Non-local observables and lightcone-averaging in relativistic thermodynamics

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The unification of relativity and thermodynamics has been a subject of considerable debate over the past 100 years. The reasons for this are twofold. First, thermodynamic variables are non-local quantities and therefore single out a preferred class of hyperplanes in spacetime. Second, there exist different ways of defining heat and work in relativistic systems and all of them seem equally plausible. These ambiguities have led, for example, to various proposals for the Lorentz-transformation law of temperature. However, traditional 'isochronous' formulations of relativistic thermodynamics are neither theoretically satisfactory nor experimentally feasible. Here, we demonstrate how these deficiencies can be resolved by defining thermodynamic quantities with respect to the backward-lightcone of an observation event. This approach yields new predictions that are, in principle, testable and allows for a straightforward extension of thermodynamics to general relativity. Our theoretical considerations are illustrated through three-dimensional relativistic many-body simulations.

Thermodynamics, in the traditional sense, aims at describing the state of a macroscopic system by means of a few characteristic parameters $\{A_i\}$ (refs 1–4). Typical candidates for thermodynamic state variables $\{A_i\}$ are either conserved (extensive) quantities, for example, the particle number N and internal energy U, or external control parameters that quantify the breaking of symmetries[1]. Examples of the last of these include the volume V of a confining vessel, indicating the violation of translational invariance, or external magnetic fields, which may break the spatial isotropy. Each extensive state variable is accompanied by an intensive quantity $a_i = \partial S/A_i$, derived from a suitably defined entropy function(al) $S(\{A_i\})$. Representing an abstract mathematical theory of differential forms[4], thermodynamic concepts have been successfully applied to vastly different areas, ranging from microscopic many-particle systems[2,3,5], where S is usually interpreted as an information measure (canonical ensemble) or integrated phase-space volume (microcanonical ensemble), to exotic objects such as black holes[6], where S is related to the black hole's surface area.

As a coarse-grained macroscopic theory, thermodynamics is inherently non-local in that it considers only certain global, or averaged, properties of a physical system[2,3]. This is rather unproblematic within non-relativistic Newtonian physics, where statements such as 'the total energy of a system at time t' are unambiguously defined for arbitrary observers. In contrast—owing to the absence of a universal time parameter—the non-local character of thermodynamics has caused considerable confusion[7–15] within Einstein's theory of relativity[16–19].

To illustrate the conceptual difficulties in relativistic thermodynamics, consider a confined gas described by a particle current density $j^\mu(t,\mathbf{x})$ and an energy–momentum tensor density $\theta^{\mu\nu}(t,\mathbf{x})$. If the gas is stationary in some inertial frame Σ, then j^μ is conserved, that is, $\partial_\mu j^\mu \equiv 0$, but the divergence of $\theta^{\mu\nu}$ does not identically vanish (owing to the pressure arising from the spatial confinement, see the example below):

$$\partial_\mu \theta^{\mu i} \not\equiv 0, \qquad i=1,2,3 \tag{1}$$

This means that space-like surface integrals over j^μ are independent of the underlying three-dimensional hypersurface $\mathcal{H}$ in $(1+3)$-dimensional Minkowski spacetime $\mathbb{M}_4$, whereas those over $\theta^{\mu\nu}$ do depend on $\mathcal{H}$. The latter fact is problematic because thermodynamic state variables such as energy U^0 or momentum $\mathbf{U}=(U^1,U^2,U^3)$ are usually defined as surface integrals over the energy–momentum tensor (see the Methods section)[16], that is,

$$U^\nu[\mathcal{H}] := \int_{\mathcal{H}} \mathrm{d}\sigma_\mu\, \theta^{\mu\nu}, \qquad \mu,\nu=0,1,2,3 \tag{2}$$

where, for a finite thermodynamic system, $\theta^{\mu\nu}$ is assumed to vanish outside a bounded spatial region. Hence, the first task in relativistic thermodynamics is to identify those hypersurfaces $\{\mathcal{H}\}$ that are suitable for defining state variables. Subsequently, one still needs to settle for appropriate definitions of entropy, heat and so on.

We shall begin by reviewing how these problems are tackled in the most popular, competing versions of relativistic thermodynamics, originally proposed by Planck[7] and Einstein[8], and Ott[10] and Van Kampen[9], respectively. A careful analysis shows that the traditional approaches are neither conceptually satisfactory nor experimentally feasible. The deficiencies can be cured by defining thermodynamic quantities in terms of lightcone integrals. To clarify these aspects, we consider a weakly interacting relativistic gas[20]. Notwithstanding, the main conclusions apply to any confined system that can be described by tensor densities $j^\mu, \theta^{\alpha\beta}, \ldots$. In the second part, we shall discuss observable consequences such as the apparent drift of distant objects that are, in fact, at rest relative to the observer. This surprising effect—which should be accounted for when estimating the velocities of very hot astrophysical objects from photographic data—will be illustrated by relativistic many-particle simulations.

Model (Jüttner gas)

We consider an enclosed, dilute gas consisting of N relativistic particles (rest mass m; velocity $\mathbf{v}$; momentum $\mathbf{p}=m\mathbf{v}(1-\mathbf{v}^2)^{-1/2}$; speed of light $c=1$). Let us assume the gas is stationary in the ('lab'-)frame

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Σ, and can be described by a Σ-time-independent, normalized one-particle phase-space probability density function (PDF)

$$f(t,\mathbf{x},\mathbf{p}) = \varphi(\mathbf{x},\mathbf{p}) = \varrho(\mathbf{x})\,\phi_{\mathrm{J}}(\mathbf{p}) \tag{3a}$$

with Jüttner momentum distribution[20,21]

$$\phi_{\mathrm{J}}(\mathbf{p}) = Z^{-1}\exp(-\beta p^0), \qquad \beta > 0 \tag{3b}$$

