le t.
\end{aligned}
\tag{3.18}
$$

Having an upper bound for absolute values of the jumps of $\tilde{Z}^2(x_1,x_2)$, we can give some estimate for $\tilde{Z}_t^2(x_1,x_2)$ itself.

LEMMA 3.6. *Fix arbitrary (deterministic)* $x_1, x_2 \in (0,1)$, *and* $\eta \in (\eta_{\mathrm{c}}, \bar{\eta}_{\mathrm{c}})$.

(a) *Then there exists a constant* $C_{(3.19)}$, *such that for any* $N, J \ge 1$,

$$
\begin{aligned}
&\mathbf{P}\big(\big|\tilde{Z}_t^{2,\eta}(x_1,x_2)\big| \ge 2C_{(3.7)}(J)|x_1-x_2|^{\eta_c-2\alpha\gamma}2^{-N(\eta-\eta_c)}, A^{\varepsilon}, \\
&\qquad\qquad (x_1,x_2)\in\tilde{S}_\eta(N,J)\big) \\
&\qquad \le \big(C_{(3.19)}2^{-\alpha\gamma N}\big)^{|x_1-x_2|^{-\alpha\gamma}}.
\end{aligned}
\tag{3.19}
$$

(b) *Let* $\bar\eta_c > 1$ *and assume* $\eta \in (1, \bar\eta_c)$. *Then there exists a constant* $C_{(3.20)}$, *such that for any* $J \geq 1$,

$$\begin{aligned}
&\mathbf{P}\big(\big|\tilde Z_t^{2,\eta,\prime}(x_1,x_2)\big| \geq 2C_{(3.18)}(J)|x_1-x_2|^{\eta-1-2\alpha\gamma}, A^\varepsilon, (x_1,x_2) \in \tilde S'_\eta(J)\big)\\
&\qquad \leq \big(C_{(3.20)}|x_1-x_2|^{\alpha\gamma}\big)^{|x_1-x_2|^{-\alpha\gamma}}.
\end{aligned} \tag{3.20}$$

PROOF. (a) According to Lemma 2.15 from [4], there exist spectrally positive $(1+\beta)$-stable processes L^+ and L^- such that

$$\tilde Z_s^{2,\eta}(x_1,x_2) = L^+_{T^\eta_+(s)} - L^-_{T^\eta_-(s)}, \qquad s \leq t, \tag{3.21}$$

where

$$T^\eta_\pm(s) := \int_0^s du \int_{\mathbb{R}} X_u(dy)\big(\big(\tilde p^{\alpha,\eta}_{t-u}(x_1-y,x_2-y)\big)^\pm\big)^{1+\beta}, \qquad 0 \leq s \leq t. \tag{3.22}$$

(Note that L^+, L^- also depend on η; however, we omit the corresponding superindex η to simplify the notation.) Therefore, we get

$$\begin{aligned}
&\mathbf{P}\big(\big|\tilde Z_t^{2,\eta}(x_1,x_2)\big| \geq 2C_{(3.7)}(J)|x_1-x_2|^{\eta_c-2\alpha\gamma}2^{-N(\eta-\eta_c)}, A^\varepsilon,\\
&\qquad\qquad \big(x_1,x_2 \in \tilde S_\eta(N,J)\big)\big)\\
&\quad \leq \mathbf{P}\big(\big|L^+_{T^\eta_+(s)}\big| \geq C_{(3.7)}(J)|x_1-x_2|^{\eta_c-2\alpha\gamma}2^{-N(\eta-\eta_c)}, A^\varepsilon,\\
&\qquad\qquad \big(x_1,x_2 \in \tilde S_\eta(N,J)\big)\big)\\
&\qquad + \mathbf{P}\big(\big|L^-_{T^\eta_-(s)}\big| \geq C_{(3.7)}(J)|x_1-x_2|^{\eta_c-2\alpha\gamma}2^{-N(\eta-\eta_c)}, A^\varepsilon,\\
&\qquad\qquad \big(x_1,x_2 \in \tilde S_\eta(N,J)\big)\big).
\end{aligned} \tag{3.23}$$

By going through the derivation of (3.43) in [5], one can easily get that in our setting, on the event A^ε, for any $\eta \in (\eta_c, \bar\eta_c)$ and any $\varepsilon_1 \in (0, \alpha\beta\gamma)$, there exists a constant $C_{(3.24)} = C_{(3.24)}(\varepsilon, \varepsilon_1, \eta)$ such that

$$T^\eta_\pm(t) \leq C_{(3.24)}|x_1-x_2|^{\alpha-\beta-\varepsilon_1} =: \hat T(x_1,x_2). \tag{3.24}$$

From this bound, Lemmas 2.5 and 3.4 we get

$$\begin{aligned}
&\mathbf{P}\big(\big|L^\pm_{T^\eta_\pm(t)}\big| \geq C_{(3.7)}(J)|x_1-x_2|^{\eta_c-2\alpha\gamma}2^{-N(\eta-\eta_c)}, A^\varepsilon, \big(x_1,x_2 \in \tilde S_\eta(N,J)\big)\big)\\
&\quad \leq \mathbf{P}\Big(\big|L^\pm_{T^\eta_\pm(t)}\big| \geq C_{(3.7)}(J)|x_1-x_2|^{\eta_c-2\alpha\gamma}2^{-N(\eta-\eta_c)}, A^\varepsilon,\\
&\qquad\qquad \sup_{s \leq T^\eta_\pm} \mathbf{\Delta} L^\pm_s \leq C_{(3.7)}(J)|x_1-x_2|^{\eta_c-\alpha\gamma}2^{-N(\eta-\eta_c)}\Big)\\
&\quad \leq \mathbf{P}\Big(\sup_{0\leq s\leq \hat T(x_1,x_2)} |L^\pm_s| 1\Big\{\sup_{0\leq v\leq s} \mathbf{\Delta} L^\pm_v \leq C_{(3.7)}(J)|x_1-x_2|^{\eta_c-\alpha\gamma}2^{-N(\eta-\eta_c)}\Big\}\\
&\qquad\qquad\qquad \geq C_{(3.7)}(J)|x_1-x_2|^{\eta_c-2\alpha\gamma}2^{-N(\eta-\eta_c)}\Big)
\end{aligned}$$

$$\le \left(\frac{C_{(2.6)} C_{(3.24)} |x_1 - x_2|^{\alpha - \beta - \varepsilon_1}}{|x_1 - x_2|^{-\alpha\gamma} (C_{(3.7)}(J) |x_1 - x_2|^{\eta_c - \alpha\gamma} 2^{-N(\eta - \eta_c)})^{1+\beta}} \right)^{|x_1 - x_2|^{-\alpha\gamma}}$$

$$+ \exp\left\{ - \frac{(C_{(3.7)}(J) |x_1 - x_2|^{\eta_c - 2\alpha\gamma} 2^{-N(\eta - \eta_c)})^{(1+\beta)/\beta}}{C_{(2.6)} (C_{(3.24)} |x_1 - x_2|^{\alpha - \beta - \varepsilon_1})^{1/\beta}} \right\}$$

$$= \left(C \frac{|x_1 - x_2|^{\alpha\gamma(2+\beta) - \varepsilon_1 + 1}}{2^{-N(\eta - \eta_c)(1+\beta)}} \right)^{|x_1 - x_2|^{-\alpha\gamma}}$$

$$+ \exp\{ -c |x_1 - x_2|^{-1/\beta + \varepsilon_1/\beta - 2\alpha\gamma(1+\beta)/\beta} 2^{-N(\eta - \eta_c)(1+\beta)/\beta} \}$$

$$= O\big((2^{-\alpha\gamma N})^{|x_1 - x_2|^{-\alpha\gamma}} \big),$$

where the last equality follows since $(\eta - \eta_c)(1 + \beta) \le 1$, $\varepsilon_1 \le \alpha\beta\gamma$, $|x_1 - x_2| \le 2^{-N+1}$. (We omit here some elementary arithmetic calculations.)

The claim follows now from (3.23).

(b) The proof goes along the similar lines. □

LEMMA 3.7. *Let* $J \ge 1$, $\bar{\eta}_c > 1$, $\eta \in (1, \bar{\eta}_c)$. *For almost every* $\omega \in A^\varepsilon$, *there exists* $N_2 = N_2(\omega)$ *such that for all* $n \ge N_2$, *and*

$$(x_1, x_2) \in \tilde{S}'_\eta(J) \cap \{ (i2^{-n}, j2^{-n}), i, j \in \mathbb{Z}, |i - j| = 1 \}$$

the following holds:

$$\big| \tilde{Z}_t^{2,\eta,\prime}(x_1, x_2) \big| \le 2C_{(3.18)}(J) |x_1 - x_2|^{\eta - 1 - 2\alpha\gamma}.$$

PROOF. Define

$$M_n := \max\{ \big| \tilde{Z}_t^{2,\eta,\prime}(x_1, x_2) \big| : (x_1, x_2) \in \{ (i2^{-n}, j2^{-n}), i, j \in \mathbb{Z}, |i - j| = 1 \} \cap (0, 1) \cap \tilde{S}'_\eta(J) \}.$$

Applying Lemma 3.6(b), we obtain

$$\mathbf{P}\big(M_n \ge 2C_{(3.18)}(J) 2^{-n(\eta - 1 - 2\alpha\gamma)}; A^\varepsilon \big) \le 2^n \big(C_{(3.20)} 2^{-n\alpha\gamma} \big)^{2^{n\alpha\gamma}}.$$

Let

$$A_N := \{ M_n \ge 2C_{(3.18)}(J) 2^{-n(\eta - 1 - 2\alpha\gamma)} \text{ for some } n \ge N \}.$$

It is clear that

$$\sum_{N=1}^{\infty} \mathbf{P}(A_N \cap A^\varepsilon) \le \sum_{N=1}^{\infty} \sum_{n=N}^{\infty} \mathbf{P}\big(M_n \ge 2C_{(3.18)}(J) 2^{-n(\eta - 1 - 2\alpha\gamma)}; A^\varepsilon \big)$$

$$\le \sum_{N=1}^{\infty} \sum_{n=N}^{\infty} 2^n \big(C_{(3.20)} 2^{-n\alpha\gamma} \big)^{2^{n\alpha\gamma}}$$

$$\le \sum_{N=1}^{\infty} \sum_{n=N}^{\infty} 2^n \left(C_{(3.20)} 2^{-n\alpha\gamma}\right)^{2^{n\alpha\gamma}}$$

$$\le C \sum_{N=1}^{\infty} 2^{-\alpha\gamma N} < \infty$$

and we are done by the Borel–Cantelli lemma. □

LEMMA 3.8. *Let $J \ge 1$, $\eta \in (\eta, \bar{\eta}_c)$. For almost every $\omega \in A^\varepsilon$, there exists $N_1 = N_1(\omega)$ such that for all $n \ge N \ge N_1$,*

$$(x_1, x_2) \in \tilde{S}_\eta(N, J) \cap \{(i2^{-n}, j2^{-n}), i, j \in \mathbb{Z}\}$$

with $|x_1 - x_2| \le 2^{-\log^2 n}$ we have the inequality

$$|\tilde{Z}_t^{2,\eta}(x_1, x_2)| \le 2C_{(3.7)}(J)|x_1 - x_2|^{\eta_c - 2\alpha\gamma} 2^{-N(\eta - \eta_c)}.$$

PROOF. Define events

$$B(x_1, x_2) := \{\tilde{Z}_t^{2,\eta}(x_1, x_2) \ge 2C_{(3.7)}(J)|x_1 - x_2|^{\eta_c - 2\alpha\gamma} 2^{-N(\eta - \eta_c)}\},$$

and

$$A_{n,N} := \bigcup_{\substack{x_1, x_2 \in \{i2^{-n}, i \in \mathbb{Z}\} \\ \cap (0,1) \cap \tilde{S}_\eta(N, J): \\ \{|x_1 - x_2| \le 2^{-\log^2 n}\}}} B(x_1, x_2).$$

Applying Lemma 3.6(a), we obtain

$$\mathbf{P}(A_{n,N} \cap A^\varepsilon) \le 2^{2n} \left(C_{(3.19)} 2^{-\alpha\gamma N}\right)^{2^{\alpha\gamma \log^2 n}}.$$

Let

$$A_N := \bigcup_{n \ge N} A_{n,N}.$$

It is clear that

$$\sum_{N=1}^{\infty} \mathbf{P}(A_N \cap A^\varepsilon) \le \sum_{N=1}^{\infty} \sum_{n=N}^{\infty} \mathbf{P}(A_{n,N} \cap A^\varepsilon)$$

$$\le \sum_{N=1}^{\infty} \sum_{n=N}^{\infty} 2^{2n} \left(C_{(3.19)} 2^{-\alpha\gamma N}\right)^{2^{\alpha\gamma \log^2 n}}$$

$$\le C \sum_{N=1}^{\infty} 2^{-\alpha\gamma N} < \infty$$

and we are done by the Borel–Cantelli lemma. □

LEMMA 3.9. *Let $J \geq 1$, $\bar{\eta}_c > 1$, $\eta \in (1, \bar{\eta}_c)$. Fix an integer $k_0 > \max\{1 + \frac{1}{\alpha-1-\beta}, 3\}$. For almost every $\omega \in A^{\varepsilon}$ and for all $x \in (0,1) \setminus S_\eta(J)$, there exists $V'(x) = V'(x,\omega)$ and $C_{(3.25)}(J,\omega)$ such that*

$$\begin{aligned} &\left|Z^2(y) - Z^2(x) - (y-x)V'(x)\right| \\ &\quad \leq C_{(3.25)}(J)|y-x|^{\eta-2\alpha\gamma} \qquad \forall y \in B\left(x, 2^{-N_3}\right), \end{aligned} \tag{3.25}$$

where

$$N_3 = \max\left\{N_2(\omega), N_1(\omega), \log_2\left(\left|V'(x)\right|\right), (k_0)^{10}\right\} + 2$$

and N_2, N_1 are from Lemmas 3.7 and 3.8.

REMARK 3.10. $Z^2(x)$ [similarly for $Z^2(y)$] at a random point x is defined via $Z^2(x) = X_t(x) - Z^1(x) - Z^3(x)$, where all the terms on the right-hand side are well defined.

REMARK 3.11. The lemma shows that $V'(x)$ is in fact a spatial derivative of $Z^2(x)$ at the point x.

