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Stable Hamiltonian structures in dimension 3 are supported by open books

K. Cieliebak and E. Volkov

Abstract

We prove that every stable Hamiltonian structure on a closed oriented three-manifold is stably homotopic to one which is supported (with suitable signs) by an open book.

1. *Introduction*

In dimension 3, (cooriented) contact structures are closely related to open books via the following results, the first due to Thurston–Winkelnkemper [**24**] and the other ones due to Giroux [**13**].

(C1) Every open book supports a contact structure.
(C2) Any two contact structures supported by the same open book are connected by a contact isotopy supported by the open book.
(C3) Every contact structure is supported by an open book.
(C4) Two open books supporting the same contact structure are isotopic after finitely many stabilizations.

In this paper, we prove analogues of the first three results for stable Hamiltonian structures in dimension 3.

Theorem 1.1. *Every open book supports a stable Hamiltonian structure realizing a given cohomology class and given signs at the binding components.*

Theorem 1.2. *Any two stable Hamiltonian structures supported by the same open book in the same cohomology class and with the same signs at the binding components are connected by a stable homotopy supported by the open book.*

Theorem 1.3. *Every stable Hamiltonian structure on a closed oriented* 3*-manifold is stably homotopic to one which is supported by an open book.*

Definitions. Let us first explain the notions appearing in the statement; see [**8**] for more details and background.

Let M denote a closed oriented 3-manifold. A Hamiltonian structure (HS) on M is a closed nowhere zero 2-form ω on M. A stable Hamiltonian structure (SHS) on M is a pair (ω, λ)

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consisting of a closed 2-form ω and a 1-form λ such that

$$\lambda \wedge \omega > 0 \quad \text{and} \quad \ker(\omega) \subset \ker(d\lambda). \tag{1}$$

Note that in this case ω is nowhere zero (since $\lambda \wedge \omega > 0$), hence an HS. We say that λ *stabilizes* ω. An SHS induces a canonical *Reeb vector field* R generating $\ker(\omega)$ and normalized by $\lambda(R) = 1$. Note that since we are in dimension 3, the second condition in (1) is equivalent to

$$d\lambda = f\omega,$$

for some $f \in C^\infty(M)$ which we write as $f = d\lambda/\omega$. It is important to note that f is constant on the flowlines of the vector field R. Indeed, the last displayed equation implies that $df \wedge \omega = 0$; contraction with R gives $i_R df = 0$, so Cartan's formula implies $L_R f = 0$. A *stable homotopy* is a smooth family of SHS (ω_t, λ_t) *such that the cohomology class of* ω_t *remains constant*. (The condition on the cohomology class is natural for various reasons that are explained in [**8**].) An SHS (ω, λ) with $0 = [\omega] \in H^2(M)$ is called *exact*. (Throughout this paper, $H^*(M)$ always denotes de Rham cohomology, and $H^*(M, \partial M)$ denotes de Rham cohomology with compact support in $M \setminus \partial M$.) Note that each positive contact form λ (that is, satisfying $\lambda \wedge d\lambda > 0$) induces an exact stable Hamiltonian structure $(d\lambda, \lambda)$.

Stable Hamiltonian structures were introduced by Hofer and Zehnder [**15**] as a condition on hypersurfaces for which the Weinstein conjecture can be proved. Later, they attained importance as the structure on a manifold needed for the compactness result in symplectic field theory [**4**, **7**, **9**]. Further interest in stable Hamiltonian structures arises from the recent proof of the Weinstein conjecture in dimension 3 by Hutchings and Taubes [**16**] (see also Rechtman [**22**, **23**]), and from their relation to Mañé's critical values [**5**] and other dynamical properties [**6**, **19**]. Stable Hamiltonian structures also appear in work by Eliashberg, Kim and Polterovich [**10**] on contact non-squeezing, and by Wendl and coauthors on symplectic fillings [**17**, **18**, **20**, **26**].

An *open book decomposition* of M is a pair (B, π), where $B \subset M$ is an oriented link (called the *binding*) and $\pi : M \setminus B \to S^1$ is a fibration satisfying the following condition near B: Each connected component B_l of B has a tubular neighbourhood $S^1 \times D^2 \hookrightarrow M$ with orienting coordinates (ϕ, r, θ), where ϕ is an orienting coordinate along $S^1 \cong B_l$ and (r, θ) are polar coordinates on D^2, such that on $(S^1 \times D^2) \setminus B_l$ we have $\pi(\phi, r, \theta) = \theta$. It follows that the closure of each fibre $\pi^{-1}(\theta)$ is an embedded compact oriented surface Σ_θ (called a *page*) with boundary B.

We say that an SHS (ω, λ) is *supported by the open book* (B, π) if ω is positive on each fibre $\pi^{-1}(\theta)$. It follows that λ is nowhere vanishing on the binding. Thus, for every binding component B_l we have a sign $s(B_l)$ which is $+1$ if and only if λ induces the orientation of B_l as the boundary of a page. We emphasize that this *differs from the common notion in contact topology* where one requires that all signs are $+1$. The results (C1–4) above in the contact case have to be understood for this more restrictive notion to which we will refer as *positively supported*.

Discussion and examples. (1) Theorem 1.1 implies the following existence result (see [**20**, Proposition 2.6] or [**8**, Proposition 2.18]): Every closed oriented 3-manifold admits a stable Hamiltonian structure realizing a given cohomology class and a given homotopy class of oriented plane fields. Indeed, by the classical results of Lutz and Martinet (see, for example, [**12**]), any oriented plane field is homotopic to a contact structure. By (C3), the contact structure is positively supported by an open book (B, π). Now the existence result follows from Theorem 1.1 applied to the open book (B, π) and the easy observation that any two stable Hamiltonian structures positively supported by the same open book are homotopic as oriented plane fields.

(2) Any positive stabilization of an open book (B, π) positively supports a contact structure in the same isotopy class (cf. [**13**]). By taking successive positive stabilizations, we obtain countably many open books (B_i, π_i) positively supporting contact structures ξ_i, all the ξ_i being

isotopic (in particular homotopic as plane fields) to each other. Applying Theorem 1.1 (with positively supported) to each (B_i, π_i), we obtain countably many stable Hamiltonian structures, all of them homotopic as oriented plane fields and defining the same cohomology class η. The following question arises naturally: For $\eta \neq 0$, are these stable Hamiltonian structures all stably homotopic?

(3) Let us fix a closed oriented 3-manifold M and a cohomology class $\eta \in H^2(M)$. Theorems 1.1 and 1.2 provide a map that associates to each *decorated* (that is, with given signs at the binding components) open book decomposition of M the homotopy class of SHS supported by the open book with the given signs and representing the class η. By Theorem 1.3, this map is surjective, so homotopy classes of SHS representing η correspond to certain equivalence classes of decorated open books. Unfortunately, we do not understand at all this equivalence relation, that is, when two decorated open books support homotopic stable Hamiltonian structures. Note that for contact structures the corresponding equivalence relation on open books is described by (C4). This raises the question of whether there is a notion of stabilization and an analogue of (C4) for stable Hamiltonian structures (with given signs at the binding components).

Example 1.4 (see Section 8.1). The 3-sphere S^3 has a natural open book decomposition whose pages are annuli. According to Theorems 1.1 and 1.2, different choices of signs at the two boundary components give rise to four homotopy classes of SHS supported by this open book which we write as $\omega_{(+,+)}$, $\omega_{(+,-)}$, $\omega_{(-,+)}$, $\omega_{(-,-)}$ (suppressing the stabilizing 1-forms). The SHS $\omega_{(+,-)}$ and $\omega_{(-,+)}$ are both induced by negative contact forms and are stably homotopic. The SHS $\omega_{(+,+)}$ corresponds to the positive tight contact form and $\omega_{(-,-)}$ to the negative overtwisted contact form defining the same homotopy class of oriented plane fields. We conjecture that $\omega_{(+,+)}$ and $\omega_{(-,-)}$ are not stably homotopic. In fact, they should be distinguished by their rational symplectic field theory, which is known to vanish for overtwisted contact structures and to be non-vanishing for the standard contact structure on S^3. However, rational symplectic field theory is not known to be invariant under homotopies through stable Hamiltonian structures.

A similar discussion applies to $S^1 \times S^2$, see Section 8.2.

(4) Note that if an exact SHS is *positively* supported by an open book, then by (C1) and Theorem 1.2 it is stably homotopic to a positive contact structure (that is, to an SHS of the form $(d\lambda, \lambda)$ for a positive contact form λ). In [**8**], we give evidence for the following conjecture.

Conjecture 1.5. There exists a stable Hamiltonian structure on S^3 which is not stably homotopic to a positive contact structure.

This conjecture would imply that in Theorem 1.3 we cannot achieve 'positively supported': A stable Hamiltonian structure as in the conjecture would not be stably homotopic to one that is positively supported by an open book.

Example 1.6. Consider an exact SHS (ω, λ) for which λ defines a *confoliation*, that is, $\lambda \wedge d\lambda \geqslant 0$ (see [**11**]). Then (ω, λ) is stably homotopic to a positive contact structure. To see this, note that the confoliation condition is equivalent to $f = d\lambda/\omega \geqslant 0$. This allows us to achieve positive signs at all binding components of the open book in Theorem 1.3 (see Remark 7.2), so the resulting SHS is positively supported by the open book and hence stably homotopic to a positive contact structure. The stable homotopy can also be constructed explicitly as follows: Write $\omega = d\alpha$. Then $\lambda_t := \lambda + t\alpha,\ t \in [0, \varepsilon]$, defines for small $\varepsilon > 0$ a homotopy of stabilizing forms from λ to the positive contact form λ_ε, and $\big((1-t)\omega + t\,d\lambda_\varepsilon, \lambda_\varepsilon\big)$ yields a stable homotopy from $(\omega, \lambda_\varepsilon)$ to $(d\lambda_\varepsilon, \lambda_\varepsilon)$.

(5) An attempt to classify exact SHS up to homotopy leads to the following question in contact topology: *Given a decorated open book on a closed oriented* 3*-manifold, does there exist a positive/negative contact form* α *such that the SHS* $(d\alpha, \pm\alpha)$ *is supported by the open book with the given signs?* In view of the three theorems above, an affirmative answer to this question would reduce the homotopy classification of exact SHS to a question about contact structures. However, the answer to this question is not always affirmative (see Section 8) and seems to depend in a subtle way on the monodromy of the open book.