$Z = 4\pi m^3 K_2(\beta m)/(\beta m)$ is the normalization constant, $K_n(z)$ is the nth modified Bessel function of the second kind[22] and $p^0 = (m^2 + \mathbf{p}^2)^{1/2}$ is the particle energy. Later, the distribution parameter β will be identified with the inverse thermodynamic (rest) temperature of the gas. The exact functional form of the spatial density ϱ in equation (3a) is irrelevant, as long as ϱ is normalizable (that is, restricted to a finite spatial volume set $\mathbb{V} \subset \mathbb{R}^3$ in Σ). For simplicity, we may consider a spatially homogeneous gas enclosed in a stationary cubic box $\mathbb{V} = [-L/2, L/2]^3$ in Σ, corresponding to

$$\varrho(\mathbf{x}) = \begin{cases} V^{-1}, & \text{if } \mathbf{x} \in \mathbb{V} \\ 0, & \text{if } \mathbf{x} \notin \mathbb{V} \end{cases} \tag{3c}$$

Here, $V = L^3$ is the Σ-simultaneously measured (Lebesgue) volume of $\mathbb{V}$ in Σ.

The phase-space PDF f is a Lorentz scalar[23]. Thus, the current density j^μ and energy–momentum tensor $\theta^{\mu\nu}$ can be constructed from f by:

$$j^\mu(t,\mathbf{x}) = N \int \frac{\mathrm{d}^3 p}{p^0}\, f\, p^\mu \tag{4a}$$

$$\theta^{\mu\nu}(t,\mathbf{x}) = N \int \frac{\mathrm{d}^3 p}{p^0}\, f\, p^\mu p^\nu \tag{4b}$$

where $\mathrm{d}^3 p/p^0$ is the Lorentz-invariant integration measure in relativistic momentum space. Concretely, for equation (3) we have $(j^\alpha) = (N\varrho, \mathbf{0})$ and

$$\theta^{\mu\nu} = N\,\varrho \begin{cases} \langle p^0 \rangle, & \mu = \nu = 0 \\ \beta^{-1}, & \mu = \nu = 1,2,3 \\ 0, & \mu \neq \nu \end{cases} \tag{5}$$

where $\langle p^0 \rangle = 3\beta^{-1} + m\, K_1(\beta m)/K_2(\beta m)$ is the mean energy per particle. One readily verifies that $\partial_\alpha j^\alpha \equiv 0$, whereas $\partial_\mu \theta^{\mu i} = N\beta^{-1}\partial_i \varrho \neq 0$ at the boundary of $\mathbb{V}$, in agreement with equation (1). Confinement generates stress—the importance of this seemingly trivial statement shall be seen immediately.

Isochronous state variables

The traditional versions of relativistic thermodynamics[7–12,14] are recovered from equations (3)–(5) by inserting isochronous spacetime hypersurfaces into equation (2). To see this, consider an inertial frame Σ', moving at velocity w along the x^1-axis of the lab-frame Σ. An event $\mathscr{E}$ with coordinates $(\xi^0, \boldsymbol{\xi})$ in Σ and $(\xi'^0, \boldsymbol{\xi}')$ in Σ' defines isochronous hyperplanes $\mathcal{I}(\xi^0)$ and $\mathcal{I}'(\xi'^0)$ in Σ and Σ', respectively, by

$$\mathcal{I}(\xi^0) := \{\, (t,\mathbf{x}) \mid t = \xi^0 \,\} \tag{6a}$$

$$\mathcal{I}'(\xi'^0) := \{\, (t',\mathbf{x}') \mid t' = \xi'^0 \,\} \tag{6b}$$

If Σ and Σ' are in relative motion, these hyperplanes differ from each other, $\mathcal{I}(\xi^0) \neq \mathcal{I}'(\xi'^0)$, see Fig. 1. Inserting $\mathcal{H} = \mathcal{I}(\xi^0)$ into equation (2), we obtain the lab-isochronous energy–momentum vector $U^\mu[\mathcal{I}]$ in Σ:

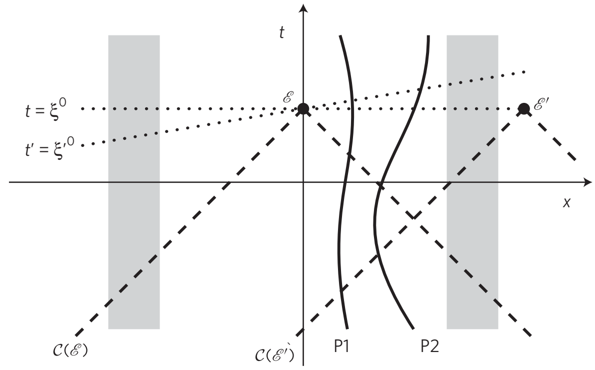

Figure 1 | Non-local thermodynamic quantities depend on the underlying hypersurface in Minkowski spacetime. When particles 'P1' and 'P2' interact with each other or with a confining structure (grey) they change their momentum. Hence, different hyperplanes sample different many-particle momentum states. Traditional formulations of relativistic thermodynamics introduce global state variables as integrals over isochronous hyperplanes (dotted), whereas a photograph taken at the spacetime event $\mathscr{E}$ or $\mathscr{E}'$ records the state of the system along the corresponding backward-lightcone $\mathcal{C}(\mathscr{E})$ or $\mathcal{C}(\mathscr{E}')$.

