Local limit theorem for the maximum of asymptotically stable random walks

Vitali Wachtel

Abstract Let $\{S_n; n \geq 0\}$ be an asymptotically stable random walk and let M_n denote it's maximum in the first n steps. We show that the asymptotic behaviour of local probabilities for M_n can be approximated by the density of the maximum of the corresponding stable process if and only if the renewal mass-function based on ascending ladder heights is regularly varying at infinity. We also give some conditions on the random walk, which guarantee the desired regularity of the renewal mass-function. Finally, we give an example of a random walk, for which the local limit theorem for M_n does not hold.

Keywords Limit theorems · Random walks · Renewal theorem

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1 Introduction and statement of results

Let $\{S_n, n \geq 0\}$ denote the random walk with increments X_i, that is,

$$S_0 := 0, \quad S_n := \sum_{i=1}^{n} X_i, \quad n \geq 1.$$

We shall assume that $X_1, X_2, \ldots$ are independent copies of a random variable X. Moreover, we shall assume that X is taken from the domain of attraction of a stable

V. Wachtel
Mathematical Institute, University of Munich, Theresienstrasse 39, 80333 Munich, Germany
e-mail: wachtel@math.lmu.de

law with characteristic function

$$G_{\alpha,\beta}(t) := \exp\left\{-|t|^{\alpha}\left(1 - i\beta\frac{t}{|t|}\tan\frac{\pi\alpha}{2}\right)\right\} \tag{1}$$

with $(\alpha, \beta) \in \mathcal{A} := (0, 1) \times (-1, 1) \cup \{(1, 1/2)\} \cup (1, 2] \times [-1, 1]$. If $\mathbf{E}X$ exists, we assume that $\mathbf{E}X = 0$. In this case we write $X \in \mathcal{D}(\alpha, \beta)$.

It is well known, that for every $X \in \mathcal{D}(\alpha, \beta)$ there exists a sequence c_n regularly varying of index $1/\alpha$ such that $\{S_{[nt]}/c_n,\ t \in [0, 1]\}$ converges weakly to the stable process $\{Y_{\alpha,\beta}(t)\, t \in [0, 1]\}$ characterised by (1), i.e., $\mathbf{E}e^{itY_{\alpha,\beta}(1)} = G_{\alpha,\beta}(t)$.

Denote $M_n := \max_{0\le k\le n} S_k$. It follows from the invariance principle for asymptotically stable random walks that appropriately rescaled M_n converges towards the maximum of the corresponding stable law:

$$\mathbf{P}(M_n \le c_n x) \to \mathbf{P}\left(\max_{0\le t\le 1} Y_{\alpha,\beta}(t) \le x\right) \quad \text{as } n \to \infty.$$

In the present note we pose a question on whether a local version of this convergence is valid. More precisely, we investigate the conditions, under which the convergence

$$c_n\mathbf{P}(M_n \in [c_n x, c_n x + 1)) \to m_{\alpha,\beta}(x) \quad \text{as } n \to \infty$$

is true for every $x > 0$. ($m_{\alpha,\beta}$ stands for the density function of $\max_{0\le t\le 1} Y_{\alpha,\beta}(t)$.) Moreover, we study the asymptotic behaviour of $\mathbf{P}(M_n \in [x, x+1))$ for $x = o(c_n)$. It is well known that by investigating local probabilities, one has to distinguish between lattice and non-lattice distributions. We concentrate here on the case of aperiodic lattice distributions, that is, $\mathbf{P}(X \in \mathbb{Z}) = 1$ and $\mathbf{P}(X \in r\mathbb{Z}) < 1$ for all $r \ge 2$.

To formulate our results we need some additional notation. Define ladder epochs

$$\tau^{+} := \min\{k \ge 1 : S_k > 0\} \quad \text{and} \quad \tau^{-} := \min\{k \ge 1 : S_k \le 0\}.$$

Let χ^{+} and χ^{-} denote the corresponding ladder heights, that is,

$$\chi^{+} := S_{\tau^{+}} \quad \text{and} \quad \chi^{-} := -S_{\tau^{-}}.$$

We finally introduce the following renewal functions:

$$H^{+}(x) = \sum_{y=0}^{x} h^{+}(y), \quad h^{+}(y) := 1\{y = 0\} + \sum_{k=1}^{\infty}\mathbf{P}\left(\chi_1^{+} + \chi_2^{+} + \cdots + \chi_k^{+} = y\right)$$

and

$$H^{-}(x) = \sum_{y=0}^{x} h^{-}(y), \quad h^{-}(y) := 1\{y = 0\} + \sum_{k=1}^{\infty}\mathbf{P}\left(\chi_1^{-} + \chi_2^{-} + \cdots + \chi_k^{-} = y\right),$$

where $\chi_1^{+}, \chi_2^{+}, \ldots$ and $\chi_1^{-}, \chi_2^{-}, \ldots$ are independent copies of χ^{+} and χ^{-} respectively.

Theorem 1 *Suppose $X \in \mathcal{D}(\alpha, \beta)$. Then, as $n \to \infty$,*

$$\mathbf{P}(M_n = x) \sim h^+(x)\mathbf{P}(\tau^+ > n) \tag{2}$$

uniformly in $x \in (0, \delta_n c_n)$, where $\delta_n \to 0$ arbitrary slowly. Furthermore,

$$c_n \mathbf{P}(M_n = x) - m_{\alpha,\beta}(x/c_n) \to 0 \tag{3}$$

uniformly in $x \geq ac_n$, $a > 0$, if and only if h^+ is regularly varying at infinity.

In the case of finite variance the asymptotic behaviour of $\mathbf{P}(M_n = x)$ has been studied by Aleshkyavichene [1] and by Nagaev and Eppel [10]. Moreover, in the case when $\mathbf{E}\chi^+ < \infty$, (2) has been obtained by Alili and Doney [2].