Sketch of proof. Theorems 1.1 and 1.2 are rather easy consequences of the techniques in [**8**] and will be proved in Sections 2 and 3.

The proof of Theorem 1.3 departs from a structure theorem proved in [**8**] (see Section 7.1): For each SHS (ω, λ) on a closed oriented 3-manifold M, we can modify the 1-form λ to obtain a new SHS, still denoted by (ω, λ), with the following property: $M = \bigcup_i N_i \cup \bigcup_j U_j$ is a union of compact regions such that (ω, λ) is T^2-invariant on $U_j \cong [0,1] \times T^2$, and $d\lambda = c_i\omega$ on N_i with constants $c_i \in \mathbb{R}$. We refer to the regions U_j as *integrable regions*, and to the regions N_i with $c_i = 0$ (respectively, > 0, < 0) as *flat* (*respectively*, *positive/negative contact*) *regions*.

On a flat region, we perturb and rescale λ to make it integral and obtain a fibration over S^1. On a (positive or negative) contact region, we use a relative version of Giroux's existence theorem (C3) above to produce an open book supporting the contact form λ (and hence the SHS (ω, λ)) which induces a fibration $T^2 \to S^1$ on each boundary torus. Finally, we use techniques from [**8**] to extend the SHS and open books over the integrable regions U_j to an SHS and supporting open book on M (Section 7).

To prove the relative version of (C3), we collapse a circle direction transverse to the Reeb direction ∂N_i to obtain a closed contact manifold $(\bar{N}_i, \bar{\lambda})$ (Section 6). Each boundary torus T_j gives rise to a transverse knot L_j. We use Giroux's existence theorem (C3) to find a supporting open book for $(\bar{N}_i, \bar{\lambda})$ and apply a result of Pavelescu [**21**] to braid the link $L = \bigcup L_j$ around its binding (Section 4). After standardizing $\bar{\lambda}$ near the resulting link (Section 5), we replace its components back by 2-tori to obtain the desired open book on N_i. □

2. *Existence of stable Hamiltonian structures on open books*

2.1. *Cohomology of mapping tori and open books*

For a continuous map $f : X \to X$ of a topological space, denote by $X_f := [0,1] \times X/(0,x) \sim (1, f(x))$ its mapping torus. Denote by $i : X \to X_f$ the inclusion $x \mapsto [0,x]$ and by $\pi : X_f \to S^1$ the projection onto $S^1 = \mathbb{R}/\mathbb{Z}$. There exists a long exact sequence in singular homology (see [**14**, Section 2.2])

$$\cdots \longrightarrow H_k(X) \xrightarrow{\mathbb{1}-f_*} H_k(X) \xrightarrow{i_*} H_k(X_f) \xrightarrow{\partial} H_{k-1}(X) \longrightarrow \cdots.$$

Moreover, the construction of this sequence shows that the boundary map ∂ is given by intersection with the fibre X over 0. Now suppose that f maps a subset $A \subset X$ to itself. Then the corresponding sequence in relative cohomology is

$$\cdots \longrightarrow H^k(X_f, A_f) \xrightarrow{i^*} H^k(X, A) \xrightarrow{\mathbb{1}-f^*} H^k(X, A) \xrightarrow{d} H^{k+1}(X_f, A_f) \longrightarrow \cdots.$$

To describe the coboundary map d, pick a function $\rho : (-\frac{1}{2}, \frac{1}{2}) \to \mathbb{R}$ with compact support and integral 1 and set

$$\Theta := \rho(\theta)\, d\theta \in \Omega^1(S^1).$$

Then $\pi^*\Theta$ is dual to the fibre over 0, and therefore d is given by the cup product with $[\pi^*\Theta] \in H^1(X_f)$.

Now suppose that $f : \Sigma \to \Sigma$ is an orientation-preserving diffeomorphism of a compact connected surface with boundary. Then $f^* = \mathbb{1}$ on $H^2(\Sigma, \partial\Sigma)$ and the exact sequence simplifies to

$$\begin{aligned} 0 \longrightarrow H^1(\Sigma_f, \partial\Sigma_f) &\xrightarrow{i^*} H^1(\Sigma, \partial\Sigma) \xrightarrow{\mathbb{1}-f^*} H^1(\Sigma, \partial\Sigma) \\ &\xrightarrow{\Theta\wedge} H^2(\Sigma_f, \partial\Sigma_f) \xrightarrow{i^*} H^2(\Sigma, \partial\Sigma) \longrightarrow 0. \end{aligned} \tag{2}$$

In particular, we have

$$\begin{aligned} H^1(\Sigma_f, \partial\Sigma_f) &\cong \ker(\mathbb{1} - f^*) \subset H^1(\Sigma, \partial\Sigma), \\ H^2(\Sigma_f, \partial\Sigma_f) &= V \oplus \Theta \wedge H^1(\Sigma, \partial\Sigma), \end{aligned} \tag{3}$$

where $V \subset H^2(\Sigma_f, \partial\Sigma_f)$ is a subspace such that the restriction $i^*|_V : V \to H^2(\Sigma, \partial\Sigma)$ is an isomorphism. Let us discuss the two terms in $H^2(\Sigma_f, \partial\Sigma_f)$. The first term is a one-dimensional $\mathbb{R}$-linear space generated by a preimage (under the map i^*) of a cohomology class in $H^2(\Sigma, \partial\Sigma)$ represented by a compactly supported 2-form on Σ of positive area. The second term is generated by compactly supported closed 1-forms on Σ spanning a complement of $\mathrm{im}(\mathbb{1} - f^*)$ in $H^1(\Sigma, \partial\Sigma)$. Each such 1-form α induces a closed 2-form

$$\Theta \wedge \alpha = d\theta \wedge (\rho(\theta)\alpha)$$

on Σ_f. Note that $\rho(\theta)\alpha$ is a global 1-form on Σ_f, but in general it is not closed. This completes our discussion of mapping tori.

Now consider an open book decomposition (B, π) of a closed oriented 3-manifold M with page Σ and binding B. Recall (see, for example, [**12**]) that to the open book decomposition we can associate a diffeomorphism $f : \Sigma \to \Sigma$ which equals the identity near $\partial\Sigma$ (called the *monodromy* and uniquely defined up to isotopy rel $\partial\Sigma$) such that M is obtained from the mapping torus Σ_f by collapsing $\partial\Sigma_f = S^1 \times \partial\Sigma$ to $\partial\Sigma = B$.

Let (ϕ_l, r_l, θ) be standard coordinates as above near each binding component B_l, $l = 1, \ldots, n$. Consider the following part of the exact cohomology sequence of the pair (M, B):

$$\mathbb{R}^n \cong H^1(B) \xrightarrow{d^*} H^2(M, B) \xrightarrow{j^*} H^2(M) \longrightarrow H^2(B) = 0, \tag{4}$$

where d^* is the connecting homomorphism and j^* is the map induced by the natural forgetful map.

Lemma 2.1. (a) *Any de Rham cohomology class $\eta \in H^2(M)$ has a representative of the form $d\theta \wedge \beta$ for some 1-form β with compact support in $M \setminus B$.*

(b) *Any de Rham cohomology class $\eta \in \ker(j^*) \subset H^2(M, B)$ has a representative of the form*

$$\sum_{l=1}^{n} c_l\, d(\sigma(r_l)\, d\phi_l),$$

with constants $c_l \in \mathbb{R}$ and a non-increasing function $\sigma : [0, 1] \to [0, 1]$ which equals 1 near 0 and 0 near 1.

Proof. Let $f : \Sigma \to \Sigma$ be the monodromy of the open book. Note that $H^2(M, B) = H^2(\Sigma_f, \partial\Sigma_f)$. For part (b), note that $\ker(j^*) = \mathrm{im}(d^*)$ in the exact sequence (4). Now $H^1(B_l)$

is generated by $[d\phi_l]$ and $d^*[d\phi_l]$ is represented by the 2-form $d(\sigma(r_l)\, d\phi_l)$. For part (a), recall that the space V in equation (3) was not uniquely defined. The above description of $\mathrm{im}(d^*)$ implies that we can take $V := d^*(H^1(B))$. The discussion of the term $\Theta \wedge H^1(\Sigma, \partial\Sigma)$ after equation (3) concludes the proof. $\square$

2.2. T^2-*Invariant stable Hamiltonian structures*

In this subsection, we discuss a very special class of SHS that will appear repeatedly throughout the paper. Let I be an interval in $\mathbb{R}$ (open or closed or half open). Consider $I \times T^2$ with coordinates (r, θ, ϕ) and the T^2-action by shift in (θ, ϕ). We orient $I \times T^2$ by the volume form $dr \wedge d\theta \wedge d\phi$.

Hamiltonian structures. For an *immersion* $h = (h_1, h_2) : I \to \mathbb{R}^2$, consider the T^2-invariant 1-form

$$\alpha_h := h_1(r)\, d\theta + h_2(r)\, d\phi,$$

and the 2-form

$$\omega_h := d\alpha_h = h_1'(r)\, dr \wedge d\theta + h_2'(r)\, dr \wedge d\phi.$$

Since h is an immersion, ω_h is an HS.

Note that α_h is a positive contact form if and only if h avoids the origin and always turns clockwise. To see this, we are viewing $\mathbb{R}^2$ as $\mathbb{C}$. Then the contact condition $\alpha_h \wedge d\alpha_h > 0$ reads as $\langle h, ih'\rangle > 0$, where $\langle\cdot,\cdot\rangle$ denotes the standard scalar product on $\mathbb{R}^2$. Writing $h = \rho(r)\, e^{i\sigma(r)}$, we find $\langle h, ih'\rangle = -|h|^2\sigma'(r)$, so the contact condition is equivalent to $\sigma' < 0$.

Stabilizing 1-*forms.* For another path (now not necessarily an immersion) $g = (g_1, g_2) : I \to \mathbb{R}^2$, consider a 1-form

$$\lambda_g := g_1(r)\, d\theta + g_2(r)\, d\phi,$$

we have

$$d\lambda_g = g_1'(r)\, dr \wedge d\theta + g_2'(r)\, dr \wedge d\phi, \quad \lambda_g \wedge \omega_h = (h_1' g_2 - h_2' g_1)\, dr \wedge d\theta \wedge d\phi.$$

So λ_g stabilizes ω_h if and only if

$$g_1' h_2' - g_2' h_1' = 0, \quad h_1' g_2 - h_2' g_1 > 0.$$

Viewing $\mathbb{R}^2$ as $\mathbb{C}$, this can also be written as

$$\langle g', ih'\rangle = 0, \quad \langle g, ih'\rangle > 0. \tag{5}$$

We turn to the question of stabilizability. Given h, when does there exist g such that λ_g stabilizes ω_h and g has prescribed values near ∂I? It turns out that if the *slope function* $h'/|h'| : I \to S^1 \subset \mathbb{C}$ is not constant, then the answer is always positive. The following result is proved in [**8**].