$$(U^\mu[\mathcal{I}]) = N(\langle p^0 \rangle, \mathbf{0}) \tag{7}$$

On the other hand, choosing $\mathcal{H} = \mathcal{I}'[\xi'^0]$ yields the Σ'-isochronous energy–momentum vector $U'^\mu[\mathcal{I}']$ in Σ':

$$\left(U'^\mu[\mathcal{I}']\right) = N \begin{cases} \gamma(\langle p^0 \rangle + w^2\beta^{-1}), & \mu = 0 \\ -\gamma w(\langle p^0 \rangle + \beta^{-1}), & \mu = 1 \\ 0, & \mu > 1 \end{cases} \tag{8}$$

where $\gamma := (1 - w^2)^{-1/2}$. Applying a Lorentz transformation with $-w$ to equation (8), we find the corresponding energy–momentum in Σ

$$(U^\mu[\mathcal{I}']) = N(\langle p^0 \rangle, -w\beta^{-1}, 0, 0)$$

Hence, the energy–momentum vectors equations (7) and (8) are not related by a Lorentz transformation. In fact, $U^\mu[\mathcal{I}]$ and $U^\mu[\mathcal{I}']$ are connected by

$$(U^\mu[\mathcal{I}']) = (U^\mu[\mathcal{I}]) + N\beta^{-1}(0, -w, 0, 0)$$

reflecting the underlying hypersurface and observer velocity. As mentioned earlier, the difference between $U^\mu[\mathcal{I}]$ and $U^\mu[\mathcal{I}']$ arises because the energy–momentum tensor of a spatially confined gas is not conserved. It is also a reason for the existence of various temperature Lorentz transformation laws.

Entropy

Having at hand the state variables 'energy' U^0 and 'momentum' $\mathbf{U}$, one still needs 'entropy'. For a Jüttner gas, one can define the entropy density four-current[24,25] by (h = Planck's constant, units $k_{\mathrm{B}} = 1$)

$$s^\mu(t,\mathbf{x}) = -N \int \frac{\mathrm{d}^3 p}{p^0}\, p^\mu f \ln(h^3 f) \tag{9}$$

The specific shape of equation (9) is tightly linked to the exponential form of the Jüttner distribution equation (3). In fact, this combination (f, s^{μ}) is just one among several probabilistic models of thermodynamics; that is, there exist other pairings, for example, based on Renyi-type entropies, that yield consistent thermodynamic relations as well[26]. However, inserting equations (3) into (9), we find

$$s^{\mu}(t,\mathbf{x}) = N\varrho \begin{cases} \ln(VZ/h^3) + \beta\langle p^0\rangle, & \mu = 0 \\ 0, & \mu > 0 \end{cases} \tag{10}$$

Hence, the current equation (10) is stationary in Σ and satisfies the conservation law

$$\partial_{\nu} s^{\nu} \equiv 0 \tag{11}$$

The associated thermodynamic entropy S is obtained by integrating s^{μ} over some space-like or light-like hyperplane $\mathcal{H}$, yielding the Lorentz-invariant quantity

$$S[\mathcal{H}] := \int_{\mathcal{H}} \mathrm{d}\sigma_{\nu}\, s^{\nu}(t,\mathbf{x}) \tag{12}$$

Equation (11) implies that the integral equation (12) is the same for the hyperplanes $\mathcal{I}(\xi^0)$ and $\mathcal{I}'(\xi'^0)$,

$$S[\mathcal{I}] = S'[\mathcal{I}] = S[\mathcal{I}'] = S'[\mathcal{I}'] \tag{13}$$

Thus, there is little or no room for controversy about the transformation laws of entropy in this example. The integral equation (12) is most conveniently calculated along $\mathcal{H} = \mathcal{I}(\xi^0)$ in Σ, yielding

$$S = \int \mathrm{d}^3x\, s^0 = N\ln(VZ/h^3) + \beta N\langle p^0\rangle$$

This can also be rewritten as

$$\begin{aligned} S' &= N\ln(\gamma V'Z/h^3) + \beta U'^0[\mathcal{I}]/\gamma \\ &= N\ln(\gamma V'Z/h^3) + \beta\gamma(U'^0[\mathcal{I}'] + wU'^1[\mathcal{I}']) \end{aligned} \tag{14}$$

where $V' = V/\gamma$ denotes the Lorentz-contracted (that is, Σ'-simultaneously measured) volume. More precisely, one should write $V' = V'[\mathcal{I}']$ and $V = V[\mathcal{I}]$ to reflect how volume is measured (defined) in either frame.

Einstein–Planck theory

We are now ready to summarize the most common versions of relativistic thermodynamics. Planck[7] and Einstein[8] propose to use the Σ'-synchronous four-vector $U'^{\mu}[\mathcal{I}']$ from equation (8) as thermodynamic energy–momentum state variables. Furthermore, they choose to define heat $Q'[\mathcal{I}']$ and, thus, temperature T' in Σ' by the following postulated form of the first law of thermodynamics (see equation (23) in Einstein's paper[8])

$$đQ'[\mathcal{I}'] := T'\mathrm{d}S' := \mathrm{d}U'^0[\mathcal{I}'] - w'\mathrm{d}U'^1[\mathcal{I}'] + P'\mathrm{d}V' \tag{15a}$$

where the intensive variable $w' = -w$ is the constant x'^1-velocity of the gas (container) in Σ' and P' is the pressure. Considering the special case $w' = 0$ first, we see that equation (15a) is consistent with the second line of equation (14) on identifying $T = \beta^{-1}$ and $PV = N\beta^{-1}$; that is, the parameter β of the Jüttner distribution equals the inverse rest temperature. Furthermore, for moving systems with $w' \neq 0$, we find that thermodynamic quantities in Σ and Σ' are related by[9]

$$V' = V/\gamma, \qquad P' = P, \qquad S' = S \tag{15b}$$

$$U'^0[\mathcal{I}'] = \gamma\left(U^0[\mathcal{I}] + w'^2\, PV\right) \tag{15c}$$

$$U'^1[\mathcal{I}'] = \gamma w'\left(U^0[\mathcal{I}] + PV\right) \tag{15d}$$

$$T' = \gamma^{-1}\, T = (1 - w'^2)^{1/2}\, T \tag{15e}$$

so that

$$T'\mathrm{d}S' = đQ'[\mathcal{I}'] = \gamma^{-1}đQ[\mathcal{I}] = \gamma^{-1}T\,\mathrm{d}S \tag{15f}$$

that is, within the Einstein–Planck formalism a moving body appears cooler (although it seems that, in the later stages of his life, Einstein changed[27,28] his opinion about the transformation laws of thermodynamic quantities). Equations (15) were criticized in a posthumously published paper by Ott[10] and, later, also by Van Kampen[9,29] and Landsberg[11,12].