PROOF OF LEMMA 3.9. First, we will define $V'(y)$ for fixed points y. For any $y \in \mathsf{R}$, let

$$V'(y) := \int_0^t \int_{\mathsf{R}} M\left(d(u,z)\right) \frac{\partial}{\partial y} p_{t-u}^{\alpha}(y-z).$$

Let x and ω be as in the statement of the lemma. For any $n \geq 1$, take $x_n \in \{i2^{-n}, i \in \mathbb{Z}\}$ satisfying the following conditions:

$$|x_n - x| \leq 2^{-n}, \qquad |x_n - x_{n+1}| = 2^{-n-1} \qquad \forall n \geq 1. \tag{3.26}$$

Applying Lemma 3.7, we get for every $n \geq N_3$ the bound

$$\begin{aligned} \left|V'(x_n) - V'(x_{n+1})\right| &= \left|\tilde{Z}_t^{2,\eta,\prime}(x_n, x_{n+1})\right| \\ &\leq 2C_{(3.18)}(J)2^{-n(\eta-1-2\alpha\gamma)}. \end{aligned}$$

Then for any $m > n \geq N_3$ we have

$$\begin{aligned} \left|V'(x_n) - V'(x_m)\right| &\leq \sum_{k=n}^{m-1} \left|V'(x_k) - V'(x_{k+1})\right| \\ &\leq C_{(3.27)}(J)2^{-n(\eta-1-2\alpha\gamma)}. \end{aligned} \tag{3.27}$$

This implies that $\{V'(x_n)\}_{n\geq 1}$ is a Cauchy sequence and we denote the limit by $V'(x)$. Moreover, it is easy to check that

$$\left|V'(x_n) - V'(x)\right| \leq C_{(3.27)}(J)2^{-n(\eta-1-2\alpha\gamma)}, \qquad n \geq N_3. \tag{3.28}$$

Now let us check (3.25). Let

$$y \in B(x, 2^{-N_3}) \setminus \{x\}.$$

Then we can fix an integer $N^* \geq N_3$ such that

$$2^{-N^*-1} \leq |x-y| \leq 2^{-N^*}. \tag{3.29}$$

Fix a sequence $\{x_n\}_{n\geq 1}$ satisfying (3.26), and $\{y_n\}_{n\geq 1}$ satisfying the same condition with y instead of x. Then for any $n \geq N_3$ we have

$$\begin{aligned}
&\left|Z^2(y) - Z^2(x) - (y-x)V'(x)\right| \\
&\quad \leq \left|Z^2(y_n) - Z^2(x_n) - (y_n - x_n)V'(x_n)\right| \\
&\qquad + \left|Z^2(y_n) - Z^2(y)\right| + \left|Z^2(x_n) - Z^2(x)\right| \\
&\qquad + |y_n - y| \times \left|V'(x)\right| + |x_n - x| \times \left|V'(x)\right| \\
&\qquad + |y_n - x_n| \times \left|V'(x) - V'(x_n)\right|.
\end{aligned} \tag{3.30}$$

In what follows fix

$$n = k_0 N^*. \tag{3.31}$$

Then we have

$$\begin{aligned}
&|y_n - y| \times \left|V'(x)\right| + |x_n - x| \times \left|V'(x)\right| \\
&\quad \leq 2 \cdot 2^{-k_0 N^*}\left|V'(x)\right| \\
&\quad \leq 2 \cdot 2^{-N^*(\eta - 2\alpha\gamma)} 2^{-N^*}\left|V'(x)\right| \qquad (\text{since } \eta \leq 2, k_0 \geq 3) \\
&\quad \leq \left(2|x-y|\right)^{\eta - 2\alpha\gamma} 2^{1-N^*}\left|V'(x)\right| \qquad [\text{by } (3.29)] \\
&\quad \leq \left(|x-y|\right)^{\eta - 2\alpha\gamma} 2^{2-N^*}\left|V'(x)\right| \\
&\quad \leq |x-y|^{\eta - 2\alpha\gamma}, \qquad \forall n \geq N_1',
\end{aligned} \tag{3.32}$$

where the last inequality follows from $N^* \geq \log_2(|V'(x)|) + 2$. Now by triangle inequality and (3.31) we get

$$\begin{aligned}
|x_n - y_n| &\leq 2^{1-n} + |x-y| \\
&\leq 2^{1-N^*} \\
&\leq 4|x-y|.
\end{aligned} \tag{3.33}$$

This, (3.31) and (3.28) imply

$$\begin{aligned}
|y_n - x_n| \cdot \left|V'(x) - V'(x_n)\right| &\leq 4|x-y| \cdot C_{(3.27)}(J) 2^{-N^* k_0(\eta - 1 - 2\alpha\gamma)} \\
&\leq 8C(J)|x-y| \cdot |x-y|^{\eta - 1 - 2\alpha\gamma} \\
&\leq 8C(J)|x-y|^{\eta - 2\alpha\gamma}.
\end{aligned} \tag{3.34}$$

Now recall that Z^2 is Hölder continuous with any exponent less than η_c (see Theorem 2 in [4]) to get that there exists $C = C(\omega)$ such that

$$\begin{aligned}&|Z^2(y_n) - Z^2(y)| + |Z^2(x_n) - Z^2(x)|\\&\quad \le C(\omega)(|y_n - y|^{\eta_c-(2\alpha\gamma)/k_0} + |x_n - x|^{\eta_c-(2\alpha\gamma)/k_0}) \qquad \forall y \in B(x, 2^{-N_3}).\end{aligned}$$

Recalling that

$$|y_n - y|, |x_n - x| \le 2^{-n} \tag{3.35}$$

and (3.31) we get

$$\begin{aligned}|Z^2(y_n) - Z^2(y)| + |Z^2(x_n) - Z^2(x)| &\le 2C(\omega)2^{-n(\eta_c-(2\alpha\gamma)/k_0)}\\&\le 2C(\omega)2^{-N^*(k_0\eta_c-2\alpha\gamma)}\\&\le 2C(\omega)2^{-N^*(\eta-2\alpha\gamma)},\end{aligned}$$

where the last inequality follows since by assumption $k_0 > 1 + 1/(\alpha - 1 - \beta)$ and hence $k_0\eta_c > \bar{\eta}_c > \eta$. By (3.29), we immediately get

$$|Z^2(y_n) - Z^2(y)| + |Z^2(x_n) - Z^2(x)| \le 8C(\omega)|x - y|^{\eta-2\alpha\gamma}. \tag{3.36}$$

Now use again (3.35) and triangle inequality to get that

$$|y_n - x| \le |y_n - y| + |y - x| \le 2^{-(N^*-1)}.$$

This together with the (3.35) and the definition of $\tilde{S}_\eta(N, J)$ implies that

$$(x_n, y_n) \in \tilde{S}_\eta(N^* - 1, J) \cap \{(i2^{-n}, j2^{-n}), i, j \in \mathbb{Z}\}. \tag{3.37}$$

Note that

$$\begin{aligned}|x_n - y_n| &\le 2^{-(N^*-1)} \qquad \text{[by (3.33)]}\\&\le 2^{-\log^2(k_0N^*)}\\&= 2^{-\log^2(n)},\end{aligned} \tag{3.38}$$

where the first inequality follows by (3.33) and the second inequality follows easily by our assumption $N^* \ge (k_0)^{10}$, $k_0 > 3$. By (3.37), (3.38) and since $N^* - 1 \ge N_1$, we can apply Lemma 3.8 to get

$$\begin{aligned}&|Z^2(y_n) - Z^2(x_n) - (y_n - x_n)V'(x_n)|\\&\quad = |Z_t^{2,\eta}(x_n, y_n)|\\&\quad \le C|y_n - x_n|^{\eta-2\alpha\gamma}\\&\quad \le C'|y - x|^{\eta-2\alpha\gamma},\end{aligned} \tag{3.39}$$

where the last inequality follows by (3.33).

By (3.30) and the bounds (3.32), (3.34), (3.36), (3.39), we complete the proof. □

LEMMA 3.12. *Let $J \geq 1$, $\eta \in (\eta_c, \min\{\bar{\eta}_c, 1\})$. For almost every $\omega \in A^\varepsilon$ and for all $x \in (0,1) \setminus S_\eta(J)$,*

$$\left|Z^2(y) - Z^2(x)\right| \leq C_1(J)|y-x|^{\eta - 2\alpha\gamma} \qquad \forall y \in B\left(x, 2^{-N_1}\right),$$

where $N_1 = N_1(\omega)$ is from Lemma 3.8.

PROOF. For any $n \geq 1$, take $x_n, y_n \in \{i2^{-n}, i \in \mathbb{Z}\}$ satisfying the following conditions:

$$|x_n - x| \leq 2^{-n}, \qquad |x_n - x_{n+1}| \leq 2^{-n-1},$$
$$|y_n - y| \leq 2^{-n}, \qquad |y_n - y_{n+1}| \leq 2^{-n-1}.$$

Applying Lemma 3.8, we get for every $N \geq N_1$ the bound

$$\begin{aligned} &\left|Z_t^{2,\eta}(y,x)\right| \\ &\quad \leq \left|Z_t^{2,\eta}(y_N, x_N)\right| + \sum_{n=N}^{\infty}\left(\left|Z_t^{2,\eta}(y_{N+1}, y_N)\right| + \left|Z_t^{2,\eta}(x_{N+1}, x_N)\right|\right) \\ &\quad \leq 2C(J)2^{-N(\eta-\eta_c)}\left(|x_N - y_N|^{\eta_c - 2\alpha\gamma} + 2\sum_{n=N}^{\infty} 2^{-n(\eta_c - \alpha\gamma)}\right) \\ &\quad \leq C'(J)2^{-N(\eta-\eta_c)}\left(|x_N - y_N|^{\eta_c-2\alpha\gamma} + 2^{-N(\eta_c - 2\alpha\gamma)}\right). \end{aligned} \tag{3.40}$$

Choosing N so that $|y-x| \in [2^{-N-1}, 2^{-N}]$, we complete the proof. □

Now we are able to complete the following.

PROOF OF LEMMA 3.2. Lemmas 3.9, 3.12 imply that

$$\mathbf{P}\left(H_{Z^2}(x) \geq \eta - 2\alpha\gamma, \forall x \in (0,1) \setminus S_\eta(J); A^\varepsilon\right) \geq 1 - \varepsilon.$$

Letting here $\varepsilon \to 0$, we complete the proof of the lemma. □

4. Lower bound for the Hausdorff dimension. The aim of this section is to prove the following proposition.

PROPOSITION 4.1. *For every $\eta \in (\eta_c, \overline{\eta}_c) \setminus \{1\}$,*

$$\dim(\mathcal{E}_{Z^2,\eta}) \geq (1+\beta)(\eta - \eta_c), \qquad \mathbf{P}\text{-}a.s.\ on\ \left\{X_t\left((0,1)\right) > 0\right\}.$$

REMARK 4.2. Clearly, the above proposition together with Proposition 3.1 completes the proof of Theorem 1.2.

As we have already mentioned in the Introduction, the proof of the lower bound is much more involved then the proof of the upper one. Due to the mentioned complexity of the proof we give, for the reader's convenience, a short description of our strategy. Section 4.1 is devoted to deriving some uniform estimates on "masses" of X_s of dyadic intervals at times s close to t. In Section 4.2, we construct a set $\tilde{J}_{\eta,1}$ with $\dim(\tilde{J}_{\eta,1}) \geq (\beta+1)(\eta-\eta_{\mathrm{c}})$, on which we show existence of "big" jumps of X that occur close to time t. These jumps are "encoded" in the jumps of the auxiliary processes $L^{+}_{n,l,r}$ and they, in fact, "may" destroy the Hölder continuity of $X_t(\cdot)$ on $\tilde{J}_{\eta,1}$ for any index greater or equal to η (see Lemma 4.7 and Lemma 4.8). However, there are also other jumps of the process X (they will be encoded in processes $L^{-}_{n,l,r}$) which may compensate the impact of the jumps of X encoded in L^{+}. The most difficult part of the proof is to show that there is no such compensation, and this is done in Section 4.3. More precisely, we prove in Section 4.3 that such a compensation is possible on a set of the Hausdorff dimension strictly smaller than $(\beta+1)(\eta-\eta_{\mathrm{c}})$, and hence does not influence the dimension result. It is done in Lemmata 4.9, 4.10 and 4.13.

4.1. *Uniform estimates for values of X_s on dyadic intervals.* In this subsection, we derive some bounds for $X_s(I_k^{(n)})$, where

$$I_k^{(n)} := [k2^{-n}, (k+1)2^{-n}).$$

In what follows, fix some

$$m > 3/\alpha, \tag{4.1}$$

and let $\theta \in (0,1)$ be arbitrarily small. Define

$$O_n := \Big\{\omega : \text{there exists } k \in [0, 2^n-1] \text{ such that } \sup_{s\in(t-2^{-\alpha n}n^{\alpha^2 m/3},t)} X_s(I_k^{(n)}) \geq 2^{-n}n^{2m\alpha/3}\Big\}$$

and

$$B_n = B_n(\theta) := \Big\{\omega : \text{there exists } k \in [0, 2^n-1] \text{ such that } I_k^{(n)} \cap \{x : X_t(x) \geq \theta\} \neq \varnothing \text{ and } \inf_{s\in(t-2^{-\alpha n}n^{-\alpha m},t)} X_s(I_k^{(n)}) \leq 2^{-n}n^{-2m}\Big\}.$$

LEMMA 4.3. *There exists a constant C such that*

$$\mathbf{P}(O_n) \leq Cn^{-m\alpha/3}, \qquad n \geq 1.$$

The proof is an almost word-by-word repetition of the proof of Lemma 5.5 in [4], and we omit it.

LEMMA 4.4. *There exists a constant $C = C(m)$ such that, for every $\theta \in (0,1)$,*

$$\mathbf{P}(B_n(\theta) \cap A^{\varepsilon}) \le C\theta^{-1} n^{-\alpha m/3}, \qquad n \ge \tilde{n}(\theta),$$

for some $\tilde{n}(\theta)$ sufficiently large.

