Qubit-oscillator dynamics in the dispersive regime: Analytical theory beyond the rotating-wave approximation

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We generalize the dispersive theory of the Jaynes-Cummings model beyond the frequently employed rotating-wave approximation (RWA) in the coupling between the two-level system and the resonator. For a detuning sufficiently larger than the qubit-oscillator coupling, we diagonalize the non-RWA Hamiltonian and discuss the differences to the known RWA results. Our results extend the regime in which dispersive qubit readout is possible. If several qubits are coupled to one resonator, an effective qubit-qubit interaction of Ising type emerges, whereas RWA leads to isotropic *XY* interaction. This impacts on the entanglement characteristics of the qubits.

I. INTRODUCTION

More than 40 years ago, Jaynes and Cummings [1] introduced a fully quantum-mechanical model for the interaction of light and matter, which are represented by a single harmonic oscillator and a two-level system, respectively. Within dipole approximation for the interaction that model is expressed by the Hamiltonian

$$H = \frac{\hbar\epsilon}{2}\sigma^z + \hbar\omega a^\dagger a + \hbar g\sigma^x(a^\dagger + a), \qquad (1)$$

where $\hbar\epsilon$ is the level splitting of the two-level system, henceforth "qubit," ω is the frequency of the electromagnetic field mode, and g is the dipole interaction strength. The Pauli matrices σ^α, $\alpha = x, y, z$, refer to the two-level system, while $a^\dagger$ and a denote the bosonic creation and annihilation operators of the electromagnetic field mode. This model describes a wealth of physical phenomena rather well and by now is a "standard model" of quantum optics. A particular experimental realization of the Hamiltonian is an atom interacting with the field inside an optical cavity, usually referred to as cavity quantum electrodynamics. Corresponding experiments have demonstrated quantum coherence between light and matter manifest in phenomena such as Rabi oscillations and entanglement [2,3].

Despite its simplicity, the Hamiltonian (1) cannot be diagonalized exactly and, thus, is often simplified by a rotating-wave approximation (RWA). There, one expresses the qubit-cavity interaction in terms of the ladder operators $\sigma^\pm = \frac{1}{2}(\sigma^x \pm i\sigma^y)$. In the interaction picture with respect to the uncoupled Hamiltonian, the coupling operators $\sigma^+ a$, $\sigma^- a^\dagger$ and $\sigma^- a$, $\sigma^+ a^\dagger$ oscillate with the phase factors $\exp[\pm i(\omega - \epsilon)t]$ and $\exp[\pm i(\omega + \epsilon)t]$, respectively. Operating at or near resonance, the cavity-qubit detuning is small, $|\epsilon - \omega| \ll \epsilon + \omega$, so that the former operators oscillate slowly, whereas the latter exhibit fast "counter-rotating" oscillations. If in addition, the coupling is sufficiently weak $g \ll \min\{\epsilon, \omega\}$, one can separate time scales and replace the counter-rotating terms by their vanishing time average. Then one obtains the Jaynes-Cummings Hamiltonian [1]

$$H_{\text{RWA}} = \frac{\hbar\epsilon}{2}\sigma^z + \hbar\omega a^\dagger a + \hbar g(\sigma^- a^\dagger + \sigma^+ a). \qquad (2)$$

Lately, new interest in Jaynes-Cummings physics has emerged in the solid-state realm. There, one implements artificial atoms with Cooper-pair boxes (charge qubits) [4] or superconducting loops (flux qubits) [5]. The role of the cavity is played now by a transmission line or a superconducting quantum interference device [6,7] or even a nanomechanical oscillator [8]. Since the first experimental realizations in 2004 [4,5], a plethora of results has been obtained, such as quantum-non-demolition-like readout of a qubit state [9], the generation of Fock states [10], the observation of Berry phases [11], multiphoton resonances [7], entanglement between two qubits inside one cavity [12,13], and the demonstration of a two-qubit algorithm [14].

These experiments have in common that they operate in the strong-coupling limit, that is, the coupling g is larger than the linewidth of the resonator. On the other hand, g is typically two orders of magnitude less than the qubit and resonator frequencies. In this scenario, the Jaynes-Cummings model (2) has been shown to describe the experiments faithfully.