The restriction $x > ac_n$ in the second statement of the theorem means that we exclude "small" values of M_n. This seems to be quite natural, because the density function $m_{\alpha,\beta}$ is not bounded near zero in general. More exactly, it has been proven by Doney and Savov [6] that $m_{\alpha,\beta}(u) \sim u^{\alpha\rho-1}$ as $u \to 0$, where ρ is a constant defined in (4). So, $m_{\alpha,\beta}$ remains bounded if and only if $\alpha\rho = 1$. We expect that in this case one can get the uniform convergence on the set $x > x_n$ for any $x_n \to \infty$. For random walks with finite variance it has been shown in [10].

Theorem 1 establishes the direct connection between local probabilities of the maximum and the mass-function of the renewal function based on ladder heights. First, to use (2) one has to know the asymptotic behaviour of $h^+(x)$ and that of $\mathbf{P}(\tau^+ > n)$. The behaviour of the latter probability is well known: If

$$\mathbf{P}(S_n > 0) \to \rho \in (0, 1), \tag{4}$$

which is always true for $X \in \mathcal{D}(\alpha, \beta)$, then

$$\mathbf{P}(\tau^+ > n) = l(n)n^{-\rho}, \tag{5}$$

where $l(x)$ is a regularly varying function. Second, to approximate local probabilities for M_n by the density of a stable law, one needs to know that h^+ is regularly varying. Thus, we need to understand, under which restrictions on the distribution of X we have the desired property of h^+. It is worth mentioning that the regular variation of h^+ appears as a restriction in some further situations. Caravenna and Chaumont [4], for example, have imposed this restriction in proving an invariance principle for random walks conditioned to stay positive.

It is well known that $\mathbf{P}(\chi^+ \geq x)$ is regularly varying of index $-\alpha\rho$, if $\alpha\rho < 1$, and χ^+ is relative stable if $\alpha\rho = 1$. Then, using the result of Garsia and Lamperti [7], we have that h^+ is regularly varying for any random walk with $\alpha\rho \in (1/2, 1)$. It is also known, see Williamson [12], that if the tail of a positive random variable is regularly varying of some index less than $1/2$, then the mass-function of the corresponding renewal function is not regularly varying in general. Doney [5] has shown that the mass-function of renewal function is regularly varying under certain assumptions on the local probabilities of underlying random variables. In order to apply his result to ladder heights, we need an information on $\mathbf{P}(\chi^+ = x)$.

Theorem 2 *Suppose $X \in \mathcal{D}(\alpha, \beta)$ with $\alpha < 2$ and $|\beta| < 1$. If*

$$\mathbf{P}(X = x) = O\left(x^{-1}\mathbf{P}(X \ge x)\right), \tag{6}$$

then

$$\mathbf{P}(\chi^+ = x) \le C\frac{\mathbf{P}(\chi^+ \ge x)}{x}. \tag{7}$$

If, additionally, $\alpha(1-\rho) > 1/2$ or

$$\mathbf{P}(X = -x) = O\left(x^{-1}\mathbf{P}(X \le -x)\right), \tag{8}$$

then, as $x \to \infty$,

$$\mathbf{P}(\chi^+ = x) \sim \frac{\alpha\rho}{x}\mathbf{P}(\chi^+ \ge x). \tag{9}$$

Applying Doney's result mentioned above, we obtain the following statement.

Corollary 3 *If $X \in \mathcal{D}(\alpha, \beta)$ with $\alpha < 2$ and $|\beta| < 1$ and (6) holds, then, as $x \to \infty$,*

$$h^+(x) \sim \alpha\rho\frac{H^+(x)}{x}. \tag{10}$$

Consequently, in view of Theorem 1,

$$c_n\mathbf{P}(M_n = x) - m_{\alpha,\beta}(x/c_n) \to 0$$

uniformly in $x \ge ac_n$, $a > 0$.

To the best of our knowledge, the behaviour of $\mathbf{P}(\chi^+ = x)$ for oscillating random walks has been studied in the case $\beta = 1$ only. Namely, Bertoin and Doney [3] have proven the following result: If $\mathbf{E}\chi^- < \infty$, which is a particular case of $\beta = 1$, and the right tail of X is long-tailed, then

$$\mathbf{P}(\chi^+ = x) \sim \frac{1}{\mathbf{E}\chi^-}\mathbf{P}(X \ge x) \quad \text{as } x \to \infty. \tag{11}$$

In this special case the behaviour of $\mathbf{P}(\chi^+ = x)$ is resistant against all kinds of irregularity in the local structure of the distribution of X. But this is not true in the case when $\beta < 1$. We demonstrate it with the following example.

Example 4 Assume that

$$\mathbf{P}(X = x) = \frac{C2^{\gamma n}}{2^{n(\alpha+1)}}, \quad x \in [2^n,\ 2^n + 2^{(1-\gamma)n}/n], \quad n \ge 1$$

for some $\gamma \in (0, 1)$, and that $\mathbf{P}(X = x) = Cx^{-\alpha-1}$ for all other values of $x > 0$. The negative part of X is such that $\mathbf{P}(X \le -x) = O(x^{-\alpha})$ and $\beta < 1$. One can easily see, that $\mathbf{P}(X \ge x) \sim Cx^{-\alpha}$. But

$$\limsup_{x\to\infty} \frac{x\mathbf{P}(\chi^+ = x)}{\mathbf{P}(\chi^+ \ge x)} = \infty. \tag{12}$$

If, furthermore, $\alpha(1+\rho) < 1$, then

$$\limsup_{x\to\infty} \frac{xh^+(x)}{H^+(x)} = \infty. \tag{13}$$

We postpone the proof of these relations to the end of the paper.