PROOF. Define

$$\tau_n := \inf\{s \in (t - 2^{-\alpha n} n^{-\alpha m}, t) : X_s(I_k^{(n)}) \le 2^{-n} n^{-2m} \text{ for some } k \in [0, 2^n - 1]\}.$$

Fix an arbitrary $\theta \in (0,1)$. If $\omega \in B_n = B_n(\theta)$, then there exists a sequence $\{(s_j, I_{k_j}^{(n)})\}$ such that $X_{s_j}(I_{k_j}^{(n)}) \le 2^{-n} n^{-2m}$ for all $j \ge 1$, and $s_j \downarrow \tau_n$, as $j \to \infty$. Since for each $n \ge 1$ the number of intervals $I_k^{(n)}$ is finite, there exist $\tilde{k}_n$ and a subsequence j_r such that $k_{j_r} = \tilde{k}_n$ for all $r \ge 1$. Therefore,

$$\lim_{r\to\infty} X_{s_{j_r}}(I_{\tilde{k}_n}^{(n)}) \le 2^{-n} n^{-2m}.$$

By the right continuity of the measure valued process $\{X_t\}_{t\ge 0}$, we get that

$$X_{\tau_n}(I_{\tilde{k}_n}^{(n)} \setminus \{\tilde{k}_n 2^{-n}\}) \le 2^{-n} n^{-2m}.$$

Since X has only positive jumps in the form of atomic measures and these jumps do not occur with probability one at dyadic rational points of space, we immediately deduce that, in fact,

$$X_{\tau_n}(I_{\tilde{k}_n}^{(n)}) \le 2^{-n} n^{-2m}, \qquad \mathbf{P}\text{-a.s.}$$

Put

$$\tilde{B}_n := \left[\frac{\tilde{k}_n}{2^n} + \frac{1}{2^{n+1}} - 2^{-n} n^{-m}, \frac{\tilde{k}_n}{2^n} + \frac{1}{2^{n+1}} + 2^{-n} n^{-m}\right],$$

and

$$\tilde{E}^{(n)} := \{\omega : I_{\tilde{k}_n}^{(n)} \cap \{x : X_t(x) > \theta\} \ne \varnothing\}.$$

Recall that, on A^{ε}, $X_t(\cdot)$ is locally Hölder continuous on $(0,1)$ with exponent $\eta_c - \varepsilon$ and Hölder constant $C_{(3.3)}$ [see (3.4)]. Therefore, on the event $A^{\varepsilon} \cap \tilde{E}^{(n)}$, we have $X_t(x) \ge \theta/2$ for all $x \in \tilde{B}_n$ and all $n \ge \tilde{n}(\theta)$, where $\tilde{n}(\theta)$ is chosen sufficiently large.

Thus, for $n \ge \tilde{n}(\theta)$,

$$\begin{aligned} \theta 2^{-n} n^{-m} \mathbf{P}(\tau_n < t, O_n^c, \tilde{E}^{(n)}, A^{\varepsilon}) &= \frac{\theta}{2} |\tilde{B}_n| \mathbf{P}(\tau_n < t, O_n^c, \tilde{E}^{(n)}, A^{\varepsilon}) \\ &\le \mathbf{E}[X_t(\tilde{B}_n) 1_{\{\tau_n < t, O_n^c, \tilde{E}^{(n)}, A^{\varepsilon}\}}] \qquad (4.2)\\ &\le \mathbf{E}[X_t(\tilde{B}_n) 1_{\{\tau_n < t, \tilde{O}_n^c\}}], \end{aligned}$$

where

$$\tilde{O}_n^c := \{X_{\tau_n}(I_k^{(n)}) \le 2^{-n} n^{2m\alpha/3}, \text{for all } k = 0, \dots, 2^n - 1\},$$

and the last inequality in (4.2) follows since $O_n^c \subset \tilde{O}_n^c$.

Using the strong Markov property, we then obtain

$$\theta 2^{-n} n^{-m} \mathbf{P}(\tau_n < t, O_n^c, \tilde{E}^{(n)}, A^\varepsilon) \le \mathbf{E}[S_{t-\tau_n} X_{\tau_n}(\tilde{B}_n) 1_{\{\tau_n < t, \tilde{O}_n^c\}}] e^{|a|t}, \tag{4.3}$$

for all $n \ge \tilde{n}(\theta)$. It is clear that

$$\mathbf{E}[S_{t-\tau_n} X_{\tau_n}(\tilde{B}_n) 1_{\{\tau_n < t, \tilde{O}_n^c\}}] \tag{4.4}$$

$$= \mathbf{E}\left[\int_{\mathsf{R}} X_{\tau_n}(dz) \int_{\tilde{B}_n} p_{t-\tau_n}^\alpha(y-z)\, dy 1_{\{\tau_n < t, \tilde{O}_n^c\}}\right] \qquad \forall n \ge 1. \tag{4.5}$$

Since $X_{\tau_n}(I_{\tilde{k}_n}^{(n)}) \le 2^{-n} n^{-2m}$ on the event $\{\tau_n < t\}$, we have

$$\mathbf{E}\left[\int_{I_{\tilde{k}_n}^{(n)}} X_{\tau_n}(dz) \int_{\tilde{B}_n} p_{t-\tau_n}^\alpha(y-z)\, dy 1_{\{\tau_n < t, \tilde{O}_n^c\}}\right] \le 2^{-n} n^{-2m} \qquad \forall n \ge 1. \tag{4.6}$$

Recalling that $\tau_n \ge t - 2^{-\alpha n} n^{-\alpha m}$ and using the scaling property of the kernel p^α together with the bound (3.11) we get

$$\begin{aligned} p_{t-\tau_n}^\alpha(y-z) &= (t-\tau_n)^{-1/\alpha} p_1^\alpha\left(\frac{y-z}{(t-\tau_n)^{1/\alpha}}\right) \\ &\le C(t-\tau_n)|y-z|^{-\alpha-1} \\ &\le C 2^{-\alpha n} n^{-\alpha m} |y-z|^{-\alpha-1}. \end{aligned}$$

Further, if $z \in I_{\tilde{k}_n \pm j}^{(n)}$ and $y \in \tilde{B}_n$, then

$$\begin{aligned} |y-z| &\ge (j-1)2^{-n} + (1/2 - n^{-m})2^{-n} = (j - 1/2 - n^{-m})2^{-n} \\ &\ge \tfrac{1}{10} j 2^{-n} \qquad \forall n \ge 2, j \ge 1. \end{aligned}$$

Combining the last two bounds, we get

$$\begin{aligned} &\int_{I_{\tilde{k}_n \pm j}^{(n)}} X_{\tau_n}(dz) \int_{\tilde{B}_n} p_{t-\tau_n}^\alpha(y-z)\, dy \\ &\quad \le \int_{I_{\tilde{k}_n \pm j}^{(n)}} X_{\tau_n}(dz) \int_{\tilde{B}_n} C j^{-\alpha-1} 2^{(\alpha+1)n} 2^{-\alpha n} n^{-\alpha m}\, dy \\ &\quad = C j^{-\alpha-1} n^{-(\alpha+1)m} X_{\tau_n}(I_{\tilde{k}_n \pm j}^{(n)}) \qquad \forall n \ge 2, j \ge 1. \end{aligned}$$

On the event $\{\tau_n < t\} \cap \tilde{O}_n^c$ we then have

$$\begin{aligned}
&\int_{I^{(n)}_{\tilde{k}_n-j} \cup I^{(n)}_{\tilde{k}_n+j}} X_{\tau_n}(dz) \int_{\tilde{B}_n} p^{\alpha}_{t-\tau_n}(y-z)\,dy \\
&\qquad \le Cj^{-\alpha-1}2^{-n}n^{-(\alpha+1)m+2m\alpha/3} \\
&\qquad = Cj^{-\alpha-1}2^{-n}n^{-((1/3)\alpha+1)m} \qquad \forall n \ge 2, j \ge 1.
\end{aligned} \tag{4.7}$$

Consequently, by summing up (4.7) over $j \ge 1$, we get

$$\begin{aligned}
&\mathbf{E}\biggl[\int_{\mathsf{R}\setminus I^{(n)}_{\tilde{k}_n}} X_{\tau_n}(dz) \int_{\tilde{B}_n} p^{\alpha}_{t-\tau_n}(y-z)\,dy \mathbf{1}_{\{\tau_n<t,\tilde{O}_n^c\}}\biggr] \le C2^{-n}n^{-((1/3)\alpha+1)m}, \\
&\qquad\qquad n \ge 2.
\end{aligned} \tag{4.8}$$

This and (4.6) imply that (4.4) is bounded by

$$C2^{-n}\bigl(n^{-((1/3)\alpha+1)m} + n^{-2m}\bigr), \qquad n \ge 2. \tag{4.9}$$

Combining (4.3), (4.9) and using the trivial bound for $n = 1$, we obtain

$$\theta\mathbf{P}\bigl(\tau_n < t, O_n^c, \tilde{E}^{(n)}, A^{\varepsilon}\bigr) \le C\bigl(n^{-(1/3)\alpha m} + n^{-m}\bigr) \le Cn^{-(1/3)\alpha m}, \qquad n \ge \tilde{n}(\theta).$$

In view of Lemma 4.3,

$$\begin{aligned}
\mathbf{P}\bigl(\tau_n < t, \tilde{E}^{(n)}, A^{\varepsilon}\bigr) &\le \mathbf{P}(O_n) + \mathbf{P}\bigl(\tau_n < t, O_n^c, \tilde{E}^{(n)}, A^{\varepsilon}\bigr) \\
&\le C\theta^{-1}n^{-\alpha m/3}, \qquad n \ge \tilde{n}(\theta).
\end{aligned}$$

This completes the proof of the lemma. □

4.2. *Analysis of the set of jumps which destroy the Hölder continuity.* In this subsection, we construct a set $\tilde{J}_{\eta,1}$ such that its Hausdorff dimension is bounded from below by $(\beta+1)(\eta-\eta_{\mathrm{c}})$ and in the vicinity of each $x \in \tilde{J}_{\eta,1}$ there are jumps of X which destroy the Hölder continuity at x for any index greater than η.

We first introduce $\tilde{J}_{\eta,1}$ and prove the lower bound for its dimension. Set

$$q := \frac{(\alpha+3)m}{(\beta+1)(\eta-\eta_c)}$$

and define

$$\begin{aligned}
A_k^{(n)} &:= \bigl\{\mathbf{\Delta} X_s\bigl(I^{(n)}_{k-2n^q-2}\bigr) \ge 2^{-(\eta+1)n} \\
&\qquad \text{for some } s \in \bigl[t-2^{-\alpha n}n^{-\alpha m}, t-2^{-\alpha(n+1)}(n+1)^{-\alpha m}\bigr)\bigr\}, \\
J^{(n)}_{k,r} &:= \biggl[\frac{k}{2^n} - \bigl(n^q 2^{-n}\bigr)^r, \frac{k+1}{2^n} + \bigl(n^q 2^{-n}\bigr)^r\biggr].
\end{aligned}$$

Let us introduce the following notation. For a Borel set B and an event E, define a random set

$$B1_E(\omega) := \begin{cases} B, & \omega \in E, \\ \varnothing, & \omega \notin E. \end{cases}$$

Now we are ready to define random sets

$$\tilde{J}_{\eta,r} := \limsup_{n\to\infty} \bigcup_{k=2n^q+2}^{2^n-1} J_{k,r}^{(n)} 1_{A_k^{(n)}}, \qquad r > 0.$$

As we have mentioned already, we are interested in getting the lower bound on Hausdorff dimension of $\tilde{J}_{\eta,1}$. The standard procedure for this is as follows. First, show that a bit "inflated" set $\tilde{J}_{\eta,r}$, for certain $r \in (0, 1)$, contains open intervals. This would imply a lower bound r on the Hausdorff dimension of $\tilde{J}_{\eta,1}$ (see Lemma 4.5 and Theorem 2 from [7] where a similar strategy was implemented). Thus, to get a sharper bound on Hausdorff dimension of $\tilde{J}_{\eta,1}$, one should try to take r as large as possible. In the next lemma, we show that, in fact, one can choose $r = (\beta + 1)(\eta - \eta_c)$.

LEMMA 4.5. *On the event* A^ε,

$$\{x \in (0, 1) : X_t(x) \geq \theta\} \subseteq \tilde{J}_{\eta,(\beta+1)(\eta-\eta_c)}, \qquad \mathbf{P}\text{-}a.s.$$

for every $\theta \in (0, 1)$.

PROOF. Fix an arbitrary $\theta \in (0, 1)$. We estimate the probability of the event $E_n \cap A^\varepsilon$, where

$$E_n := \left\{\omega : \{x \in (0, 1) : X_t(x) \geq \theta\} \subseteq \bigcup_{k=2n^q+2}^{2^n-1} J_{k,(\beta+1)(\eta-\eta_c)}^{(n)} 1_{A_k^{(n)}}\right\}.$$

It follows from Lemma 4.4 that, for all $n \geq \tilde{n}(\theta)$,

$$\begin{aligned} \mathbf{P}(E_n^c \cap A^\varepsilon) &\leq \mathbf{P}(E_n^c \cap B_n \cap A^\varepsilon) + \mathbf{P}(E_n^c \cap B_n^c \cap A^\varepsilon) \\ &\leq C\theta^{-1} n^{-\alpha m/3} + \mathbf{P}(E_n^c \cap B_n^c \cap A^\varepsilon). \end{aligned} \tag{4.10}$$

For any $k = 0, \dots, 2^n - 1$, the compensator measure $\widehat{N}(dr, dy, ds)$ of the random measure $\mathcal{N}(dr, dy, ds)$ [the jump measure for X—see discussion after (1.8)], on

$$\begin{aligned} &\mathcal{J}_1^{(n)} \times I_k^{(n)} \times \mathcal{J}_2^{(n)} \\ &\quad := [2^{-(\eta+1)n}, \infty) \times I_k^{(n)} \times [t - 2^{-\alpha n} n^{-\alpha m}, t - 2^{-\alpha(n+1)}(n+1)^{-\alpha m}), \end{aligned}$$

is given by the formula

$$1\{(r, y, s) \in \mathcal{J}_1^{(n)} \times I_k^{(n)} \times \mathcal{J}_2^{(n)}\} \varrho r^{-2-\beta}\, dr\, X_s(dy)\, ds. \tag{4.11}$$

If

$$k \in K_\theta := \{l : I_l^{(n)} \cap \{x \in (0,1) : X_t(x) \geq \theta\} \neq \varnothing\},$$

then, by the definition of B_n, we have

$$X_s(I_k^{(n)}) \geq 2^{-n} n^{-2m}, \qquad \text{for } s \in \mathcal{J}_2^{(n)}, \text{ on the event } A^\varepsilon \cap B_n^c. \tag{4.12}$$

Define the measure $\widehat{\Gamma}(dr, dy, ds)$ on $\mathsf{R}_+ \times (0,1) \times \mathsf{R}_+$, as follows:

$$\widehat{\Gamma}(dr, dy, ds) := \varrho r^{-2-\beta}\, dr n^{-2m}\, dy\, ds. \tag{4.13}$$

Then, by (4.11) and (4.12), on $A^\varepsilon \cap B_n^c$, and on the set

$$\mathcal{J}_1^{(n)} \times \{y \in (0,1) : X_t(y) \geq \theta\} \times \mathcal{J}_2^{(n)}$$

we have the following bound:

$$\widehat{\Gamma}(dr, I_k^{(n)}, \mathcal{J}_2^{(n)}) \leq \widehat{\mathcal{N}}(dr, I_k^{(n)}, \mathcal{J}_2^{(n)}), \qquad k \in K_\theta.$$

By standard arguments, it is easy to construct the Poisson point process $\Gamma(dr, dx, ds)$ on $\mathsf{R}_+ \times (0,1) \times \mathsf{R}_+$ with intensity measure $\widehat{\Gamma}$ given by (4.13) on the whole space $\mathsf{R}_+ \times (0,1) \times \mathsf{R}_+$ such that on $A^\varepsilon \cap B_n^c$,

$$\Gamma(dr, I_k^{(n)}, \mathcal{J}_2^{(n)}) \leq \mathcal{N}(dr, I_k^{(n)}, \mathcal{J}_2^{(n)})$$

for $r \in \mathcal{J}_1^{(n)}$ and $k \in K_\theta$.

