# Potential analysis for positive recurrent Markov chains with asymptotically zero drift: Power-type asymptotics☆

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**Abstract**

We consider a positive recurrent Markov chain on $\mathbb{R}^+$ with asymptotically zero drift which behaves like $-c_1/x$ at infinity; this model was first considered by Lamperti. We are interested in tail asymptotics for the stationary measure. Our analysis is based on construction of a harmonic function which turns out to be regularly varying at infinity. This harmonic function allows us to perform non-exponential change of measure. Under this new measure Markov chain is transient with drift like $c_2/x$ at infinity and we compute the asymptotics for its Green function. Applying further the inverse transform of measure we deduce a power-like asymptotic behaviour of the stationary tail distribution. Such a heavy-tailed stationary measure happens even if the jumps of the chain are bounded. This model provides an example where possibly bounded input distributions produce non-exponential output.

*Keywords:* Markov chain; Invariant distribution; Lamperti problem; Asymptotically zero drift; Test (Lyapunov) function; Regularly varying tail behaviour; Convergence to $\Gamma$-distribution; Renewal function; Harmonic function; Non-exponential change of measure; Martingale technique

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## 1. Introduction, main results and discussion

Let $X = \{X_n, n \geq 0\}$ be a time homogeneous Markov chain taking values in $\mathbb{R}^+$. Denote by $\xi(x)$, $x \in \mathbb{R}^+$, a random variable corresponding to the jump of the chain at point $x$, that is, a random variable with distribution

$$\begin{aligned}\mathbb{P}\{\xi(x) \in B\} &= \mathbb{P}\{X_{n+1} - X_n \in B \mid X_n = x\} \\ &= \mathbb{P}_x\{X_1 \in x + B\}, \quad B \in \mathcal{B}(\mathbb{R});\end{aligned}$$

hereinafter the subscript $x$ denotes the initial position of the Markov chain $X$, that is, $X_0 = x$.

Denote the $k$th moment of the jump at point $x$ by $m_k(x) := \mathbb{E}\xi^k(x)$. We say that a Markov chain has *asymptotically zero drift* if $m_1(x) = \mathbb{E}\xi(x) \to 0$ as $x \to \infty$. The study of processes with asymptotically zero drift was initiated by Lamperti in a series of papers [15–17].

Processes with asymptotically zero drift naturally appear in various stochastic models, here we mention only some of them: branching processes, Klebaner [10] and Küster [14]; random billiards, Menshikov et al. [20]; random polymers, Alexander [1], Alexander and Zygouras [2], De Coninck et al. [4].

We assume that the Markov chain $X_n$ possesses a stationary (invariant) distribution and denote this distribution by $\pi$. If we consider an irreducible aperiodic Markov chain on $\mathbb{Z}^+$, then the existence of probabilistic invariant distribution is equivalent to finiteness of $\mathbb{E}_0\tau_0$ where $\tau_0 := \min\{n \geq 1 : X_n = 0\}$. For the state space $\mathbb{R}^+$, we assume that $X_n$ is a positive Harris recurrent and strongly aperiodic chain; see related definitions in [21]. In particular, there exists a sufficiently large $x_0$ such that

$$\mathbb{E}_x\tau_B < \infty \quad \text{for all } x > x_0, \tag{1}$$

where $\tau_B := \min\{n \geq 1 : X_n \in B\}$ and $B := [0, x_0]$. We assume that the chain makes excursions from any compact set, in the following sense. We suppose that, for every fixed $x_1 > x_0$, there exists an $\varepsilon = \varepsilon(x_1) > 0$ such that, for every $x > x_0$,

$$\mathbb{P}_x\{X_{n(x)} > x_1, \tau_B > n(x)\} \geq \varepsilon \quad \text{for some } n(x). \tag{2}$$

We consider the case where $\pi$ has unbounded support, that is, $\pi(x, \infty) > 0$ for every $x$. Our main goal is to describe the asymptotic behaviour of its tail, $\pi(x, \infty)$, for a class of Markov chains with asymptotically zero drift.

As it was shown in [13, Theorem 1] any Markov chain with asymptotically zero drift has heavy-tailed invariant distribution provided

$$\liminf_{x\to\infty} \mathbb{E}\{\xi^2(x); \xi(x) > 0\} > 0;$$

that is, all positive exponential moments of the invariant distribution are infinite. The present paper is devoted to the precise asymptotic behaviour of the invariant tail distribution in the critical case where $m(x)$ behaves like $-c/x$ for large $x$. The existence of invariant distribution in critical case was studied by Lamperti [17]; this study is based on considering the test function $V(x) = x^2$. Then the drift of $V$ at point $x$ is equal to $\mathbb{E}\{V(X_{n+1}) - V(X_n) \mid X_n = x\} = 2xm_1(x) + m_2(x)$ and if $2xm_1(x) + m_2(x) < -\varepsilon$ for all sufficiently large $x$, then the chain is positive recurrent and, under mild technical conditions, it has unique invariant distribution (see [21, Chapter 11]).

There are two types of Markov chains for which the invariant distribution is explicitly calculable. Both are related to skip-free processes, either on lattice or on continuous state space $\mathbb{R}^+$.

The first case where the stationary distribution is explicitly known is diffusion processes on $\mathbb{R}^+$ (slotted in time if we need just a Markov chain). Let $m_1(x)$ and $m_2(x)$ be the drift and diffusion coefficients at state $x$, respectively. In the case of stable diffusion, the invariant density function $p(x)$ solves the Kolmogorov forward equation

$$0 = -\frac{d}{dx}(m_1(x)p(x)) + \frac{1}{2}\frac{d^2}{dx^2}(m_2(x)p(x)),$$

which has the following solution:

$$p(x) = \frac{2c}{m_2(x)} e^{\int_0^x \frac{2m_1(y)}{m_2(y)}dy}, \quad c > 0. \tag{3}$$

The second case is the Markov chain on $\mathbb{Z}^+$ with $\xi(x)$ taking values $-1$, 1 and 0 only, with probabilities $p_-(x)$, $p_+(x)$ and $1 - p_-(x) - p_+(x)$ respectively, $p_-(0) = 0$. Then the stationary probabilities $\pi(x)$, $x \in \mathbb{Z}^+$, satisfy the equations

$$\pi(x) = \pi(x-1)p_+(x-1) + \pi(x)(1 - p_+(x) - p_-(x)) + \pi(x+1)p_-(x+1),$$

which have the following solution:

$$\pi(x) = \pi(0)\prod_{k=1}^{x}\frac{p_+(k-1)}{p_-(k)} = \pi(0)e^{\sum\limits_{k=1}^{x}(\log p_+(k-1) - \log p_-(k))}, \tag{4}$$

where under some regularity conditions the sum may be approximated by the integral like in the diffusion case.

To the best of our knowledge there are no other results in the literature on the exact asymptotic behaviour for the measure $\pi$.

**Theorem 1.** *Suppose that, as $x \to \infty$,*

$$m_1(x) \sim -\frac{\mu}{x}, \qquad m_2(x) \to b \quad \text{and} \quad 2\mu > b. \tag{5}$$

*Suppose also that there exists a differentiable function $r(x) > 0$ such that $r'(x) \sim -\frac{2\mu}{bx^2}$ and*

$$\frac{2m_1(x)}{m_2(x)} = -r(x) + O(1/x^{2+\delta}) \tag{6}$$

*for some $\delta > 0$. Suppose also that*

$$\sup_x \mathbb{E}|\xi(x)|^{3+\delta} < \infty, \tag{7}$$

$$\mathbb{E}\xi^3(x) \to m_3 \in (-\infty, \infty) \quad \text{as } x \to \infty \tag{8}$$

*and, for some $A < \infty$,*

$$\mathbb{E}\{\xi^{2\mu/b+3+\delta}(x); \xi(x) > Ax\} = O(x^{2\mu/b}). \tag{9}$$

*Then there exists a constant $c > 0$ such that*

$$\pi(x,\infty) \sim cxe^{-\int_0^x r(y)dy} = cx^{-2\mu/b+1}\ell(x) \quad \text{as } x \to \infty,$$

*where $\ell(x) := x^{2\mu/b}/e^{\int_0^x r(y)dy}$ is a slowly varying function.*

In paper [19], Menshikov and Popov investigated behaviour of the invariant distribution $\{\pi(x), x \in \mathbb{Z}^+\}$ for countable Markov chains with asymptotically zero drift and with bounded jumps (see also Aspandiiarov and Iasnogorodski [3]). Some rough theorems for the local probabilities $\pi(x)$ were proved; if the condition (5) holds then for every $\varepsilon > 0$ there exist constants $c_- = c_-(\varepsilon) > 0$ and $c_+ = c_+(\varepsilon) < \infty$ such that

$$c_- x^{-2\mu/b-\varepsilon} \le \pi(x) \le c_+ x^{-2\mu/b+\varepsilon}.$$

The paper [13] is devoted to the existence and non-existence of moments of invariant distribution. In particular, it was proven that if (5) holds and the families of random variables $\{(\xi^+(x))^{2+\gamma}, x \ge 0\}$ for some $\gamma > 0$ and $\{(\xi^-(x))^2, x \ge 0\}$ are uniformly integrable then the moment of order $\gamma$ of the invariant distribution $\pi$ is finite if $\gamma < 2\mu/b - 1$, and infinite if $\pi$ has unbounded support and $\gamma > 2\mu/b - 1$. This result implies that for every $\varepsilon > 0$ there exists $c(\varepsilon)$ such that

$$\pi(x, \infty) \le c(\varepsilon) x^{-2\mu/b+1+\varepsilon}. \tag{10}$$

It is clear that the convergence of third moments in Theorem 1 is a technical condition because the asymptotic behaviour of the stationary measure depends on $m_1(x)$ and $m_2(x)$ only and does not depend on $m_3$. Also as follows from the moments existence results [13], it is likely that the statement of Theorem 1 should follow under less restrictive condition than (9), with $2\mu/b+1+\delta$ moments instead. Unfortunately, we cannot just remove restriction (8) from the theorem, but we can weaken it by introducing some structural restrictions, the main of which is the left-continuity of $X_n$.