Proposition 2.2. *Let $h_t : [0,1] \to \mathbb{C}$, $t \in [a,b]$ be a homotopy of immersions such that for each t the slope function $h_t'/|h_t'|$ is not constant on $[\varepsilon, 1-\varepsilon]$. Let $\bar{g}_t : [0,\varepsilon] \cup [1-\varepsilon, 1]$, $t \in [a,b]$ be a homotopy such that $(h_t, \bar{g}_t)$ satisfies* (5) *on $[0,\varepsilon] \cup [1-\varepsilon,1]$ for all $t \in [a,b]$. Then there exists a homotopy g_t which agrees with $\bar{g}_t$ on $[0,\varepsilon] \cup [1-\varepsilon,1]$ such that (h_t, g_t) satisfies* (5) *on $[0,1]$ for all $t \in [a,b]$.*

We will refer to an HS ω_h, respectively, an SHS (ω_h, λ_g) as above as a T^2-*invariant HS, respectively, SHS*. (This is a slight abuse of language because this condition is slightly more restrictive than just T^2-invariance.)

2.3. *Proof of Theorem* 1.1

Consider an open book (B, π), a function $s : \pi_0(B) \to \{+, -\}$ and a cohomology class $\eta \in H^2(M)$. To prove Theorem 1.1, we proceed in two steps: First we construct some SHS supported by (B, π) whose Reeb vector field and stabilizing 1-form are given by ∂_θ, respectively, $d\theta$ outside a neighbourhood of the binding; in the second step, we arrange the desired cohomology class η.

Step 1. Since we are in dimension 3, stable Hamiltonian structures (ω, λ) are in one-to-one correspondence with triples (R, V, λ) consisting of a vector field R, a volume form V and a 1-form λ such that

$$L_R V = 0, \quad \lambda(R) = 1, \quad i_R\, d\lambda = 0.$$

(In one direction, set $\omega = i_R V$, and in the other direction $V = \lambda \wedge \omega$.) We begin by constructing the volume form V. Pick a positive area form $\tilde{\omega}$ on the page Σ satisfying $\tilde{\omega} = d\phi \wedge r\, dr$ in standard coordinates (ϕ, r, θ) near each binding component B_l. A simple application of Moser's trick shows that after a compactly supported isotopy of the monodromy map f, we may assume that $f^*\tilde{\omega} = \tilde{\omega}$. Thus, $\tilde{\omega}$ induces a maximally non-degenerate form $i_*\tilde{\omega}$ on $M \setminus B$ whose kernel is spanned by the natural vector field ∂_θ, and which agrees with $d\phi \wedge r\, dr$ near each binding component B_l. Therefore, the volume form $d\theta \wedge i_*\tilde{\omega}$ on $M \setminus B$ coincides with

$$V_{\mathrm{loc}} := d\theta \wedge d\phi \wedge r\, dr$$

near B_l and thus extends by V_{loc} to a volume form V on M. Note that the vector field ∂_θ on $M \setminus B$ preserves the form $V|_{M\setminus B} = d\theta \wedge i_*\tilde{\omega}$, so the triple $(\partial_\theta, V|_{M\setminus B}, d\theta)$ defines an SHS on $M \setminus B$.

Unfortunately, the vector field ∂_θ on $M \setminus B$ extends as 0 over B. So we need to modify ∂_θ in a neighbourhood of the binding. This works as follows. Let $W_l = \{r \leqslant r_0\}$ denote a neighbourhood of B_l where $V = V_{\mathrm{loc}}$. We set $R = s(B_l)\partial_\phi + \partial_\theta$ (so $\omega = s(B_l) r\, dr \wedge d\theta - r\, dr \wedge d\phi$) and $\lambda = s(B_l)\, d\phi$ near B_l. We set $R = \partial_\theta$ and $\lambda = d\theta$ near ∂W_l. We extend R to the whole of W_l such that it is always linear on the tori $\{r = \mathrm{const}\}$ (that is, of the form $a(r)\partial_\theta + b(r)\partial_\phi$) and thus preserves the volume form V_{loc}. Moreover, we require that the ∂_θ component of R is always positive. We apply Proposition 2.2 (for the constant homotopy) to find λ stabilizing $i_R V_{\mathrm{loc}}$ and respecting the conditions near B_l and ∂W_l. This way we have constructed an SHS $(i_R V, \lambda)$ supported by (B, π) such that λ restricts to B_l positively or negatively according to $s(B_l)$. Moreover, $R = \partial_\theta$ and $\lambda = d\theta$ outside the neighbourhood $W := \bigcup_{l=1}^n W_l$ of B.

Step 2. We still have to take care of the cohomology class. Set $\tilde{\eta} := [i_R V] \in H^2(M)$. According to Lemma 2.1(a), the cohomology class $\eta - \tilde{\eta}$ can be represented by $d\theta \wedge \beta$ with a 1-form β supported in $M \setminus B$. By modifying β slightly if necessary, we can achieve that its support is contained in $M \setminus W$. Define the closed 2-form

$$\omega := i_R V + d\theta \wedge \beta.$$

Its restriction to each page agrees with that of $i_R V$ and is hence a positive area form. Moreover, ω represents the desired cohomology class:

$$[\omega] = [i_R V] + [d\theta \wedge \beta] = \tilde{\eta} + (\eta - \tilde{\eta}) = \eta.$$

We claim that the 1-form λ from Step 1 stabilizes ω. Indeed, on $M \setminus \mathrm{supp}(\beta)$ we have that $\omega = i_R V$ and so λ stabilizes ω by construction. On $\mathrm{supp}(\beta) \subset M \setminus W$ the vector field R and the 1-form λ are given by

$$R|_{\mathrm{supp}(\beta)} = \partial_\theta, \quad \lambda|_{\mathrm{supp}(\beta)} = d\theta.$$

In particular, we have $d\lambda = 0$ and

$$\lambda \wedge \omega = d\theta \wedge (i_{\partial_\theta} V + d\theta \wedge \beta) = V.$$

This shows that (ω, λ) is an SHS supported by (B, π) with the given signs and thus completes the proof of Theorem 1.1.

REMARK 2.3. Note that the SHS (ω, λ) constructed in the preceding proof has the following property: Near each component B_l of the binding, (ω, λ) is T^2-invariant with respect to the T^2 action given by shifts in ϕ and θ.

3. *Uniqueness of stable Hamiltonian structures on open books*

3.1. *Hamiltonian structures on mapping tori*

As before, we denote by θ the coordinate on S^1, and by $d\theta$, the corresponding 1-form as well as its pullback under the projection $X \to S^1$ of circle bundles.

LEMMA 3.1. *Let Σ be a compact oriented surface with boundary and let $X \to S^1$ be a fibration over the circle with fibre Σ. Let (ω_0, λ_0) and (ω_1, λ_1) be two SHS on X with the following properties:*

(i) *on some neighbourhood N of ∂X, we have $\omega_0 = \omega_1$ and $\lambda_0 = c_0\, d\theta$, $\lambda_1 = c_1\, d\theta$ for some positive constants $c_0,\ c_1$;*
(ii) *$d\theta \wedge \omega_0 > 0$ and $d\theta \wedge \omega_1 > 0$;*
(iii) *$[\omega_1 - \omega_0] = 0 \in H^2(X, \partial X)$.*

Then (ω_0, λ_0) and (ω_1, λ_1) are homotopic via a stable homotopy (ω_t, λ_t) satisfying $d\theta \wedge \omega_t > 0$ and $[\omega_t - \omega_0] = 0 \in H^2(X, \partial X)$ for all t. Moreover, on N we have $\omega_t = \omega_0 = \omega_1$ and $\lambda_t = \rho(t)\, d\theta$ for a positive function $\rho : [0, 1] \to \mathbb{R}$ with $\rho(0) = c_0$ and $\rho(1) = c_1$.

Proof. As both λ_0 and $d\theta$ stabilize ω_0, we can homotope (ω_0, λ_0) to $(\omega_0, d\theta)$ by interpolating linearly between λ_0 and $d\theta$. Since on N we have $\lambda_0 = c_0\, d\theta$, this homotopy has the desired form when restricted to N. Similarly for (ω_1, λ_1). So it remains to homotope $(\omega_0, d\theta)$ to $(\omega_1, d\theta)$. Set

$$\omega_t := (1-t)\omega_0 + t\omega_1, \quad t \in [0, 1].$$

This a homotopy of closed 2-forms rel ∂X joining ω_0 and ω_1. Now $d\theta \wedge \omega_0 > 0$ and $d\theta \wedge \omega_1 > 0$ imply that $d\theta \wedge \omega_t > 0$ for all t, so in particular $\{(\omega_t, d\theta)\}_{t\in[0,1]}$ is a homotopy of SHS. The homological condition holds since $[\omega_t - \omega_0] = [t(\omega_1 - \omega_0)] = 0 \in H^2(X, \partial X)$. □

3.2. *Standardization near the binding*

We will need the following result from [8].

COROLLARY 3.2. *Let (ω, λ) be an SHS on a closed 3-manifold M and set $f := d\lambda/\omega$. Let $a \in \mathrm{im}(f)$ be any (singular or regular) value of f. Then there exists a stabilizing form $\tilde{\lambda}$ for ω such that $\tilde{f} := d\tilde{\lambda}/\omega$ satisfies $\tilde{f} \equiv a$ on an open neighbourhood of $f^{-1}(a)$. Moreover, $\tilde{f} = \sigma \circ f$ for a smooth function $\sigma : \mathbb{R} \to \mathbb{R}$.*

For the remainder of this section, the following terminology will prove useful. Consider a solid torus $U := S^1 \times D^2$ with orienting coordinates (ϕ, r, θ), where (r, θ) are polar coordinates on the disc $D^2 = \{r \leqslant b\}$ for some $b > 0$.

Definition 3.3. Let $s \in \{+,-\}$ be a sign. We call an SHS (ω, λ) on $S^1 \times D^2$ *s-special* (in words: positively or negatively special) if

$$\omega = r\,dr \wedge (s\,d\theta - d\phi),$$

and $\lambda(\partial_\theta + s\partial_\phi)$ is a positive constant.