Now, define

$$\xi_k^{(n)} = \mathbf{1}_{\{\Gamma(\mathcal{J}_1^{(n)} \times I_{k-2n^q-2}^{(n)} \times \mathcal{J}_2^{(n)}) \geq 1\}}, \qquad k \geq 2n^q + 2.$$

Clearly, on $A^\varepsilon \cap B_n^c$ and for k such that $k - 2n^q - 2 \in K_\theta$,

$$\xi_k^{(n)} \leq \mathbf{1}_{A_k^{(n)}}.$$

Moreover, by construction $\{\xi_k^{(n)}\}_{k=2n^q+2}^{2^n+2n^q+1}$ is a collection of independent identically distributed Bernoulli random variables with success probabilities

$$\begin{aligned} p^{(n)} &:= \widehat{\Gamma}(\mathcal{J}_1^{(n)} \times I_{k-2n^q-2}^{(n)} \times \mathcal{J}_2^{(n)}) \\ &= C 2^{(\eta-\eta_c)(1+\beta)n-n} n^{-(\alpha+2)m}. \end{aligned}$$

From the above coupling with the Poisson point process Γ, it is easy to see that

$$\mathbf{P}(E_n^c \cap B_n^c \cap A^\varepsilon) \leq \mathbf{P}(\tilde{E}_n^c), \tag{4.14}$$

where

$$\tilde{E}_n := \left\{(0,1) \subseteq \bigcup_{k=2n^q+2}^{2^n+2n^q+1} J_k^{(n)} \mathbf{1}_{\{\xi_k^{(n)}=1\}}\right\}.$$

Let $L(n)$ denote the length of the longest run of zeros in the sequence $\{\xi_k^{(n)}\}_{k=2n^q+2}^{2^n+2n^q+1}$. Clearly,

$$\mathbf{P}(\tilde{E}_n^c) \le \mathbf{P}(L^{(n)} \ge 2^{n-(\beta+1)(\eta-\eta_c)n} n^{m(\alpha+3)})$$

and it is also obvious that

$$\mathbf{P}(L^{(n)} \ge j) \le 2^n p^{(n)}(1-p^{(n)})^j \qquad \forall j \ge 1.$$

Use this with the fact that, by (4.1), $m > 1$, to get that

$$\mathbf{P}(\tilde{E}_n^c) \le \exp\{-\tfrac{1}{2} n^m\} \tag{4.15}$$

for all n sufficiently large. Combining (4.10), (4.14) and (4.15), we conclude that the sequence $\mathbf{P}(E_n^c \cap A^\varepsilon)$ is summable. Applying Borel–Cantelli, we complete the proof of the lemma. □

Define

$$h_\eta(x) := x^{(\beta+1)(\eta-\eta_c)} \log^2 \frac{1}{x}$$

and

$$\mathcal{H}_\eta(A) := \lim_{\epsilon\to 0} \inf\left\{\sum_{j=1}^{\infty} h_\eta(|I_j|), A \in \bigcup_{j=1}^{\infty} I_j \text{ and } |I_j| \le \epsilon\right\}.$$

Combining Lemma 4.5 and Theorem 2 from [7], one can easily get.

COROLLARY 4.6. *On the event* $A^\varepsilon \cap \{X_t((0,1)) > 0\}$,

$$\mathcal{H}_\eta(\tilde{J}_{\eta,1}) > 0, \qquad \mathbf{P}\text{-}a.s.$$

and, consequently, on $A^\varepsilon \cap \{X_t((0,1)) > 0\}$,

$$\dim(\tilde{J}_{\eta,1}) \ge (\beta+1)(\eta-\eta_c), \qquad \mathbf{P}\text{-}a.s.$$

PROOF. Fix any $\theta \in (0,1)$. If $\omega \in A^\varepsilon$ is such that $B_\theta := \{x \in (0,1) : X_t(x) \ge \theta\}$ is not empty, then by the local Hölder continuity of $X_t(\cdot)$ there exists an open interval $(x_1(\omega), x_2(\omega)) \subset B_{\theta/2}$. Moreover, in view of Lemma 4.5,

$$(x_1(\omega), x_2(\omega)) \subset \tilde{J}_{\eta,(\beta+1)(\eta-\eta_c)}(\omega), \qquad \mathbf{P}\text{-a.s.}$$

on the event $A^\varepsilon \cap \{B_\theta$ is not empty$\}$. Thus, we may apply Theorem 2 from [7] to the set $(x_1(\omega), x_2(\omega))$, which gives

$$\mathcal{H}_\eta((x_1(\omega), x_2(\omega)) \cap \tilde{J}_{\eta,1}) > 0, \qquad \mathbf{P}\text{-a.s.}$$

on the event $A^\varepsilon \cap \{B_\theta$ is not empty$\}$. Thus,

$$\dim((x_1(\omega), x_2(\omega)) \cap \tilde{J}_{\eta,1}) \ge (\beta+1)(\eta-\eta_c), \qquad \mathbf{P}\text{-a.s.}$$

on the event $A^\varepsilon \cap \{B_\theta$ is not empty$\}$. Due to the monotonicity of $\mathcal{H}_\eta(\cdot)$ and $\dim(\cdot)$, we conclude that $\mathcal{H}_\eta(\tilde{J}_{\eta,1}) > 0$ and $\dim(\tilde{J}_{\eta,1}) \geq (\beta+1)(\eta-\eta_c)$, $\mathbf{P}$-a.s. on the event $A^\varepsilon \cap \{B_\theta$ is not empty$\}$. Noting that $1_{\{B_\theta \text{ is not empty}\}} \uparrow 1_{\{X_t(0,1)>0\}}$ as $\theta \downarrow 0$, $\mathbf{P}$-a.s., we complete the proof. $\square$

Now we turn to the second part of the present subsection. By construction of $\tilde{J}_{\eta,1}$, we know that to the left of every point $x \in \tilde{J}_{\eta,1}$ there exist big jumps of X at time s "close" to t: such jumps are defined by the events $A_k^{(n)}$. We would like to show that these jumps will result in destroying the Hölder continuity of any index greater than η at the point x. To this end, we will introduce auxiliary processes $L_{n,l,r}^{\pm}$ that are indexed by a grid *finer* than $\{k2^{-n}, k=0,1,\ldots\}$. That is, take some integer $Q>1$ (note, that eventually Q will be chosen large enough, depending on η). According to Lemma 2.15 from [4] [see also (3.21), (3.22)], there exist spectrally positive $(1+\beta)$-stable processes $L_{n,l,r}^{\pm}$ such that

$$\tilde{Z}_s^{2,\eta}(l2^{-Qn}, r2^{-Qn}) = L_{n,l,r}^{+}(T_+^{n,l,r}(s)) - L_{n,l,r}^{-}(T_-^{n,l,r}(s)), \qquad 0 \leq l < r \leq 2^{Qn}, 0 \leq s \leq t, \tag{4.16}$$

where

$$T_{\pm}^{n,l,r}(s) = \int_0^s du \int_{\mathbb{R}} X_u(dy)\big((\tilde{p}_{t-u}^{\alpha,\eta}(l2^{-Qn}-y, r2^{-Qn}-y))^{\pm}\big)^{1+\beta}, \qquad s \leq t.$$

The goal of the remaining part of this subsection is to show that, in fact, "big" jumps of X defined via $A_k^{(n)}$ imply "big" values of $L_{n,l,r}^{+}$ for certain l, r.

We need to introduce additional notation related to the event $A_k^{(n)}$. If $A_k^{(n)}$ occurs, then there is a jump of the process X at time s_k^n such that

$$\mathbf{\Delta} X_{s_k^n}(I_{k-2n^q-2}^{(n)}) \geq 2^{-(\eta+1)n}, \tag{4.17}$$

and

$$s_k^n \in [t - 2^{-\alpha n} n^{-\alpha m}, t - 2^{-\alpha(n+1)}(n+1)^{-\alpha m}). \tag{4.18}$$

Let $y_k^n \in I_{k-2n^q-2}^{(n)}$ denote the spatial position of that jump. Now put

$$l_k^n = \lfloor 2^{Qn} y_k^n \rfloor,$$

and for every $x \in (0,1)$ define

$$\tilde{k}_n(x) = \lfloor 2^{Qn} x \rfloor.$$

To simplify notation, in what follows, for any n, l, r, we denote by $\mathbf{\Delta} L_{n,l,r}^{+}$ the maximal jump of $L_{n,l,r}^{+}$, that is,

$$\mathbf{\Delta} L_{n,l,r}^{+} := \sup_{s \leq t} \mathbf{\Delta} L_{n,l,r}^{+}(T_+^{n,l,r}(s)).$$

Also set

$$\mathbf{L}^{\pm}_{n,l,r} := L^{\pm}_{n,l,r}\big(T^{n,l,r}_{\pm}(t)\big). \tag{4.19}$$

Now we explain briefly why we look at fine dyadic intervals $(l2^{-Qn}, r2^{-Qn})$. First, we can not work directly with increments of Z^2 at random points $x \in \tilde{J}_{\eta,1}$. However, if we show that Hölder continuity is destroyed at $2^{-Qn}\tilde{k}_n(x)$, then we will be able to infer that, at the point x, the Hölder continuity of any index greater than η is destroyed as well: to show this we need Q to be sufficiently large. Also, on this finer scale, we can show that $L^{+}_{n,l^n_k,r}$ has "large" jumps for all r which are close to $\tilde{k}_n(x)$. This property is quite important because of a possible compensation effect, which will be investigated in the next subsection.

In the next two lemmas, we start to fulfill the above program. In Lemma 4.7, we show that on the event $A^{(n)}_k$, $L^{+}_{n,l^n_k,r}$ has "large" jumps of order $2^{-\eta n}n^m$ for all r's sufficiently close to $\tilde{k}_n(x)$ with $x \in J^{(n)}_{k,1}$. As a consequence, in Lemma 4.8, we obtain, by pretty standard arguments, that $\mathbf{L}^{+}_{n,l^n_k,r}$ can also take "big" values of the same order, for certain n, l^n_k, r.

LEMMA 4.7. *Let $\eta \in (\eta_c, \bar{\eta}_c)$, and fix an arbitrary integer $R > 0$. There exist constants $C_{(4.20)}$ and $N_{4.7}$ sufficiently large, such that for all $n \geq N_{4.7}$ and all $r^n_k \in \{\tilde{k}_n(x) - R, \tilde{k}_n(x) - R + 1, \dots, \tilde{k}_n(x)\}$,*

$$A^{(n)}_k \subset \big\{\mathbf{\Delta} L^{+}_{n,l^n_k,r^n_k} \geq C_{(4.20)} 2^{-\eta n} n^m, \forall x \in J^{(n)}_{k,1}\big\}, \qquad k = 2n^q + 2, \dots, 2^n. \tag{4.20}$$

PROOF. Fix n sufficiently large (to be chosen later) and $k \in \{2n^q + 2, \dots, 2^n\}$. In what follows, we assume that $A^{(n)}_k$ occurs. Then we have to show that

$$\mathbf{\Delta} L^{+}_{n,l^n_k,r^n_k} \geq C_{(4.20)} 2^{-\eta n} n^m \qquad \forall x \in J^{(n)}_{k,1}.$$

Fix an arbitrary $x \in J^{(n)}_{k,1}$. Recall that (s^n_k, y^n_k) denotes a space-time location of a jump of X that appears in the definition of $A^{(n)}_k$. To simplify notation, to the end of the lemma, we will suppress the superindex n in l^n_k, r^n_k, y^n_k. If $A^{(n)}_k$ occurred, then

$$\mathbf{\Delta} L^{+}_{n,l_k,r_k} \geq 2^{-(\eta+1)n}\big(\tilde{p}^{\alpha,\eta}_{t-s_k}\big(l_k 2^{-Qn} - y_k, r_k 2^{-Qn} - y_k\big)\big)_+. \tag{4.21}$$

So to verify the lemma, we have to obtain a suitable strictly positive lower bound for $\tilde{p}^{\alpha,\eta}_{t-s_k}(l_k 2^{-Qn} - y_k, r_k 2^{-Qn} - y_k)$.