Ott's versus Van Kampen's theory

Ott[10] and Van Kampen[9] choose to formulate thermodynamic relations in the moving frame Σ' in terms of the Σ-isochronous energy–momentum vector $U'^{\mu}[\mathcal{I}] = \Lambda^{\mu}{}_{\nu}U^{\nu}[\mathcal{I}]$. They differ, however, as to how heat and work should be defined. Van Kampen[9,29] replaces Planck's version of the first law, equation (15a), by introducing a covariant thermal energy–momentum transfer four-vector Q^{μ} by means of

$$đQ^{\mu}[\mathcal{I}] := \mathrm{d}U^{\mu}[\mathcal{I}] - đA^{\mu}[\mathcal{I}] \tag{16}$$

where, in the (lab) frame Σ, the non-thermal work vector $A^{\mu}[\mathcal{I}]$ is determined by $(đA^{\mu}[\mathcal{I}]) := (-P\mathrm{d}V, \mathbf{0})$. Accordingly, in a moving frame Σ', one then finds $đQ'^{\mu}[\mathcal{I}] = \mathrm{d}U'^{\mu}[\mathcal{I}] - đA'^{\mu}[\mathcal{I}]$, where by means of a Lorentz transformation

$$\mathrm{d}U'^{\mu}[\mathcal{I}] = w'^{\mu}\,\mathrm{d}U^0[\mathcal{I}], \qquad đA'^{\mu} = -w'^{\mu}\,P\mathrm{d}V \tag{17}$$

Here, $(w'^{\mu}) = (\gamma, \gamma w', 0, 0)$ denotes the velocity four-vector of the gas (container) in Σ'. Although essentially agreeing on equations (16), (17) and on the scalar character of entropy, $S' = S$, Van Kampen and Ott postulate different formulations of the second law, respectively. Specifically, Ott[10] defines the temperature T' in Σ' by means of

$$T'\,\mathrm{d}S' := đQ'^0 = \gamma\, đQ^0 = \gamma\, T\,\mathrm{d}S \tag{18a}$$

which implies the modified temperature transformation law[30–32]

$$T' = \gamma\, T = (1 - w'^2)^{-1/2}\, T \tag{18b}$$

that is, according to Ott's definition of heat and temperature, a moving body appears hotter. Van Kampen[9] argues that the equations (18) are not well suited if one wishes to describe heat and energy–momentum exchange between systems that move at different velocities (hetero-tachic processes). To achieve a more convenient description, he proposes to characterize the heat transfer by means of a heat scalar $Q' = Q$, defined by[9,29]

$$đQ' := -w'_{\mu}đQ'^{\mu} = -w_{\mu}đQ^{\mu} = đQ = đQ^0$$

He then goes on to define temperature with respect to Q,

$$T'\mathrm{d}S' := \text{đ}Q' = \text{đ}Q = T\,\mathrm{d}S$$

yielding yet another temperature transformation law:

$$T' = T$$

that is, according to Van Kampen's definition, a moving body appears neither hotter nor colder. Adopting this convention, one can define an inverse-temperature four-vector $\beta'_\mu := T'^{-1}\,w'_\mu = T^{-1}\,w'_\mu$ and rewrite the second law in the compact covariant form

$$\mathrm{d}S' = -\beta'_\mu \text{đ}Q'^\mu$$

Discussion

Evidently, whether a moving body appears hotter or not depends solely on how one defines thermodynamic quantities. The formalisms of Ott[10] and Van Kampen[9,29] are based on the same (lab-)isochronous hyperplane $\mathcal{I}$—they merely differ in their respective temperature definitions[16]. In contrast, the Einstein–Planck theory[7,8] is based on an observer-dependent isochronous hyperplane $\mathcal{I}'$. Although, in principle, there is nothing wrong with this, a conceptual downside of the latter approach is that the state variables energy and momentum, when measured in different frames, are not connected by Lorentz transformations—or, put differently: to experimentally determine, for example, the energies $U^0[\mathcal{I}]$ and $U'^0[\mathcal{I}']$, two observers need to carry out non-equivalent measurements[18], because measurements must be carried out Σ-simultaneously in the first case, but Σ'-simultaneously in the second case. This might seem sufficient for regarding either Ott's or Van Kampen's (more elegant) approach as preferable. However, before adopting this point of view, it is worthwhile to ask the following questions. Which formulation is feasible from an experimental point of view? Which formalism provides a suitable conceptual framework for extensions to general relativity[33,34]?

Unfortunately, from an objective perspective, neither of the above proposals fulfils these criteria. The reason is that either formulation is based on simultaneously defined integrals. On the one hand, this means that it is virtually impossible to directly measure the quantities appearing in the theory; for example, to determine $U^0[\mathcal{I}]$ one would have to determine the velocities of the particles at time $t = \xi^0$ in Σ, which requires either superluminal information transport[18] or unrealistic efforts of trying to reconstruct isochronous velocity data from recorded trajectories. On the other hand, it is very difficult, if not impossible, to transfer the concept of global isochronicity to general relativity owing to the absence of global inertial frames in curved spacetime.