The following remark will simplify the construction of s-special SHS.

Remark 3.4. If for a positive real number C the SHS $(C\omega, \lambda)$ is s-special on $S^1 \times D^2$, then we can stably homotope (ω, λ) rel $\partial(S^1 \times D^2)$, keeping the same kernel foliation and the same λ, to an SHS that is s-special on a smaller solid torus. To see this, note that in the notation of Section 2.2 we have $\omega = (1/C)\omega_h$ for the immersion $h(r) := \frac{1}{2}r^2(s,-1)$. Take a diffeomorphism ρ of $(0, b]$ supported away from b and such that for some $\varepsilon > 0$, we have $\rho(r) = C^{1/2}r$ for $r \leqslant \varepsilon$. Since the linear interpolation between any two orientation-preserving diffeomorphisms of $(0, b]$ supported away from the b is running through diffeomorphisms of the same type, we set $\rho_t(r) := (1-t)r + t\rho(r)$ and $h_t := h \circ \rho_t$. Now $\omega_t = (1/C)\omega_{h_t}$, $t \in [0,1]$, is the required homotopy, since $\omega_0 = \omega$ and (ω_1, λ) is s-special on $\{r \leqslant \varepsilon\}$.

Now we show that an SHS can be deformed to be special near the binding of an open book, or more generally, near any finite collection of periodic orbits.

Proposition 3.5 (Standardization near the binding). *Let (ω, λ) be an SHS supported by an open book (B, π). Then there exists a stable homotopy of SHS (ω_t, λ_t) supported by (B, π) such that $(\omega_0, \lambda_0) = (\omega, \lambda)$ and (ω_1, λ_1) is $s(B_l)$-special near each binding component B_l.*

Proof. Let $f := d\lambda/\omega$. Recall from Section 1 that f is constant on Reeb orbits, so in particular on binding components. For each binding component B_l, we distinguish three cases.

Case 1: $f \equiv a \neq 0$ on a neighbourhood of B_l.

Case 2: $f \equiv 0$ on a neighbourhood of B_l.

Case 3: f is not constant on any neighbourhood of B_l.

In Case 3, we apply Corollary 3.2 to the value $a := f(B_l)$. So we find a new stabilizing 1-form $\tilde\lambda$ for ω such that $\tilde f = d\tilde\lambda/\omega$ is constant on a neighbourhood of B_l. Since ω has not changed, $(\omega, \tilde\lambda)$ is still supported by (B, π) and so are the SHS $(\omega_t, \lambda_t) = (\omega, (1-t)\lambda + t\tilde\lambda)$, $t \in [0,1]$. Thus, Case 3 reduces to Cases 1 and 2, which are treated in Lemmas 3.8 and 3.11.

Note that in Case 1 the sign of a coincides with the sign of λ as a contact form. If $a > 0$, then λ is positive. We apply Lemma 3.8 to the form λ on a neighbourhood U of $\gamma = B_l$ and note that the sign of $T = \int_\gamma \lambda$ is just $s(B_l)$. This way we obtain a homotopy rel ∂U of positive contact forms $\{\lambda_t\}_{\{t \in [0,1]\}}$ on U from $\lambda_0 = \lambda$ to λ_1 which agrees near B_l with a positive multiple of $r^2 d\theta + (1-r^2)\,d\phi$ if $s(B_l) = +$, or $-r^2 d\theta - (1+r^2)\,d\phi$ if $s(B_l) = -$. Then $(\omega_t = (1/a)\,d\lambda_t, \lambda_t)$ on U defines a stable homotopy rel ∂U supported by (B, π).

If $a < 0$, then λ is negative. As before, the sign of $T = \int_\gamma \lambda$ is just $s(B_l)$. Let Rev_ϕ denote the map on a small tubular neighbourhood U of B_l sending ϕ to $-\phi$. This is just reversing the direction along this binding component. Then the pullback $\mathrm{Rev}_\phi^*\lambda$ is a positive contact form. Moreover, $d\theta \wedge d\lambda < 0$ (since a is negative) implies that $d\theta \wedge d(\mathrm{Rev}_\phi^*\lambda) > 0$. This means that we can apply Lemma 3.8 to the form $\tilde\lambda := \mathrm{Rev}_\phi^*\lambda$ with $T = \int_\gamma \tilde\lambda$ having the sign opposite to $s(B_l)$. Applying Rev_ϕ^* to the homotopy provided by Lemma 3.8, we obtain a homotopy rel ∂U of negative contact forms $\{\lambda_t\}_{\{t \in [0,1]\}}$ on U from $\lambda_0 = \lambda$ to λ_1 which agrees near B_l with a positive multiple of $-r^2 d\theta + (1+r^2)\, d\phi$ if $s(B_l) = +$, or $r^2 d\theta - (1-r^2)\, d\phi$ if $s(B_l) = -$. Then $(\omega_t = (1/a)\, d\lambda_t, \lambda_t)$ on U defines a stable homotopy rel ∂U supported by (B, π).

In Case 2, Lemma 3.11 provides a stable homotopy (ω_t, λ_t) on U rel ∂U such that near B_l, λ_1 is a positive multiple of $s\, d\phi$ and ω_1 is a positive multiple of $r\, dr \wedge (s\, d\theta - d\phi)$, where $s = s(B_l)$. Hence, for both signs of a in Case 1, and also in Case 2, we get that ω_1 equals near B_l a positive multiple of $r\, dr \wedge (s\, d\theta - d\phi)$. Moreover, the 1-forms $r^2 d\theta + (1-r^2)\, d\phi$ if $s = +$, $-r^2 d\theta - (1+r^2)\, d\phi$ if $s = -$, and $s\, d\phi$ for both s all evaluate to 1 on the vector field $\partial_\theta + s\partial_\phi$. Hence, by Remark 3.4 we can deform the SHS further to make it $s(B_l)$-special. □

Corollary 3.6 (Standardization near periodic orbits). *Let (ω, λ) be an SHS on a closed 3-manifold M and $\gamma_1, \ldots, \gamma_k$ periodic orbits of the Reeb vector field. Let $s_1, \ldots, s_k$ be a finite sequence of signs. Then there exists a stable homotopy of SHS (ω_t, λ_t) having the γ_i as periodic orbits such that $(\omega_0, \lambda_0) = (\omega, \lambda)$ and (ω_1, λ_1) is s_i-special near the orbits $\{\gamma_i\}_{i=1,\ldots,k}$.*

Proof. It suffices to consider the case of one periodic orbit γ and one sign $s \in \{-, +\}$. If $f := d\lambda/\omega$ is not constant on any neighbourhood of γ, then we modify λ as in the preceding proof. So we may assume that f is constant on a neighbourhood $U \cong S^1 \times D^2$ of γ.

Assume that $s = +$. Let ϕ be the coordinate along $\gamma = S^1 \times \{0\}$ and (r, θ) be polar coordinates on D^2. For any $k \in \mathbb{Z}$, the map $\pi_k(\phi, r, \theta) := k\phi + \theta$ (for $r > 0$) defines an open book decomposition near γ. A short computation shows that the Reeb vector field of (ω, λ) is transverse to the pages of this open book for k sufficiently large. Applying Lemmas 3.8 and 3.11 to this open book decomposition on U yields the result.

If $s = -$. Let ϕ be the coordinate along $\gamma = S^1 \times \{0\}$ in the reversed direction. Let r, θ, π_k be as above. The Reeb vector field of (ω, λ) is transverse to the pages of this open book for k negative and sufficiently large in the absolute value. □

For the remainder of this subsection, we consider a neighbourhood U of a component γ of the binding. We fix coordinates (ϕ, r, θ) on

$$U \cong S^1 \times D^2$$

supported by the open book, so ϕ is the coordinate along $\gamma = S^1 \times \{0\}$ and (r, θ) are polar coordinates on D^2 such that the open book projection is given for $r > 0$ by $\pi(\phi, r, \theta) = \theta$. We will also sometimes use cartesian coordinates (x, y) on D^2. Note that an SHS (ω, λ) on U is supported by the open book if and only if

$$d\theta \wedge \omega > 0 \quad \text{on the set } \{r > 0\}. \tag{6}$$

(It follows that the 3-form extends continuously to $r = 0$ but may become zero there.) We consider an SHS (ω, λ) satisfying (6) with $f = d\lambda/\omega$ being *constant*.

In subsequent constructions, we will repeatedly use the following technical lemma.

Lemma 3.7. *For all $\delta, \varepsilon > 0$, there exists a smooth function $\rho : [0, \delta] \to [0, 1]$ with the following properties: ρ is non-increasing, constant 1 in a neighbourhood of 0, constant 0 in a neighbourhood of δ, and*

$$|x\rho'(x)| < \varepsilon,$$

for all $x \in [0, \delta]$.

Proof. The function $f(x) := \varepsilon \ln(\delta/x)$ satisfies $x\, f'(x) = -\varepsilon$, $f(\delta) = 0$ and $f(\delta\, e^{-1/\varepsilon}) = 1$. Now shift the function $\max(f, 1)$ slightly to the left, extend by 0 to the right and smoothen it. ☐

The contact case.

LEMMA 3.8. *Let λ be a positive contact form on $U = S^1 \times D^2$ satisfying (6) (with $\omega = d\lambda$). Then there exists a homotopy rel ∂U of contact forms λ_t satisfying (6) such that $\lambda_0 = \lambda$ and $\lambda_1 = |T|\lambda_{\rm st}$ near $\gamma = S^1 \times \{0\}$, where $T = \int_\gamma \lambda$ and*

$$\lambda_{\rm st} := \begin{cases} r^2 d\theta + (1 - r^2)\, d\phi & \text{if } T > 0, \\ -r^2 d\theta - (1 + r^2)\, d\phi & \text{if } T < 0. \end{cases}$$

As preparation for the proof, let us derive under which conditions the interpolation between two contact forms is again contact.