The relation (12) shows that (9) and (7) can not be true if one simply removes the condition (6) from Theorem 2. And (13) shows that (10) can not be valid for all random variables $X \in \mathcal{D}(\alpha, \beta)$. Combining (13) with the second statement of Theorem 1, we see that the local limit theorem for M_n takes place not for all asymptotically stable random walks.

There is another interesting observation connected to Example 4. Williamson's counterexample to the local renewal theorem can be seen as a special case of (13) for random walks with positive increments. Since $\rho = 1$ for positive increments, the condition $\alpha(1+\rho) < 1$ changes to $\alpha < 1/2$. But we know that the local renewal theorem holds for all random walks with $\alpha > 1/2$. Therefore, the following conjecture seems to be quite plausible: (10) holds for all random walks with $\alpha(1+\rho) > 1$.

In conclusion we mention that we expect that analogous results are true in the non-lattice case. There one has to replace $\mathbf{P}(M_n = x)$ and $h^+(x)$ by $\mathbf{P}(M_n \in [x, x+1))$ and $H^+(x+1) - H^+(x)$ respectively.

2 Proofs

2.1 Some results from fluctuation theory

In this paragraph we collect some known results from the fluctuation theory for random walks.

We start with a representation for $\mathbf{P}(M_n = x)$, which will be used in the proof of Theorem 1.

Lemma 5 *For all* $n, x > 0$,

$$\mathbf{P}(M_n = x) = \sum_{k=0}^{n} \mathbf{P}(S_k = x, \tau^- > k)\mathbf{P}(\tau^+ > n-k). \tag{14}$$

Proof Denote $\theta_n := \min\{k \ge 0 : \ S_k = M_n\}$. By the Markov property,

$$\mathbf{P}(M_n = k, \theta_n = k) = \mathbf{P}(S_k = x, \theta_k = k)\mathbf{P}(\theta_{n-k} = 0).$$

Furthermore, it follows from the duality lemma for random walks that

$$\mathbf{P}(S_k = x, \theta_k = k) = \mathbf{P}(S_k = x, \tau^- > k).$$

Noting also that $\mathbf{P}(\theta_{n-k} = 0) = \mathbf{P}(\tau^+ > n-k)$, we have

$$\begin{aligned}\mathbf{P}(M_n = x) &= \sum_{k=0}^{n} \mathbf{P}(M_n = k, \theta_n = k) \\ &= \sum_{k=0}^{n} \mathbf{P}(S_k = x, \tau^- > k)\mathbf{P}(\tau^+ > n-k).\end{aligned}$$

Thus, the lemma is proved. □

Remark 6 Representation (14) has been derived in [2]. The authors have used it by proving an analog of (2) for random walks with $\mathbf{E}\chi^+ < \infty$. It is worth mentioning that [1] contains another representation for $\mathbf{P}(M_n = x)$, which is based on a recursive formula for the characteristic function of M_n. The latter is due to Nagaev [9].

We next note that h^+ can be written as an infinite sum of $\mathbf{P}(S_k = x, \tau^- > k)$. Indeed, using the duality lemma once again, we have

$$\begin{aligned}h^+(x) &= 1\{x=0\} + \sum_{k=1}^{\infty} \mathbf{P}\left(\chi_1^+ + \chi_2^+ + \cdots + \chi_k^+ = x\right) \\ &= 1\{x=0\} + \sum_{j=1}^{\infty} \mathbf{P}\left(S_j = x; S_j > S_0, S_j > S_1, \ldots, S_j > S_{j-1}\right) \\ &= 1\{x=0\} + \sum_{j=1}^{\infty} \mathbf{P}\left(S_j = x; \tau^- > j\right). \qquad (15)\end{aligned}$$

If $X \in \mathcal{D}(\alpha, \beta)$, then the norming sequence c_n can be specified by the relation

$$c_n := \inf\left\{u > 0 : u^{-2}\int_{-u}^{u} x^2 \mathbf{P}(X \in dx) > n\right\}, \quad n \geq 1.$$

If, furthermore, $\alpha < 2$ and $\beta > -1$, then $\mathbf{P}(X \geq x)$ is regularly varying of index $-\alpha$ and there exists a positive constant $C(\alpha, \beta)$ such that

$$\mathbf{P}(X \geq c_n) \sim \frac{C(\alpha, \beta)}{n}, \quad \text{as } n \to \infty.$$

As it has been mentioned in the introduction, see (4), $\mathbf{P}(S_n > 0) \to \rho$ for every asymptotically stable random walk. We have also mentioned, that ρ determines the asymptotic behaviour of τ^+, see (5). Besides this relation one has

$$\lim_{n\to\infty} \mathbf{P}(\tau^- > n)\mathbf{P}(\tau^+ > n) = \frac{\sin \pi\rho}{\pi}. \tag{16}$$

We next state a result, which establishes a connection between the tail of τ^+ and $H^+(x)$.

Lemma 7 *If $X \in \mathcal{D}(\alpha, \beta)$, then there exists a constant $C(\alpha, \beta)$ such that*

$$H^+(c_n) \sim \frac{C(\alpha, \beta)}{\mathbf{P}(\tau^+ > n)}. \tag{17}$$

This statement is contained implicitly in Lemma 13 of [11].

In our proofs we shall frequently use the following well-known properties of regularly varying sequences.

Lemma 8 *Let a_n be regularly varying of index γ.*

(i) *For every $\varepsilon > 0$,*

$$\frac{a_k}{a_n} = (k/n)^\gamma + o(1) \quad \text{as } n \to \infty$$

uniformly in $k \in [\varepsilon n, (1-\varepsilon)n]$.