'Photographic' thermodynamics

To overcome these drawbacks, we propose to define relativistic thermodynamic quantities by means of surface integrals along the backward-lightcone $\mathcal{C}[\mathscr{E}]$, where $\mathscr{E}$ is the event of the observation, see Fig. 1. This is motivated by the following facts. (1) A photograph taken by an observer $\mathcal{O}$ at the event $\mathscr{E}$ reflects the state of the system along the lightcone $\mathcal{C}[\mathscr{E}]$. (2) The hyperplane $\mathcal{C}[\mathscr{E}]$ is a relativistically invariant object that is equally accessible for any inertial observer; that is, if another observer $\mathcal{O}'$, who moves relative to $\mathcal{O}$, takes a snapshot at the same event $\mathscr{E}$, then the resulting picture will reflect the same state of the system—although the 'colours' will be different owing to the Doppler effect caused by the observers' relative motion[34]. (3) The concept of lightcone integration can be readily extended to general relativity, which for sufficiently well-behaved spacetime models amounts to replacing the globally flat Minkowski metric $\eta_{\mu\nu}$ with a curved metric field $g_{\mu\nu}(x^\lambda)$. For a gas in the vicinity of a black hole or in a galaxy, the gravitational contribution of the thermodynamic system is usually negligible and $g_{\mu\nu}$ can be considered as a background metric. In other models, it may be necessary to include the thermodynamic energy–momentum tensor $\theta^{\mu\nu}(x^\lambda)$ as an extra source in Einstein's field equations[33,34]. (4) In the non-relativistic limit $c \to \infty$, the lightcone flattens so that photographic measurements become isochronous in any frame in this limit. Thus, lightcone integrals seem to be the best-suited candidates if one wishes to characterize relativistic many-particle systems by means of non-locally defined, macroscopic variables.

Mathematically, the backward-lightcones $\mathcal{C}[\mathscr{E}]$ in Σ and $\mathcal{C}'[\mathscr{E}]$ in Σ' are given by

$$\mathcal{C}(\mathscr{E}) := \{\,(t,\mathbf{x}) \mid t = \xi^0 - |\mathbf{x} - \boldsymbol{\xi}|\,\} \tag{19a}$$

$$\mathcal{C}'(\mathscr{E}) := \{\,(t',\mathbf{x}') \mid t' = \xi'^0 - |\mathbf{x}' - \boldsymbol{\xi}'|\,\} \tag{19b}$$

Unlike the isochronous hyperplanes $\mathcal{I}(\xi^0)$ and $\mathcal{I}'(\xi'^0)$, the lightcones describe the same set of spacetime events, $\mathcal{C}(\mathscr{E}) = \mathcal{C}'(\mathscr{E})$. Fixing $\mathcal{H} = \mathcal{C}(\mathscr{E})$ in equation (2), we find (see the Methods section)

$$U^0[\mathcal{C}] = N\langle p^0\rangle \tag{20a}$$

$$U^i[\mathcal{C}] = \frac{N}{\beta}\int \mathrm{d}^3x\,\frac{x^i - \xi^i}{|\mathbf{x} - \boldsymbol{\xi}|}\,\varrho(\mathbf{x}) \tag{20b}$$

On dividing by the invariant particle number N, the lightcone integral equation (20b) can be interpreted as a lightcone average, and we shall use both terms synonymously from now on. Unlike $\mathbf{U}[\mathcal{I}]$ and $\mathbf{U}[\mathcal{I}']$, the three-vector $\mathbf{U}[\mathcal{C}]$ depends on the space coordinates $\boldsymbol{\xi}$ of the observation event $\mathscr{E}$. Lightcones are Lorentz-invariant objects, implying that $U^\mu[\mathcal{C}]$ and $U'^\mu[\mathcal{C}']$ are directly linked by a Lorentz transformation, that is, $U'^\mu[\mathcal{C}'] = \Lambda^\mu_\nu U^\nu[\mathcal{C}]$. Moreover, for a spatially homogeneous Jüttner gas, it is straightforward to compute the entropy $S[\mathcal{C}]$ as (see equation (14))

$$\begin{aligned} S[\mathcal{C}] &= N\ln(VZ/h^3) + \beta U^0[\mathcal{C}] \\ &= N\ln(\gamma V'Z/h^3) + \beta\gamma(U'^0[\mathcal{C}] - w'U'^1[\mathcal{C}]) \\ &= S'[\mathcal{C}] \end{aligned}$$

where, furthermore, $S[\mathcal{C}] = S[\mathcal{I}]$ owing to the conservation law equation (11). Thus, depending on which definition of heat we choose, we again end up with either Ott's or Van Kampen's temperature transformation law. In our opinion, Van Kampen's approach is more appealing as it defines temperature (similar to rest mass) as an intrinsic property of the thermodynamic system, whereas Ott's formalism treats temperature as a 'dynamic' quantity similar to the zero-component of the energy–momentum four-vector.

Observable consequences

As, unlike their isochronous counterparts, the state variables $U^\mu[\mathcal{C}]$ are experimentally accessible, it is worthwhile to discuss implications for present and future astrophysical observations. Let us assume that an idealized photograph, taken by an observer $\mathcal{O}$ at $\mathscr{E}$, encodes both the positions and velocities (for example, from Doppler shifts) of a confined gas. If $\mathcal{O}$ is at rest relative to the gas (corresponding to $\mathbf{w} = \mathbf{0}$), then the mean values of the energy and momentum sampled from the photographic data will converge to $U^\mu[\mathcal{C}]$ given by equation (20). Equation (20a) implies that it does not matter for an observer at rest in Σ whether energy values are

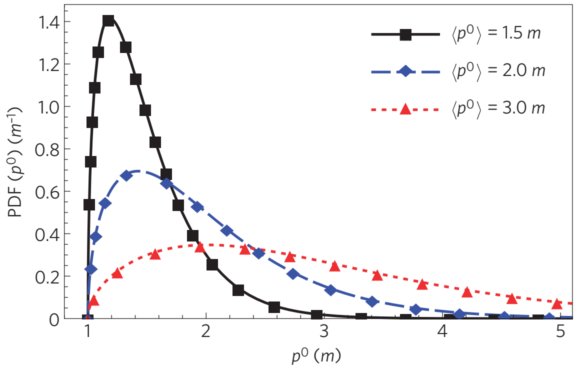

Figure 2 | Equilibrium energy distribution of relativistic gas particles in the lab frame. Our numerical results (symbols) are in good agreement with the theoretically expected Jüttner distribution (lines) from equation (3b) for various values of the mean particle energy $\langle p^0 \rangle$, which confirms the adequacy of the algorithm used (see the Methods section).