Of practical interest is the dispersive limit, in which the qubit and the resonator are far detuned compared to the coupling strength $g \ll |\epsilon - \omega|$ [6,15]. In this regime, a non-demolition-type measurement of the qubit can be performed by probing the resonator [4,16]. Moreover, it is possible to simulate quantum spin chains with two or more qubits that are dispersively coupled to one resonator [12,14,17]. The complementary architecture of two cavities dispersively coupled to one qubit allows building a quantum switch [18]. All these ideas have been developed from the RWA model (2) in the dispersive limit or from according generalizations to many qubits or many oscillators. Thus, these theories are restricted to the range

$$g \ll |\epsilon - \omega| \ll \epsilon + \omega, \qquad (3)$$

where the first inequality refers to the dispersive limit, while the second one has been used to derive the RWA Hamiltonian (2) from the original model (1).

In recent experiments, efforts are made to reach an even stronger qubit-cavity coupling g. Thus, it will eventually be no longer possible to fulfill both inequalities in Eq. (3) [19]. In particular, when trying to operate in the dispersive limit, the second inequality may be violated, so that RWA is no longer applicable. Non-RWA effects of the model Hamiltonian (1) have already been studied in the complementary adiabatic limits $\epsilon \ll \omega$ and $\omega \ll \epsilon$ [20–22]. Furthermore, Van Vleck perturbation theory has been used in the resonant and close-to-resonant cases [23]. Polaron transformation [24], cluster methods [25], wave-packet approach [26], and even generalized RWA approximations [27] have also been considered. Finally, with the method of small oscillations [28], effective models were found for the different resonances beyond the RWA approximation [29]. Motivated by the importance of the dispersive regime and in view of the experimental tendency toward stronger qubit-oscillator coupling, we present in this work a dispersive theory beyond RWA, so that the second condition in Eq. (3) can be dropped. This means that our approach is valid under the less stringent condition

$$g \ll |\epsilon - \omega|, \qquad (4)$$

which implies that the detuning is not necessarily smaller than ϵ and ω. In order to set the stage, we briefly review in Sec. II the dispersive theory within RWA. In Sec. III, we derive a dispersive theory for Hamiltonian (1) beyond RWA, which we generalize in Sec. IV to the presence of several qubits.

II. DISPERSIVE THEORY WITHIN RWA

The dispersive limit is characterized by a large detuning $\Delta = \epsilon - \omega$ as compared to the qubit-oscillator coupling g. Thus,

$$\lambda = \frac{g}{\Delta} \qquad (5)$$

represents a small parameter, while the RWA Hamiltonian (2) is valid for $|\epsilon - \omega| \ll \epsilon + \omega$. Then it is convenient to separate the coupling term from the RWA Hamiltonian, i.e., to write $H_{\text{RWA}} = H_0 + \hbar g X_+$ with the contributions

$$H_0 = \frac{\hbar \epsilon}{2} \sigma^z + \hbar \omega a^\dagger a, \qquad (6)$$

$$X_\pm = \sigma^- a^\dagger \pm \sigma^+ a. \qquad (7)$$

Applying the unitary transformation,

$$D_{\text{RWA}} = e^{\lambda X_-}, \qquad (8)$$

one obtains for the transformed Hamiltonian $H_{\text{RWA,disp}} = D_{\text{RWA}}^\dagger H_{\text{RWA}} D_{\text{RWA}}$ to second order in λ: $H_{\text{RWA,disp}} = H_{\text{RWA}} + \lambda [H_{\text{RWA}}, X_-] + \frac{1}{2} \lambda^2 \{[H_{\text{RWA}}, X_-], X_-\}$, which can be evaluated to read as

$$H_{\text{RWA,disp}} = \frac{\hbar \epsilon}{2} \sigma^z + \frac{\hbar g^2}{2\Delta} \sigma^z + \left(\hbar \omega + \frac{\hbar g^2}{\Delta} \sigma^z \right) a^\dagger a. \qquad (9)$$