(ii) *If $\gamma > -1$, then, for every $r > 0$,*

$$\sum_{k=1}^{rn} a_k \sim \frac{r^{1+\gamma}}{1+\gamma} n a_n \quad \text{as } n \to \infty.$$

(iii) *If $\gamma < -1$, then, for every $r > 0$,*

$$\sum_{k=rn}^{\infty} a_k \sim -\frac{r^{1+\gamma}}{1+\gamma} n a_n \quad \text{as } n \to \infty.$$

2.2 Small deviations for the maximum: Proof of (2)

Fix any $\varepsilon \in (0, 1)$. It is easy to see that

$$\mathbf{P}(\tau^+ > n) \le \frac{\sum_{k=0}^{\varepsilon n} \mathbf{P}(S_k = x, \tau^- > k)\mathbf{P}(\tau^+ > n-k)}{\sum_{k=0}^{\varepsilon n} \mathbf{P}(S_k = x, \tau^- > k)} \le \mathbf{P}(\tau^+ > n - \varepsilon n). \tag{18}$$

Using (15), we have

$$\sum_{k=0}^{\varepsilon n}\mathbf{P}(S_k = x, \tau^- > k) = \sum_{k=0}^{\infty}\mathbf{P}(S_k = x, \tau^- > k) - \sum_{\varepsilon n}^{\infty}\mathbf{P}(S_k = x, \tau^- > k)$$
$$= h^+(x) - \sum_{\varepsilon n}^{\infty}\mathbf{P}(S_k = x, \tau^- > k).$$

According to Lemma 20 from [11],

$$\mathbf{P}(S_k = x, \tau^- > k) \le C\frac{H^+(x)}{kc_k}, \quad x > 0. \tag{19}$$

Therefore,

$$\sum_{\varepsilon n}^{\infty}\mathbf{P}(S_k = x, \tau^- > k) \le CH^+(x)\sum_{\varepsilon n}^{\infty}\frac{1}{kc_k} \le C\varepsilon^{-1/\alpha}\frac{H^+(x)}{c_n},$$

where in the last step we have applied Lemma 8(iii) to the sequence $(nc_n)^{-1}$. Consequently,

$$h^+(x) - C\varepsilon^{-1/\alpha}\frac{H^+(x)}{c_n} \le \sum_{k=0}^{\varepsilon n}\mathbf{P}(S_k = x, \tau^- > k) \le h^+(x). \tag{20}$$

Applying Theorem 1.1 from [7] to h^+, we have

$$\liminf_{x\to\infty}\frac{xh^+(x)}{H^+(x)} = \alpha\rho.$$

This implies that

$$H^+(x) \le Cxh^+(x), \quad x > 0. \tag{21}$$

From this bound and (20) we conclude that

$$\left|\frac{1}{h^+(x)}\sum_{k=0}^{\varepsilon n}\mathbf{P}(S_k = x, \tau^- > k) - 1\right| \le C\varepsilon^{-1/\alpha}\frac{x}{c_n}.$$

Therefore,

$$\frac{1}{h^+(x)}\sum_{k=0}^{\varepsilon n}\mathbf{P}(S_k = x, \tau^- > k) \to 1$$

uniformly in $x \le \delta_n c_n$. Applying this to the inequalities in (18), we obtain

$$\liminf_{n\to\infty} \min_{x\le\delta_n c_n} \frac{\sum_{k=0}^{\varepsilon n} \mathbf{P}(S_k = x, \tau^- > k)\mathbf{P}(\tau^+ > n-k)}{h^+(x)\mathbf{P}(\tau^+ > n)} \ge 1 \tag{22}$$

and

$$\limsup_{n\to\infty} \max_{x\le\delta_n c_n} \frac{\sum_{k=0}^{\varepsilon n} \mathbf{P}(S_k = x, \tau^- > k)\mathbf{P}(\tau^+ > n-k)}{h^+(x)\mathbf{P}(\tau^+ > n)} \le (1-\varepsilon)^{-\rho}. \tag{23}$$

Using (19) once again, we get

$$\begin{aligned}
\sum_{\varepsilon n\le k\le n} \mathbf{P}(S_k = x, \tau^- > k)\mathbf{P}(\tau^+ > n-k) &\le CH^+(x) \sum_{\varepsilon n\le k\le n} \frac{\mathbf{P}(\tau^+ > n-k)}{kc_k} \\
&\le \frac{C\varepsilon^{-1-1/\alpha} H^+(x)}{nc_n} \sum_{j=0}^{n} \mathbf{P}(\tau^+ > j) \\
&\le C\varepsilon^{-1-1/\alpha} \frac{H^+(x)\mathbf{P}(\tau^+ > n)}{c_n},
\end{aligned}$$

where in the last two steps we have used Lemma 8 (i) and (ii) respectively. It follows now from (21) that

$$\frac{\sum_{\varepsilon n\le k\le n} \mathbf{P}(S_k = x, \tau^- > k)\mathbf{P}(\tau^+ > n-k)}{h^+(x)\mathbf{P}(\tau^+ > n)} \to 0 \tag{24}$$

uniformly in $x \le \delta_n c_n$. Combining (14) and (22)–(24), we obtain the desired relation.

2.3 Local limit theorem for the maximum: Proof of (3)

Let ε be any fixed number from the interval $(0, 1/2)$. Then, using the bound (see Lemma 19 in [11])

$$\mathbf{P}(S_k = x, \tau^- > k) \le \frac{C}{c_k}\mathbf{P}(\tau^- > k),\ \ x > 0,$$

and applying Lemma 8(ii) to the sequence $\mathbf{P}(\tau^+ > n)$, we get

$$\begin{aligned}
\sum_{k=(1-\varepsilon)n}^{n} \mathbf{P}(S_k = x, \tau^- > k)\mathbf{P}(\tau^+ > n-k) &\le \frac{C}{c_n}\mathbf{P}(\tau^- > n) \sum_{k=1}^{\varepsilon n} \mathbf{P}(\tau^+ > k) \\
&\le C\varepsilon^{1-\rho} \frac{n\mathbf{P}(\tau^- > n)\mathbf{P}(\tau^+ > n)}{c_n}.
\end{aligned}$$

In view of (16), the sequence $n\mathbf{P}(\tau^- > n)\mathbf{P}(\tau^+ > n)$ remains bounded. Therefore,

$$c_n \sum_{k=(1-\varepsilon)n}^{n} \mathbf{P}(S_k = x, \tau^- > k)\mathbf{P}(\tau^+ > n-k) \le C\varepsilon^{1-\rho} \tag{25}$$

for all $x > 0$.