LEMMA 3.9. *Let μ, λ be two positive contact forms with Reeb vector fields R_μ, R_λ on a compact oriented 3-manifold M (possibly with boundary). Fix a metric on M to define norms of differential forms. Let $f : M \to [0, 1]$ be a smooth function. Then*

$$\tilde\lambda := (1 - f)\lambda + f\mu$$

is again a positive contact form provided that $\lambda(R_\mu) \geqslant 0$, $\mu(R_\lambda) \geqslant 0$, and

$$|df|\, |\lambda|\, |\mu - \lambda| < \tfrac{1}{2} \min(\lambda \wedge d\lambda, \mu \wedge d\mu). \tag{7}$$

Proof. We compute

$$d\tilde\lambda = df \wedge (\mu - \lambda) + (1 - f)\, d\lambda + f d\mu,$$
$$\tilde\lambda \wedge d\tilde\lambda = (1 - f)^2 \lambda \wedge d\lambda + f^2 \mu \wedge d\mu + f(1 - f)[\lambda \wedge d\mu + \mu \wedge d\lambda] + df \wedge \mu \wedge \lambda.$$

On the right-hand side of the last equation, the sum of the first two terms is greater than or equal to $\frac{1}{2} \min(|\lambda \wedge d\lambda|, |\mu \wedge d\mu|)$. The third term in non-negative under the hypothesis $\lambda(R_\mu) \geqslant 0$, $\mu(R_\lambda) \geqslant 0$. The last term can be rewritten as $df \wedge (\mu - \lambda) \wedge \lambda$, so its norm is at most $|df|\, |\lambda|\, |\mu - \lambda|$ and the lemma follows. ☐

In the following arguments, the role of M is played by $S^1 \times D^2$ and we choose the standard Euclidean metric as the reference one. The corresponding Riemannian volume form is $r\, dr \wedge d\phi \wedge d\theta$ and we identify 3-forms with smooth functions via this form.

LEMMA 3.10. *Let λ and μ be two positive contact forms on $S^1 \times D^2$. Let ϕ be the angular coordinate on S^1 and (r, θ) be polar coordinates on D^2. Let $\sigma : [0, 1] \to \mathbb{R}$ be a radial function, constant near $r = 0$ and supported in $[0, \delta]$, with $|r\sigma'(r)| \leqslant \varepsilon$. Define the 1-form $\tilde\lambda := (1 - \sigma(r))\lambda + \sigma(r)\mu$.*

(i) *If the core circle $\{r = 0\}$ (oriented by $d\phi$) is a common Reeb orbit for λ and μ and $\lambda = \mu$ along $\{r = 0\}$, then $\tilde\lambda$ is again contact for δ, ε sufficiently small.*

(ii) *If $d\theta \wedge d\lambda \geqslant \beta > 0$, $d\theta \wedge d\mu \geqslant \beta > 0$ for some constant $\beta > 0$ and $dr \wedge d\theta \wedge (\mu - \lambda) = O(r)$ near $r = 0$, then $\tilde\lambda$ satisfies $d\theta \wedge d\tilde\lambda > 0$ for δ, ε sufficiently small.*

Proof. For (i), let us check the conditions in Lemma 3.9 (with $f=\sigma$ and M the region $\{r \leqslant \delta\}$). Since $\lambda=\mu$ and $R_\lambda = R_\mu$ at $r=0$, the conditions $\lambda(R_\mu)\geqslant 0$ and $\mu(R_\lambda)\geqslant 0$ are satisfied for δ sufficiently small. For condition (7), set

$$C := \tfrac{1}{2}\min(\lambda\wedge d\lambda, \mu\wedge d\mu) > 0,$$

and note that $|d\sigma| = |\sigma'(r)|$ and $|\mu-\lambda| = O(r)$, so the estimate $|r\sigma'(r)|\leqslant\varepsilon$ gives us

$$|d\sigma|\,|\lambda|\,|\mu-\lambda| = r|\sigma'(r)|O(1) = \varepsilon O(1) < C,$$

for ε sufficiently small. Thus, $\tilde\lambda$ is a contact form.

Item (ii) follows from

$$\begin{aligned} d\theta\wedge d\tilde\lambda &= (1-\sigma)\,d\theta\wedge d\lambda + \sigma\,d\theta\wedge d\mu + \sigma'(r)\,d\theta\wedge dr\wedge(\mu-\lambda)\\ &\geqslant \beta - C|\sigma'(r)r| \geqslant \beta - C\varepsilon > 0,\end{aligned}$$

for δ,ε sufficiently small. □

Proof of Lemma 3.8. *Step* 1. By assumption, $\gamma = S^1\times\{0\}$ is an orbit of the Reeb vector fields $R, R_{\rm st}$ of $\lambda,\lambda_{\rm st}$. In particular, $\lambda|_\gamma, \lambda_{\rm st}|_\gamma$ are volume forms of total volume $T=\int_\gamma\lambda$ and $\operatorname{sign}(T)$, respectively. So after pulling back λ by an isotopy rel ∂U of diffeomorphisms of the form

$$F_t(\phi,r,\theta) = (f_t(\phi,r),r,\theta),$$

we may assume that $\lambda|_\gamma = (|T|\lambda_{\rm st})|_\gamma$. Note that condition (6) is preserved because $F^*d\theta = d\theta$. To simplify notation, we will replace λ by $\lambda/|T|$ (and insert back $|T|$ at the end of the proof), so we have $\lambda|_\gamma = \lambda_{\rm st}|_\gamma$. Note that now $R = \operatorname{sign}(T)\partial_\phi$ along γ.

Step 2. Next we improve condition (6). Pick a non-increasing function $h;[0,1]\to[0,1]$ which vanishes near 1 and equals $1-r^2/2$ near 0. Since λ is a contact form, for sufficiently small $\beta>0$ the form $\lambda+\beta h(r)\,d\phi$ is also contact. Note that

$$d\theta\wedge d(\lambda+\beta h(r)\,d\phi) = d\theta\wedge d\lambda - \beta h'(r)\,dr\wedge d\theta\wedge d\phi.$$

Both terms are non-negative and near $r=0$ the second term equals $\beta r\,dr\wedge d\theta\wedge d\phi$, so the whole expression has norm greater than or equal to β. Therefore, after replacing λ by $\lambda + \beta h(r)\,d\phi$, it satisfies the following quantified version of condition (6):

$$d\theta\wedge d\lambda \geqslant \beta > 0. \tag{8}$$

Note that the linear homotopy from λ to $\lambda+\beta h(r)\,d\phi$ is contact and satisfies (6), and the condition $\lambda|_\gamma = \lambda_{\rm st}|_\gamma$ from Step 1 is preserved.

Step 3. We write

$$\lambda = l_1\,dr + l_2\,d\theta + l_3\,d\phi,$$

with functions $l_i(\phi,r,\theta)$. Recall that in cartesian coordinates

$$dr = \frac{x\,dx+y\,dy}{r},\quad d\theta = \frac{x\,dy - y\,dx}{r^2}.$$

Thus, smoothness of λ at $r=0$ implies that $l_1 = O(r)$ and $l_2 = O(r^2)$ near $r=0$. Moreover, the condition $\lambda|_\gamma = \lambda_{\rm st}|_\gamma$ from Step 1 implies $l_3 = \operatorname{sign}(T)+O(r)$. Let us compute

$$d\lambda = (l_{2r}-l_{1\theta})\,dr\wedge d\theta + (l_{3r}-l_{1\phi})\,dr\wedge d\phi + (l_{2\phi}-l_{3\theta})\,d\phi\wedge d\theta.$$

From $i_{\partial_\phi}d\lambda = 0$ along γ, we deduce $l_{3r}|_{r=0} = l_{1\phi}|_{r=0} = 0$ (since $l_1 = O(r)$), so we have found the conditions

$$l_1 = O(r),\quad l_2 = O(r^2),\quad l_3 = \operatorname{sign}(T)+O(r^2). \tag{9}$$

Step 4. Take now a cutoff function ρ as in Lemma 3.7, with small constants $\delta, \varepsilon > 0$ to be specified later, and consider the 1-form

$$\lambda_t := (1 - t\rho(r))\lambda + t\rho(r)\lambda_{\mathrm{st}}.$$

Let us check the conditions in Lemma 3.10 (with $\sigma = t\rho$, $\mu = \lambda_{\mathrm{st}}$ and M the region $\{r \leqslant \delta\}$). Since $\lambda = \lambda_{\mathrm{st}} = \operatorname{sign}(T)\, d\phi$ and $R = R_{\mathrm{st}} = \operatorname{sign}(T)\partial_\phi$ at $r = 0$, the conditions of Lemma 3.10(i) are satisfied. The conditions of Lemma 3.10(ii) hold in view of (8) (and the analogous condition for λ_{st}) and

$$\begin{aligned} dr \wedge d\theta \wedge (\lambda - \lambda_{\mathrm{st}}) &= (l_3 - \operatorname{sign}(T)(1 + r^2))\, dr \wedge d\theta \wedge d\phi \\ &= O(r^2)\, dr \wedge d\theta \wedge d\phi \\ &= O(r), \end{aligned}$$

where we have used (9). Hence, λ_t is a contact form that satisfies condition (6) and agrees with λ near ∂U. Since $\lambda_1 = \lambda_{\mathrm{st}}$ near γ, this concludes (after inserting back the constant $|T|$ from Step 1) the proof of Lemma 3.8. □

The foliation case.

LEMMA 3.11. *Let (ω, λ) be an SHS on $U = S^1 \times D^2$ with $d\lambda = 0$ and satisfying (6). Then there exists a stable homotopy rel ∂U of SHS (ω_t, λ_t) with $d\lambda_t = 0$ and satisfying (6) such that $(\omega_0, \lambda_0) = (\omega, \lambda)$ and*

$$\lambda_1 = T\, d\phi, \quad \omega_1 = kr\, dr \wedge (\operatorname{sign}(T)\, d\theta - d\phi)$$

near $\gamma = S^1 \times \{0\}$, for some constants $k > 0$ and $T = \int_\gamma \lambda$.

Proof. *Step* 1. With $T = \int_\gamma \lambda$, the forms λ and $T\, d\phi$ are cohomologous, thus $T\, d\phi = \lambda + df$ for some function $f : U \to \mathbb{R}$. Moreover, both forms stabilize ω on some neighbourhood $V \subset U$ of γ. Pick a cutoff function $g : U \to \mathbb{R}$ which vanishes near ∂U and equals 1 on V. Then $\lambda_t := \lambda + td(gf)$ is a homotopy rel ∂U of closed 1-forms stabilizing ω with $\lambda_0 = \lambda$ and $\lambda_1 = T\, d\phi$ on the neighbourhood V of γ. In the following steps, we will keep λ_1 fixed and modify ω in V through HS stabilized by λ_1, or equivalently, by $T\, d\phi$.