The physical interpretation of Eq. (9) is that the oscillator frequency is shifted as

$$\omega \to \omega \pm g^2/\Delta, \qquad (10)$$

where the sign depends on the state of the qubit. If one now probes the resonator with a microwave signal at its bare resonance frequency ω, the phase of the reflected signal possesses a shift that depends on the qubit state. This allows one to measure the low-frequency dynamics of the qubit [6]. Since, according to Eq. (9), the qubit Hamiltonian $(\hbar \epsilon/2) \sigma_z$ commutes with the dispersive coupling $(\hbar g^2/\Delta) \sigma_z a^\dagger a$, this constitutes a quantum nondemolition measurement of the qubit, which has already been implemented experimentally [4,22,30,31]. In turn, the qubit energy splitting is shifted depending on the mean photon number $n = \langle a^\dagger a \rangle$. Accordingly, one can also measure the mean photon number and even perform a full quantum state tomography of the oscillator state [32]. Note also that besides the condition of λ being small, the perturbational result (9) is accurate only if the mean photon number $\langle n \rangle$ does not exceed the critical value $n_{\text{crit}} = 1/4\lambda^2$. For larger photon numbers, higher powers of the number operator $a^\dagger a$ must be taken into account [33,34]. Henceforth, we restrict ourselves to the so-called linear dispersive regime, in which the photon number is clearly below the critical value n_{crit}.

III. DISPERSIVE THEORY BEYOND RWA

It is now our aim to treat the original Hamiltonian (1) in the dispersive limit accordingly, i.e., to derive an expression that corresponds to Eq. (9) but is valid in the full dispersive regime defined by inequality (4). Going beyond RWA, we have to keep the counter-rotating coupling terms

$$Y_\pm = \sigma^+ a^\dagger \pm \sigma^- a, \qquad (11)$$

which are relevant if either of the relations $g \ll \min\{\epsilon, \omega\}$ or $|\epsilon - \omega| \ll \epsilon + \omega$ is violated. Separating again the qubit-oscillator coupling from the bare terms, we rewrite Hamiltonian (1) as

$$H = H_0 + \hbar g X_+ + \hbar g Y_+, \qquad (12)$$

which differs from H_{RWA} by the last term. It will turn out that a unitary transformation corresponding to Eq. (8) is achieved by the operator

$$D = e^{\lambda X_- + \bar{\lambda} Y_-}. \qquad (13)$$

Here we have introduced the parameter

$$\bar{\lambda} = \frac{g}{\epsilon + \omega} = \frac{g}{2\epsilon - \Delta}, \qquad (14)$$

which obviously fulfills the relation $\bar{\lambda} < \lambda$, since ϵ and ω are positive. Thus, whenever λ is small, $\bar{\lambda}$ is small as well. Nevertheless, under condition (4), λ and $\bar{\lambda}$ may be of the same order.

Proceeding as in Sec. II, we define the dispersive Hamiltonian $H_{\text{disp}} = D^\dagger H D$. Using the commutation relations $[Y_+, Y_-] = \sigma^z(2a^\dagger a + 1) - 1$, $[\epsilon/2\sigma^z + \omega a^\dagger a, Y_-] = -(\epsilon + \omega) Y_+$, and $[Y_\pm, X_\mp] = \sigma^z \{a^2 + (a^\dagger)^2\}$, we obtain the expression

$$H_{\text{disp}} = \frac{\hbar\epsilon}{2}\sigma^z + \hbar\omega a^\dagger a + \frac{\hbar g^2}{2}\left(\frac{1}{\Delta} + \frac{1}{2\epsilon - \Delta}\right)\sigma^z(a^\dagger + a)^2,$$

$$(15)$$

which is valid up to second order in the dimensionless coupling parameters λ and $\bar{\lambda}$.

As compared to the RWA result (9), we find two differences. First, the prefactor of the coupling has a contribution that obviously stems from $\bar{\lambda}$. Second and more importantly, the coupling is no longer proportional to the number operator $a^\dagger a$ but rather to $(a^\dagger + a)^2$. Thus, the operator $Y_\pm$ has turned into the counter-rotating contributions a^2 and $(a^\dagger)^2$. For this reason, the dispersive Hamiltonian (15) is not diagonal in the eigenbasis of the uncoupled Hamiltonian H_0.