Applying Theorem 5 of [11], we have, uniformly in $x > 0$,

$$\begin{aligned}
&\sum_{k=\varepsilon n}^{(1-\varepsilon)n} \mathbf{P}(S_k = x, \tau^- > k)\mathbf{P}(\tau^+ > n-k) \\
&\quad = \sum_{k=\varepsilon n}^{(1-\varepsilon)n} \frac{p_{\alpha,\beta}(x/c_k)}{c_k}\mathbf{P}(\tau^- > k)\mathbf{P}(\tau^+ > n-k) \\
&\qquad + o\left(\sum_{k=\varepsilon n}^{(1-\varepsilon)n} \frac{\mathbf{P}(\tau^- > k)\mathbf{P}(\tau^+ > n-k)}{c_k}\right),
\end{aligned} \tag{26}$$

where $p_{\alpha,\beta}$ denotes the density of the corresponding Levy meander.

Then, using Lemma 8(i), and taking into account the fact that $p_{\alpha,\beta}$ is uniformly continuous, we get

$$\begin{aligned}
&c_n \sum_{k=\varepsilon n}^{(1-\varepsilon)n} \frac{p_{\alpha,\beta}(x/c_k)}{c_k}\mathbf{P}(\tau^- > k)\mathbf{P}(\tau^+ > n-k) \\
&\sim \mathbf{P}(\tau^- > n)\mathbf{P}(\tau^+ > n) \sum_{k=\varepsilon n}^{(1-\varepsilon)n} p_{\alpha,\beta}\left(\frac{x}{c_n}\left(\frac{k}{n}\right)^{-1/\alpha}\right)\left(\frac{k}{n}\right)^{\rho-1-1/\alpha}\left(1-\frac{k}{n}\right)^{-\rho} \\
&\sim \frac{\sin \pi\rho}{\pi} \int_{\varepsilon}^{1-\varepsilon} p_{\alpha,\beta}\left(\frac{x}{c_n}v^{1/\alpha}\right) v^{\rho-1-1/\alpha}(1-v)^{-\rho}dv
\end{aligned} \tag{27}$$

uniformly in $x > 0$. Using the same arguments, one can easily get

$$\sum_{k=\varepsilon n}^{(1-\varepsilon)n} \frac{\mathbf{P}(\tau^- > k)\mathbf{P}(\tau^+ > n-k)}{c_k} \sim \frac{\sin \pi\rho}{\pi} \int_{\varepsilon}^{1-\varepsilon} v^{\rho-1-1/\alpha}(1-v)^{-\rho}dv. \tag{28}$$

Combining (26)–(28), we have

$$
\begin{aligned}
& c_n \sum_{k=\varepsilon n}^{(1-\varepsilon)n} \mathbf{P}(S_k = x, \tau^- > k)\mathbf{P}(\tau^+ > n-k) = \\
& (1+o(1))\frac{\sin \pi\rho}{\pi} \int_{\varepsilon}^{1-\varepsilon} p_{\alpha,\beta}\left(\frac{x}{c_n} v^{-1/\alpha}\right) v^{\rho-1-1/\alpha}(1-v)^{-\rho} dv + o(1),
\end{aligned}
$$

and the $o(1)$-terms are uniform in $x > 0$. Consequently, uniformly in $x > ac_n$,

$$
\begin{aligned}
\lim_{\varepsilon \to 0} \lim_{n\to\infty} \Bigg(& c_n \sum_{k=\varepsilon n}^{(1-\varepsilon)n} \mathbf{P}(S_k = x, \tau^- > k)\mathbf{P}(\tau^+ > n-k) \\
& - \int_0^1 p_{\alpha,\beta}\left(\frac{x}{c_n} v^{-1/\alpha}\right) v^{\rho-1-1/\alpha}(1-v)^{-\rho} dv \Bigg) = 0.
\end{aligned}
\tag{29}
$$

The finiteness of the integral for $x > ac_n$ follows from the boundedness of $p_{\alpha,\beta}$ and from the relation $p_{\alpha,\beta}(y) \sim Cy^{-\alpha-1}$, $y \to \infty$. The latter has been proven in [6].

Define

$$
N_x = \max\{n : \ c_n \le x\}.
$$

Lemma 9 *Assume that* $X \in \mathcal{D}(\alpha, \beta)$. *Then*

$$
\lim_{b\to 0} \lim_{x\to\infty} \frac{x}{H^+(x)} \sum_{k \ge bN_x} \mathbf{P}(S_k = x, \tau^- > k) = \alpha\rho.
$$

Proof Using (19) and Lemma 8(ii), we have

$$
\begin{aligned}
\sum_{k \ge N_x/b} \mathbf{P}(S_k = x, \tau^- > k) &\le CH^+(x) \sum_{k\ge N_x/b} \frac{1}{kc_k} \\
&\le Cb^{1/\alpha}\frac{H^+(x)}{c_{N_x}} \le Cb^{1/\alpha}\frac{H^+(x)}{x}.
\end{aligned}
\tag{30}
$$