**Theorem 2.** *Suppose that all conditions of Theorem 1 hold except probably the condition (8). If, in addition, $X_n$ lives on $\mathbb{Z}^+$ and $\xi(x) \ge -1$, then the statement of Theorem 1 remains valid.*

To prove Theorems 1 and 2 we change the probability measure in such a way that the resulting object will be a transient Markov chain with asymptotically zero drift. We apply the following change of measure:

$$\widehat{P}(x, dy) := \frac{V(y)\mathbb{P}_x\{X_1 \in dy, \tau_B > 1\}}{V(x)},$$

where $V$ is a harmonic function for the substochastic kernel $\mathbb{P}_x\{X_1 \in dy, \tau_B > 1\}$. In this way we need to produce a suitable harmonic function $V$. Since the harmonic function for the corresponding Bessel-type process conditioned to stay positive is known, we adapt the method proposed in [6] where random walks conditioned to stay in a cone were considered. (This method allows one to construct harmonic functions for random walks from harmonic functions for corresponding limiting diffusions.) Again, the only processes, where harmonic functions were known, are diffusions and Markov chains with jumps $\pm 1$. The latter case has been considered by Alexander [1].

Investigation of large deviation probabilities for one-dimensional Markov chains with ultimately negative drift heavily depends on whether this chain is similar to the process of summation with more or less homogeneous drift (and in this case we may speak about the process with continuous statistics) or this Markov chain is close to a random walk on $\mathbb{R}^+$ with delay at the origin where the mean drift changes its sign near the origin (in this case we have the chain with discontinuous statistics). The only Markov chain which can be somehow reduced to the sums is the chain $W_n = (W_{n-1} + \xi_n)^+$ with independent identically distributed $\xi$'s which equals in distribution to $\max_{k\le n}\sum_{j=1}^{k}\xi_j$. For these two classes of Markov chains (with continuous and discontinuous statistics) the methods for investigation of large deviations are essentially different. Say, in the Cramér case where some exponential moments of jumps are bounded, an appropriate exponential change of measure preserves the measures to be probabilistic. If we apply exponential change of measures to a chain with discontinuous statistics it may lead to non-stochastic kernel. Such approach was utilised in [12] and there appears a necessity for proving limit theorems for non-stochastic transition kernels.

In the setting of the present paper one could think of applying a change of measure method with power-like weight function. Then the probability measure changes in such a way that the resulting object will be similar to a transient Markov chain with asymptotically zero drift. One may look at the following two approaches:

(a) $Q^{(1)}(x,B) := \frac{\mathbb{E}\{X_1^{\rho}\{X_1\in B\}|X_0=x\}}{x^{\rho}}$, $\rho = 2\mu/b+1$;

(b) $Q^{(2)}(x,B) := \frac{\mathbb{E}\{V(X_1)\{X_1\in B\}|X_0=x\}}{V(x)}$.

In the first case we would have measures which are not necessarily probabilistic, i.e., $Q^{(1)}(x,\mathbb{R}^+)$ can be smaller or greater than 1; this case is similar to that considered in [12] for the case of the exponential change of measure.

With $\rho = 2\mu/b+1$ one can show that the Markov evolution of masses is asymptotically equivalent to a transient Markov chain with asymptotically zero drift. And our hope is that one can adopt results, which will be proved in the present project, to Markov evolutions of masses. If this is the case, then we can translate the results for Markov evolutions of masses into results for positive recurrent Markov chains by applying the inverse change of measure.

As it was mentioned above, in this paper we develop the second possibility for the change of measure, where we get a stochastic transition kernel corresponding to a transient Markov chain. Then the main difficulties are related to the fact that the harmonic function $V$ is given implicitly. In particular, we even need to check that $V$ is a regular varying function with index $\rho$.

Having this observation in mind we face to necessity of obtaining limiting results for transient Markov chains. In Section 2 we give rather general close to necessary conditions for transience while in Section 3 we make some quantitative analyses of how fast a transient chain escapes to the infinity. Section 4 is devoted to convergence to the $\Gamma$-distribution under optimal assumptions: null-recurrence or transience of the process and minimal integrability restrictions. Section 5 contains the integral renewal theorem for transient Markov chain with drift $c/x$, $c>0$. In Section 6 general results on harmonic functions are discussed. In order to obtain results for the original positive recurrent Markov chain one needs to apply the inverse change of measure. This is done in Section 7.

## 2. Conditions for transience revised

In general, if, for some $x_0$ and $\varepsilon>0$,

$$\frac{2xm_1(x)}{m_2(x)} \ge 1+\varepsilon \quad \text{for all } x\ge x_0, \tag{11}$$

then the drift to the right dominates the diffusion and the corresponding Markov chain $X_n$ is typically transient. As an example concluding this section shows, for transience, the Markov chain should satisfy some additional conditions on jumps. In the literature, the transience in the Lamperti problem was studied by Lamperti [15], Kersting [8] and Menshikov et al. [18] under different conditions, say for the case of bounded jumps or of moments of order $2+\delta$ bounded. Our goal here is to clarify which condition in addition to (11) is responsible for transience. Surprisingly, such a condition is rather weak and is presented in (13).

**Theorem 3.** *Assume the condition* (11) *holds. In addition, let*

$$\mathbb{P}\left\{\limsup_{n\to\infty} X_n = \infty\right\} = 1 \tag{12}$$

*and, for some* $\gamma$, $0<\gamma<1-1/\sqrt{1+\varepsilon}$,

$$\mathbb{P}\{\xi(x)\le -\gamma x\} = o\left(\frac{m_2(x)}{x}p(x)\right) \quad \text{as } x\to\infty, \tag{13}$$

*where a non-increasing function* $p(x)$ *is integrable. Then* $X_n \to \infty$ *as* $n\to\infty$ *with probability* 1, *so that* $X_n$ *is transient.*

The condition (12) (which was first proposed in this framework by Lamperti [15]) may be equivalently restated as follows: for any $N$ the exit time from the set $[0, N]$ is finite with probability 1. In this way it is clear that, for a countable Markov chain, the irreducibility implies (12). For a Markov chain on general state space, the related topic is $\psi$-irreducibility; see [21, Sections 4 and 8].

**Proof of Theorem 3.** It is based on the standard approach of construction of a nonnegative bounded test function $V_*(x)\downarrow 0$ such that $V_*(X_n)$ is a supermartingale with further application of Doob's convergence theorem for supermartingales.

Since $p(x)$ is non-increasing and integrable, by Denisov [5], there exists a continuous non-increasing integrable regularly varying at infinity with index $-1$ function $V_1(x)$ such that $p(x)\le V_1(x)$. Take

$$V(x) := \int_x^\infty V_2(y)dy, \quad \text{where } V_2(x) := \int_x^\infty \frac{V_1(y)}{y}dy.$$

By Theorem 1(a) from [7, Chapter VIII, Section 9] we know that $V_2$ is regularly varying with index $-1$ and $V_2(x)\sim V_1(x)$ as $x\to\infty$. Since $V_1$ is integrable, the nonnegative non-increasing function $V(x)$ is bounded, $V(0)<\infty$, and, by the same reference, $V(x)$ is slowly varying.

Let us prove that the mean drift of $V(x)$ is negative for all sufficiently large $x$. We have

$$\begin{aligned}\mathbb{E}V(x+\xi(x))-V(x) &\le V(0)\mathbb{P}\{\xi(x)\le-\gamma x\}+\mathbb{E}\{V(x+\xi(x))-V(x);\xi(x)>-\gamma x\}\\ &= V(0)\mathbb{P}\{\xi(x)\le-\gamma x\}+V'(x)\mathbb{E}\{\xi(x);\xi(x)>-\gamma x\}\\ &\quad+\frac{1}{2}\mathbb{E}\{\xi^2(x)V''(x+\theta\xi(x));\xi(x)>-\gamma x\},\end{aligned}$$

where $0\le\theta=\theta(x,\xi(x))\le 1$, by Taylor's formula with the remainder in the Lagrange form. By the construction, $V'(x)=-V_2(x)<0$, $\mathbb{E}\{\xi(x);\xi(x)>-\gamma x\}\ge m_1(x)>0$ for $x\ge x_0$, and $V''(x)=V_1(x)/x$ is non-increasing because $V_1(x)$ is so. Hence,

$$\begin{aligned}\mathbb{E}V(x+\xi(x))-V(x) &\le V(0)\mathbb{P}\{\xi(x)\le-\gamma x\}-V_2(x)m_1(x)+\frac{V''((1-\gamma)x)}{2}m_2(x)\\ &= o\left(\frac{m_2(x)V_1(x)}{x}\right)\\ &\quad-\frac{m_2(x)V_1(x)}{2x}\left(\frac{2xm_1(x)}{m_2(x)}\frac{V_2(x)}{V_1(x)}-\frac{x}{V_1(x)}\frac{V_1((1-\gamma)x)}{(1-\gamma)x}\right),\end{aligned}$$

by the condition (13) and the inequality $p(x) \le V_1(x)$. Applying now the condition (11) together with the equivalences $V_2(x) \sim V_1(x)$ and $V_1((1-\gamma)x) \sim V_1(x)/(1-\gamma)$ we deduce that there exists a sufficiently large $x_*$ such that, for all $x \ge x_*$,

$$\mathbb{E}V(x+\xi(x)) - V(x) \le -\frac{m_2(x)V_1(x)}{2x}\varepsilon_*,$$

where $\varepsilon_* := (1+\varepsilon-(1-\gamma)^{-2})/2 > 0$. Now take $V_*(x) := \min(V(x), V(x_*))$. Then

$$\mathbb{E}V_*(x+\xi(x)) - V_*(x) \le \mathbb{E}V(x+\xi(x)) - V(x) < 0$$

for every $x \ge x_*$ and

$$\mathbb{E}V_*(x+\xi(x)) - V_*(x) = \mathbb{E}\{V(x+\xi(x)) - V(x_*); x+\xi(x) \ge x_*\} \le 0$$

for every $x < x_*$. Therefore, $V_*(X_n)$ constitutes a nonnegative bounded supermartingale and, by Doob's convergence theorem, $V_*(X_n)$ has an a.s. limit as $n \to \infty$. Due to the condition (12), this limit equals $V_*(\infty) = 0$ and the proof is complete. □

Roughly speaking, the condition (13) guarantees that large negative jumps do not make any valuable contribution to the evolution of the chain compared to the contribution of the first and second moments of jumps. Let us demonstrate by example that the condition (13) is very essential and in a sense almost necessary.