Nevertheless, it is possible to interpret the result as a qubit-state dependent frequency shift by the following reasoning. Let us interpret $\hbar\omega a^\dagger a$ as the Hamiltonian of a particle with unit mass in the potential $\frac{1}{2}\omega^2 x^2$, where $x = \sqrt{\hbar/2\omega}(a^\dagger + a)$. Then the qubit-oscillator coupling in Eq. (15) modifies the potential curvature ω^2, such that the oscillator frequency undergoes a shift according to

$$\omega \rightarrow \bar{\omega} = \omega\sqrt{1 \pm \frac{2g^2}{\omega}\left(\frac{1}{\Delta} + \frac{1}{2\epsilon - \Delta}\right)}. \qquad (16)$$

Again the sign depends on the qubit state. To be consistent with the second-order approximation in g, we have to expand also the square root to that order. This complies with the experimentally interesting parameter regime, where $g < \omega$. We finally obtain

$$\bar{\omega} = \omega \pm g^2\left(\frac{1}{\Delta} + \frac{1}{2\epsilon - \Delta}\right). \qquad (17)$$

As for the RWA Hamiltonian, we find that the qubit state shifts the resonance frequency of the oscillator. This result is not only of appealing simplicity but also has a rather important consequence. Dispersive readout is possible even when the qubit-oscillator coupling is so strong that condition (3) cannot be fulfilled, that is, when the RWA result is not valid.

For a quantitative analysis of our analytical findings, we compare the frequency shifts (10) and (17) with numerical results. In doing so, we diagonalize the Hamiltonian (1) in the subspace of the qubit state $|\downarrow\rangle$, where $\sigma^z|\downarrow\rangle = -|\downarrow\rangle$. The results are depicted in Fig. 1.

For a qubit splitting ϵ close to the cavity frequency ω, i.e., outside the dispersive regime, the analytically obtained frequency shifts diverge. This behavior is certainly expected for an expansion in g/Δ. For a relatively small coupling $g/\omega \lesssim 0.025$, the RWA result [panel (a)] agrees very well with the numerical data in the dispersive regime. In the case of larger coupling strengths $g/\omega \gtrsim 0.05$, the predictions from RWA exhibit clear differences. The general tendency is that RWA overestimates the frequency shift for blue detuning $\Delta = \epsilon - \omega < 0$, while it predicts a too small shift for red detuning.

The data shown in panel (b) demonstrate that the treatment beyond RWA yields the correct frequency shift in the entire dispersive regime, i.e., whenever the detuning significantly exceeds the coupling $|\Delta| \gg g$. Thus, as long as the
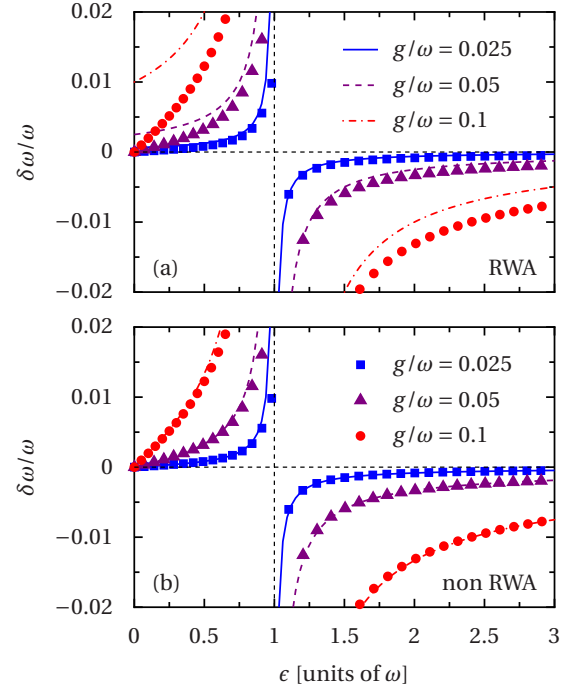

FIG. 1. (Color online) Oscillator frequency shift as function or the qubit splitting $\epsilon = \omega + \Delta$ for the spin state $|\downarrow\rangle$ obtained (a) within RWA [Eq. (10)] and (b) beyond RWA [Eq. (17)]. The lines mark the analytical results, while the symbols refer to the numerically obtained splitting between the ground state and the first-excited state in the subspace of the qubit state $|\downarrow\rangle$.

coupling remains much smaller than the oscillator frequency $g \ll \omega$, it is always possible to tune the qubit splitting ϵ into a regime in which inequality (4) is fulfilled. Moreover, the excellent quantitative agreement of our analytical result (17) with the numerically exact solution indicates the feasibility to determine g from measurements at stronger couplings [19].