First, we will obtain a lower bound for $p^{\alpha}_{t-s_k}(l_k 2^{-Qn} - y_k)$:

$$\begin{aligned} p^{\alpha}_{t-s_k}\big(l_k 2^{-Qn} - y_k\big) &= (t - s_k)^{-1/\alpha} p_1\big((t - s_k)^{-1/\alpha}\big(l_k 2^{-Qn} - y_k\big)\big) \\ &\geq 2^n n^m p_1\big((t - s_k)^{-1/\alpha}\big(l_k 2^{-Qn} - y_k\big)\big), \end{aligned} \tag{4.22}$$

where the last inequality follows by (4.18). By definition of l_k, we get

$$|l_k 2^{-Qn} - y_k| \le 2^{-Qn}, \tag{4.23}$$

and this with again (4.18) and monotonicity of $p_1(\cdot)$ implies

$$p_1\big((t-s_k)^{-1/\alpha}(l_k 2^{-Qn} - y_k)\big) \ge p_1\big(2^{-(Q-1)n+1}(n+1)^m\big) \ge p_1(1) \tag{4.24}$$

for all n sufficiently large. Let $N_{(4.24)}$ be sufficiently large such that (4.24) holds for all $n \ge N_{(4.24)}$. Then (4.22), (4.24) imply

$$\begin{aligned} p^{\alpha}_{t-s_k}(l_k 2^{-Qn} - y_k) &\ge 2^n p_1(1) n^m \\ &= C_{(4.25)} 2^n n^m \qquad \forall n \ge N_{(4.24)}. \end{aligned} \tag{4.25}$$

Next, by definition of $A_k^{(n)}$, $V_k^{(n)}$, we easily get, for all sufficiently large n,

$$-y_k + r_k 2^{-Qn} \ge 2^{-n}(2n^q + 1) - (2^{-n} n^q + R2^{-Qn}) \ge 2^{-n-1} n^q. \tag{4.26}$$

Use this and the bound on $t - s_k$ to get

$$\begin{aligned} p^{\alpha}_{t-s_k}(r_k 2^{-Qn} - y_k) &= (t-s_k)^{-1/\alpha} p_1\big((t-s_k)^{-1/\alpha}(r_k 2^{-Qn} - y_k)\big) \\ &\le C_{(3.11)}(t-s_k)^{-1/\alpha}\big((t-s_k)^{-1/\alpha}(r_k 2^{-Qn} - y_k)\big)^{-\alpha-1} \\ &= C_{(3.11)}(t-s_k)(r_k 2^{-Qn} - y_k)^{-\alpha-1} \\ &\le C_{(3.11)} 2^{-\alpha n} n^{-\alpha m} 2^{(n+1)(\alpha+1)} n^{-q(\alpha+1)} \\ &= C_{(4.27)} 2^n n^{-\alpha m - q(\alpha+1)}. \end{aligned} \tag{4.27}$$

Next, we will bound from above the quantity

$$\big|(l_k 2^{-Qn} - r_k 2^{-Qn}) p^{\alpha,\prime}_{t-s_k}(r_k 2^{-Qn} - y_k)\big|,$$

where $p_t^{\alpha,\prime}(z) := \frac{\partial p^{\alpha}(z)}{\partial z}$. It is easy to check that

$$\begin{aligned} \big|p^{\alpha,\prime}_{t-s_k}(r_k 2^{-Qn} - y_k)\big| &\le C(t-s_k)^{-2/\alpha} p_1^{\alpha}\big((t-s_k)^{-1/\alpha}(r_k 2^{-Qn} - y_k)/2\big) \\ &= C(t-s_k)^{-1/\alpha} p^{\alpha}_{t-s_k}\big((r_k 2^{-Qn} - y_k)/2\big) \\ &\le C_{(4.28)} 2^n n^m 2^n n^{-\alpha m - q(\alpha+1)}, \end{aligned} \tag{4.28}$$

where the last inequality follows by (4.27) and the bound on $t - s_k$. Since

$$|l_k 2^{-Qn} - r_k 2^{-Qn}| \le 3 \cdot 2^{-n} n^q,$$

this implies

$$\big|(l_k 2^{-Qn} - r_k 2^{-Qn}) p^{\alpha,\prime}_{t-s_k}(r_k 2^{-Qn} - y_k)\big| \le 3C_{(4.28)} 2^n n^{-m(\alpha-1)-q\alpha}. \tag{4.29}$$

Then by definition of $\tilde{p}_t^{\alpha,\eta}$, (4.25), (4.27), (4.29), we immediately get that there exists an $N_{(4.30)} \geq N_{(4.24)}$, such that, for any $\eta \in (\eta_c, \bar{\eta}_c)$,

$$\begin{aligned} &\tilde{p}^{\alpha,\eta}_{t-s_k}(l_k 2^{-Qn} - y_k, r_k 2^{-Qn} - y_k) \\ &\quad \geq C_{(4.25)} 2^n n^m - C_{(4.27)} 2^n n^{-m\alpha - q(\alpha+1)} - 3C_{(4.28)} 2^n n^{-m(\alpha-1)-q\alpha} \\ &\quad \geq \tfrac{1}{2} C_{(4.25)} 2^n n^m \qquad \forall n \geq N_{(4.30)}. \end{aligned} \tag{4.30}$$

Substitute the above lower bound into (4.21) and the result follows immediately. □

LEMMA 4.8. *Let $\eta \in (\eta_c, \bar{\eta}_c)$, and fix an arbitrary integer $R > 0$. On A^ε, for every $x \in \tilde{J}_{\eta,1}$ there exists a (random) sequence $\{(n_j, k_j)\}$, such that*

$$\mathbf{L}^+_{n_j, l^{n_j}_{k_j}, r^{n_j}_{k_j}} \geq C 2^{-\eta n_j} n_j^m$$

for all $r^{n_j}_{k_j} \in [\tilde{k}_{n_j}(x) - R, \tilde{k}_{n_j}(x) - R + 1, \ldots, \tilde{k}_{n_j}(x)]$.

PROOF. Recall (3.24) to get that on A^ε, and for any l, r,

$$T_+^{n,l,r}(t) \leq \hat{T}^{n,l,r}(l 2^{-Qn}, r 2^{-Qn}) = C_{(3.24)}(|r - l| 2^{-Qn})^{\alpha - \beta - \varepsilon_1}, \tag{4.31}$$

and take $\epsilon_1 < (\bar{\eta}_c - \eta)(\beta + 1)/2$. This, Lemma 4.7 and Lemma 2.4 from [4], imply that for all n sufficiently large, and for any $k \in [2n^q + 2, 2^n - 1]$,

$$\begin{aligned} &\mathbf{P}(A^\varepsilon \cap A_k^{(n)} \cap \{\mathbf{L}^+_{n, l_k^n, r_k^n} \leq C_{(4.20)} 2^{-\eta n - 1} n^m\}) \\ &\quad \leq \mathbf{P}(A^\varepsilon \cap \{\mathbf{\Delta} L^+_{n, l_k^n, r_k^n} \geq C_{(4.20)} 2^{-\eta n} n^m\} \cap \{\mathbf{L}^+_{n, l_k^n, r_k^n} \leq C_{(4.20)} 2^{-\eta n - 1} n^m\}) \\ &\quad \leq \sum_{\substack{l : 2^{-Qn} l \in I_k^{(n)}, \\ r : 2^{-Qn} r \in J_{k,1}^{(n)}}} \mathbf{P}(A^\varepsilon \cap \{\mathbf{\Delta} L^+_{n,l,r} \geq C_{(4.20)} 2^{-\eta n} n^m\} \\ &\qquad\qquad\qquad \cap \{\mathbf{L}^+_{n,l,r} \leq C_{(4.20)} 2^{-\eta n - 1} n^m\}) \\ &\quad \leq \sum_{\substack{l : 2^{-Qn} l \in I_k^{(n)}, \\ r : 2^{-Qn} r \in J_{k,1}^{(n)}}} \mathbf{P}\Big(\inf_{s \leq \hat{T}^{n,l,r}(l 2^{-Qn}, r 2^{-Qn})} L^+_{n,l,r}(s) \leq -C_{(4.20)} 2^{-\eta n - 1} n^m\Big) \\ &\quad \leq 2^{(Q-1)n} 5 n^q \exp\{-c 2^{(\bar{\eta}_c - \eta)(\beta+1)n/(2\beta)}\}. \end{aligned}$$

Consequently,

$$\begin{aligned}\mathbf{P}\bigl(A^{\varepsilon} \cap A_k^{(n)} \cap \{\mathbf{L}^{+}_{n,l_k^n,r_k^n} \le C_{(4.20)}2^{-\eta n-1}n^m\}\bigr)\\ \le 2^{(2Q-1)n}5n^q \exp\{-c2^{(\bar{\eta}_c-\eta)(\beta+1)n/(2\beta)}\}\\ \le \exp\{-2^{(\bar{\eta}_c-\eta)(\beta+1)n/(4\beta)}\},\end{aligned}$$

for all n sufficiently large. Using Borel–Cantelli, we get that with probability one

$$\bigcup_{k=2n^q+2}^{2^n-1} \{A^{\varepsilon} \cap A_k^{(n)} \cap \{\mathbf{L}^{+}_{n,l_k^n,r_k^n} \le C_{(4.20)}2^{-\eta n-1}n^m\}\}$$

occurs only a finite number of times. Let $x \in \tilde{J}_{\eta,1}$ be arbitrary. By definition of $\tilde{J}_{\eta,1}$, there exists a (random) sequence $\{(n_j, k_j)\}$ such that

$$x \in J_{k_j,1}^{(n_j)} \quad \text{and} \quad 1_{A_{k_j}^{(n_j)}} = 1 \qquad \forall j \ge 1.$$

Therefore, on the set A^{ε} we have

$$\mathbf{L}^{+}_{n,l_{k_j}^{n_j},r_{k_j}^{n_j}} > C_{(4.20)}2^{-\eta n_j-1}n_j^m,$$

for all j sufficiently large and all $r_{n_j}^{k_j} \in [\tilde{k}_{n_j}(x) - R, \tilde{k}_{n_j}(x)]$. $\square$

4.3. *Effect of compensation*. If we recall (4.16), then Lemma 4.8 implies that it is maybe possible to destroy the Hölder continuity, of any index greater than η, of the process on the set $\tilde{J}_{\eta,1}$. For this purpose, we use processes $L^{+}_{n,k,l}$. It is also clear from (4.16) that in addition one should show that (loosely speaking) on a "significant" part of $\tilde{J}_{\eta,1}$ there is no compensation of "big" values of L^+ by "big" values of L^-.

First, fix arbitrary positive constants ρ, c, ν such that

$$\text{(4.32)} \quad \rho < 10^{-2}\gamma, \qquad \nu \in \left(\frac{\alpha\gamma + 5\rho}{\eta_c}, 10^{-1}\right), \qquad c \in \left(\frac{10}{2-\eta}, \frac{1}{10\rho}\right).$$

Define

$$\begin{aligned} G_k^{(n)} := \Bigl\{&\text{there exist at least two jumps of } M, \text{ of the form } r\delta_{(s,y)},\\ &\text{satisfying } r \ge 2^{-(\eta+1+2\rho+2c\rho)n}, s \in [t - 2^{-\alpha(1-c\rho)n}, t - 2^{-\alpha(1+c\rho)n})\\ &\text{and } y \in \left[\frac{k}{2^n} - 2^{-n(1-c\rho)(1-\nu)}, \frac{k+1}{2^n} + 2^{-n(1-c\rho)(1-\nu)}\right]\Bigr\} \end{aligned} \tag{4.33}$$

and

$$\tilde{G}_\eta := \limsup_{n\to\infty} \bigcup_{k=0}^{2^n-1} I_k^{(n)} 1_{G_k^{(n)}}.$$

Informally, $\tilde{G}_\eta$ is such that in certain proximity of every $x \in \tilde{G}_\eta$ there are at least two "big" jumps of M. If one of the jumps, appears in L^+, and another in L^-, they may compensate each other; however, in the next lemma we will show that the Hausdorff dimension of $\tilde{G}_\eta$ is small.

LEMMA 4.9. *On A^ε,*

$$\dim(\tilde{G}_\eta) \le \big(2(\beta+1)(\eta-\eta_c)-1\big)^+ + 2\big(\nu + 8(c+1)\rho\big), \qquad \mathbf{P}\text{-}a.s.$$

PROOF. On the event O_n^c, we have the following upper bound for the intensity of the jumps in $G_k^{(n)}$:

$$\begin{aligned}
&\int_{t-2^{-\alpha(1-c\rho)n}}^{t-2^{-\alpha(1+c\rho)n}} ds\, X_s\bigg(\bigg[\frac{k}{2^n} - 2^{-n(1-c\rho)(1-\nu)}, \frac{k+1}{2^n} + 2^{-n(1-c\rho)(1-\nu)}\bigg]\bigg)\\
&\quad\times \int_{2^{-(\eta+1+2\rho+2c\rho)n}} \varrho r^{-2-\beta}\, dr\\
&\qquad \le C2^{n-n(1-c\rho)(1-\nu)}2^{-n}n^{2m}2^{-\alpha(1-\rho c)n}2^{(\beta+1)(\eta+1+2\rho+2c\rho)}\\
&\qquad \le C2^{-n+n(\beta+1)(\eta-\eta_c)+\delta n},
\end{aligned}$$

where $\delta = \nu + 7(c+1)\rho$. Since the number of such jumps can be represented by means of a time-changed standard Poisson process, the probability to have at least two such jumps is bounded by the square of the above bound, that is,

$$\mathbf{P}\big(O_n^c \cap G_k^{(n)}\big) \le C2^{-2n+2n(\beta+1)(\eta-\eta_c)+2\delta n} =: p^{(n)}.$$

Combining this bound with Lemma 4.3 and the Markov inequality, we get

$$\begin{aligned}
\mathbf{P}\Bigg(\sum_{k=0}^{2^n-1} 1_{G_k^{(n)}} \ge 2^{n+\varepsilon n} p^{(n)}\Bigg) &\le \mathbf{P}(O_n) + \mathbf{P}\Bigg(\sum_{k=0}^{2^n-1} 1_{G_k^{(n)}} \ge 2^{n+\varepsilon n} p^{(n)};\, O_n^c\Bigg)\\
&\le Cn^{-m\alpha/3} + \frac{2^n \mathbf{P}(G_1^{(n)})}{2^{n+\varepsilon n} p^{(n)}}\\
&\le Cn^{-m\alpha/3} + 2^{-\varepsilon n}.
\end{aligned}$$

If $2(\beta+1)(\eta-\eta_c)+2\delta < 1$, then, choosing ε sufficiently small, we obtain

$$\mathbf{P}\Bigg(\sum_{k=0}^{2^n-1} 1_{G_k^{(n)}} \ge 1\Bigg) \le C2^{-mn} + 2^{-\varepsilon n}.$$

Applying finally Borel–Cantelli, we conclude that $\tilde{G}_\eta = \varnothing$ almost surely. In particular, $\dim(\tilde{G}_\eta) = 0$ with probability one.