Step 2. Recall that (x, y) are cartesian coordinates on D^2. Let

$$\alpha = A\, dx + B\, dy + C\, d\phi$$

be the chosen primitive of ω in U. Consider the function

$$h(\phi, x, y) := -A(\phi, 0, 0)x - B(\phi, 0, 0)y,$$

and the cutoff function g from Step 1. Then $\alpha + d(gh)$ is a primitive of ω which agrees with α near ∂U and whose coefficients in front of dx and dy vanish at γ. We denote this new primitive again by α.

Step 3. Let $\alpha = A\, dx + B\, dy + C\, d\phi$ be the primitive of ω from Step 2 with $A|_\gamma = B|_\gamma = 0$. This implies that $A = O(r)$ and $B = O(r)$ at γ, thus $d\theta \wedge d(A\, dx + B\, dy)$ is bounded in a neighbourhood of γ. As γ is a Reeb orbit of (ω, λ), the expression $d\theta \wedge \omega$ is bounded and thus $d\theta \wedge d(C\, d\phi)$ is also bounded. Let $\sigma(r)$ be a radial cutoff function supported in V which equals 1 on a neighbourhood $W \subset V$ of γ. Consider the compactly supported homotopy $\{\alpha_t\}_{t \in [0,1]}$ of 1-forms

$$\alpha_t := \alpha - t\sigma(r) C\, d\phi$$

on V and set $\omega_t := d\alpha_t$. Since by construction $\omega_t - \omega$ is proportional to $d\phi$, it has zero wedge with $d\phi$, and we thus see that ω_t is an HS stabilized by $d\phi$ for all $t \in [0, 1]$. Note that $\alpha_0 = \alpha$

and

$$\alpha_t|_W = \alpha - tC\,d\phi, \quad \alpha_1|_W = A\,dx + B\,dy.$$

Observe that $d\theta \wedge \omega_t$ *is bounded uniformly in* t since both $d\theta \wedge d\alpha$ and $d\theta \wedge d(C\,d\phi)$ are bounded. This boundedness will be traced throughout as we go in Steps 4–5 and then used in Step 6 to get transversality to the pages.

Step 4. After the homotopy in Step 3, we can assume that

$$\alpha = A\,dx + B\,dy$$

is a primitive of ω in a neighbourhood $W \subset V$ of γ, and A and B both vanish at γ. Next we want to replace A and B with their linear (in (x, y)) parts at γ. For this, we set

$$A_1(\phi, x, y) := A_x(\phi, 0, 0)x + A_y(\phi, 0, 0)y,$$
$$B_1(\phi, x, y) := B_x(\phi, 0, 0)x + B_y(\phi, 0, 0)y.$$

Let $\rho(r)$ be a cutoff function as in Lemma 3.7, with small δ, ε to be specified later and support in W, and set

$$\alpha_{\text{lin}} := A_1\,dx + B_1\,dy, \quad \alpha_t := \alpha - t\rho(r)(\alpha - \alpha_{\text{lin}}), \quad \omega_t := d\alpha_t,$$

for $t \in [0, 1]$. Note that at γ we have the following estimates:

$$\alpha - \alpha_{\text{lin}} = O(r^2), \quad d(\alpha - \alpha_{\text{lin}}) = O(r).$$

They imply

$$d\phi \wedge d(\rho(r)(\alpha - \alpha_{\text{lin}})) = \rho\,d\phi \wedge d(\alpha - \alpha_{\text{lin}}) + \rho'(r)\,d\phi \wedge dr \wedge (\alpha - \alpha_{\text{lin}}) = O(r).$$

Thus, $d\phi \wedge \omega_t > 0$ for δ sufficiently small, which shows that the ω_t are HS stabilized by $d\phi$. Note that $\alpha_0 = \alpha$ and $\alpha_1 = \alpha_{\text{lin}}$ on a neighbourhood of γ. Moreover, $d\theta \wedge \omega_t$ remains bounded because the first jet of α_t at γ remains unchanged during the homotopy.

Step 5. After the homotopy in Step 4, we can assume that $\alpha = A\,dx + B\,dy$ is a primitive of ω in some neighbourhood $Z \subset W$ of γ, and A and B are both linear in x and y on Z, that is,

$$A = A_1 x + A_2 y, \quad B = B_1 x + B_2 y,$$

with functions A_i, B_i of ϕ. Note that

$$T\,d\phi \wedge d\alpha = T(B_1 - A_2)\,d\phi \wedge dx \wedge dy > 0,$$

thus $T(B_1 - A_2) > 0$ for all $\phi \in S^1$. We set

$$c := \min_{S^1} |B_1 - A_2| > 0, \quad M := \max_{S^1} \operatorname{sign}(T)B_1, \quad m := \min_{S^1} \operatorname{sign}(T)A_2.$$

Let ρ be a function as in Lemma 3.7, with $\delta > 0$ so small that $\{r^2 \leqslant \delta\} \subset Z$ and $\varepsilon > 0$ to be specified later. Set

$$A_t := (1 - t\rho(r^2))A_1 x + [t\rho(r^2)\operatorname{sign}(T)m + (1 - t\rho(r^2))A_2]y,$$
$$B_t := [t\rho(r^2)\operatorname{sign}(T)M + (1 - t\rho(r^2))B_1]x + (1 - t\rho(r^2))B_2 y,$$
$$\alpha_t := A_t\,dx + B_t\,dy, \quad t \in [0, 1].$$

This is a homotopy of 1-forms supported in Z. First we check that the $\omega_t := d\alpha_t$ are HS stabilized by $d\phi$. So we need to check the inequality

$$T\,d\phi \wedge d\alpha = T(B_{tx} - A_{ty})\,d\phi \wedge dx \wedge dy > 0.$$

We write out

$$\begin{aligned}T(B_{tx}-A_{ty}) = {} & [t\rho(r^2)|T|M + T(1-t\rho(r^2))B_1] \\ & + T\{2t(M-B_1)x^2\rho'(r^2) - 2tB_2xy\rho'(r^2)\} \\ & - [t\rho(r^2)|T|m + T(1-t\rho(r^2))A_2] \\ & - T\{2t(m-A_2)y^2\rho'(r^2) - 2tA_1xy\rho'(r^2)\}.\end{aligned}$$

Using $M-m \geqslant c$, we estimate terms in square brackets from below and the other terms from above to apply the triangle inequality:

$$\begin{aligned}&[t\rho(r^2)|T|M + T(1-t\rho(r^2))B_1] - [t\rho(r^2)|T|m + T(1-t\rho(r^2))A_2] \\ &\quad = t\rho(r^2)|T|(M-m) + T(1-t\rho(r^2))(B_1-A_2) \\ &\quad \geqslant |T|(t\rho(r^2)(M-m) + (1-t\rho(r^2))c) \geqslant |T|c, \\ &|\{2t(M-B_1)x^2\rho'(r^2) - 2tB_2xy\rho'(r^2)\} - \{2t(m-A_2)y^2\rho'(r^2) - 2tA_1xy\rho'(r^2)\}| \\ &\quad \leqslant 2t\varepsilon(|M-B_1| + |B_2| + |m-A_2| + |A_1|) < |T|c,\end{aligned}$$

for ε small enough.

Second, near $r=0$ the coefficients of α_t are linear in (x,y). In particular, $A_t = O(r)$ and $B_t = O(r)$ at γ, thus $d\theta \wedge \omega_t$ is bounded uniformly in t. Moreover, for $t=1$ we have

$$\alpha_1 = \operatorname{sign}(T)my\,dx + \operatorname{sign}(T)Mx\,dy, \quad \omega_1 = \operatorname{sign}(T)(M-m)\,dx\wedge dy.$$

Step 6. In Steps 2–5, we constructed a homotopy $\{\alpha_t\}_{t\in[0,1]}$ of 1-forms supported in V such that the HS $\omega_t = d\alpha_t$ are all stabilized by $d\phi$, and $d\theta\wedge\omega_t$ is bounded uniformly in $t\in[0,1]$. Note that $\omega_t = \omega$ outside the relatively compact neighbourhood $W\subset V$ of γ from Step 3. We may assume that all neighbourhoods are chosen radially symmetric.

Pick a function $F(r)$ with support in V and

$$F|_W = 1 - r^2/2.$$

Set

$$\alpha_{\mathrm{tr}} := F(r)\,d\phi, \quad \omega_{\mathrm{tr}} := d\alpha_{\mathrm{tr}} = F'(r)\,dr\wedge d\phi.$$

Note the following two crucial properties of ω_{tr}:

$$d\phi\wedge\omega_{\mathrm{tr}} = 0$$

and

$$d\theta\wedge\omega_{\mathrm{tr}}|_W = d\phi\wedge r\,dr\wedge d\theta,$$

that is, $d\theta\wedge\omega_{\mathrm{tr}}$ is positive and bounded from below. The first property shows that for any $t\in[0,1]$ and any constant k, the form $\omega_t + k\omega_{\mathrm{tr}}$ is stabilized by $d\phi$. The second property together with boundedness of $d\theta\wedge\omega_t$ uniformly in t shows that for large enough positive k we have $d\theta\wedge(\omega_t + k\omega_{\mathrm{tr}}) > 0$ on M for all t. Since $d\theta\wedge\omega_0 > 0$, there exists a non-negative function $\mathcal{A}:[0,1]\to\mathbb{R}$ with $\mathcal{A}(0)=0$ and $\mathcal{A}(1)=k$ such that the homotopy

$$\{(\tilde\omega_t = \omega_t + \mathcal{A}(t)\omega_{\mathrm{tr}}, d\phi)\}_{t\in[0,1]}$$

of SHS has the property that

$$d\theta\wedge\tilde\omega_t > 0,$$

for all $t\in[0,1]$. Finally, note that

$$\tilde\omega_1 = r\,dr\wedge(\operatorname{sign}(T)(M-m)\,d\theta - k\,d\phi)$$

in some neighbourhood $U_1 = \{r\leqslant r_1\}$ of γ.

Step 7. For r_1 as in Step 6, let $\rho(r)$ be a positive function which equals $(M-m)/k-1$ near 0 and 0 near r_1. The form $\tilde{\omega}_1$ on $U_1 = \{r \leqslant r_1\}$ from Step 6 can be linearly homotoped to

$$r\,dr \wedge (\mathrm{sign}(T)(M-m)\,d\theta - k(\rho+1)\,d\phi).$$

The resulting homotopy of HS is stabilized by $T\,d\phi$ and transverse to the pages. Moreover, the cohomology class remains constant as the form $\rho(r)r\,dr \wedge d\phi$ has a primitive compactly supported in U_1. The last 2-form agrees with

$$(M-m)r\,dr \wedge (\mathrm{sign}(T)\,d\theta - d\phi)$$

in some smaller neighbourhood of γ. This completes the proof of Lemma 3.11, and hence of Proposition 3.5. □

3.3. *Proof of Theorem* 1.2

We will need the following simple interpolation lemma.