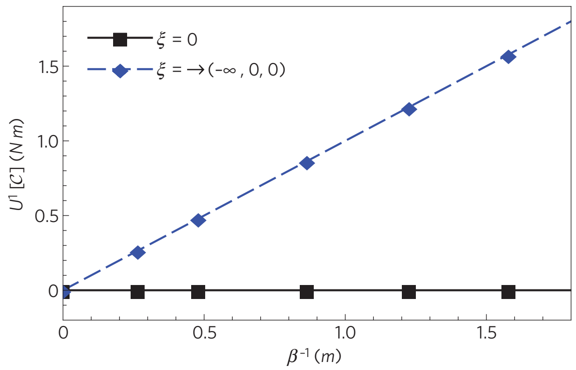

Figure 3 | Temperature-induced apparent drift effect as seen by a resting observer (w = 0). The photographically estimated momentum value $U^1[\mathcal{C}]$ depends on both the gas temperature β^{-1} and the observer's position $\boldsymbol{\xi}$. The symbols indicate values obtained from simulation and the lines indicate the corresponding theoretical predictions from equations (21) and (22).

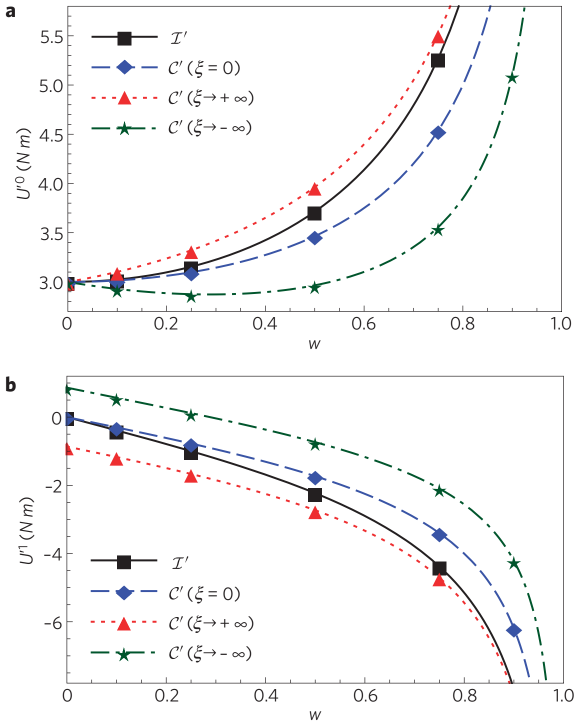

Figure 4 | Photographic energy and momentum estimates for relativistic gases depend sensitively on both the observer velocity and the observer position. a,**b**, Total energy U'^0 (**a**) and momentum U'^1 (**b**) as a function of the observer velocity w along the x^1 axis, either sampled from an isochronous hyperplane $\mathcal{I}'$ or observer backward-lightcones $\mathcal{C}'$. The observers are positioned at the centre of the gas container ($\boldsymbol{\xi} = \mathbf{0}$) and far behind/ahead of the container ($\xi^1 \to \mp\infty$), respectively. The symbols correspond to a simulation with fixed mean particle energy $\langle p^0 \rangle = 3m$ in the lab frame; the lines indicate the theoretical predictions.

sampled from a photograph or from Σ-simultaneously collected (that is, reconstructed) data.

The situation is different when estimating the mean momentum from photographic data. Equation (20b) implies that the lightcone average depends on the observer position $\boldsymbol{\xi}$. Averages in Σ do not depend on the specific value ξ^0 of the time coordinate if the gas is stationary in this frame. Then, a distinguished 'photographic centre-of-mass' position $\boldsymbol{\xi}_*$ in Σ can be defined by

$$U^i[\mathcal{C}]\big|_{\boldsymbol{\xi}=\boldsymbol{\xi}_*} = 0, \qquad i = 1,2,3 \tag{21}$$

For example, if ϱ is symmetric with respect to the origin of Σ, then $\boldsymbol{\xi}_* = \mathbf{0}$. This would correspond to a lightcone $\mathcal{C}(\mathscr{E})$ as drawn in Fig. 1. In this (and only this) case, we find $U'^1[\mathcal{C}] = w' U'^0[\mathcal{C}]$ and, thus, lightcone thermodynamics reduces to the Ott–Van Kampen formalism.