Applying Theorem 5 of [11] once again, we have

$$
\sum_{bN_x \le k \le N_x/b} \mathbf{P}(S_k = x, \tau^- > k) = (1+o(1)) \sum_{bN_x \le k \le N_x/b} \frac{\mathbf{P}(\tau^- > k)}{c_k} p_{\alpha,\beta}(x/c_k).
$$

Repeating the arguments, which have been used in deriving (27), and noting that $c_{N_x} \sim x$, one can easily see that, as $x \to \infty$,

$$\sum_{k=bN_x}^{N_x/b} \frac{\mathbf{P}(\tau^- > k)}{c_k} p_{\alpha,\beta}(x/c_k) \sim \frac{N_x \mathbf{P}(\tau^- > N_x)}{c_{N_x}} \int_b^{1/b} v^{\rho-1-1/\alpha} p_{\alpha,\beta}(v^{-1/\alpha}) dv$$
$$\sim \frac{\sin \pi\rho}{\pi} \frac{1}{x\mathbf{P}(\tau^+ > N_x)} \int_b^{1/b} v^{\rho-1-1/\alpha} p_{\alpha,\beta}(v^{-1/\alpha}) dv.$$

Taking into account (17), we get

$$\sum_{k=bN_x}^{N_x/b} \mathbf{P}(S_k = x, \tau^- > k) \sim C(\alpha,\beta) \frac{H^+(x)}{x} \int_b^{1/b} v^{\rho-1-1/\alpha} p_{\alpha,\beta}(v^{-1/\alpha}) dv. \quad (31)$$

From (30) and (31) we conclude that

$$\lim_{b\to 0} \lim_{x\to\infty} \frac{x}{H^+(x)} \sum_{k \ge bN_x} \mathbf{P}(S_k = x, \tau^- > k) = C(\alpha,\beta). \quad (32)$$

The constant on the right hand side is finite, since

$$\int_0^\infty v^{\rho-1-1/\alpha} p_{\alpha,\beta}(v^{-1/\alpha}) dv = \alpha \int_0^\infty z^{-\alpha\rho} p_{\alpha,\beta}(z) dz < \infty.$$

To finish the proof it is sufficient to show that $C(\alpha,\beta) = \alpha\rho$ for some special random walk. We consider any $X \in \mathcal{D}(\alpha,\beta)$ with the following property: $\mathbf{P}(X = x)$ is regularly varying of index $-\alpha - 1$. From Lemma 7 of Jones [8] we have

$$\mathbf{P}(S_k = x, \tau^- > k) \le Ck\mathbf{P}(X = x)\mathbf{P}(\tau^- > k).$$

Thus, using Lemma 8(ii), we get

$$\sum_{k=1}^{bN_x} \mathbf{P}(S_k = x, \tau^- > k) \le C\mathbf{P}(X = x) \sum_{k=1}^{bN_x} k\mathbf{P}(\tau^- > k)$$
$$\le Cb^{1+\rho} N_x^2 \mathbf{P}(\tau^- > N_x)\mathbf{P}(X = x)$$
$$\le Cb^{1+\rho} N_x \mathbf{P}(\tau^- > N_x) N_x \mathbf{P}(X \ge x)/x$$
$$\le Cb^{1+\rho} \frac{H^+(x)}{x}.$$

Combining this bound with (32), we get

$$h^+(x) = \sum_{k=1}^{\infty} \mathbf{P}(S_k = x, \tau^- > k) \sim C(\alpha, \beta)\frac{H^+(x)}{x}.$$

Recalling that $H^+(x) = \sum_{y=0}^{x} h^+(y)$ is regularly varying of index $\alpha\rho$, $h^+(x) \sim \alpha\rho H^+(x)/x$. Thus, the proof of the lemma is finished. □

We now continue the proof of the local limit theorem. Assume first that $h^+(x)$ is regularly varying. Then $h^+(x) \sim \alpha\rho H^+(x)/x$. From this relation and Lemma 9 we infer that

$$\lim_{b\to 0}\lim_{x\to\infty} \frac{x}{H^+(x)} \sum_{k=1}^{bN_x} \mathbf{P}(S_k = x, \tau^- > k) = 0.$$

The latter yields

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} \frac{c_n}{H^+(c_n)} \sum_{k=1}^{\varepsilon n} \mathbf{P}(S_k = x, \tau^- > k) = 0$$

uniformly in $x \geq ac_n$. Using (17) once again, we have

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} c_n \sum_{k=1}^{\varepsilon n} \mathbf{P}(S_k = x, \tau^- > k)\mathbf{P}(\tau^+ > n-k) = 0. \tag{33}$$

Combining (25), (28), (29) and (33), we obtain

$$c_n\mathbf{P}(M_n = x) - \int_0^1 p_{\alpha,\beta}\left(\frac{x}{c_n}v^{-1/\alpha}\right) v^{\rho-1-1/\alpha}(1-v)^{-\rho}dv = o(1)$$

uniformly in $x \geq ac_n$. It was shown in [6] that the integral in the latter formula is equal to $m_{\alpha,\beta}(x/c_n)$. Thus, we have proven that the regular variation of h^+ is sufficient for the local limit theorem.

To get the reversed statement, we note that the convergence

$$c_n\mathbf{P}(M_n = [c_n]) \to m_{\alpha,\beta}(1)$$

implies, in view of (25), (28) and (29), that

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} c_n \sum_{k=1}^{\varepsilon n} \mathbf{P}(S_k = [c_n], \tau^- > k)\mathbf{P}(\tau^+ > n-k) = 0.$$

(Here $[x]$ denotes the integer part of x.) The latter is equivalent to

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{c_n}{H^+(c_n)}\sum_{k=1}^{\varepsilon n}\mathbf{P}(S_k=[c_n],\tau^->k)=0.$$

Substituting $x=[c_n]$, we get

$$\lim_{\varepsilon\to 0}\lim_{x\to\infty}\frac{x}{H^+(x)}\sum_{k=1}^{\varepsilon N_x}\mathbf{P}(S_k=x,\tau^->k)=0.$$

This, together with Lemma 9, implies that

$$h^+(x)\sim\alpha\rho\frac{H^+(x)}{x}.$$

Thus, the proof of the second part of Theorem 1 is completed.