LEMMA 3.12. *Let $\lambda, \tilde{\lambda}$ be two stabilizing* 1*-forms for an HS ω on M^3 such that $\lambda(R) = \tilde{\lambda}(R)$ for a vector field R generating* $\ker\omega$. *Then for any function $\rho : M \to \mathbb{R}$ satisfying $d\rho(R) = 0$, the* 1*-form $(1-\rho)\lambda + \rho\tilde{\lambda}$ also stabilizes ω.*

Proof. This follows immediately from $((1-\rho)\lambda + \rho\tilde{\lambda})(R) = \lambda(R) > 0$ and

$$i_R d((1-\rho)\lambda + \rho\tilde{\lambda}) = i_R(d\rho \wedge (\tilde{\lambda} - \lambda)) = 0.$$ □

Now we can prove Theorem 1.2. Let (ϕ_l, r_l, θ) be standard coordinates as above near each binding component B_l, $l = 1, \ldots, n$. Let (ω, λ) and $(\tilde{\omega}, \tilde{\lambda})$ be two SHS supported by the open book (B, π) such that $[\omega] = [\tilde{\omega}]$ and $s_l(\omega, \lambda) = s_l(\tilde{\omega}, \tilde{\lambda}) =: s_l$.

Step 1. We apply Proposition 3.5 to the SHS (ω, λ) and $(\tilde{\omega}, \tilde{\lambda})$ to homotope them to (ω_1, λ_1), respectively, $(\tilde{\omega}_1, \tilde{\lambda}_1)$ which are s_l-special near each B_l.

Step 2. By Step 1, we may assume that (ω, λ) and $(\tilde{\omega}, \tilde{\lambda})$ are both s_l-special when restricted to some tubular neighbourhood V_l of a binding component B_l. In particular, $\omega = \tilde{\omega}$ on V_l. Moreover, the 1-forms evaluate to constants $\lambda(R) \equiv n_l > 0$ and $\tilde{\lambda}(R) \equiv \tilde{n}_l > 0$ on the vector field $R := \partial_\theta + s_l\partial_\phi$ generating $\ker\omega$. Note that any function $\rho(r)$ depending only on the radial coordinate r satisfies $d\rho(R) = 0$. Using such functions in Lemma 3.12, we can thus interpolate between $\tilde{\lambda}$ and $m_l\lambda$, where $m_l := \tilde{n}_l/n_l > 0$. Hence, after a further homotopy of stabilizing forms rel ∂V_l we may assume that $\tilde{\lambda} = m_l\lambda$ on some smaller tubular neighbourhood (called V_l again) of B_l. Since $d\theta(R) \equiv 1$, another application of Lemma 3.12 allows us to homotope λ and $\tilde{\lambda}$ rel ∂V_l to achieve that $\lambda|_{\{r_l \in [a,b]\}} = n_l d\theta$ and $\tilde{\lambda}|_{\{r_l \in [a,b]\}} = m_l n_l d\theta$. Here, the parameters a and b depend on l, but we suppress the dependence from the notation.

Step 3. Next we adjust the relative cohomology class $\eta := [\tilde{\omega} - \omega] \in H^2(M, B)$. Since $[\tilde{\omega} - \omega] = 0 \in H^2(M)$, it follows from Lemma 2.1(b) that η has a representative of the form

$$\sum_{l=1}^{n} c_l d(\sigma(r_l)\,d\phi_l),$$

with constants $c_l \in \mathbb{R}$ and a non-increasing function $\sigma : [a, b] \to [0, 1]$ which equals 1 near a and 0 near b. For $t \in [0, 1]$, set

$$\omega_t := \omega + t\sum_{c_l \geqslant 0} c_l d(\sigma(r_l)\,d\phi_l), \quad \tilde{\omega}_t := \tilde{\omega} + t\sum_{c_l < 0} (-c_l)\,d(\sigma(r_l)\,d\phi_l).$$

Now $\sigma' \leqslant 0$ implies

$$d\theta \wedge \omega_t = d\theta \wedge \omega - t \sum_{c_l \geqslant 0} c_l \sigma'(r_l)\, dr_l \wedge d\theta \wedge d\phi_l > 0,$$

which in view of $\lambda|_{\{r_l \in [a,b]\}} = n_l\, d\theta$ shows that (ω_t, λ) is a homotopy of SHS supported by (B, π). Similarly for $(\tilde{\omega}_t, \tilde{\lambda})$. The resulting HS ω_1 and $\tilde{\omega}_1$ coincide in some neighbourhood of $\{r \leqslant a\}$ and satisfy $[\tilde{\omega}_1 - \omega_1] = 0 \in H^2(M, B)$. Moreover, $\tilde{\lambda} = m_l \lambda$ on $\{r \leqslant a\}$ and $\lambda = n_l\, d\theta$, $\tilde{\lambda} = n_l m_l\, d\theta$ near $\{r = a\}$.

Step 4. Now we restrict (ω_1, λ) and $(\tilde{\omega}_1, \tilde{\lambda})$ from Step 3 to the mapping torus

$$X := M \setminus \{r < a\} \xrightarrow{\pi} S^1,$$

with fibre $\Sigma \setminus \{r < a\}$. Let N be a neighbourhood of ∂X on which $\omega_1 = \tilde{\omega}_1$ and $\tilde{\lambda} = m_l \lambda = m_l n_l\, d\theta$. Since $[\tilde{\omega}_1 - \omega_1] = 0 \in H^2(X, \partial X)$, Lemma 3.1 allows us to connect (ω_1, λ) and $(\tilde{\omega}_1, \tilde{\lambda})$ by a stable homotopy $(\hat{\omega}_t, \hat{\lambda}_t)$ on X supported by (B, π). Moreover, on N we have $\hat{\omega}_t = \omega_1 = \tilde{\omega}_1$ and $\hat{\lambda}_t = \rho(t)\lambda$ for a positive function $\rho : [0,1] \to \mathbb{R}$ with $\rho(0) = 1$ and $\rho(1) = m_l$. Thus, we can extend homotopy $(\hat{\omega}_t, \hat{\lambda}_t)$ by $(\omega_1, \rho(t)\lambda)$ over $\{r < a\}$ to a stable homotopy on M connecting (ω_1, λ) to $(\tilde{\omega}_1, \tilde{\lambda})$ and supported by (B, π). This completes the proof of Theorem 1.2.

4. *Braiding transverse knots around the binding of a contact open book*

We will use the following terminology. A contact form α is *supported by an open book* (B, π) if α restricts positively to the binding and $d\alpha$ restricts positively to the pages. A contact structure ξ is *supported by an open book* if there exists a contact form defining ξ which is supported by the open book. An oriented link L is *(positively) transverse to the contact structure* ξ if a defining contact form α restricts positively to L. An oriented link L is *braided around an open book* (B, π) if L is disjoint from the binding and positively transverse to the pages, that is, $d\pi$ restricts positively to L.

The goal of this section is to explain the following result from Pavelescu's thesis [**21**].

Theorem 4.1. *Let M be a closed oriented 3-manifold and ξ be a cooriented contact structure on M supported by an open book (B, π), and $L \subset M$ be a link positively transverse to ξ. Then there exist isotopies $(B_t, \pi_t, L_t)_{t \in [0,1]}$ of open books (B_t, π_t) supporting ξ and links L_t transverse to ξ such that $(B_0, \pi_0, L_0) = (B, \pi, L)$ and L_1 is braided around (B_1, π_1).*

Moreover, for every collection of sufficiently large natural numbers $k_1, \ldots, k_\ell \geqslant K$, where ℓ is the number of components of L and K a constant depending on (B, π, ξ, L), we can arrange that the intersection number of the ith component of L_1 with a page of (B_1, π_1) equals k_i.

Remark 4.2. Pavelescu claims the stronger result that the open book $(B_t, \pi_t) = (B, \pi)$ can be fixed. As the proof in [**21**] contains some gaps, we repeat it below with some more details. The deformation of the open book is needed for technical reasons; it can be made C^1-small, and presumably be avoided with more work.

The proof in [**21**] is based on the following construction. Consider a contact form α supported by an open book (B, π). Suppose that

$$\alpha = T(1 - r^2)(d\phi + r^2\, d\theta)$$

in a tubular neighbourhood $W = \{r \leqslant r_0\}$ of B, where $T > 0$ is some constant and (ϕ, r, θ) are (polar) coordinates near the respective binding components in which $\pi = \theta$. We define another neighbourhood

$$V := \{r \leqslant r_0/2\} \subset \{r \leqslant r_0\} = W$$

of B. Let $f : [0, r_0] \to [0, 1]$ be a non-decreasing function which equals $T(1 - r^2)r^2$ for $r \leqslant r_0/2$ and 1 near $r = r_0$. It extends by 1 over $M \setminus W$ to a function on M that we also denote by f. Consider the family of contact structures

$$\xi_t := \ker \alpha_t, \quad \alpha_t := \alpha + tf\, d\theta,\ t \in [0, \infty). \tag{10}$$

By Gray's stability theorem, we know that there is a family of diffeomorphisms $\{\Psi_t\}_{t\in[0,\infty)}$ such that Ψ_t pulls back α_t to a multiple of $\alpha_0 = \alpha$. We need to analyse $\{\Psi_t\}$ more closely. Recall that Ψ_t is given as the flow of the time-dependent vector field $X_t \in \xi_t$ defined by

$$i_{X_t} d\alpha_t = h_t\alpha_t - \dot{\alpha}_t, \quad h_t := \dot{\alpha}_t(R_t), \tag{11}$$

where R_t denotes the Reeb vector field of α_t and $\dot{\alpha}_t$ denotes the time derivative of α_t.