Assume now that $2(\beta+1)(\eta-\eta_c)+2\delta \geq 1$. Applying Borel–Cantelli once again, we see that the number of indices k with $1_{G_k^{(n)}} = 1$ is bounded by $2^{n+\varepsilon n} p^{(n)}$. Noting that $\tilde{G}_\eta$ can be covered by $\bigcup_{k=1}^{2^n} I_k^{(n)} 1_{G_k^{(n)}}$ and

$$\sum_{n=1}^{\infty} 2^{n+\varepsilon n} p^{(n)} 2^{-\theta n} < \infty \qquad \text{for all } \theta > 2(\beta+1)(\eta-\eta_c)+2\delta-1+\varepsilon,$$

we infer that

$$\dim(\tilde{G}_\eta) \leq 2(\beta+1)(\eta-\eta_c)+2\delta-1+\varepsilon.$$

Letting $\varepsilon \to 0$, we get the desired result. □

The remaining part of the subsection is devoted to the proof of the fact that, on the set $(\tilde{J}_{\eta,1} \setminus S_{\eta-2\rho}) \setminus \tilde{G}_\eta$ "enough" of big values of $\mathbf{L}^+_{n,l,k}$ cannot be compensated by $\mathbf{L}^-_{n,l,k}$. This will lead later to the desired upper bound for Hölder exponents for points from $(\tilde{J}_{\eta,1} \setminus S_{\eta-2\rho}) \setminus \tilde{G}_\eta$.

As usual, we start with the analysis of jumps. For $N \geq 1$, define

$$F_k^{(N)} := \{\mathbf{\Delta} X_s(y) \geq (t-s)^{1/(1+\beta)-\gamma} |y-(k+1)2^{-N}|^{\eta-\eta_c} \quad \text{for some } s \geq t-2^{-\alpha N}, y \in I_k^{(N)}\}, \qquad k=0,\dots,2^N-1,$$

and

$$F^{(N)} := \bigcup_{k=0}^{2^N-1} \Bigl(\bigcap_{j<R} F_{k+j}^{(N)}\Bigr).$$

Note that we use N (and not n) above, and at some point we will take $N = Qn$.

LEMMA 4.10. *For every $R > (1-(1+\beta)(\eta-\eta_c))^{-1}$, there exists a constant $C = C(R)$ such that*

$$\mathbf{P}(F^{(N)}) \leq C N^{-m\alpha/3+1}.$$

PROOF. It follows from Lemma 4.3 that

$$\mathbf{P}\Bigl(\bigcup_{n\geq N} O_n\Bigr) \leq \sum_{n=N}^{\infty} \mathbf{P}(O_n) \leq C N^{-m\alpha/3+1}.$$

Therefore,

$$\begin{aligned} \mathbf{P}(F^{(N)}) &\leq \mathbf{P}\Bigl(F^{(N)} \cap \Bigl(\bigcap_{n\geq N} O_n^c\Bigr)\Bigr) + \mathbf{P}\Bigl(\bigcup_{n\geq N} O_n\Bigr) \\ &\leq \sum_{k=0}^{2^N-R-1} \mathbf{P}\Bigl(\Bigl(\bigcap_{j<R} F_{k+j}^{(N)}\Bigr) \cap \Bigl(\bigcap_{n\geq N} O_n^c\Bigr)\Bigr) + C N^{-m\alpha/3+1}. \end{aligned} \tag{4.34}$$

Consider a jump characterized by the triple (y,s,r). We first assume that

$$-(t-s)^{1/\alpha} < y-(k+1)2^{-n} < 0.$$

This jump affects $F_k^{(N)}$ if and only if $r > (t-s)^{1/(1+\beta)-\gamma+(\eta-\eta_c)/\alpha}$. If we consider $s \in [t-2^{-\alpha j}, t-2^{-\alpha(j+1)})$, then r should be greater than $2^{-\alpha(j+1)(1/(1+\beta)-\gamma+(\eta-\eta_c)/\alpha)}$. Since

$$\sup_{s\geq t-2^{-\alpha j}} X_u\big(\big[(k+1)2^{-n}-2^{-j},(k+1)2^{-n}\big)\big) \leq j^{2m}2^{-j}$$

on the event O_j^c, we have the following bound for the intensity of jumps described above:

$$\begin{aligned}
&\int_{t-2^{-\alpha j}}^{t-2^{-\alpha(j+1)}} du\, X_u\big(\big[(k+1)2^{-n}-2^{-j},(k+1)2^{-n}\big)\big)\\
&\quad\times\int_{2^{-\alpha(j+1)(1/(1+\beta)-\gamma+(\eta-\eta_c)/\alpha)}}^{\infty} \varrho r^{-2-\beta}\,dr\\
&\quad\leq Cj^{2m}2^{-j(\alpha+1)}2^{j(\alpha-\gamma\alpha(\beta+1)+(\eta-\eta_c)(\beta+1)}\\
&\quad= Cj^{2m}2^{-j(1-(\eta-\eta_c)(\beta+1))-j\gamma\alpha(\beta+1)}.
\end{aligned}\tag{4.35}$$

If $(k+1)2^{-N}-y \in [a2^{-j},(a+1)2^{-j})$ with some $a\geq 1$, then r should be bigger than $2^{-\alpha(j+1)(1/(1+\beta)-\gamma)}a^{\eta-\eta_c}2^{-j(\eta-\eta_c)}$. Then, on the event O_j^c,

$$\begin{aligned}
&\int_{t-2^{-\alpha j}}^{t-2^{-\alpha(j+1)}} du\, X_u\big(\big[a2^{-j},(a+1)2^{-j}\big)\big)\\
&\quad\times\int_{2^{-\alpha(j+1)(1/(1+\beta)-\gamma)}a^{\eta-\eta_c}2^{-j(\eta-\eta_c)}}^{\infty} \varrho r^{-2-\beta}\,dr\\
&\quad\leq Cj^{2m}2^{-j(\alpha+1)}2^{j(\alpha-\gamma\alpha(\beta+1)+(\eta-\eta_c)(\beta+1)}a^{-(\eta-\eta_c)(\beta+1)}\\
&\quad= Cj^{2m}2^{-j(1-(\eta-\eta_c)(\beta+1))-j\gamma\alpha(\beta+1)}a^{-(\eta-\eta_c)(\beta+1)}.
\end{aligned}\tag{4.36}$$

Combining (4.35), (4.36) and noting that we can cover the interval $I_k^{(N)}$ by the union of intervals $[(k+1)2^{-N}-(a+1)2^{-j},(k+1)2^{-N}-a2^{-j})$ with $a<2^{j-N}$, we see that the intensity of jumps with $y\in I_k^{(N)}$, $s\geq t-2^{-\alpha N}$ is bounded by

$$\begin{aligned}
&\sum_{j=N}^{\infty} Cj^{2m}2^{-j(1-(\eta-\eta_c)(\beta+1))-j\gamma\alpha(\beta+1)}\Bigg(1+\sum_{a=1}^{2^{j-N}-1}a^{-(\eta-\eta_c)(\beta+1)}\Bigg)\\
&\quad\leq C\sum_{j=N}^{\infty} j^{2m}2^{-j(1-(\eta-\eta_c)(\beta+1))-j\gamma\alpha(\beta+1)}\big(2^{j-N}\big)^{1-(\eta-\eta_c)(\beta+1)}
\end{aligned}$$

$$= C(2^{-N})^{1-(\eta-\eta_c)(\beta+1)} \sum_{j=N}^{\infty} j^{2m} 2^{-j\gamma\alpha(\beta+1)}$$
$$\le C(2^{-N})^{1-(\eta-\eta_c)(\beta+1)}.$$

This implies that

$$\mathbf{P}\Big(F_k^{(N)} \cap \Big(\bigcap_{n\ge N} O_n^c\Big)\Big) \le C(2^{-N})^{1-(\eta-\eta_c)(\beta+1)}.$$

Since the jumps can be represented by a time-changed Poisson process, we then get

$$\mathbf{P}\Big(\Big(\bigcap_{j<R} F_{k+j}^{(N)}\Big) \cap \Big(\bigcap_{n\ge N} O_n^c\Big)\Big) \le C(2^{-N})^{R(1-(\eta-\eta_c)(\beta+1))}.$$

Applying this bound to summands in (4.34), we complete the proof of the lemma. □

From this lemma and the Borel–Cantelli lemma, we obtain:

COROLLARY 4.11. *Let R be as in the previous lemma. For $\mathbf{P}$-a.s. $\omega \in A^\varepsilon$ there exists $N_{4.11} = N_{4.11}(\omega)$ such that for every $N \ge N_{4.11}$ and every $k : R \le k < 2^N$ there exists $j = j(k, N) \in \{1, \dots, R\}$ with $1_{F_{k-j}^{(N)}} = 0$.*

We need to introduce additional notation. Let

$$k_n(x) := \lfloor 2^n x \rfloor. \tag{4.37}$$

Recall $A_{k_n(x)}^{(n)}$, and let $\tilde{s} = s_{k_n(x)}^n$ [see (4.17), (4.18)] be the time and $\tilde{y} = y_{k_n(x)}^n$ [defined below (4.18)] be the spatial position of a jump described in the definition of the event $A_{k_n(x)}^{(n)}$. Then on $A_{k_n(x)}^{(n)}$, fix

$$\tilde{l}_n(x) := \lfloor 2^{Qn} \tilde{y} \rfloor. \tag{4.38}$$

Moreover, since $Q > 1$, for every $n \ge N_{4.11}$ we can define

$$\tilde{r}_n(x) = \tilde{k}_n(x) - j(\tilde{k}_n(x), Qn), \tag{4.39}$$

where $j(\cdot,\cdot)$ is defined in Corollary 4.11, and recall that $\tilde{k}_n(x) = k_{Qn}(x)$.

REMARK 4.12. Note that above definition of $\tilde{l}_n(x)$ and especially the construction of $\tilde{r}_n(x)$ are crucial for the proof of the lower bound. In the sequel, we will show that for $x \in (\tilde{J}_{\eta,1} \setminus S_{\eta-2\rho}) \setminus \tilde{G}_\eta$, there exists subsequence $\{n_j\}$ such that big values of $\mathbf{L}^+_{n_j,\tilde{l}_{n_j}(x),\tilde{r}_{n_j}(x)}$ are not compensated by $\mathbf{L}^-_{n_j,\tilde{l}_{n_j}(x),\tilde{r}_{n_j}(x)}$ (see Lemma 4.14 below and Lemma 4.8 above).

We also will define three sets in $[0,t)\times\mathsf{R}$. For any $x\in\mathsf{R}$, set

$$S^1_{n,x} := \{(s,y)\in[0,t)\times\mathsf{R}: |y-x|\geq (t-s)^{(1-\nu)/\alpha}\},$$

$$S^2_{n,x} := \{(s,y)\in([0,t)\setminus(t-2^{-\alpha(1-c\rho)n}, t-2^{-\alpha(1+c\rho)n}))\times\mathsf{R}: |y-x|\leq (t-s)^{(1-\nu)/\alpha}\},$$

$$S^3_{n,x} := \{(s,y)\in[t-2^{-\alpha(1-c\rho)n}, t-2^{-\alpha(1+c\rho)n}]\times\mathsf{R}: |y-x|\leq (t-s)^{(1-\nu)/\alpha}, \mathbf{\Delta}X_s(y)\leq (t-s)^{(\eta+1+3\rho)/\alpha}\},$$

and note that the last set is random. In the next lemma, we will show that, under certain conditions, the jumps of $L^-_{n,\tilde{l}_n(x),\tilde{r}_n(x)}$ are small on the above sets. We will also need an additional piece of notation. Let

$$S^4 := \{(s,y)\in[0,t)\times\mathsf{R}: \mathbf{\Delta}X_s(\{y\})>0\}$$

be the set of points in space×time where the jumps of X, or equivalently of M, occur.

LEMMA 4.13. *Let $\eta\in(\eta_c,\bar{\eta}_c)\setminus\{1\}$, and ρ,ν,c be as in (4.32). For $\mathbf{P}$-a.s. $\omega\in A^\varepsilon$, there exists $N_{4.13}=N_{4.13}(\omega)$ such that for every $n\geq N_{4.13}$ the following holds. Fix arbitrary $x\in(0,1)\setminus S_{\eta-2\rho}$ such that $1_{A^{(n)}_{k_n(x)}}=1$. Then there exists a constant $C_{(4.40)}=C_{(4.40)}(\rho,\nu,c)$ such that for any $(s,y)\in(\bigcup_{i=1}^3 S^i_{n,x})\cap S^4$, we have*

$$\mathbf{\Delta}L^-_{n,\tilde{l}_n(x),\tilde{r}_n(x)}(T_-^{n,\tilde{l}_n(x),\tilde{r}_n(x)}(s))\leq C_{(4.40)}2^{-(\eta+2\rho)n}. \tag{4.40}$$

PROOF. Fix some $\omega\in A^\varepsilon$ and choose $N_{4.13}\geq N_{4.11}$; the choice of $N_{4.13}$ will be clear from the proof. Take arbitrary $n\geq N_{4.13}$. Fix also some $x\in(0,1)\setminus S_{\eta-2\rho}$ satisfying $1_{A^{(n)}_{k_n(x)}}=1$. Recall (4.38), (4.39) and in what follows to simplify the notation denote

$$l=\tilde{l}_n(x),\qquad r=\tilde{r}_n(x).$$

It is clear that a jump, which appears in the definition of $A^{(n)}_{k_n(x)}$, does not produce a jump of $L^-_{n,l,r}$.

Recall that $x\in(0,1)\setminus S_{\eta-2\rho}$ means that, for any $y\in\mathsf{R}$,

$$\mathbf{\Delta}X_s(y)\leq (t-s)^{1/(\beta+1)-\gamma}|y-x|^{\eta-2\rho-\eta_c}. \tag{4.41}$$

We will treat the three regions $S^i_{n,x}$, $i=1,2,3$ separately.

(i) Let $(s,y)\in S^1_{n,x}\cap S^4$, that is,

$$|y-x|\geq (t-s)^{(1-\nu)/\alpha}. \tag{4.42}$$

First assume $|y-x|\leq n^q2^{-n}$. We will consider the cases of $\eta<1$ and $\eta>1$ separately. We start with the following.