Consider a Markov chain $X_n$ on $\mathbb{R}^+$ satisfying the following conditions: for some function $f(x) \ge 0$, $m_1(x) \le f(x)$ and

$$\mathbb{P}\{\xi(x) = -x\} = f(x)p(x) \tag{14}$$

for all sufficiently large $x$, where $p(x)$ is a non-increasing function satisfying $p(x) = O(1/x)$ and

$$V(x) := \int_0^x p(y)dy \to \infty \quad \text{as } x \to \infty.$$

In this example the high probability of the large negative jump $-x$ leads to recurrence of the chain (note that if $f(x) = m_2(x)/x$ then the condition (13) fails to hold).

Indeed, decompose the mean drift of the increasing concave test function $V$ at state $x$ separating the jump to the origin:

$$\begin{aligned}\mathbb{E}V(x+\xi(x)) - V(x) &= -V(x)\mathbb{P}\{\xi(x) = -x\} \\ &\qquad + \mathbb{E}\{V(x+\xi(x)) - V(x); \xi(x) > -x\}.\end{aligned} \tag{15}$$

Since $V(x)$ is concave and $V'(x) = p(x)$, by Jensen's inequality,

$$\begin{aligned}\mathbb{E}\{V(x+\xi(x)) - V(x); \xi(x) > -x\} &\le p(x)\mathbb{E}\{\xi(x); \xi(x) > -x\} \\ &= p(x)(m_1(x) + x\mathbb{P}\{\xi(x) = -x\}) \\ &= O(p(x)f(x)) \quad \text{as } x \to \infty,\end{aligned}$$

because $xp(x)$ is bounded. Substituting this together with (14) into (15), we obtain the following upper bound for the drift:

$$\mathbb{E}V(x+\xi(x)) - V(x) \le p(x)f(x)(-V(x) + O(1)).$$

Since $V(x) \to \infty$ as $x \to \infty$, the drift becomes asymptotically negative and the chain $X_n$ is recurrent; see e.g. [21, Theorem 8.4.3].

## 3. Quantitative analysis of escaping to infinity for a transient chain

First we give an upper bound for the return probability for a transient Markov chain.

**Lemma 1.** *Assume the condition (11) holds and, for some $\delta$, $\gamma > 0$ satisfying $(1+\delta)/(1-\gamma)^{2+\delta} < 1+\varepsilon$,*

$$\mathbb{P}\{\xi(x) \le -\gamma x\} = o\left(\frac{m_2(x)}{x^{2+\delta}}\right) \quad \text{as } x \to \infty. \tag{16}$$

*Then there exists $x_0$ such that*

$$\mathbb{P}\{X_n \le x \text{ for some } n \ge 1 \mid X_0 = y\} \le (x/y)^\delta \quad \text{for all } y > x > x_0.$$

**Proof.** Fix $y > 0$. Consider the test function $V(x) := \min(x^{-\delta}, 1)$. The mean drift of $V(x)$ is negative for all sufficiently large $x$. Indeed,

$$\begin{aligned}
\mathbb{E}V(x+\xi(x)) - V(x) &\le \mathbb{P}\{\xi(x) \le -\gamma x\} + \mathbb{E}\{V(x+\xi(x)) - V(x); \xi(x) > -\gamma x\} \\
&= \mathbb{P}\{\xi(x) \le -\gamma x\} - \frac{\delta}{x^{1+\delta}}\mathbb{E}\{\xi(x); \xi(x) > -\gamma x\} \\
&\quad + \frac{\delta(1+\delta)}{2}\mathbb{E}\left\{\frac{\xi^2(x)}{(x+\theta\xi(x))^{2+\delta}}; \xi(x) > -\gamma x\right\},
\end{aligned}$$

where $0 \le \theta = \theta(x, \xi(x)) \le 1$, by Taylor's formula. Therefore,

$$\begin{aligned}
\mathbb{E}V(x+\xi(x)) - V(x) &\le \mathbb{P}\{\xi(x) \le -\gamma x\} - \frac{\delta}{x^{1+\delta}}m_1(x) + \frac{\delta(1+\delta)}{2((1-\gamma)x)^{2+\delta}}m_2(x) \\
&= o\left(\frac{m_2(x)}{x^{2+\delta}}\right) - \frac{\delta m_2(x)}{2x^{2+\delta}}\left(\frac{2xm_1(x)}{m_2(x)} - \frac{1+\delta}{(1-\gamma)^{2+\delta}}\right),
\end{aligned}$$

by the condition (16). Then the condition (11) implies that there exists sufficiently large $x_*$ such that, for all $x \ge x_*$,

$$\mathbb{E}V(x+\xi(x)) - V(x) \le -\frac{\delta m_2(x)}{2x^{2+\delta}}\varepsilon_*,$$

where $\varepsilon_* := (1+\varepsilon-(1+\delta)/(1-\gamma)^{2+\delta})/2 > 0$. Now take $V_*(x) := \min(V(x), V(x_*))$ so that $V_*(X_n)$ is a nonnegative bounded supermartingale. Hence we may apply Doob's inequality for a nonnegative supermartingale and deduce that, for every $y > x \ge x_*$ (so that $V_*(y) < V_*(x)$),

$$\mathbb{P}\left\{\sup_{n\ge1} V_*(X_n) \ge V_*(x) \mid V_*(X_0) = V_*(y)\right\} \le \frac{\mathbb{E}V_*(X_0)}{V_*(x)} = \left(\frac{x}{y}\right)^\delta,$$

which is equivalent to the lemma conclusion. □

In the next lemma we estimate from above the mean value $\mathbb{E}_y T(x)$ of the first up-crossing time

$$T(x) := \min\{n \ge 1 : X_n > x\}.$$

**Lemma 2.** *Assume that, for some $x_0 \ge 0$, $\varepsilon_0 \ge 0$, and $\varepsilon > 0$,*

$$2xm_1(x) + m_2(x) \ge \begin{cases} \varepsilon, & \text{if } x > x_0, \\ -\varepsilon_0, & \text{if } x \le x_0. \end{cases} \tag{17}$$

*Then, for every $x > y$,*

$$\mathbb{E}_y T(x) \le \frac{x^2 - y^2 + c(x) + (\varepsilon + \varepsilon_0) H_y(x_0)}{\varepsilon},$$

*where*

$$c(x) := \sup_{z \le x} (2z m_1(z) + m_2(z)) \tag{18}$$

*and*

$$H_y(x_0) := \sum_{n=0}^{\infty} \mathbb{P}_y\{X_n \le x_0\}.$$

**Proof.** Consider the following random sequence:

$$Y_n := X_n^2 + (\varepsilon_0 + \varepsilon) \sum_{k=0}^{n-1} \mathbb{I}\{X_k \le x_0\}.$$

First, $Y_n$ is a submartingale with respect to the filtration $\mathcal{F}_n := \sigma(X_k, k \le n)$. Indeed,

$$Y_{n+1} - Y_n = X_{n+1}^2 - X_n^2 + (\varepsilon_0 + \varepsilon)\mathbb{I}\{X_n \le x_0\},$$

so that

$$\begin{aligned} \mathbb{E}\{Y_{n+1} - Y_n \mid \mathcal{F}_n\} &= 2X_n m_1(X_n) + m_2(X_n) + (\varepsilon_0 + \varepsilon)\mathbb{I}\{X_n \le x_0\} \\ &\ge \varepsilon > 0, \end{aligned} \tag{19}$$

by the condition (17). Thus, for any $x > y$,

$$\mathbb{E}_y Y_{T(x)} \ge y^2 + \varepsilon \mathbb{E}_y T(x), \tag{20}$$

due to the adapted version of the proof of Dynkin's formula (see, e.g. [21, Theorem 11.3.1]):

$$\begin{aligned} \mathbb{E}_y Y_{T(x)} &= \mathbb{E}_y Y_0 + \mathbb{E}_y \sum_{n=1}^{\infty} \mathbb{I}\{n \le T(x)\}(Y_n - Y_{n-1}) \\ &= y^2 + \mathbb{E}_y \sum_{n=1}^{\infty} \mathbb{E}\{\mathbb{I}\{n \le T(x)\}(Y_n - Y_{n-1}) \mid \mathcal{F}_{n-1}\} \\ &= y^2 + \mathbb{E}_y \sum_{n=1}^{\infty} \mathbb{I}\{T(x) \ge n\} \mathbb{E}\{Y_n - Y_{n-1} \mid \mathcal{F}_{n-1}\}, \end{aligned}$$

because $\mathbb{I}\{n \le T(x)\} \in \mathcal{F}_{n-1}$. Hence, it follows from (19) that

$$\begin{aligned} \mathbb{E}_y Y_{T(x)} &\ge y^2 + \varepsilon \mathbb{E}_y \sum_{n=1}^{\infty} \mathbb{I}\{T(x) \ge n\} \\ &= y^2 + \varepsilon \sum_{n=1}^{\infty} \mathbb{P}_y\{T(x) \ge n\}, \end{aligned}$$

and the inequality (20) follows.