2.4 Proof of Theorem 2

Using Wiener–Hopf factorisation one can get, see formula (5) in [3],

$$\mathbf{P}(\chi^+=x)=\sum_{y=0}^{\infty}\mathbf{P}(X=x+y)h^-(y). \tag{34}$$

It follows from (6) that

$$\max_{0\le y\le x}\mathbf{P}(X=jx+y)\le C\frac{\mathbf{P}(X\ge jx)}{jx}\le Cj^{-\alpha-1}\frac{\mathbf{P}(X\ge x)}{x},\quad j\ge 1. \tag{35}$$

Applying this bound to the summands on the right hand side of (34), we get

$$\begin{aligned}\mathbf{P}(\chi^+=x)&=\sum_{j=1}^{\infty}\sum_{y=0}^{x-1}\mathbf{P}(X=jx+y)h^-((j-1)x+y)\\&\le C\frac{\mathbf{P}(X\ge x)}{x}\sum_{j=1}^{\infty}j^{-\alpha-1}\left(H^-(jx-1)-H^-((j-1)x-1)\right).\end{aligned}$$

Since H^- is regularly varying of index $\alpha(1-\rho)$,

$$\lim_{x\to\infty}\frac{H^-(jx-1)-H^-((j-1)x-1)}{H^-(x)}=\left(j^{\alpha(1-\rho)}-(j-1)^{\alpha(1-\rho)}\right). \tag{36}$$

As a result we have

$$\mathbf{P}(\chi^+ = x) \le CH^-(x)\frac{\mathbf{P}(X \ge x)}{x}. \tag{37}$$

We next note that (34) yields

$$\mathbf{P}(\chi^+ \ge x) = \sum_{y=0}^{\infty} \mathbf{P}(X \ge x + y)h^-(y) = \sum_{j=1}^{\infty}\sum_{y=0}^{x-1} \mathbf{P}(X \ge jx + y)h^-((j-1)x + y)$$

Using (36) and the fact that $\mathbf{P}(X \ge x)$ is regularly varying, one can easily obtain

$$C_1\mathbf{P}(X \ge x)H^-(x) \le \mathbf{P}(\chi^+ \ge x) \le C_2\mathbf{P}(X \ge x)H^-(x).$$

Combining the lower bound with (37), we have

$$\mathbf{P}(\chi^+ = x) \le C\frac{\mathbf{P}(\chi^+ \ge x)}{x},$$

i.e., the first statement is proven.

We now turn to the proof of (9). First we note that $h^-(x)$ is regularly varying of index $-1-\alpha(1-\rho)$. Indeed, if $\alpha(1-\rho) > 1/2$, then it follows from the renewal theorem of Garsia and Lamperti. And if (8) holds, then, in view of (7), it is a consequence of Theorem 3 in [5].

Fix any $\varepsilon \in (0, 1)$. In view of (6),

$$\begin{aligned}\sum_{0 \le y < \varepsilon x} \mathbf{P}(X = x + y)h^-(y) &\le C\frac{\mathbf{P}(X \ge x)}{x}H^-(\varepsilon x)\\ &\le C\varepsilon^{\alpha(1-\rho)}\frac{\mathbf{P}(X \ge x)}{x}H^-(x).\end{aligned} \tag{38}$$

(In the last step we used the fact that H is regularly varying of index $\alpha(1-\rho)$.)

It is easy to see that $\mathbf{P}(X \ge ux | X \ge x) \to u^{-\alpha}$, i.e. given $X > x$, X/x converges weakly to the Pareto distribution. Moreover, as we have already proven, h^- is regularly varying. This implies that $h^-(ux - x)/h^-(x)$ is bounded on $(1+\varepsilon, \infty)$ and, moreover, converges to $(u-1)^{-1-\alpha(1-\rho)}$ pointwise. Therefore,

$$\begin{aligned}&\sum_{y \ge \varepsilon x} \mathbf{P}(X = x + y)h^-(y)\\ &= h^-(x)\mathbf{P}(X \ge x)\mathbf{E}\left[\frac{h^-(X - x)}{h^-(x)}1\{X \ge (1+\varepsilon)x\}\Big| X \ge x\right]\\ &= (1 + o(1))h^-(x)\mathbf{P}(X \ge x)\int_{1+\varepsilon}^{\infty}(u-1)^{-1-\alpha(1-\rho)}\alpha u^{-\alpha-1}du.\end{aligned} \tag{39}$$

Applying (38) and (39) to the right hand side of (34), and letting $\varepsilon \to 0$, we obtain

$$\mathbf{P}(\chi^+ = x) \sim h^-(x)\mathbf{P}(X \geq x) \int_1^\infty (u-1)^{-1-\alpha(1-\rho)} \alpha u^{-\alpha-1} du.$$

In particular, $\mathbf{P}(\chi^+ = x)$ is regularly varying, as a product of two regularly varying functions. Since $\mathbf{P}(\chi^+ \geq x)$ is also regularly varying, we conclude that

$$\mathbf{P}(\chi^+ = x) \sim \alpha\rho \mathbf{P}(\chi^+ \geq x)/x.$$

Thus, the proof of the theorem is completed.