To illustrate how $U^i[\mathcal{C}]$ generally depends on the observer's position, let us consider a gas with density profile equation (3c). For a stationary observer at a position $\boldsymbol{\xi}$ far away from $\mathbb{V}$, we can approximate $|\mathbf{x} - \boldsymbol{\xi}| \simeq |\boldsymbol{\xi}|$ in equation (20b), yielding

$$U^i[\mathcal{C}] = -\frac{\xi^i}{|\boldsymbol{\xi}|} NT \tag{22}$$

Hence, a distant observer $\mathcal{O}$ who naively estimates $U^i[\mathcal{C}]$ from photographic data could erroneously conclude that the gas is moving away with a momentum vector proportional to the temperature. Reinstating constants c and k_B, this relativistic effect becomes negligible if $mc^2 \gg k_B T$, but—given the rapid improvement of telescopes and spectrographs[35]—it should be taken into consideration when estimating the velocities of astrophysical objects from photographs in the future. In particular, as Lorentz transformations mix energy and spatial momentum components, for a moving observer $\mathcal{O}'$ both $U'^0[\mathcal{C}]$ and $\mathbf{U}'^i[\mathcal{C}]$ will be affected, see numerical results below. In principle, similar phenomena arise whenever one is limited to photographic observations of partial components in distant compound systems (for example, the gas in a galaxy), if the energy–momentum tensor of this partial component is not conserved. At present, it is an open problem whether or not these effects may even be amplified in curved spacetime geometries. Gravity not only affects the energy–momentum tensor of the thermodynamic system but also the propagation of emitted photons[33,34]. The latter effect presents the basis of inhomogeneous-universe models that have been recently investigated as a potential alternative to the dark-energy paradigm[36]. The methods developed in such cosmological studies may provide helpful guidance for identifying observable thermodynamic effects within a general relativity framework.

Numerical results

Returning to flat Minkowski spacetime, the preceding theoretical considerations can be illustrated by (1+3)-dimensional relativistic many-body simulations. Compared with the non-relativistic case, simulations of relativistic many-particle systems are more difficult because particle collisions cannot be modelled by simple interaction potentials anymore[37,38]. Generalizing recently proposed lower-dimensional algorithms[21,39], our computer experiments are based on hard-sphere-type interactions in the two-particle centre-of-mass frame (see the Methods section for details). This model can be considered as 'fully' relativistic at low-to-intermediate particle densities.

Figures 2–4 depict results of simulations with $N = 1{,}000$ particles (volume filling fraction 0.5%). Initially, all particles are randomly distributed in a cubic box with the same energy p^0, but random velocity directions. After a few collisions per particle, the energy distribution relaxes to the Jüttner distribution, see Fig. 2, which confirms that our collision algorithm works correctly in this density regime.

The thermodynamic energy–momentum vector $U'^{\mu}[\mathcal{H}]$ is determined by recording each particles' momentum as its trajectory passes through the corresponding hyperplane. We first consider an observer $\mathcal{O}$ who is at rest relative to the gas. As predicted by equations (20b) and (22), we find that, if the location of $\mathcal{O}$ deviates from the centre of the box, a photo made by $\mathcal{O}$ yields a non-zero momentum $U^i \propto \beta^{-1}$, see the blue line in Fig. 3. For comparison, Fig. 4 shows the results for a moving observer $\mathcal{O}'$ (speed w), obtained by isochronous sampling along different hyperplanes $\mathcal{I}'(w)$ or photographic sampling along the backward-lightcones $\mathcal{C}(\mathscr{E})$. Again, as predicted by the theory, the resulting overall energy–momentum does not depend only on the observer velocity, but also on the underlying hypersurface and, in particular, on the observer's position. Although our study still neglects quantum processes and gravity, which have an important role in real astrophysical systems, the results suggest that one needs to be careful when reconstructing the velocities of very hot, relativistic objects from integrated photographic measurements.

Conclusions

Sometimes, discussions of relativistic thermodynamics start by postulating a set of macroscopic state variables, for which the thermodynamics relations (and Lorentz transformations laws) are subsequently deduced by plausibility considerations. Unfortunately, this approach—although quite successful in non-relativistic physics—is intrinsically limited in a relativistic framework, as it conceals the actual source for conceptual difficulties, namely, the non-local character of thermodynamic quantities. The above analysis may provide guidance for constructing consistent relativistic thermodynamic theories for more complex systems. Relevant (non-)conserved tensor densities can be derived from relativistic classical or quantum Lagrangians[40]. Our discussion has focused on equilibrium systems, which can be characterized by time-independent tensor densities in distinguished reference frames. In principle, the formalism can also be extended to non-equilibrium cases, when the tensorial currents are time-dependent in any frame. In this case, the corresponding lightcone-integrated observables will become explicitly time dependent and one faces the difficult task of extracting useful information from their temporal fluctuations.

Generally, care is required when integrating tensor densities to obtain global thermodynamic state variables, because conservation laws may be violated owing to confinement, so that averages may vary depending on the underlying hyperplane(s). Within a conceptually satisfying and experimentally feasible framework, thermostatistical averaging procedures should be defined over lightcones rather than isochronous hypersurfaces. To put it provocatively, the isochronous definition of non-local quantities, as adopted in traditional formulations of relativistic thermodynamics, can be viewed as a relic of our accustomed non-relativistic 'thinking'. With regard to present and future astrophysical observations, it will be important to better understand how the temperature-dependent, apparent drift effects due to lightcone averaging (that is, photographic measurements) become modified in curved spacetime, because this might affect velocity estimates for astronomical objects, which are pivotal for our understanding of the cosmological evolution[35].

Methods

Notation. We adopt units such that the speed of light $c = 1$ and the Boltzmann constant $k_{\mathrm{B}} = 1$, and the metric convention $(\eta_{\mu\nu}) = \mathrm{diag}(-1,1,1,1) = (\eta'_{\mu\nu})$. Einstein's sum rule is applied throughout. If an event $\mathscr{E}$ has coordinates $(\xi^\mu) = (\xi^0, \xi^i) = (\xi^0, \xi^1, \xi^2, \xi^3) = (t, \boldsymbol{\xi})$ in the inertial spacetime frame Σ, then its coordinates in another frame Σ', moving at constant relative velocity w along the x^1 axis of Σ, are given by $(\xi'^\mu) = (\gamma(\xi^0 - w\xi^1), \gamma(\xi^1 - w\xi^0), \xi^2, \xi^3)$ with $\gamma = (1 - w^2)^{-1/2}$. In short, $\xi'^\mu = \Lambda^\mu{}_\nu \xi^\nu$, where $(\Lambda^\mu{}_\nu)$ is the corresponding Lorentz transformation matrix.