A particular limit is $\Delta \rightarrow -\omega$, which corresponds to a vanishing qubit splitting $\epsilon \rightarrow 0$. In this case, it is obvious from Hamiltonian (1) that the coupling to the qubit merely entails a linear displacement of the oscillator coordinate, while the oscillator frequency remains unaffected. This limit is perfectly reproduced by our non-RWA result (17), irrespective of the coupling strength. The RWA result, by contrast, predicts a spurious frequency shift, indicating the failure of RWA.

IV. SEVERAL QUBITS IN A CAVITY

An experimentally relevant generalization of model (1) is the case of several qubits coupling to the same oscillator. The corresponding Hamiltonian reads as [35]

$$H = \hbar\sum_j \frac{\epsilon_j}{2}\sigma_j^z + \hbar\omega a^\dagger a + \hbar\sum_j g_j\sigma_j^x(a^\dagger + a), \qquad (18)$$

where the index j labels the qubits. As for the one-qubit case, the RWA is frequently applied and yields the Tavis-Cummings Hamiltonian [36]

$$H = \hbar \sum \frac{\epsilon_j}{2}\sigma_j^z + \hbar\omega a^\dagger a + \hbar \sum g_j X_+^j, \qquad (19)$$

where $X_\pm^j = \sigma_j^- a_j^\dagger \pm \sigma_j^+ a_j$ [cf. Eq. (6)].

A. Dispersive theory within RWA

We obtain for each qubit the dimensionless coupling parameter $\lambda_j = g_j/(\epsilon_j - \omega)$. The dispersive limit is now determined by $|\lambda_j| \ll 1$ for all j. Effective decoupling of the qubits and the cavity to second order is then achieved via a transformation with the unitary operator $\exp(-\Sigma_j \lambda_j X_-^j)$ [cf. Eq. (8)]. The resulting dispersive Hamiltonian reads as [35]

$$H_{\text{disp}} = \hbar\omega a^\dagger a + \frac{\hbar}{2}\sum_j \left(\epsilon_j + \frac{g^2}{\Delta_j} \right)\sigma_i^z + \hbar\sum_j \frac{g^2}{\Delta_j} a^\dagger a \sigma_z^j$$
$$+ \hbar\sum_{j>k} J_{jk}(\sigma_j^-\sigma_k^+ + \sigma_j^+\sigma_k^-). \qquad (20)$$

Remarkably, the oscillator entails an effective coupling between the qubits with the strength

$$J_{jk} = g_j g_k \left(\frac{1}{\Delta_j} + \frac{1}{\Delta_k} \right), \qquad (21)$$

which has already been observed experimentally [12]. It has been proposed to employ this interaction for building qubit networks [17] and for generating qubit-qubit entanglement [37–39]. Moreover, quantum tomography of a two-qubit state has been implemented by probing the cavity at its bare resonance frequency [40]. In this scenario, the oscillator frequency exhibits a shift depending on a collective coordinate of all qubits. Consequently, the cavity response experiences a phase shift from the ingoing signal, which in turn contains information about that collective qubit coordinate.

B. Dispersive theory beyond RWA

As in Sec. III for the one-qubit case, we now extend the dispersive theory of the Tavis-Cummings model beyond RWA, taking into account the counter-rotating terms of the Hamiltonian (18). In analogy to transformation (13), we employ the ansatz

$$D = \exp\left(\sum \lambda_j X_-^j + \bar{\lambda}_j Y_-^j \right), \qquad (22)$$

where $Y_-^j = \sigma_j^- a - \sigma_j^+ a^\dagger$ and $\bar{\lambda}_j = g/(2\epsilon - \Delta_j)$. Following the lines of Sec. III, i.e., expanding the transformed Hamiltonian to second order in λ and $\bar{\lambda}$, we obtain the dispersive Hamiltonian

$$H_{\text{disp}} = D^\dagger H D = \hbar\omega a^\dagger a + \frac{\hbar}{2}\sum_j \epsilon_j \sigma_j^z$$
$$+ \frac{\hbar}{2}\sum_j g_j^2\left(\frac{1}{\Delta_j} + \frac{1}{2\epsilon - \Delta_j} \right)(a^\dagger + a)^2 \sigma_j^z$$
$$+ \hbar\sum_{j>k} \bar{J}_{jk}\sigma_j^x\sigma_k^x. \qquad (23)$$