On the other hand,

$$\mathbb{E}_y Y_{T(x)} = \mathbb{E}_y X^2_{T(x)} + (\varepsilon_0 + \varepsilon)\mathbb{E}_y \sum_{k=0}^{T(x)-1} \mathbb{I}\{X_k \le x_0\}. \tag{21}$$

Further,

$$\begin{aligned}\mathbb{E}\{X^2_{T(x)} \mid X_{T(x)-1}\} &= \mathbb{E}\{(X_{T(x)-1} + \xi(X_{T(x)-1}))^2 \mid X_{T(x)-1}\} \\ &= X^2_{T(x)-1} + \mathbb{E}\{2X_{T(x)-1}m_1(X_{T(x)-1}) + m_2(X_{T(x)-1}) \mid X_{T(x)-1}\} \\ &\le x^2 + c(x),\end{aligned}$$

by the definition (18) of $c(x)$. Substituting this into (21) we deduce

$$\mathbb{E}_y Y_{T(x)} \le x^2 + c(x) + (\varepsilon_0 + \varepsilon) H_y(x_0),$$

which together with (20) yields the lemma conclusion. The proof is complete. □

**Lemma 3.** *Let the conditions of Lemma 2 hold and $c(x) = O(x^2)$ in the condition (18) and*

$$\sup_{y \le x_0} H_y(x_0) = \sup_{y \le x_0} \sum_{n=0}^{\infty} \mathbb{P}_y\{X_n \le x_0\} < \infty. \tag{22}$$

*Then there exist $c > 0$ and $t_0$ such that, for any $t > 0$ and $y < x$,*

$$\mathbb{P}_y\{T(x) > tx^2\} \le e^{-c(t-t_0)}.$$

**Proof.** Considering the first visit to the interval $[0, x_0]$ we deduce from the condition (22) that

$$\sup_{y \ge 0} \sum_{n=0}^{\infty} \mathbb{P}_y\{X_n \le x_0\} < \infty.$$

Thus, by Lemma 2, there exists $c_1 < \infty$ such that, for all $x$,

$$\sup_y \mathbb{E}_y T(x) \le c_1(x^2 + 1). \tag{23}$$

Next, by the Markov property, for every $t$ and $s > 0$,

$$\begin{aligned}\mathbb{P}_y\{T(x) > t + s\} &= \int_0^x \mathbb{P}_y\{T(x) > t, X_t \in dz\}\mathbb{P}_z\{T(x) > s\} \\ &\le \mathbb{P}_y\{T(x) > t\} \sup_{z \le x} \mathbb{P}_z\{T(x) > s\}.\end{aligned}$$

Therefore, the monotone function $q(t) := \sup_{y \le x} \mathbb{P}_y\{T(x) > tx^2\}$ satisfies the relation $q(t+s) \le q(t)q(s)$. Then the increasing function $r(t) := \log(1/q(t))$ is convex and $r(0) = 0$. By the bound (23) and Chebyshev's inequality, there exists $t_0$ such that $q(t_0) < 1$ so that $q(t_0) = e^{-c}$ with $c > 0$, and $r(t_0) = c > 0$. Then, by $r(0) = 0$ and convexity of $r$, $r(t) \ge c(t - t_0)$ which implies $q(t) \le e^{-c(t-t_0)}$. The proof is complete. □

## 4. Convergence to $\Gamma$-distribution for transient and null-recurrent chains

In this section we are interested in the asymptotic growth rate of a Markov chain $X_n$ that goes to infinity in distribution as $n \to \infty$. It happens if this chain is either transient or null recurrent.

First time a limit theorem for the Markov chain with asymptotically zero drift was produced by Lamperti in [16] where the convergence to $\Gamma$-distribution was proven for the case of jumps with all moments finite. The proof is based on the method of moments. The results from [16] have been generalised by Klebaner [11] and later by Kersting [9]. The author of the latter paper works under the assumption that the moments of order $2+\delta$ are bounded. But the convergence is proven on the event $\{X_n \to \infty\}$ only which restricts generality; for example, Lamperti's result allows $X_n$ to be null-recurrent, and for null-recurrent processes we have $\mathbb{P}\{X_n \to \infty\} = 0$.

**Theorem 4.** *Assume that, for some $b > 0$ and $\mu > -b/2$,*

$$\mathbb{E}\xi(x) \sim \mu/x \quad \textit{and} \quad \mathbb{E}\xi^2(x) \to b \textit{ as } x \to \infty \tag{24}$$

*and that the family $\{\xi^2(x), x \geq 0\}$ possesses an integrable majorant $\Xi$, that is, $\mathbb{E}\Xi < \infty$ and*

$$\xi^2(x) \leq_{st} \Xi \quad \textit{for all } x. \tag{25}$$

*If $X_n \to \infty$ in probability as $n \to \infty$, then $X_n^2/n$ converges weakly to the $\Gamma$-distribution with mean $2\mu + b$ and variance $(2\mu + b)2b$.*

**Proof.** For any $n \in \mathbb{N}$, consider a new Markov chain $Y_k(n)$, $k = 0, 1, 2, \dots$, with transition probabilities depending on the parameter $n$, whose jumps $\eta(n, x)$ are just truncations of the original jumps $\xi(x)$ at level $x \vee \sqrt{n}$ depending on both point $x$ and time $n$, that is,

$$\eta(n, x) = \min\{\xi(x), x \vee \sqrt{n}\}.$$

Given $Y_0(n) = X_0$, the probability of discrepancy between the trajectories of $Y_k(n)$ and $X_k$ until time $n$ is at most

$$\begin{aligned} \mathbb{P}\{Y_k(n) \neq X_k \text{ for some } k \leq n\} &\leq \sum_{k=0}^{n-1} \mathbb{P}\{X_{k+1} - X_k \geq \sqrt{n}\} \\ &\leq n\mathbb{P}\{\Xi \geq n\} \\ &\leq \mathbb{E}\{\Xi; \Xi \geq n\} \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{26}$$

Since $X_n \to \infty$ in probability, (26) implies that, for every $c$,

$$\inf_{n > n_0, k \in [n_0, n]} \mathbb{P}\{Y_k(n) > c\} \to 1 \quad \text{as } n_0 \to \infty. \tag{27}$$

By the choice of the truncation level,

$$\xi(x) \geq \eta(n, x) \geq \xi(x) - \xi(x)\mathbb{I}\{\xi(x) > x\}.$$

Therefore, by the condition (25),

$$\mathbb{E}\eta(n, x) = \mathbb{E}\xi(x) + o(1/x) \quad \text{as } x \to \infty \text{ uniformly in } n \tag{28}$$

and

$$\mathbb{E}\eta^2(n, x) = \mathbb{E}\xi^2(x) + o(1) \quad \text{as } x \to \infty \text{ uniformly in } n, \tag{29}$$

hereinafter we write $f_1(x, n) = o(f_2(x, n))$ as $x \to \infty$ uniformly in $n$ if $\sup_n |f_1(x, n)/f_2(x, n)| \to 0$ as $x \to \infty$. In addition, the inequality $\eta(n, x) \leq x \vee \sqrt{n}$ and the condition (25) imply that, for every $j \geq 3$,

$$\mathbb{E}\eta^j(n, x) = o(x^{j-2} + n^{(j-2)/2}) \quad \text{as } x \to \infty \text{ uniformly in } n. \tag{30}$$

Let us compute the mean of the increment of $Y_k^j(n)$. For $j=2$ we have

$$\begin{aligned}\mathbb{E}\{Y_{k+1}^2(n)-Y_k^2(n)|Y_k(n)=x\} &= \mathbb{E}(2x\eta(n,x)+\eta^2(n,x))\\ &= 2\mu+b+o(1)\end{aligned}$$

as $x\to\infty$ uniformly in $n$, by (28) and (29). Applying now (27) we get

$$\mathbb{E}(Y_{k+1}^2(n)-Y_k^2(n))\to 2\mu+b \quad \text{as } k,n\to\infty, k\le n.$$

Hence,

$$\mathbb{E}Y_n^2(n)\sim(2\mu+b)n \quad \text{as } n\to\infty. \tag{31}$$

Let now $j=2i$, $i\ge 2$. We have

$$\begin{aligned}&\mathbb{E}\{Y_{k+1}^{2i}(n)-Y_k^{2i}(n)|Y_k(n)=x\}\\ &\quad=\mathbb{E}\left(2ix^{2i-1}\eta(n,x)+i(2i-1)x^{2i-2}\eta^2(n,x)+\sum_{l=3}^{2i}x^{2i-l}\eta^l(n,x)\binom{2i}{l}\right)\\ &\quad=i[2\mu+(2i-1)b+o(1)]x^{2i-2}+\sum_{l=3}^{2i}x^{2i-l}\mathbb{E}\eta^l(n,x)\binom{2i}{l}\end{aligned} \tag{32}$$

as $x\to\infty$ uniformly in $n$, by (28) and (29). Owing to (30),

$$\begin{aligned}\sum_{l=3}^{2i}x^{2i-l}\mathbb{E}\eta^l(n,x)\binom{2i}{l} &= \sum_{l=3}^{2i}x^{2i-l}o(x^{l-2}+n^{(l-2)/2})\\ &= o(x^{2i-2})+\sum_{l=3}^{2i}x^{2i-l}o(n^{(l-2)/2})\end{aligned}$$

as $x\to\infty$ uniformly in $n$. Substituting this into (32) with $x=Y_k(n)$ and taking into account (27), we deduce that

$$\begin{aligned}\mathbb{E}\{Y_{k+1}^{2i}(n)-Y_k^{2i}(n)\} &= i[2\mu+(2i-1)b+o(1)]\mathbb{E}Y_k^{2i-2}(n)\\ &\quad+\sum_{l=3}^{2i}\mathbb{E}Y_k^{2i-l}(n)o(n^{(l-2)/2}).\end{aligned} \tag{33}$$

In particular, for $j=2i=4$ we get

$$\begin{aligned}\mathbb{E}\{Y_{k+1}^4(n)-Y_k^4(n)\} &= 2(2\mu+3b)\mathbb{E}Y_k^2(n)+\mathbb{E}Y_k(n)o(\sqrt{n})+o(n)\\ &\sim 2(2\mu+3b)(2\mu+b)n,\end{aligned}$$

due to (31). It implies that

$$\mathbb{E}Y_n^4(n)\sim(2\mu+3b)(2\mu+b)n^2 \quad \text{as } n\to\infty.$$

By induction arguments, we deduce from (33) that, as $n\to\infty$,

$$\mathbb{E}Y_n^{2i}(n)\sim n^i\prod_{k=1}^{i}(2\mu+(2k-1)b),$$

which yields that $Y_n^2(n)/n$ weakly converges to Gamma distribution with mean $2\mu+b$ and variance $2b(2\mu+b)$. Together with (26) this completes the proof. $\square$

## 5. Integral renewal theorem for a transient chain

If the Markov chain $X_n$ is transient then it visits any bounded set at most finitely many times. The next result is devoted to the asymptotic behaviour of the renewal functions

$$H_y(x) := \sum_{n=0}^{\infty} \mathbb{P}_y\{X_n \le x\},$$

$$H(x) := \sum_{n=0}^{\infty} \mathbb{P}\{X_n \le x\} = \int H_y(x)\mathbb{P}\{X_0 \in dy\}.$$