2.5 Calculations related to Example 4

It follows from (34) that

$$\mathbf{P}(\chi^+ = 2^n + z) \geq \sum_{y=0}^{r_n - z} \mathbf{P}(X = 2^n + z + y) h^-(y),$$

where $r_n = 2^{(1-\gamma)n}/n$. And according to our choice of the law of X,

$$\mathbf{P}(\chi^+ = 2^n + z) \geq \frac{C 2^{\gamma n}}{2^{(\alpha+1)n}} H^-(r_n - z), \quad z < r_n.$$

Recalling that H^- is regularly varying of index $\alpha(1-\rho)$, we conclude that

$$\begin{aligned}\mathbf{P}(\chi^+ = 2^n + z) &\geq \frac{C 2^{\gamma n}}{2^{(\alpha+1)n}} \left(\frac{2^{(1-\gamma)n}}{n}\right)^{\alpha(1-\rho)} L^-(r_n) \\ &\geq C 2^{-n(1+\alpha\rho)} \frac{2^{\gamma n(1-\alpha(1-\rho))}}{n^{\alpha(1-\rho)}} L^-(r_n), \quad z < r_n/2.\end{aligned}$$

Using that $\mathbf{P}(\chi^+ \geq x)$ is also regularly varying, we get finally the bound

$$\mathbf{P}(\chi^+ = x) \geq C 2^{n\gamma'} \frac{\mathbf{P}(\chi^+ \geq x)}{x}, \quad x \in [2^n, 2^n + r_n/2], \ \gamma' < \gamma(1-\alpha(1-\rho)). \tag{40}$$

This implies (12).

We obtain (13) as a particular case of a more general observation, which can be seen as a generalisation of the well-known example of Williamson, see [12].

Assume that X is positive and that there exists a sequence $R_n < 2^n$ such that

$$\mathbf{P}(X = x) \geq \frac{2^{\gamma n}}{2^{(\alpha+1)n}} \ell(2^n), \quad x \in [2^n, 2^n + R_n]$$

and

$$\mathbf{P}(X = x) = \frac{\ell(x)}{x^{\alpha+1}}, \quad x \in (2^n + R_n, 2^{n+1})$$

for some $\gamma, \alpha \in (0, 1)$ and for some slowly varying function ℓ. One can easily verify that the additional restriction $R_n \ll 2^{(1-\gamma)n}$ yields that we can choose X in a such way that

$$\mathbf{P}(X \geq x) \sim \frac{\ell(x)}{x^{\alpha}} \quad \text{as } x \to \infty,$$

i.e., $X \in \mathcal{D}(\alpha, 1)$.

We next derive a lower bound for $\mathbf{P}(S_k = 2^n + R_n)$. It is clear that

$$\begin{aligned}\mathbf{P}(S_k = 2^n + R_n) &\geq k\mathbf{P}(X_1 \geq 2^n, S_k = 2^n + R_n)\\ &\geq k \min_{y \leq R_n} \mathbf{P}(X = 2^n + y)\mathbf{P}(S_{k-1} \leq R_n)\\ &= k\mathbf{P}(S_{k-1} < R_n)\frac{2^{\gamma n}}{2^{(\alpha+1)n}}\ell(2^n).\end{aligned}$$

Since $h^+(x) = \sum_{k=1}^{\infty} \mathbf{P}(S_k = x)$ in the case of positive random variables, we have the inequality

$$h^+(2^n + R_n) \geq \left(\sum_{k=1}^{\infty} k\mathbf{P}(S_{k-1} < R_n)\right)\frac{2^{\gamma n}}{2^{(\alpha+1)n}}\ell(2^n).$$

It follows from the convergence to a stable law, that $\mathbf{P}(S_{k-1} < R_n) \geq C > 0$ for all k such that $c_k \leq R_n$, say $k \leq T_n$. Then

$$\sum_{k=1}^{\infty} k\mathbf{P}(S_{k-1} < R_n) \geq CT_n^2 \geq \frac{C}{(\mathbf{P}(X \geq R_n))^2},$$

in the last step we have used the relation $T_n \sim 1/\mathbf{P}(X \geq R_n)$, which follows from the properties of the norming sequence c_n.

As a result we have

$$h^+(2^n + R_n) \geq C\frac{2^{\gamma n}}{2^{(\alpha+1)n}}R_n^{2\alpha}\frac{\ell(2^n)}{(\ell(R_n))^2} = C\frac{2^{(\alpha-1)n}}{\ell(n)}\frac{2^{\gamma n}R_n^{2\alpha}}{2^{2\alpha n}}\frac{(\ell(2^n))^2}{(\ell(R_n))^2}.$$

from this bound we conclude that

$$\limsup_{x\to\infty} \ell(x)x^{1-\alpha}h^+(x) = \infty \tag{41}$$

provided that R_n satisfies the condition

$$\frac{2^{(\gamma/2)n} R_n^{\alpha}}{2^{\alpha n}} \frac{\ell(2^n)}{\ell(R_n)} \to \infty. \tag{42}$$

If $\alpha > 1/2$ it is not possible. But if $\alpha < 1/2$, then we can choose $R_n = 2^{(1-\gamma)n-\delta n}$ with some $0 < \delta < \gamma(1-2\alpha)$.

We now come back to random variables which take values of both signs. It follows from (40) that (42) holds with $\alpha\rho$ instead of α and $R_n = r_n/2$ if $1-\alpha-\alpha\rho > 0$. Then, (41) yields (13).

We finish the paper with the following remark. The additional restriction $\alpha(1+\rho) < 1$ in (13) reflects the fact that the local behaviour of χ^+ is much smoother in the case when $\mathbf{P}(X < 0) > 0$ than in the case of positive summands [note that (41) holds without any additional assumption]. This effect appears due to convolution of $\mathbf{P}(X = x)$ with h^-, see formula (34). This gives rise to the following question: Is it possible, at least for some α and β with $\alpha\rho \le 1/2$, to infer the regular behaviour of h^+ from that of h^-?

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