Case $\eta < 1$. For all $x_1, x_2 \in \mathsf{R}$,

$$\big(p^{\alpha,\eta}_{t-s}(x_1,x_2)\big)^- = \big(p^{\alpha}_{t-s}(x_1) - p^{\alpha}_{t-s}(x_1)\big)^- \le \frac{p^{\alpha}_1(0)}{(t-s)^{1/\alpha}}.$$

Combining this inequality with (4.41) and (4.42), we get

$$\begin{aligned}\mathbf{\Delta} L^-_{n,l,r}\big(T^{n,l,r}_-(s)\big) &\le p^{\alpha}_1(0)(t-s)^{1/(\beta+1)-\gamma}|y-x|^{\eta-2\rho-\eta_c}(t-s)^{-1/\alpha}\\ &= p^{\alpha}_1(0)(t-s)^{(\eta_c-\alpha\gamma)/\alpha}|y-x|^{\eta-2\rho-\eta_c}\\ &\le p^{\alpha}_1(0)|y-x|^{(\eta_c-\alpha\gamma)/(1-\nu)+\eta-\eta_c-2\rho}.\end{aligned}$$

We choose $N_{4.13}$ sufficiently large such that, for any $\nu \in (\frac{\alpha\gamma+5\rho}{\eta_c}, 10^{-1})$ and $|y-x| \le n^q 2^{-n}$, (4.40) holds, for all $n \ge N_{4.13}$.

Case $\eta > 1$. Here, we have

$$\begin{aligned}&p^{\alpha,\eta}_{t-s}(l2^{-Qn}-y, r2^{-Qn}-y)\\ &\quad= p^{\alpha}_{t-s}(l2^{-Qn}-y) - p^{\alpha}_{t-s}(r2^{-Qn}-y) + (r-l)2^{-Qn}p^{\alpha,\prime}_{t-s}(r2^{-Qn}-y).\end{aligned}$$

Note that $p^{\alpha,\prime}_{t-s}(z) \ge 0$ for all $z \le 0$. Consequently,

$$\begin{aligned}&\big(p^{\alpha,\eta}_{t-s}(l2^{-Qn}-y, r2^{-Qn}-y)\big)^-\\ &\quad\le \big(p^{\alpha}_{t-s}(l2^{-Qn}-y) - p^{\alpha}_{t-s}(r2^{-Qn}-y)\big)^-\\ &\quad\le \frac{p^{\alpha}_1(0)}{(t-s)^{1/\alpha}}\end{aligned}\tag{4.43}$$

for all $y \ge r2^{-Qn}$.

In the complementary case $y < r2^{-Qn}$, one can easily get

$$\begin{aligned}&\big(p^{\alpha,\eta}_{t-s}(l2^{-Qn}-y, r2^{-Qn}-y)\big)^-\\ &\quad\le \frac{p^{\alpha}_1(0)}{(t-s)^{1/\alpha}} + \big|(r-l)2^{-Qn}p^{\alpha,\prime}_{t-s}(r2^{-Qn}-y)\big|.\end{aligned}\tag{4.44}$$

If $y \le (r-1)2^{-Qn}$, then $r2^{-Qn} - y \ge (x-y)/(R+1)$. Thus, using the bound $p^{\alpha,\prime}_1(z) \le C|z|^{-\alpha-2}$ and the scaling property, we obtain

$$\begin{aligned}-p^{\alpha,\prime}_{t-s}(r2^{-Qn}-y) &\le C(t-s)^{-2/\alpha}\left(\frac{r2^{-Qn}-y}{(t-s)^{1/\alpha}}\right)^{-\alpha-2}\\ &\le CR^{\alpha+2}(t-s)|x-y|^{-\alpha-2}.\end{aligned}\tag{4.45}$$

From this, (4.43) and (4.44) we conclude that, for all y satisfying $|y-r2^{-Qn}| > 2^{-Qn}$,

$$\begin{aligned}&\big(p^{\alpha,\eta}_{t-s}(l2^{-Qn}-y, r2^{-Qn}-y)\big)^-\\ &\quad\le C\big((t-s)^{-1/\alpha} + (r-l)2^{-Qn}(t-s)|x-y|^{-\alpha-2}\big).\end{aligned}$$

Combining this with (4.41) we conclude that the corresponding jump $\mathbf{\Delta} L^-_{n,l,r}(T^{n,l,r}_-(s))$ is bounded by

$$C(t-s)^{1/(\beta+1)-\gamma}|y-x|^{\eta-2\rho-\eta_c}\big((t-s)^{-1/\alpha}+(r-l)2^{-Qn}(t-s)|x-y|^{-\alpha-2}\big).$$

Taking into account (4.42), we see that the expression on the right-hand side does not exceed

$$C\big(|y-x|^{(\eta_c-\alpha\gamma)/(1-\nu)+\eta-\eta_c-2\rho}+(r-l)2^{-Qn}|y-x|^{(\eta-1)/(1-\nu)+\nu\alpha-2\rho}\big).$$

Now it is easy to see that (4.40) remain valid for $\eta>1$ under additional assumption $y\le(r-1)2^{-Qn}$, or $y\ge r2^{-Qn}$.

Now we will take care of the case $y\in((r-1)2^{-Qn},r2^{-Qn})$. By Corollary 4.11 and our definition of $r=\tilde r_n(x)$ [recall again (4.39) and $n\ge N_{4.13}\ge N_{4.11}$, $Q>1$], we obtain

$$\mathbf{\Delta} X_s(y)\le(t-s)^{1/(\beta+1)-\gamma}\big|y-r2^{-Qn}\big|^{\eta-\eta_c} \tag{4.46}$$

for all $y\in((r-1)2^{-Qn},r2^{-Qn})$ and $s\ge t-2^{-\alpha Qn}$. It is clear that $y\in((r-1)2^{-Qn},r2^{-Qn})$ implies that $|y-x|\le(R+1)2^{-Qn}$. From this and (4.42), we infer that $(t-s)\le(R+1)^{\alpha/(1-\nu)}2^{-\alpha Qn/(1-\nu)}$ and, consequently, $s\ge t-2^{-\alpha Qn}$ for all sufficiently large n. Repeating all the arguments after (4.44) and using (4.46) instead of (4.41), we obtain (4.40).

Summarizing, (4.40) is valid for all $|y-x|\le n^q2^{-n}$.

In case $|y-x|\ge n^q2^{-n}$, we apply Corollary 2.3 if $\eta>1$, or Lemma 2.1 if $\eta<1$ with $\delta=\eta+3\rho$ (recall the bounds on ρ and γ to get that $\delta<1$ if $\eta<1$ and $\delta<2$ if $\eta>1$) to get

$$\big|\tilde p^{\alpha,\eta}_{t-s}(l2^{-Qn}-y,r2^{-Qn}-y)\big|\le C\frac{(r-l)^\delta 2^{-Q\delta n}}{(t-s)^{(\delta+1)/\alpha}}p^\alpha_1\bigg(\frac{y-r2^{-Qn}}{(t-s)^{1/\alpha}}\bigg).$$

Since $(r-l)2^{-Qn}\le 4n^q2^{-n}$ and $r2^{-Qn}\le x$, we then have

$$\big|\tilde p^{\alpha,\eta}_{t-s}(l2^{-Qn}-y,r2^{-Qn}-y)\big|\le C\frac{n^{q\delta}2^{-\delta n}}{(t-s)^{(1+\delta)/\alpha}}p^\alpha_1\bigg(\frac{y-x}{(t-s)^{1/\alpha}}\bigg).$$

From this bound and (3.11), we obtain

$$\begin{aligned}
&\mathbf{\Delta} L^-_{n,l,r}(T^{n,l,r}_-(s))\\
&\quad\le C(t-s)^{1/(\beta+1)-\gamma}|y-x|^{\eta-\eta_c-2\rho}\frac{n^{q\delta}2^{-\delta n}}{(t-s)^{(\delta+1)/\alpha}}p^\alpha_1\bigg(\frac{y-x}{(t-s)^{1/\alpha}}\bigg)\\
&\quad\le Cn^{q\delta}2^{-\delta n}(t-s)^{1/(\beta+1)-\gamma-(\delta+1)/\alpha+(\alpha+1)/\alpha}|y-x|^{\eta-\eta_c-2\rho-\alpha-1}.
\end{aligned}$$

As a result, for $|y-x|\ge (t-s)^{(1-\nu)/\alpha}$ we have

$$\begin{aligned}
&\mathbf{\Delta} L^{-}_{n,l,r}\big(T^{n,l,r}_{-}(s)\big)\\
&\quad\le Cn^{q\delta}2^{-\delta n}\big((t-s)^{1/\alpha}\big)^{\alpha/(\beta+1)-\alpha\gamma-\delta+\alpha+(\eta-\eta_c-2\rho-\alpha-1)(1-\nu)}\\
&\quad\le C2^{-(\eta+2\rho)n}\big((t-s)^{1/\alpha}\big)^{\eta_c+1-\alpha\gamma-\eta-3\rho+\alpha+(\eta-\eta_c-2\rho-\alpha-1)(1-\nu)}\\
&\quad= C2^{-(\eta+2\rho)n}\big((t-s)^{1/\alpha}\big)^{\nu(\alpha+1-\eta+\eta_c+2\rho)-\alpha\gamma-5\rho}.
\end{aligned}$$

Finally, recall that $\nu\ge \frac{5\rho+\alpha\gamma}{\eta_c}$, and then (4.40) holds with an appropriate constant $C_{(4.40)}$.

(ii) Let $(s,y)\in S^2_{n,x}\cap S^4$. We start with the subset of $S^2_{n,x}$ where $|y-x|\le (t-s)^{(1-\nu)/\alpha}$ and $s\le t-2^{-\alpha(1-c\rho)n}$.

If $\eta<1$ then, applying Lemma 2.1 with $\delta=1$, we obtain

$$\begin{aligned}
\mathbf{\Delta} L^{-}_{n,l,r}\big(T^{n,l,r}_{-}(s)\big) &\le C(t-s)^{1/(\beta+1)-\gamma}|y-x|^{\eta-\eta_c-2\rho}\frac{4n^q2^{-n}}{(t-s)^{2/\alpha}}\\
&\le Cn^q2^{-n}(t-s)^{1/(\beta+1)-\gamma-2/\alpha}(t-s)^{(1-\nu)(\eta-\eta_c-2\rho)/\alpha}\\
&= Cn^q2^{-n}\big((t-s)^{1/\alpha}\big)^{\alpha/(\beta+1)-\alpha\gamma-2+(1-\nu)(\eta-\eta_c-2\rho)}\\
&= Cn^q2^{-n}\big((t-s)^{1/\alpha}\big)^{\eta-1-\alpha\gamma-2\rho-\nu(\eta-\eta_c-2\rho)}\\
&\le Cn^q2^{-n}\big(2^{-n(1-c\rho)}\big)^{\eta-1-\alpha\gamma-2\rho-\nu(\eta-\eta_c-2\rho)}.
\end{aligned}$$

If $\eta>1$, then we can apply Corollary 2.3 with $\delta=2$, which gives

$$\begin{aligned}
\mathbf{\Delta} L^{-}_{n,l,r}\big(T^{n,l,r}_{-}(s)\big) &\le C(t-s)^{1/(\beta+1)-\gamma}|y-x|^{\eta-\eta_c-2\rho}\frac{16n^{2q}2^{-2n}}{(t-s)^{3/\alpha}}\\
&\le Cn^{2q}2^{-2n}(t-s)^{1/(\beta+1)-\gamma-3/\alpha}(t-s)^{(1-\nu)(\eta-\eta_c-2\rho)/\alpha}\\
&= Cn^{2q}2^{-2n}\big((t-s)^{1/\alpha}\big)^{\alpha/(\beta+1)-\alpha\gamma-3+(1-\nu)(\eta-\eta_c-2\rho)}\\
&= Cn^{2q}2^{-2n}\big((t-s)^{1/\alpha}\big)^{\eta-2-\alpha\gamma-2\rho-\nu(\eta-\eta_c-2\rho)}\\
&\le Cn^{2q}2^{-2n}\big(2^{-n(1-c\rho)}\big)^{\eta-2-\alpha\gamma-2\rho-\nu(\eta-\eta_c-2\rho)}.
\end{aligned}$$

Hence, for $\eta\in(\eta_c,\bar\eta_c)\setminus\{1\}$, and with c,ρ,ν as in (4.32), we immediately get (4.40) with an appropriate constant $C_{(4.40)}$.

Now we consider the complimentary subset of $S^2_{n,x}$, where

$$|y-x|\le (t-s)^{(1-\nu)/\alpha}\quad\text{and}\quad s\ge t-2^{-\alpha(1+c\rho)n}.$$

It follows from the definition of $\tilde p^{\alpha,\eta}_{t-s}$ that, for $\eta>1$,

$$\begin{aligned}
&\big|\tilde p^{\alpha,\eta}_{t-s}(l2^{-Qn}-y,r2^{-Qn}-y)\big|\\
&\qquad\le \frac{2p^\alpha_1(0)}{(t-s)^{1/\alpha}}+(r-l)2^{-Qn}\sup_z\left|\frac{\partial}{\partial z}p^\alpha_1(z)\right|\Big/(t-s)^{2/\alpha}.
\end{aligned}\tag{4.47}$$

With this and recalling that $(r-l)2^{-Qn} \le 4n^q 2^{-n}$, we obtain

$$\begin{aligned}\mathbf{\Delta}L^{-}_{n,l,r}\big(T^{n,l,r}_{-}(s)\big) &\le C(t-s)^{1/(\beta+1)-\gamma}|y-x|^{\eta-\eta_c-2\rho} \\ &\quad \times \big((t-s)^{-1/\alpha}+n^q2^{-n}(t-s)^{-2/\alpha}\big) \\ &\le Cn^q2^{-n}(t-s)^{1/(\beta+1)-\gamma-2/\alpha}|y-x|^{\eta-\eta_c-2\rho} \\ &\le Cn^q2^{-n}\big((t-s)^{1/\alpha}\big)^{\eta_c-1-\alpha\gamma+(1-\nu)(\eta-\eta_c-2\rho)} \\ &\le Cn^q2^{-n}\big(2^{-(1+c\rho)n}\big)^{\eta-1-\alpha\gamma-2\rho-\nu(\eta-\eta_c-2\rho)}.\end{aligned}$$

Again, with c, ρ, ν as in (4.32), we immediately get (4.40) with an appropriate constant $C_{(4.40)}$. If $\eta < 1$ then, instead of (4.47), we have a simpler inequality

$$\big|\tilde{p}^{\alpha,\eta}_{t-s}\big(l2^{-Qn}-y, r2^{-Qn}-y\big)\big| \le \frac{2p^{\alpha}_{1}(0)}{(t-s)^{1/\alpha}}.$$

Thus,

$$\mathbf{\Delta}L^{-}_{n,l,r}\big(T^{n,l,r}_{-}(s)\big) \le C\big(2^{-(1+c\rho)n}\big)^{\eta-\alpha\gamma-2\rho-\nu(\eta-\eta_c-2\rho)}.$$

Consequently, (4.40) holds also for $\eta < 1$.