**Lemma 4.** *Let the conditions (16) and (22) hold. If*

$$\sup_x (2xm_1(x) + m_2(x)) < \infty, \tag{34}$$

$$2xm_1(x) + m_2(x) \ge \varepsilon > 0 \quad \textit{ultimately in } x, \tag{35}$$

*then there exists $c < \infty$ such that $H_y(x) \le c(1 + x^2)$ for all $y$ and $x$.*

**Proof.** Fix $A > 1$. After the stopping time $T(Ax) = \min\{n \ge 1 : X_n > Ax\}$ the chain falls down below the level $x$ with probability not higher than $1/A^\delta$, provided $x > x_0$; see Lemma 1 (where the condition (11) follows from (34) and (35)). Hence, by the Markov property, for any $y$ we have the following upper bound

$$H_y(x) \le \mathbb{E}_y \sum_{n=0}^{T(Ax)-1} \mathbb{I}\{X_n \le x\} + \frac{1}{A^\delta} \sup_{z \le x} H_z(x). \tag{36}$$

Therefore,

$$\begin{aligned}\sup_{y \ge 0} H_y(x) &\le (1 - 1/A^\delta)^{-1} \sup_y \mathbb{E}_y T(Ax)\\ &\le (1 - 1/A^\delta)^{-1} c_1(1 + x^2)\end{aligned}$$

for some $c_1 < \infty$, by Lemma 2 (where the condition (17) follows from (34) and (35); also $c(x)$ is bounded in (18)). The conclusion of the lemma is proven. $\square$

**Theorem 5.** *Let the conditions (16), (17), (22) and (25) hold. If $m_1(x) \sim \mu/x$ and $m_2(x) \to b > 0$ as $x \to \infty$, and $2\mu > b$, then, for any initial distribution of the chain $X$,*

$$H(x) \sim \frac{x^2}{2\mu - b} \quad \textit{as } x \to \infty.$$

**Proof.** Fix an arbitrary $y$. It follows from Lemma 2 that $T(x)$ is finite a.s. for every $x$, so that the condition (12) holds and, by Theorem 3, $X_n \to \infty$ a.s. as $n \to \infty$. Then we may apply Theorem 4 and state that $X_n^2/n$ weakly converges to the $\Gamma$-distribution with mean $2\mu + b$ and variance $(2\mu + b)2b$. Thus, for every fixed $B$,

$$\begin{aligned}\sum_{n=0}^{[Bx^2]} \mathbb{P}_y\{X_n \le x\} &= \sum_{n=0}^{[Bx^2]} (\Gamma(x^2/n) + o(1))\\ &= \sum_{n=0}^{[Bx^2]} \Gamma(x^2/n) + o(x^2) \quad \text{as } x \to \infty,\end{aligned}$$

here $\Gamma(t)$ denotes the distribution function of the $\Gamma$-distribution. Since

$$\sum_{n=0}^{[Bx^2]} \Gamma(x^2/n) \sim x^2 \int_0^B \Gamma(1/z)dz \quad \text{as } x \to \infty$$

and

$$\int_0^B \Gamma(1/z)dz \to \frac{1}{2\mu - b} \quad \text{as } B \to \infty,$$

we conclude the lower bound

$$\liminf_{x\to\infty} \frac{H_y(x)}{x^2} \ge \frac{1}{2\mu - b}. \tag{37}$$

Let us now prove the upper bound

$$\limsup_{x\to\infty} \frac{H_y(x)}{x^2} \le \frac{1}{2\mu - b}. \tag{38}$$

Applying the upper bound of Lemma 4 on the right side of (36) we deduce that

$$H_y(x) \le \mathbb{E}_y \sum_{n=0}^{T(Ax)-1} \mathbb{I}\{X_n \le x\} + \frac{c}{A^\delta}(1 + x^2). \tag{39}$$

For any $B$, the mean of the sum on the right of (39) may be estimated as follows:

$$\mathbb{E}_y \sum_{n=0}^{T(Ax)-1} \mathbb{I}\{X_n \le x\} \le \mathbb{E}_y \left\{ \sum_{n=0}^{T(Ax)-1} \mathbb{I}\{X_n \le x\}; T(Ax) \le Bx^2 \right\}$$
$$+ \mathbb{E}_y\{T(Ax); T(Ax) > Bx^2\}.$$

To estimate the second term we apply Lemma 3 which yields

$$\mathbb{E}_y\{T(Ax); T(Ax) > Bx^2\} = (Ax)^2 \mathbb{E}_y \left\{ \frac{T(Ax)}{(Ax)^2}; \frac{T(Ax)}{(Ax)^2} > \frac{B}{A^2} \right\}$$
$$\le (Ax)^2 (B/A^2 + 1/c) e^{-c(B/A^2 - t_0)}.$$

Taking $B = A^3$ we can ensure that

$$\mathbb{E}_y\{T(Ax); T(Ax) > Bx^2\} \le c_1 A^3 e^{-cA} x^2.$$

Hence,

$$H_y(x) \le \mathbb{E}_y \left\{ \sum_{n=0}^{T(Ax)-1} \mathbb{I}\{X_n \le x\}; T(Ax) \le Bx^2 \right\} + x^2 O(A^{-\delta})$$
$$\le \sum_{n=0}^{[Bx^2]} \mathbb{P}_y\{X_n \le x\} + x^2 O(A^{-\delta}).$$

As already shown,

$$\sum_{n=0}^{[Bx^2]} \mathbb{P}_y\{X_n \le x\} = x^2 \int_0^B \Gamma(1/z)dz + o(x^2) \quad \text{as } x \to \infty,$$

which implies the required upper bound (38). The lower (37) and upper (38) bounds yield the equivalence, for every fixed $y$,

$$H_y(x) \sim \frac{x^2}{2\mu - b} \quad \text{as } x \to \infty.$$

Together with the uniform in $y$ estimate of Lemma 4 this completes the proof. $\square$

## 6. Construction of a harmonic function

The Markov chain $X_n$ is assumed to be positive recurrent with invariant measure $\pi$. Let $B$ be a Borel set in $\mathbb{R}^+$ with $\pi(B) > 0$; in our applications we consider an interval $[0, x_0]$. Denote $\tau_B := \min\{n \ge 1 : X_n \in B\}$. Since $X_n$ is positive recurrent and $\pi(B) > 0$, $\mathbb{E}_x \tau_B < \infty$ for every $x$.

In this section we construct a *harmonic function* for $X_n$ killed at the time of the first visit to $B$, that is, such a function $V(x)$ that, for every $x$,

$$V(x) = \mathbb{E}_x\{V(X_1); X_1 \notin B\} \, (= \mathbb{E}\{V(x + \xi(x)); x + \xi(x) \notin B\}).$$

If $V$ is harmonic then

$$V(x) = \mathbb{E}_x\{V(X_n); \tau_B > n\} \quad \text{for every } n. \tag{40}$$

For any function $U(x) : \mathbb{R}^+ \to \mathbb{R}$, denote its mean drift function by

$$u(x) := \mathbb{E}_x U(X_1) - U(x) = \mathbb{E} U(x + \xi(x)) - U(x).$$

**Lemma 5.** *Let $U \ge 0$, $U$ be zero on $B$, and*

$$\mathbb{E}_x \sum_{n=0}^{\tau_B - 1} (u(X_n))^+ < \infty \quad \textit{for every } x. \tag{41}$$

*Then the function*

$$V(x) := U(x) + \mathbb{E}_x \sum_{n=0}^{\tau_B - 1} u(X_n)$$

*is well-defined, nonnegative and harmonic.*

**Proof.** The condition (41) and the finiteness of $\mathbb{E}_x \tau_B$ ensure that

$$\mathbb{E}_x \sum_{n=0}^{\tau_B - 1} u(X_n) = \lim_{N \to \infty} \mathbb{E}_x \sum_{n=0}^{(\tau_B - 1) \wedge N} u(X_n). \tag{42}$$

Let $\mathcal{F}_n = \sigma\{X_0, \ldots, X_n\}$. We have

$$\begin{aligned}
\mathbb{E}_x \sum_{n=0}^{(\tau_B - 1) \wedge N} u(X_n) &= \mathbb{E}_x \sum_{n=0}^{N} u(X_n) \mathbb{I}\{\tau_B > n\} \\
&= \mathbb{E}_x \sum_{n=0}^{N} \mathbb{E}\{U(X_{n+1}) - U(X_n) \mid \mathcal{F}_n\} \mathbb{I}\{\tau_B > n\} \\
&= \mathbb{E}_x \sum_{n=0}^{N} \mathbb{E}\{(U(X_{n+1}) - U(X_n)) \mathbb{I}\{\tau_B > n\} \mid \mathcal{F}_n\},
\end{aligned}$$

because $\mathbb{I}\{\tau_B > n\} \in \mathcal{F}_n$. By the fact that $U$ is zero on $B$, we deduce that $U(X_{n+1})\mathbb{I}\{\tau_B = n+1\} = 0$ so that

$$\begin{aligned}\mathbb{E}_x \sum_{n=0}^{(\tau_B-1)\wedge N} u(X_n) &= \mathbb{E}_x \sum_{n=0}^{N} (U(X_{n+1})\mathbb{I}\{\tau_B > n+1\} - U(X_n)\mathbb{I}\{\tau_B > n\}) \\ &= \mathbb{E}_x U(X_{N+1})\mathbb{I}\{\tau_B > N+1\} - U(x),\end{aligned}$$

which together with (42) implies that

$$U(x) + \mathbb{E}_x \sum_{n=0}^{\tau_B-1} u(X_n) = \lim_{N\to\infty} \mathbb{E}_x U(X_{N+1})\mathbb{I}\{\tau_B > N+1\}. \tag{43}$$

The latter limit is nonnegative, since $U \ge 0$. Together with the condition (41) it implies that the mean of the left of (42) is finite and the function $V$ is well-defined and, as the representation (43) shows, nonnegative. (Also, nonnegativity follows from Theorem 14.2.2 from [21] but we here produced self-contained short proof.)