We have introduced the modified coupling strength

$$\bar{J}_{jk} = g_i g_k \left(\frac{1}{\Delta_j} + \frac{1}{\Delta_k} - \frac{1}{2\epsilon - \Delta_j} - \frac{1}{2\epsilon - \Delta_k} \right), \qquad (24)$$

which describes the effective interaction between qubits j and k, and represents the extension of Eq. (21) beyond RWA. The dispersive shifts of the qubit and cavity frequencies given by the second and third terms of Eq. (23) are equally modified as compared to the RWA result (20).

Interestingly enough, the effective qubit-qubit interaction in Eq. (23) is of the Ising type $\sigma_j^x\sigma_k^x$, whereas RWA predicts the isotropic *XY* interaction $\sigma_j^+\sigma_k^- + \sigma_j^-\sigma_k^+$ [see Eq. (20)]. Thus, the treatment beyond RWA predicts a qualitatively different effective model and not merely a renormalization of parameters. The Ising term even persists in the limit $1/\Delta_j \gg 1/(2\epsilon - \Delta_j)$. Nevertheless, one can recover the RWA Hamiltonian (20) by writing the interaction term as $\sigma_j^x\sigma_k^x = \sigma_j^+\sigma_k^- + \sigma_j^+\sigma_k^+ + \text{H.c.}$ and performing a RWA for the Ising coupling. This corresponds to discarding small-weighted rapidly oscillating terms of the type $\bar{J}_{jk}\sigma_j^+\sigma_k^+ + \text{H.c.}$

The difference between the effective models (20) and (23) has some physically relevant consequences. First, in contrast to the RWA result (20), Hamiltonian (23) does not conserve the number of qubit excitations, which will affect the design of two-qubit gates [35]. Moreover, both models possess different spectra, which influences entanglement creation. For instance, the ground state of Hamiltonian (23) for two degenerate qubits ($\epsilon_1 = \epsilon_2$) coupled to one cavity is $|0\rangle[|\downarrow\downarrow\rangle - (J/2\epsilon)|\uparrow\uparrow\rangle]$ except for normalization and thus exhibits qubit-qubit entanglement. By contrast, the corresponding ground state of the multiqubit RWA Hamiltonian (20) is the product state $|0\rangle|\downarrow\downarrow\rangle$. For the case of Hamiltonian (23), thermal qubit-qubit entanglement will consequently be present at zero temperature and even at thermal equilibrium [41,42].

V. SUMMARY

We have generalized the dispersive theory for a qubit coupled to a harmonic oscillator to the case of far detuning. In this limit, it is no longer possible to treat the qubit-oscillator interaction Hamiltonian within the RWA. Therefore, previous derivations need some refinement. It has turned out that diagonalizing the Hamiltonian analytically up to second order in the coupling constant is possible as well beyond RWA. The central result is that as within RWA, the oscillator experiences a shift of its resonance frequency; the sign of the shift depending on the qubit state. In this respect, the difference between both approaches seems to be merely quantitative. Nevertheless, our result implies an important fact for currently devised qubit-oscillator experiments with ultrastrong cavity-qubit coupling, namely, that dispersive qubit readout is possible as well in that regime. The comparison with numerical results has confirmed that our approach is quantitatively satisfactory in the whole dispersive regime.

The corresponding treatment of many qubits coupled to the same oscillator is equally possible. In such architectures, the oscillator mediates an effective qubit-qubit interaction,

which may be used for gate operations and entanglement creation. We have revealed that the form of the effective interaction depends on whether or not one employs RWA. While RWA predicts an isotropic *XY* interaction, the inclusion of the counter-rotating terms yields an interaction of Ising type. This difference impacts on various proposed entanglement creation protocols as soon as they operate in the far-detuned dispersive regime.

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