(iii) Let $(s, y) \in S^3_{n,x} \cap S^4$.

Recall that on this set, $\mathbf{\Delta}X_s(y) \le (t-s)^{(\eta+1+3\rho)/\alpha}$. Then applying Corollary 2.3 (if $\eta > 1$) or Lemma 2.1 (if $\eta < 1$) with $\delta = \eta + 3\rho$, and by using that c, ρ, ν are as in (4.32), one can easily get (4.40) in this case as well. □

Recall that $\mathbf{L}^{-}_{n,l,r} = L^{-}_{n,l,r}(T^{n,l,r}_{-}(t))$. In the next lemma, we will deal with regions where $\{\mathbf{L}^{-}_{n,\tilde{l}_n(x),\tilde{r}_n(x)}\}_{n\ge1}$ may take "big" values infinitely often.

LEMMA 4.14. *For $x \in (0,1)$ define events*

$$\mathbf{B}_n(x) \equiv \big\{\mathbf{L}^{-}_{n,\tilde{l}_n(x),\tilde{r}_n(x)} \ge 2^{-\eta n-1}\big\} \cap \{1_{A^{(n)}_{k_n(x)}} = 1\}, \qquad n \ge 1.$$

For any $\eta \in (\eta_c, \bar{\eta}_c) \setminus \{1\}$, we have

$$\mathbf{P}\big(\{x \in (0,1) \setminus S_{\eta-2\rho} : \mathbf{B}_n(x) \text{ i.o.}\} \subset \tilde{G}_\eta | A^{\varepsilon}\big) = 1.$$

PROOF. First, we will show that on $\omega \in A^{\varepsilon}$

$$\{x \in (0,1) \setminus S_{\eta-2\rho} : \mathbf{B}^*_n(x) \text{ i.o.}\} \subset \tilde{G}_\eta, \tag{4.48}$$

where for $x \in (0,1)$,

$$\mathbf{B}^*_n(x) \equiv \big\{\mathbf{\Delta}L^{-}_{n,\tilde{l}_n(x),\tilde{r}_n(x)} \ge C_{(4.40)}2^{-(\eta+2\rho)n}\big\} \cap \{1_{A^{(n)}_{k_n(x)}} = 1\}, \qquad n \ge 1.$$

On $\omega \in A^{\varepsilon}$, take $n \ge N_{4.13}$, and fix some $x \in (0,1) \setminus S_{\eta-2\rho}$ satisfying $1_{A^{(n)}_{k_n(x)}} = 1$. First of all, by definition of $A^{(n)}_{k_n(x)}$, if $1_{A^{(n)}_{k_n(x)}} = 1$ then there exists a jump of M

of the form $\tilde{r}\delta_{(\tilde{s},\tilde{y})}$ with $\tilde{r},\tilde{s},\tilde{y}$ as in $G^{(n)}_{k_n(x)}$ [see (4.33)]. Moreover, again by the definition of $A^{(n)}_{k_n(x)}$, the spatial position of the jump, $\tilde{y}$, is in $I^{(n)}_{k_n(x)-2n^q-2}$. Hence, it is easy to see that this jump does not contribute to the jumps of $L^-_{n,\tilde{l}_n(x),\tilde{r}_n(x)}$ that is, $\mathbf{\Delta} L^-_{n,\tilde{l}_n(x),\tilde{r}_n(x)}(T_-^{n,\tilde{l}_n(x),\tilde{r}_n(x)}(\tilde{s}))=0$. Thus, we have to show that, if $\mathbf{\Delta} L^-_{n,\tilde{l}_n(x),\tilde{r}_n(x)}\geq C_{(4.40)}2^{-(\eta+2\rho)n}$, and $1_{A^{(n)}_{k_n(x)}}=1$, then there exists at least one another big jump of M with properties described in $G^{(n)}_k$.

By Lemma 4.13, we get that if there exists s such that

$$\mathbf{\Delta} L^-_{n,\tilde{l}_n(x),\tilde{r}_n(x)}\big(T_-^{n,\tilde{l}_n(x),\tilde{r}_n(x)}(s)\big)\geq C_{(4.40)}2^{-(\eta+2\rho)n},$$

then the corresponding jump of M, of the form $r\delta_{(s,y)}$, has to satisfy

$$|y-x|\leq (t-s)^{(1-\nu)/\alpha},\qquad s\in\big[t-2^{-\alpha(1-c\rho)n},t-2^{-\alpha(1+c\rho)n}\big],$$
$$r\geq (t-s)^{(\eta+1+3\rho)/\alpha}.$$

This yields that on A^ε (4.48) holds.

Second, it follows from the second inequality in Lemma 2.5 and (4.31), that

$$\mathbf{P}\big(\mathbf{L}^-_{n,l,r}\geq 2^{-\eta n-1};\mathbf{\Delta} L^-_{n,l,r}\leq C_{(4.40)}2^{-(\eta+2\rho)n}\big)\leq \exp\{-c2^{2\rho n}\}$$

for all l,r satisfying $(r-l)2^{-Qn}\leq 4n^q2^{-n}$. (Recall that $(\tilde{r}_n(x)-\tilde{l}_n(x))2^{-Qn}\leq 4n^q2^{-n}$.)

Applying now Borel–Cantelli, we conclude that, with probability one,

$$\bigcup_{0\leq l<r\leq 2^{Qn}-1,(r-l)2^{-Qn}\leq 4n^q2^{-n}}\{\mathbf{L}^-_{n,l,r}\geq 2^{-\eta n-1};\mathbf{\Delta} L^-_{n,l,r}\leq C_{(4.40)}2^{-(\eta+2\rho)n}\}$$

occurs only finite number of times. This completes the proof of the lemma. □

4.4. *Proof of Proposition* 4.1. Fix arbitrary $\eta\in(\eta_c,\bar{\eta}_c)\setminus\{1\}$. Also fix

$$Q=\bigg[\max\bigg\{4\frac{\eta}{\eta_c},4\frac{\eta}{|\eta-1|}\bigg\}+2\bigg],$$

where $[x]$ denotes the integer part of x.

It follows from Lemmas 4.8 and 4.14, that for every $x\in(\tilde{J}_{\eta,1}\setminus S_{\eta-2\rho})\setminus\tilde{G}_\eta$, for $\mathbf{P}$-a.s. ω on A^ε, there exists a (random) sequence $\{n_j\}_{j\geq1}$ such that

$$\mathbf{L}^+_{n_j,\tilde{l}_{n_j}(x),\tilde{r}_{n_j}(x)}\geq n_j^m2^{-\eta n_j},\qquad \mathbf{L}^-_{n_j,\tilde{l}_{n_j}(x),\tilde{r}_{n_j}(x)}\leq 2^{-(\eta n_j-1)},$$

for all n_j sufficiently large. This implies that, on the event A^ε, we have

$$\liminf_{j\to\infty}2^{(\eta+\delta)n_j}\bigg|\tilde{Z}_t^{2,\eta}\bigg(\frac{\tilde{l}_{n_j}(x)}{2^{Qn_j}},\frac{\tilde{r}_{n_j}(x)}{2^{Qn_j}}\bigg)\bigg|=\infty,\tag{4.49}$$

for any $\delta > 0$. Recall that, $X_t(\cdot)$ and $Z_t^2(\cdot)$ are Hölder continuous with any exponent less than η_c at every point of $(0,1)$. Therefore, recalling that $Q > 4\frac{\eta}{\eta_c}$, we have

$$\begin{aligned}\lim_{j\to\infty} \sup_{x\in(0,1)} & 2^{(\eta+\delta)n_j}\big|X_t(x) - X_t\big(\tilde{r}_{n_j}(x)2^{-Qn_j}\big)\big| \\ &= \lim_{j\to\infty} C(\omega)2^{-(1/2)Q\eta_c n_j}2^{(\eta+\delta)n_j} = 0, \qquad \mathbf{P}\text{-a.s. on } A^{\varepsilon}.\end{aligned} \tag{4.50}$$

If $\eta < 1$, then $\tilde{Z}_t^{2,\eta}(x_1,x_2) = Z_t^2(x_1) - Z_t^2(x_2)$. Therefore, combining (4.49) and (4.50), we conclude that

$$H_{Z^2}(x) \le \eta \qquad \text{for all } x \in (\tilde{J}_{\eta,1} \setminus S_{\eta-2\rho}) \setminus \tilde{G}_\eta \ \mathbf{P}\text{-a.s. on } A^{\varepsilon}. \tag{4.51}$$

Assume now that $\eta > 1$. In this case, we infer from (3.28) that $\mathbf{P}$-a.s. on A^{ε},

$$\limsup_{j\to\infty} \sup_{x\in(0,1)\setminus S_{\eta-2\rho}} 2^{Q(\eta-1-2\rho-2\alpha\gamma)n_j}\big|V'\big(\tilde{r}_{n_j}2^{-Qn_j}\big) - V'(x)\big| = 0. \tag{4.52}$$

Combining (4.49), (4.50) and (4.52), and recalling that $Q > 4\eta/(\eta-1)$, we get

$$\liminf_{j\to\infty} 2^{(\eta+\delta)n_j}\big|\tilde{Z}_t^{2,\eta}\big(2^{-Qn_j}\tilde{l}_{n_j}(x), x\big)\big| = \infty \qquad \text{on } A^{\varepsilon}, \mathbf{P}\text{-a.s.}$$

This implies that (4.51) holds.

We know, by Lemma 3.2, that

$$H_{Z^2}(x) \ge \eta - \alpha\gamma - 2\rho \qquad \text{for all } x \in (0,1) \setminus S_{\eta-2\rho}, \mathbf{P}\text{-a.s.}$$

This and (4.51) imply that on A^{ε}, $\mathbf{P}$-a.s.,

$$\begin{aligned}&\eta - \alpha\gamma - 2\rho \le H_{Z^2}(x) \le \eta \\ &\qquad \text{for all } x \in (\tilde{J}_{\eta,1} \setminus S_{\eta-2\rho}) \setminus \tilde{G}_\eta, \forall \eta \in (\eta_c, \bar{\eta}_c) \setminus \{1\}.\end{aligned} \tag{4.53}$$

It follows easily from Lemma 3.3, Corollary 4.6 and Lemma 4.9 that on A^{ε}

$$\dim\big((\tilde{J}_{\eta,1} \setminus S_{\eta-2\rho}) \setminus \tilde{G}_\eta\big) \ge (\beta+1)(\eta - \eta_c), \qquad \mathbf{P}\text{-a.s.}$$

Thus, by (4.53),

$$\dim\big\{x : H_{Z^2}(x) \le \eta\big\} \ge (\beta+1)(\eta - \eta_c) \qquad \text{on } A^{\varepsilon}, \mathbf{P}\text{-a.s.}$$

It is clear that

$$\begin{aligned}\big\{x : H_{Z^2}(x) = \eta\big\} &\cup \bigcup_{n=n_0}^{\infty} \big\{x : H_{Z^2}(x) \in \big(\eta - n^{-1}, \eta - (n+1)^{-1}\big]\big\} \\ &= \big\{x : \eta - n_0^{-1} \le H_{Z^2}(x) \le \eta\big\}.\end{aligned}$$

Consequently,

$$\begin{aligned}\mathcal{H}_\eta\big(\{x : \eta - n_0^{-1} \le H_{Z^2}(x) \le \eta\}\big) \\ &= \mathcal{H}_\eta\big(\{x : H_{Z^2}(x) = \eta\}\big) \\ &\quad + \sum_{n=n_0}^{\infty} \mathcal{H}_\eta\big(\{x : H_{Z^2}(x) \in \big(\eta - n^{-1}, \eta - (n+1)^{-1}\big]\}\big).\end{aligned}$$

Since the dimensions of $S_{\eta-2\rho}$ and $\tilde{G}_\eta$ are smaller than η, the $\mathcal{H}_\eta$-measure of these sets equals zero. Applying Corollary 4.6, we then conclude that on A^ε

$$\mathcal{H}_\eta\big((\tilde{J}_{\eta,1} \setminus S_{\eta-2\rho}) \setminus \tilde{G}_\eta\big) > 0, \qquad \mathbf{P}\text{-a.s.}$$

And in view of (4.53), $\mathcal{H}_\eta(\{x : \eta - n_0^{-1} \le H_{Z^2}(x) \le \eta\}) > 0$. Furthermore, it follows from Proposition 3.1, that dimension of the set $\{x : H_{Z^2}(x) \in (\eta - n^{-1}, \eta - (n+1)^{-1}]\}$ is bounded from above by $(\beta+1)(\eta - (n+1)^{-1} - \eta_c)$. Hence, the definition of $\mathcal{H}_\eta$ immediately yields

$$\mathcal{H}_\eta\big(\{x : H_{Z^2}(x) \in \big(\eta - n^{-1}, \eta - (n+1)^{-1}\big]\}\big) = 0 \qquad \text{on } A^\varepsilon, \mathbf{P}\text{-a.s.},$$

for all $n \ge n_0$. As a result, we have

$$\mathcal{H}_\eta\big(\{x : H_{Z^2}(x) = \eta\}\big) > 0 \qquad \mathbf{P}\text{-a.s. on } A^\varepsilon. \tag{4.54}$$

Since $\varepsilon > 0$ was arbitrary, this implies that (4.54) is satisfied on the whole probability space $\mathbf{P}$-a.s. From this, we get that

$$\dim\{x : H_{Z^2}(x) = \eta\} \ge (\beta+1)(\eta - \eta_c), \qquad \mathbf{P}\text{-a.s.}$$

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