Now let us prove that $V$ is harmonic. Since $U$ is zero on $B$,

$$\mathbb{E}_x\{U(X_1); X_1 \notin B\} = \mathbb{E}_x U(X_1) = U(x) + u(x).$$

Therefore,

$$\begin{aligned}\mathbb{E}_x\{V(X_1); X_1 \notin B\} &= \mathbb{E}_x\{U(X_1); X_1 \notin B\} + \mathbb{E}_x \left\{ \mathbb{E}\left\{ \sum_{n=1}^{\tau_B-1} u(X_n) \middle| X_1 \right\}; X_1 \notin B \right\} \\ &= U(x) + u(x) + \mathbb{E}_x \left\{ \mathbb{E}\left\{ \sum_{n=1}^{\tau_B-1} u(X_n)\mathbb{I}\{X_1 \notin B\} \middle| X_1 \right\}\right\} \\ &= U(x) + u(x) + \mathbb{E}_x \sum_{n=1}^{\tau_B-1} u(X_n)\mathbb{I}\{X_1 \notin B\} \\ &= U(x) + u(x) + \mathbb{E}_x \sum_{n=1}^{\tau_B-1} u(X_n) = V(x),\end{aligned}$$

so that $V$ is harmonic which completes the proof. $\square$

**Lemma 6.** *Suppose the functions $U_1$ and $U_2$ are both locally bounded, equal to zero on $B$, positive on the complement of $B$ and $U_1(x) \sim U_2(x)$ as $x \to \infty$. If both satisfy the condition (41), then $V_1(x) = V_2(x)$ for all $x$.*

**Proof.** As stated in the previous proof, the condition (41) and the finiteness of $\mathbb{E}_x\tau_B$ ensure that

$$V_k(x) = \lim_{N\to\infty} \mathbb{E}_x\{U_k(X_{N+1}); \tau_B > N+1\}, \quad k = 1, 2. \tag{44}$$

It suffices to prove that the limit in (44) is the same for $k = 1, 2$. Indeed, for every $A$,

$$\begin{aligned}\mathbb{E}_x\{U_k(X_{N+1}); \tau_B > N+1\} = \mathbb{E}_x\{U_k(X_{N+1}); \tau_B > N+1, X_{N+1} \le A\} \\ + \mathbb{E}_x\{U_k(X_{N+1}); \tau_B > N+1, X_{N+1} > A\}.\end{aligned}$$

The first expectation on the right is not greater than

$$\sup_{x\le A} U_k(x)\mathbb{P}_x\{\tau_B > N+1\} \to 0 \quad \text{as } N \to \infty,$$

because $U_k$ is locally bounded. As far as we consider the second expectation, for every $\varepsilon > 0$ there exists a sufficiently large $A$ such that

$$(1-\varepsilon)U_1(x) \le U_2(x) \le (1+\varepsilon)U_1(x)$$

and then

$$\begin{aligned}&(1-\varepsilon)\mathbb{E}_x\{U_1(X_{N+1}); \tau_B > N+1, X_{N+1} > A\}\\ &\quad \le \mathbb{E}_x\{U_2(X_{N+1}); \tau_B > N+1, X_{N+1} > A\}\\ &\quad \le (1+\varepsilon)\mathbb{E}_x\{U_1(X_{N+1}); \tau_B > N+1, X_{N+1} > A\}.\end{aligned}$$

These observations prove that the limits in (44) are equal for $k = 1, 2$ and the proof is complete. $\square$

## 7. Proof of Theorem 1

Fix $x_0$ as in (1). Consider the following function $U$: $U = 0$ on $[0, x_0]$ and

$$U(x) := \int_{x_0}^{x} e^{R(y)}dy \quad \text{for } x \ge x_0, \text{ where } R(y) = \int_0^y r(z)dz. \tag{45}$$

Note that the function $U$ solves the equation $U'' - rU' = 0$. In other words, $U$ is a harmonic function for a diffusion with drift $-r(x)$ and diffusion coefficient 1 killed at leaving $(x_0, \infty)$. According to our assumptions,

$$r(z) = \frac{2\mu}{b}\frac{1}{z} + \frac{\varepsilon(z)}{z},$$

where $\varepsilon(z) \to 0$ as $z \to \infty$. In view of the representation theorem, there exists a slowly varying at infinity function $\ell(x)$ such that $e^{R(x)} = x^{\rho-1}\ell(x)$ and $U(x) \sim xe^{R(x)}/\rho \sim x^\rho \ell(x)/\rho$ where $\rho = 2\mu/b + 1 > 2$.

For every $C \in \mathbb{R}$, define $U_C(x) = 0$ on $[0, x_0]$ and

$$U_C(x) = U(x) + Ce^{R(x)} \quad \text{for } x > x_0.$$

**Lemma 7.** *Assume the conditions of Theorem 1 hold. Then*

$$\mathbb{E}U_C(x+\xi(x)) - U_C(x) = ((\rho-1)b(C_0 - C)/2 + o(1))\, e^{R(x)}/x^2 \quad \text{as } x \to \infty,$$

*where* $C_0 := m_3(\rho-2)/3b$.

**Proof.** We start with the following decomposition:

$$\begin{aligned}\mathbb{E}U(x+\xi(x)) - U(x) = \;& \mathbb{E}\{U(x+\xi(x)) - U(x); |\xi(x)| \le \varepsilon x\}\\ &+\mathbb{E}\{U(x+\xi(x)) - U(x); \varepsilon x \le \xi(x) \le Ax\}\\ &+\mathbb{E}\{U(x+\xi(x)) - U(x); \xi(x) > Ax\}\\ &+\mathbb{E}\{U(x+\xi(x)) - U(x); \xi(x) < -\varepsilon x\}\\ =:\;& E_1 + E_2 + E_3 + E_4.\end{aligned} \tag{46}$$

The second and fourth terms on the right may be bounded as follows:

$$\begin{aligned}E_2 + E_4 &\le U((1+A)x)\mathbb{P}\{|\xi(x)| > \varepsilon x\}\\ &\le c_1 U(x)\mathbb{P}\{|\xi(x)| > \varepsilon x\}\\ &= o(U(x)/x^3) \quad \text{as } x \to \infty,\end{aligned} \tag{47}$$

by the regular variation of $U$ and by the condition (7). For the third term we have

$$
\begin{aligned}
E_3 &\le \mathbb{E}\{U((1/A+1)\xi(x));\xi(x)>Ax\} \\
&\le c_1\mathbb{E}\{\xi^{2\mu/b+1+\delta/2}(x);\xi(x)>Ax\} \\
&\le c_1(Ax)^{-2-\delta/2}\mathbb{E}\{\xi^{2\mu/b+3+\delta}(x);\xi(x)>Ax\} \\
&= o(U(x)/x^3) \quad \text{as } x\to\infty, 
\end{aligned} \tag{48}
$$

due to the regular variation of $U$ and (9). To estimate the first term on the right side of (46), we apply Taylor's formula:

$$
\begin{aligned}
E_1 &= U'(x)\mathbb{E}\{\xi(x);|\xi(x)|\le\varepsilon x\}+\frac{U''(x)}{2}\mathbb{E}\{\xi^2(x);|\xi(x)|\le\varepsilon x\} \\
&\quad +\frac{1}{6}\mathbb{E}\{U'''(x+\theta\xi(x))\xi^3(x);|\xi(x)|\le\varepsilon x\}
\end{aligned} \tag{49}
$$

where $0\le\theta=\theta(x,\xi(x))\le 1$. By the construction of $U$ and the condition (6),

$$
\begin{aligned}
U'(x)m_1(x)+\frac{U''(x)}{2}m_2(x) &= \frac{m_2(x)e^{R(x)}}{2}\left(\frac{2m_1(x)}{m_2(x)}+r(x)\right) \\
&= O(e^{R(x)}/x^{2+\delta}).
\end{aligned} \tag{50}
$$

Notice that

$$|m_1(x)-\mathbb{E}\{\xi(x);|\xi(x)|\le\varepsilon x\}|\le c_2\mathbb{E}|\xi(x)|^{3+\delta}/x^{2+\delta},$$

and

$$0\le m_2(x)-\mathbb{E}\{\xi^2(x);|\xi(x)|\le\varepsilon x\}\le c_2\mathbb{E}|\xi(x)|^{3+\delta}/x^{1+\delta}.$$

Applying now the condition (7), the relations (50), $U'(x)=e^{R(x)}$ and $U''(x)=O(e^{R(x)}/x)$, we obtain

$$U'(x)\mathbb{E}\{\xi(x);|\xi(x)|\le\varepsilon x\}+\frac{U''(x)}{2}\mathbb{E}\{\xi^2(x);|\xi(x)|\le\varepsilon x\}=o(e^{R(x)}/x^2). \tag{51}$$

We next note that (8), our assumptions on $r(x)$ and the convergence

$$\left|\mathbb{E}\{\xi^3(x);|\xi(x)|\le\varepsilon x\}-\mathbb{E}\xi^3(x)\right|\to 0 \quad \text{as } x\to\infty,$$

imply that

$$
\begin{aligned}
U'''(x)\mathbb{E}\{\xi^3(x);|\xi(x)|\le\varepsilon x\} &= (r^2(x)+r'(x))e^{R(x)}(\mathbb{E}\xi^3(x)+o(1)) \\
&= ((\rho-1)(\rho-2)m_3+o(1))e^{R(x)}/x^2,
\end{aligned} \tag{52}
$$

and

$$|\mathbb{E}\{(U'''(x+\theta\xi(x))-U'''(x))\xi^3(x);|\xi(x)|\le\varepsilon x\}|\le c_3\varepsilon e^{R(x)}/x^2. \tag{53}$$

Substituting (51)–(53) into (49) we get, for sufficiently large $x$,

$$\left|E_1-\frac{(\rho-1)(\rho-2)}{6}m_3e^{R(x)}/x^2\right|\le(c_3+1)\varepsilon e^{R(x)}/x^2. \tag{54}$$

In its turn, (48) and (54) being implemented in (46) lead to

$$\mathbb{E}U(x+\xi(x))-U(x)=\frac{(\rho-1)(\rho-2)m_3}{6}e^{R(x)}/x^2+o(e^{R(x)}/x^2), \tag{55}$$

since $\varepsilon>0$ may be chosen as small as we please.