Surface integrals in Minkowski spacetime. To define non-local thermodynamic quantities by means of surface integrals in spacetime, one needs to specify two mathematical structures: (1) the relevant tensor density field $\theta^{\mu\alpha\beta\cdots}(t, \mathbf{x})$ and (2) a suitably defined surface measure $\mathrm{d}\sigma_\mu$ for the hyperplane in the chosen coordinate frame. Although tensor densities can be obtained from Lagrangian field theories by standard methods[33,34,40], it is worthwhile to briefly illustrate how the surface measure can be determined in practice. We wish to calculate

$$G^{\alpha\beta\cdots}[\mathcal{H}] := \int_{\mathcal{H}} \mathrm{d}\sigma_\mu \, \theta^{\mu\alpha\beta\cdots}(t, \mathbf{x}) \tag{23}$$

where $\mathcal{H}$ is a three-dimensional hyperplane in the $(1+3)$-dimensional Minkowski frame Σ and, for a finite thermodynamic system, $\theta^{\mu\alpha\beta\cdots}$ is assumed to vanish outside a bounded spatial region. If $\theta^{\mu\alpha\beta\cdots}$ is a tensor of rank n, then $G^{\alpha\beta\cdots}[\mathcal{H}]$ has rank $(n-1)$. Considering Cartesian spacetime coordinates, the surface element $\mathrm{d}\sigma_\mu$ may be expressed in terms of the alternating differential form[41]

$$\mathrm{d}\sigma_\mu = (3!)^{-1} \varepsilon_{\mu\alpha\beta\gamma} \, \mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta \wedge \mathrm{d}x^\gamma$$

where $\varepsilon_{\mu\alpha\beta\gamma}$ is the Levi-Cevita tensor[42] and $\mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta = -\mathrm{d}x^\beta \wedge \mathrm{d}x^\alpha$ is the antisymmetric product. With regard to thermodynamics, we are interested in integrating over space-like or light-like surfaces $\mathcal{H}$ given in the form $x^0 = t = h(\mathbf{x})$. Examples are the isochronous hyperplane $\mathcal{I}(\xi^0)$ from equation (6) and the lightcone $\mathcal{C}[\mathscr{E}]$ from equation (19). Given the function h, we may write $\mathrm{d}x^0 = \partial_i h \, \mathrm{d}x^i$ (in general relativity h will be more complicated depending on the underlying metric). Inserting this expression into equation (23) yields

$$\begin{aligned} G^{\alpha\cdots} &= \int_{\mathcal{H}} \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \left[\theta^{0\alpha\cdots} - (\partial_i h)\, \theta^{i\alpha\cdots}\right] \\ &:= \int \mathrm{d}^3x \left[\theta^{0\alpha\cdots}(h(\mathbf{x}), \mathbf{x}) - (\partial_i h)\, \theta^{i\alpha\cdots}(h(\mathbf{x}), \mathbf{x})\right] \end{aligned}$$

In particular, for the isochronous hyperplane $\mathcal{I}(\xi^0)$ from equation (6), we have $h(\mathbf{x}) = \xi^0$ and $\partial_i h = 0$ in Σ, leading to

$$G^{\alpha\cdots}[\mathcal{I}] = \int \mathrm{d}^3x \, \theta^{0\alpha\cdots}(\xi^0, \mathbf{x})$$

For the lightcone $\mathcal{C}(\mathscr{E})$ with $\partial_i h = -(x^i - \xi^i)/|\mathbf{x} - \boldsymbol{\xi}|$, we find

$$\begin{aligned} G^{\alpha\cdots}[\mathcal{C}] = \int \mathrm{d}^3x \Bigg\{ & \theta^{0\alpha\cdots}(\xi^0 - |\mathbf{x} - \boldsymbol{\xi}|, \mathbf{x}) \\ & + \frac{x^i - \xi^i}{|\mathbf{x} - \boldsymbol{\xi}|} \, \theta^{i\alpha\cdots}(\xi^0 - |\mathbf{x} - \boldsymbol{\xi}|, \mathbf{x}) \Bigg\} \end{aligned}$$

Figures 3 and 4 illustrate that the above mathematical construction is consistent with the intuitive averaging procedures used in experiments and computer simulations.

Numerical simulations. Fully relativistic N-body simulations would require one to also compute the interaction fields generated by particles, which is numerically expensive. For dilute gases with short-range interactions, one can obtain reliable results by considering simplified models based on quasi-elastic collisions. In our computer experiments, we simulate a three-dimensional gas of relativistic hard spheres in a cubic box. In a particle–particle collision, momentum is transferred instantaneously at the moment of closest encounter by taking into account the relativistic energy–momentum conservation laws.

The tasks during a simulation time step are[21,39]: (1) determine the times/distances of all particle pairs at their closest encounter; (2) advance all particles to the next collision time; (3) transfer momentum between the colliding particles; (4) record particle energies and momenta when the particles are reflected at the walls (for example, to measure the pressure on the boundaries) or their trajectories pass an observer hypersurface.

Our simulations show that details of the momentum transfer mechanism (for example, the differential cross-sections) do not affect the stationary momentum distribution. It is, however, important to use a Lorentz-invariant collision criterion (we use the minimum distance of particles in the two-particle centre-of-mass frame). Non-invariant criteria may lead to deviations from the Jüttner distribution. The most time-consuming task is to determine the closest-encounter times/distances for all particle pairs. A huge speed-up can be achieved by considering only close particle pairs, using a hash table based on a partition of the simulation box into subcubes. With this method, one can efficiently simulate 10^3 particles and 10^6 collisions in 30 min on a desktop personal computer.

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Author contributions

J.D. carried out the analytical calculations and S.H. conducted the numerical simulations. All three authors contributed extensively to discussions of the content and to writing the paper.