Applying similar arguments to the function $e^{R(x)}$, we get

$$\mathbb{E}e^{R(x+\xi(x))}-e^{R(x)}=-\frac{(\rho-1)b}{2}e^{R(x)}/x^2+o(e^{R(x)}/x^2). \tag{56}$$

Combining (55) and (56) we arrive at

$$\mathbb{E}U_C(x+\xi(x))-U_C(x)=\frac{\rho-1}{2}((\rho-2)m_3/3-bC+o(1))e^{R(x)}/x^2 \quad \text{as } x\to\infty,$$

which completes the proof of the lemma. □

**Lemma 8.** *Under the conditions of Theorem 1, the harmonic function $V$ generated by $U$ possesses the following decomposition:*

$$V(x)=U(x)+C_0e^{R(x)}+o(e^{R(x)}) \quad \textit{as } x\to\infty.$$

*In particular, $V(x)>0$ ultimately in $x$.*

**Proof.** Fix $\varepsilon>0$ and take $C:=C_0+\varepsilon$. According to Lemma 7,

$$u_C(x):=\mathbb{E}U_C(x+\xi(x))-U_C(x)=(-(\rho-1)b\varepsilon/2+o(1))e^{R(x)}/x^2.$$

Therefore, there exist $c_1<\infty$ and $x_1>x_0$ such that

$$u_C(x)\le\begin{cases}c_1 & \text{if } x\le x_1,\\ 0 & \text{if } x>x_1.\end{cases}$$

Hence,

$$\begin{aligned}\mathbb{E}_x\sum_{n=0}^{\tau_B-1}u_C(X_n)&\le c_1\mathbb{E}_x\sum_{n=0}^{\tau_B-1}\mathbb{I}\{X_n\le x_1\}\\ &\le c_1\sup_{x\le x_1}\mathbb{E}_x\tau_B=:c_2<\infty.\end{aligned}$$

Since $U_C(x)\sim U(x)$ as $x\to\infty$, by Lemma 6

$$\begin{aligned}V(x)&=U_C(x)+\mathbb{E}_x\sum_{n=0}^{\tau_B-1}u_C(X_n)\\ &\le U_C(x)+c_2\\ &=U(x)+(C_0+\varepsilon)e^{R(x)}+c_2.\end{aligned}$$

The arbitrary choice of $\varepsilon>0$ yields

$$V(x)\le U(x)+(C_0+o(1))e^{R(x)} \quad \text{as } x\to\infty.$$

Since $V\ge 0$ implies

$$\mathbb{E}_x\sum_{n=0}^{\tau_B-1}u_C(X_n)\ge -U_C(x)>-\infty$$

for every $x$, we have

$$\mathbb{E}_x \sum_{n=0}^{\tau_B-1} e^{R(X_n)}/X_n^2 < \infty. \tag{57}$$

Now take $C := C_0 - \varepsilon$. Again by Lemma 7,

$$u_C(x) := \mathbb{E}U_C(x+\xi(x)) - U_C(x) = ((\rho-1)b\varepsilon/2 + o(1))e^{R(x)}/x^2,$$

and the condition (41) holds due to (57). Then symmetric arguments lead to the lower bound

$$V(x) \ge U(x) + (C_0 + o(1))e^{R(x)} \quad \text{as } x \to \infty.$$

Combining altogether we get the stated decomposition for $V(x)$. □

Having the harmonic function $V$ generated by $U$ we can define a new Markov chain $\widehat{X}_n$ on $\mathbb{R}^+$ with the following transition kernel

$$\mathbb{P}_z\{\widehat{X}_1 \in dy\} = \frac{V(y)}{V(z)}\mathbb{P}_z\{X_1 \in dy; \tau_B > 1\}$$

if $V(z) > 0$ and $\mathbb{P}_z\{\widehat{X}_1 \in dy\}$ being arbitrarily defined if $V(z) = 0$. Since $V$ is harmonic, then we also have

$$\mathbb{P}_z\{\widehat{X}_n \in dy\} = \frac{V(y)}{V(z)}\mathbb{P}_z\{X_n \in dy; \tau_B > n\} \quad \text{for all } n. \tag{58}$$

As is well-known (see, e.g. [21, Theorem 10.4.9]) the invariant measure $\pi$ possesses the equality

$$\pi(dy) = \int_B \pi(dz) \sum_{n=0}^{\infty} \mathbb{P}_z\{X_n \in dy; \tau_B > n\}. \tag{59}$$

Combining (58) and (59), we get

$$\begin{aligned}\pi(dy) &= \frac{1}{V(y)} \int_B \pi(dz)V(z) \sum_{n=0}^{\infty} \mathbb{P}_z\{\widehat{X}_n \in dy\} \\ &= \frac{\widehat{H}(dy)}{V(y)} \int_B \pi(dz)V(z),\end{aligned}$$

where $\widehat{H}$ is the renewal measure generated by the chain $\widehat{X}_n$ with initial distribution

$$\mathbb{P}\{\widehat{X}_0 \in dz\} = \widehat{c}\pi(dz)V(z), \quad z \in B \quad \text{and} \quad \widehat{c} := \left(\int_B \pi(dz)V(z)\right)^{-1}.$$

Therefore,

$$\begin{aligned}\pi(x,\infty) &= \widehat{c}\int_x^\infty \frac{1}{V(y)} d\widehat{H}(y) \\ &\sim \widehat{c}\int_x^\infty \frac{1}{U(y)} d\widehat{H}(y) \quad \text{as } x \to \infty,\end{aligned}$$

since $V(x) \sim U(x)$ owing to Lemma 7. After integration by parts we deduce

$$\begin{aligned}\pi(x,\infty) &\sim \widehat{c}\left(-\frac{\widehat{H}(x)}{U(x)} + \int_x^\infty \frac{\widehat{H}(y)U'(y)}{U^2(y)}dy\right)\\ &\sim \widehat{c}\left(-\frac{\widehat{H}(x)}{U(x)} + \rho\int_x^\infty \frac{\widehat{H}(y)}{yU(y)}dy\right) \quad \text{as } x \to \infty.\end{aligned} \tag{60}$$

In order to apply Theorem 5 to the chain $\widehat{X}_n$, we have to show that its jumps $\widehat{\xi}(x)$ satisfy the corresponding conditions. By the construction, the absolute moments of order $2+\delta/2$ of $\widehat{\xi}(x)$ are uniformly bounded, because

$$\begin{aligned}\mathbb{E}|\widehat{\xi}(x)|^{2+\delta/2} &= \frac{1}{V(x)}\mathbb{E}|\xi(x)|^{2+\delta/2}V(x+\xi(x))\\ &= \frac{1}{V(x)}(\mathbb{E}\{|\xi(x)|^{2+\delta/2}V(x+\xi(x));\xi(x)\le Ax\}\\ &\quad + \mathbb{E}\{|\xi(x)|^{2+\delta/2}V(x+\xi(x));\xi(x)>Ax\})\\ &\le \frac{V((1+A)x)}{V(x)}\mathbb{E}|\xi(x)|^{2+\delta/2}\\ &\quad + \frac{1}{V(x)}\mathbb{E}\{|\xi(x)|^{2+\delta/2}V((1+1/A)\xi(x));\xi(x)>Ax\},\end{aligned}$$

where $A$ is from the condition (9). Here the first term on the right side is bounded due to the condition (7) and regular variation of $V$ with index $\rho$ and the second one is bounded by (9), because

$$\begin{aligned}\mathbb{E}\{|\xi(x)|^{2+\delta/2}V((1+1/A)\xi(x));\xi(x)>Ax\} &\le \frac{c_4}{x^{\delta/4}}\mathbb{E}\{|\xi(x)|^{2+\delta+\rho};\xi(x)>Ax\}\\ &\le c_5x^{\rho-1-\delta/4} = o(V(x)/x).\end{aligned}$$

Then, in particular, the condition (25) of existence of integrable majorant for the squares of jumps $\widehat{\xi}(x)$ and the condition (16) follow. Also it implies that

$$\lim_{x\to\infty}\mathbb{E}\widehat{\xi}^2(x) = \lim_{x\to\infty}\mathbb{E}\xi^2(x) = b. \tag{61}$$

Further, the boundedness of the moments of order $2+\delta/2$ of $\widehat{\xi}(x)$ yields that, for every $\varepsilon>0$,

$$\begin{aligned}\mathbb{E}\widehat{\xi}(x) &= \mathbb{E}\{\widehat{\xi}(x);|\widehat{\xi}(x)|\le\varepsilon x\}+o(1/x)\\ &= \frac{1}{V(x)}\mathbb{E}\{\xi(x)V(x+\xi(x));|\xi(x)|\le\varepsilon x\}+o(1/x).\end{aligned} \tag{62}$$

Fix $\varepsilon_1>0$. Recalling that, by Lemma 8, the function $V(x)-U(x)\sim C_0e^{R(x)}$ is regularly varying with index $\rho-1$, we may choose $\varepsilon>0$ so small that

$$|V(x+y)-U(x+y)-(V(x)-U(x))|\le\varepsilon_1e^{R(x)} \quad \text{for all } |y|\le\varepsilon x. \tag{63}$$

Then

$$\mathbb{E}\{\xi(x)(V(x+\xi(x))-V(x));|\xi(x)|\le\varepsilon x\}$$

differs from

$$\mathbb{E}\{\xi(x)(U(x+\xi(x))-U(x));\,|\xi(x)|\le\varepsilon x\}$$

by the quantity not greater than $\varepsilon_1 e^{R(x)}\mathbb{E}|\xi(x)|$. Using Taylor's formula and the relation

$$\sup_{|y|\le x/2} U''(x+y)=\sup_{|y|\le x/2} r(x+y)e^{R(x+y)}=O(e^{R(x)}/x),$$

we get

$$\begin{aligned}&\mathbb{E}\,\{\xi(x)(U(x+\xi(x))-U(x));\,|\xi(x)|\le\varepsilon x\}\\ &\quad=U'(x)\mathbb{E}\{\xi^2(x);\,|\xi(x)|\le\varepsilon x\}+O(e^{R(x)}/x).\end{aligned}$$

It follows now from the condition (7) that the asymptotics of truncated expectations of the first and the second order coincide with that of full expectations. From above calculations and from the relations $V(x)\sim U(x)$ and $U'(x)=e^{R(x)}\sim\rho U(x)/x$, we deduce that

$$\limsup_{x\to\infty}\left|\frac{x}{V(x)}\mathbb{E}\{\xi(x)V(x+\xi(x));\,|\xi(x)|\le\varepsilon x\}-(-\mu+\rho b)\right|\le\varepsilon_1\rho\sup_x\mathbb{E}|\xi(x)|.$$

Plugging this into (62) and recalling that $\rho=1+2\mu/b$, we conclude that

$$\limsup_{x\to\infty}|x\mathbb{E}\widehat{\xi}(x)-(\mu+b)|\le\varepsilon_1\rho\sup_x\mathbb{E}|\xi(x)|.$$

Since $\varepsilon_1>0$ may be chosen as small as we please,

$$x\mathbb{E}\widehat{\xi}(x)\to\mu+b\quad\text{as } x\to\infty. \tag{64}$$

Finally, we check the condition (22) for the chain $\widehat{X}_n$. As already shown,

$$2x\widehat{m}_1(x)+\widehat{m}_2(x)\to 2(\mu+b)+b=2\mu+3b>0\quad\text{as } x\to\infty.$$

It allows us to choose $x_1>x_0$ so that $U(x_1)>0$, $V(x)\ge U(x)/2$ for all $x>x_1$ (this is possible because $V(x)\sim U(x)$) and

$$\inf_{x>x_1}(2x\widehat{m}_1(x)+\widehat{m}_2(x))>0.$$

Then the condition (22) holds with $x_1$ instead of $x_0$. Indeed, by the construction, $\widehat{X}_n>x_0$ for any $n\ge 1$ which implies

$$\widehat{H}_y(x_0)=\sum_{n=0}^{\infty}\mathbb{P}_y\{\widehat{X}_n\le x_0\}\le 1.$$

Further, it follows from (40) and increase of the function $U$ that, for every $x>x_0$,

$$\begin{aligned}V(x)=\mathbb{E}_x\{V(X_n);\,\tau_B>n\}&\ge\frac{U(x)}{2}\mathbb{P}_x\{X_n>x_1,\tau_B>n\}\\ &\ge\frac{U(x_1)}{2}\mathbb{P}_x\{X_n>x_1,\tau_B>n\}.\end{aligned}$$

The role of the condition (2) is just to be applied here; it guarantees that

$$\inf_{x>x_0}V(x)>0.$$

Therefore, for every $y \in [x_0, x_1]$,

$$\begin{aligned}
\widehat{H}_y(x_1) = \sum_{n=0}^{\infty} \mathbb{P}_y\{\widehat{X}_n \le x_1\} &= \frac{1}{V(y)} \sum_{n=0}^{\infty} \int_{x_0}^{x_1} V(z)\mathbb{P}_y\{X_n \in dz, \tau_B > n\} \\
&\le \frac{\sup_{x_0<z\le x_1} V(z)}{\inf_{y>x_0} V(y)} \sum_{n=0}^{\infty} \mathbb{P}_y\{\tau_B > n\} \\
&= c\mathbb{E}_y \tau_B,
\end{aligned}$$

and the latter mean value is bounded in $y \in [x_0, x_1]$.

Now it is shown that $\widehat{X}_n$ satisfies all the conditions of Theorem 5, so that $\widehat{X}_n$ is transient and

$$\widehat{H}(x) \sim \frac{x^2}{2(\mu+b)-b} = \frac{x^2}{2\mu+b} \quad \text{as } x \to \infty.$$

Substituting this equivalence into (60) where $U(x)$ is regularly varying with index $\rho$ we arrive at the following equivalence:

$$\begin{aligned}
\pi(x,\infty) &\sim \frac{2}{(2\mu+b)(\rho-2)} \frac{x^2}{U(x)} \int_B \pi(dz)V(z) \\
&\sim \frac{2\rho}{(2\mu+b)(\rho-2)} x e^{-R(x)} \int_B \pi(dz)V(z) \quad \text{as } x \to \infty.
\end{aligned}$$

The proof of Theorem 1 is complete.

## 8. Proof of Theorem 2

In this section we work with the same function $U$ as defined in the previous section. Now we should again prove that the corresponding harmonic function $V$ is ultimately positive and that $V(x) \sim U(x)$ as $x \to \infty$. Since here we do not assume convergence of the third moments of jumps, we need to modify our approach for proving these properties.

As in the previous section, for every $C \in \mathbb{R}$, define $U_C(x) = 0$ on $[0, x_0]$ and

$$U_C(x) = U(x) + Ce^{R(x)} \quad \text{for } x > x_0.$$

**Lemma 9.** *Assume the conditions of Theorem 2 hold. Then there exist constants $C_1, C_2 \in \mathbb{R}$ such that, for all sufficiently large $x$,*

$$\begin{aligned}
&\mathbb{E}U_{C_1}(x+\xi(x)) - U_{C_1}(x) < 0, \\
&\mathbb{E}U_{C_2}(x+\xi(x)) - U_{C_2}(x) > 0.
\end{aligned}$$

**Proof.** As the calculations in Lemma 7 show, without the condition on the convergence of the third moments of jumps we still have the relation

$$\mathbb{E}U(x+\xi(x)) - U(x) = o(e^{R(x)}/x^2),$$

which together with (56) concludes the proof. $\square$

The only place where the condition that the chain if left skip-free is utilised is the following result.

**Lemma 10.** *Under the conditions of Theorem 2, the increments of the harmonic function $V$ generated by $U$ satisfy the following bounds: for $y > 0$,*

$$U(x+y) - U(x) + C_2(e^{R(x+y)} - e^{R(x)}) \le V(x+y) - V(x) \le U(x+y) - U(x) + C_1(e^{R(x+y)} - e^{R(x)})$$

*ultimately in $x$. In particular, $V(x) \sim U(x)$ as $x \to \infty$ and $V(x) > 0$ ultimately in $x$.*

**Proof.** Both functions $U_{C_1}$ and $U_{C_2}$ satisfy the conditions of Lemma 6 by the same arguments as in Lemma 8.

Let $y > 0$. Given $X_0 = x + y$, denote $\tau_x := \min\{n \ge 1 : X_n = x\}$. Since the chain is left skip-free, $\tau_x < \tau_B$. Having in mind that $u_{C_1}(X_n) < 0$ before this stopping time, we get, by the Markov property,

$$\begin{aligned} V(x+y) - V(x) &= U_{C_1}(x+y) - U_{C_1}(x) + \mathbb{E}_{x+y}\sum_{n=0}^{\tau_B} u_{C_1}(X_n) - \mathbb{E}_x \sum_{n=0}^{\tau_B} u_{C_1}(X_n) \\ &\le U_{C_1}(x+y) - U_{C_1}(x), \end{aligned}$$

and similarly $V(x+y) - V(x) \ge U_{C_2}(x+y) - U_{C_2}(x)$, which completes the proof. □

We are now able to compute the mean drift of the transformed chain $\widehat{X}_n$. We may just repeat the arguments from the proof of Theorem 1 with the inequality

$$|V(x+y) - U(x+y) - (V(x) - U(x))| \le \max\{|C_1|, |C_2|\}(e^{R(x+y)} - e^{R(x)})$$

instead of (63). As a result we see that (64) is valid under the conditions of Theorem 2.

All other parts of the derivation of the asymptotics of $\pi(x, \infty)$ can be taken from the proof of Theorem 1 without any change.

## References


[1] K.S. Alexander, Excursions and local limit theorems for Bessel-like random walks, Electron. J. Probab. 16 (2011) 1–44.

[2] K.S. Alexander, N. Zygouras, Quenched and annealed critical points in polymer pinning models, Comm. Math. Phys. 291 (2009) 659–689.

[3] S. Aspandiiarov, R. Iasnogorodski, Asymptotic behaviour of stationary distributions for countable Markov chains, with some applications, Bernoulli 5 (1999) 535–569.

[4] J. De Coninck, F. Dunlop, T. Huilett, Random walk weakly attracted to a wall, J. Stat. Phys. 133 (2008) 271–280.

[5] D.E. Denisov, On the existence of a regularly varying majorant of an integrable monotone function, Math. Notes 76 (2006) 129–133.

[6] D. Denisov, V. Wachtel, Random walks in cones, 2011. arXiv:1110.1254.

[7] W. Feller, An Introduction to Probability Theory and its Applications, Vol. 2, Wiley, New York, 1971.

[8] G. Kersting, On recurrence and transience of growth models, J. Appl. Probab. 23 (1986) 614–625.

[9] G. Kersting, Asymptotic $\Gamma$-distribution for stochastic difference equations, Stochastic Process. Appl. 40 (1992) 15–28.

[10] F.C. Klebaner, On population-size-dependent branching processes, Adv. Appl. Probab. 16 (1984) 30–55.

[11] F.C. Klebaner, Stochastic difference equations and generalized gamma distributions, Ann. Probab. 17 (1989) 178–188.

[12] D.A. Korshunov, One-dimensional asymptotically homogeneous Markov chains: Cramér transform and large deviation probabilities, Siberian Adv. Math. 14 (4) (2004) 30–70.

[13] D.A. Korshunov, Moments for stationary Markov chains with asymptotically zero drift, Sib. Math. J. 52 (2011) 655–664.

[14] P. Küster, Asymptotic growth of controlled Galton–Watson processes, Ann. Probab. 13 (1985) 1157–1178.

[15] J. Lamperti, Criteria for the recurrence or transience of stochastic processes I, J. Math. Anal. Appl. 1 (1960) 314–330.

[16] J. Lamperti, A new class of probability limit theorems, J. Math. Mech. 11 (1962) 749–772.

[17] J. Lamperti, Criteria for stochastic processes II: passage time moments, J. Math. Anal. Appl. 7 (1963) 127–145.

[18] M.V. Menshikov, I.M. Asymont, R. Yasnogorodskii, Markov processes with asymptotically zero drifts, Probl. Inf. Transm. 31 (1995) 248–261.

[19] M.V. Menshikov, S.Yu. Popov, Exact power estimates for countable Markov chains, Markov Process. Related Fields 1 (1995) 57–78.

[20] M.V. Menshikov, M. Vachkovskaia, A.R. Wade, Asymptotic behaviour of randomly reflecting billiards in unbounded domains, J. Stat. Phys. 132 (2008) 1097–1133.

[21] S. Meyn, R. Tweedie, Markov Chains and Stochastic Stability, Wiley, New York, 1993.