Consider a page Σ and note that $\alpha_t|_\Sigma = \alpha|_\Sigma$, so all the α_t define the *same* characteristic foliation $\mathcal{F} = \xi \cap T\Sigma = \xi_t \cap \Sigma$ on Σ. Since $d\alpha_t|_\Sigma = d\alpha|_\Sigma$ is positive, R_t is positively transverse to Σ and $\dot{\alpha}_t = f\, d\theta$ yields $h_t = f\, d\theta(R_t) > 0$. Contracting equation (11) with any vector $v \in \xi \cap T\Sigma$, we obtain $d\alpha(X_t, v) = 0$, and since $v, X_t \in \xi$ and $d\alpha|_\xi$ is non-degenerate this implies that v and X_t are collinear. This shows that X_t is tangent to the pages and spans the characteristic foliation. Moreover, the restriction of equation (11) to Σ gives

$$i_{X_t} d\alpha|_\Sigma = h_t\alpha|_\Sigma. \tag{12}$$

Thus, X_t is determined by equation (12). In particular, each $X_t = h_t X$ is a positive multiple of the Liouville field X tangent to the pages defined by

$$i_X d\alpha|_\Sigma = \alpha|_\Sigma.$$

A short computation shows $X = -((1 - r^2)/2r)\partial_r$ on W, so X points into V along ∂V. The key property of X is that $L_X\, d\alpha|_\Sigma = d(i_X\, d\alpha|_\Sigma) = d\alpha|_\Sigma$, that is, X expands the positive area form $d\alpha|_\Sigma$ on the page. This has the following dynamical consequences.

(i) Each closed orbit of X is repelling.

(ii) At each zero $p \in \Sigma$ of X, the linearization $d_pX|_{T\Sigma}$ has an eigenvalue with positive real part. If the eigenvalues are non-real, or both real and positive, then this implies that p is non-degenerate and X flows out of p; we call such p *elliptic*. If the eigenvalues are real with one positive and one non-positive, then we call p *hyperbolic*; in this case, there may be one or two flow lines converging to p in forward time.

(iii) As a consequence of (i–ii) and the Poincaré–Bendixson Theorem, every flow line of X which is not a zero or a closed orbit either enters V in finite time or converges to a hyperbolic zero in forward time.

Now consider the S^1-family of pages $\Sigma_\theta := \pi^{-1}(\theta)$, $\theta \in S^1$. For any $\theta \in S^1$, let S_θ denote the subset of Σ_θ consisting of zeros, closed orbits and flow lines converging to hyperbolic zeros in forward time. We set

$$S := \bigcup_{\theta\in S^1} S_\theta. \tag{13}$$

The following statement is crucial for the proof.

Lemma 4.3. *Let U_S be an open neighbourhood of S in $M \setminus B$. Then there exists $\tau \in [0, +\infty)$ such that $\Psi_\tau(M \setminus U_S) \subset V$.*

Proof. By property (iii) above and compactness, there exists a constant $\tau_0 > 0$ such that each point in $M \setminus U_S$ reaches V in time τ_0 under the flow of X. To get the corresponding statement with the time-dependent vector field X_t in place of X, we need to look at the behaviour of the length of X_t as $t \to +\infty$. Since the $d\theta$ component of α_t goes to ∞ as $t \to 0$, we get that $R_t \to 0$ as $t \to \infty$. From

$$1 = \alpha_t(R_t) = \alpha(R_t) + tf\,d\theta(R_t),$$

and $\alpha(R_t) \to 0$ as $t \to \infty$, we see that $h_t = f\,d\theta(R_t) = O(t^{-1})$ as $t \to \infty$ on $M \setminus B$. Since $\int_1^\infty (1/t)\,dt = \infty$, there exists a constant $\tau > 0$ such that each point in $M \setminus U_S$ reaches V in time τ under the flow Ψ_t generated by X_t. □

After these preparations, we now turn to the following proof.

Proof of Theorem 4.1. *Step* 0. We first put the open book (B, π) into nice position with respect to ξ. After a small transverse isotopy (fixing (B, π, ξ)), we may assume that the link L does not intersect the binding B. Let α be a contact form defining ξ and supported by (B, π). We apply Lemma 3.8 to $\lambda := \alpha$ on a neighbourhood U of each binding component $\gamma = B_l$ and with the constant $T = \int_\gamma \lambda > 0$. Thus, we find a homotopy of contact forms $\{\alpha_t\}_{t\in[0,1]}$ supported by (B, π), fixed outside U, with $\alpha_0 = \alpha$ and

$$\alpha_1|_W = T(r^2\,d\theta + (1 - r^2)\,d\phi),$$

for some neighbourhood $W = \{r \leqslant r_0\}$ of B_l, where (ϕ, r, θ) are the standard open book coordinates near B_l. We can further deform the contact form α_1, through contact forms supported by (B, π) and fixed outside a neighbourhood of B, to a contact form α_2 satisfying

$$\alpha_2|_W = T(1 - r^2)(r^2\,d\theta + d\phi)$$

(after shrinking W). This can be achieved by a homotopy of T^2-invariant forms α_{h_t} as in Subsection 2.2 for a family of immersions $h^t : [0, r_0] \to \mathbb{C}$ satisfying the contact condition $\langle h^t, i(h^t)'\rangle > 0$ and the supportedness condition $(h_2^t)' < 0$ connecting $(r^2, 1 - r^2)$ to $((1 - r^2)r^2, 1 - r^2)$; the existence of such a family is easily seen pictorially. Finally, we perturb α_2, keeping it fixed near B, to a contact form α_3 for which the characteristic foliations on the pages are sufficiently generic (in a sense that is made precise below).

After applying Gray's theorem, we may assume that ξ and L are fixed and the open book is moving by an isotopy. We rename the new open book back to (B, π) and the new contact form α_3 back to α.

After this preparatory step, we will keep (B, π, ξ) fixed and only deform the transverse link L. More precisely, we will construct a family $(\xi_t, L_t)_{t\in[0,A]}$ of contact structures ξ_t and links L_t in M with the following properties:

(a) there exists a family of diffeomorphisms $(\psi_t)_{t\in[0,A]}$ with

$$\psi_0 = \mathbb{1}, \quad \psi_t^*\xi_t = \xi, \quad \psi_t(B) = B, \quad \pi \circ \psi_t = \pi;$$

(b) $L_0 = L$ and L_t is transverse to ξ_t for all $t \in [0, A]$;

(c) L_A is braided around (B, π) and the intersection number the ith component of L_A with a page is k_i.

Then the links $L'_t := \psi_t^{-1}(L_t)$ satisfy $L'_0 = L$, L'_t is transverse to $\psi_t^*\xi_t = \xi$ for all $t \in [0, A]$, and L'_A is braided around $(\psi_A^{-1}(B), \pi \circ \psi_A) = (B, \pi)$ with intersection numbers k_i. Thus, L'_t is the desired isotopy.

We construct the family $(\xi_t, L_t)_{t\in[0,A]}$ in six steps.

Step 1. Recall that from Step 0 we have a contact form α defining ξ supported by (B, π) which is given by

$$\alpha = T(1 - r^2)(d\phi + r^2\,d\theta)$$

in a tubular neighbourhood $W = \{r \leqslant r_0\}$ of B not meeting L. Let us call an arc γ of L *good* if γ is positively transverse to the pages, and *bad* otherwise. Let L_g denote the union of all good arcs and set $L_b := L \setminus L_g$. Since transversality is an open condition, we know that L_g is a union of open arcs and (after a small isotopy of L) we may assume that L_b is a union of closed arcs. Our objective is to achieve $L_b \cap S = \emptyset$, so that we can use Lemma 4.3 to push L_b into a smaller neighbourhood

$$V := \{r \leqslant r_0/2\}$$

of B.

By the last generic perturbation in Step 0, we can achieve that the set S is a union of submanifolds of dimensions $0, 1, 2$ of the following types:

- S(1) finitely many zeros of birth–death type;
- S(2) non-degenerate (elliptic or hyperbolic) zeros varying in one-dimensional families with θ;
- S(3) isolated flow lines converging in forward time to birth–death type zeros;
- S(4) isolated flow lines converging in forward and backward time to non-degenerate hyperbolic zeros;
- S(5) non-degenerate closed orbits varying in one-dimensional families with θ;
- S(6) flow lines converging in forward time to non-degenerate hyperbolic zeros (and in backward time to elliptic zeros or closed orbits) varying in one-dimensional families with θ.

Note that, for each stratum $S(n)$, $\overline{S(n)} \setminus S(n)$ is contained in the union of strata $S(m)$ with $m < n$. We will successively make L_b disjoint from $S(1), \ldots, S(6)$. To begin, we make L_b disjoint from the zero- and one-dimensional strata $S(1), \ldots, S(4)$ simply by a small perturbation of L.

Step 2. Let $\mathcal{P} = S(5)$ be the union of closed orbits of X. Each connected component of $\mathcal{P}$ is an embedded 2-torus in M fibred over S^1 by π. Consider a point $p \in \mathcal{P}$ belonging to a page Σ. Note that the three planes $T_p\Sigma$, ξ_p and $T_p\mathcal{P}$ all contain the line $\mathbb{R}X(p)$, so their intersection with a plane $E \subset T_pM$ transverse to $X(p)$ gives three lines in E.

Recall that the contact structures $\xi_t = (\Psi_t)_*\xi$ defined above (see equation (10)) converge at p to $T_p\Sigma$ as $t \to \infty$. After applying the homotopy $(\xi_t, L_t = \Psi_t(L))$, $t \in [0, \tau]$, for sufficiently large τ and renaming (ξ_τ, L_τ) back to (ξ, L), we may hence assume that at all points $p \in \mathcal{P}$ the three lines corresponding to $\Sigma, \xi, \mathcal{P}$ (and their coorientations) are ordered as in Figure 1.

Step 3. After a perturbation of L, we may assume that the intersection $L_b \cap \mathcal{P}$ is transverse. Consider a point $p \in L_b \cap \mathcal{P}$. Pick local coordinates $(x, y, z) \in \mathbb{R}^3$ near $p = (0, 0, 0)$ in which $\pi(x, y, z) = z$ and $\mathcal{P} = \{y = 0\}$, so $X(p)$ points in the x-direction. By Step 2, the intersections of $T_p\Sigma$, ξ_p and $T_p\mathcal{P}$ with the plane $\{x = 0\}$ are ordered as in Figure 1. After a further coordinate change near p, we may assume that near p the curve L_b is a straight line segment contained in the plane $\{x = 0\}$. After zooming in near 0 and rescaling, we may assume that the contact planes in the neighbourhood are C^0-close to ξ_p. Now we modify L_b near p within the plane $\{x = 0\}$ as shown in Figure 1. The new arc L_b' is transversely isotopic to L_b, by an isotopy fixed outside the neighbourhood of p and always intersecting $\mathcal{P}$ transversely at the only point p, and L_b' is good near the intersection point p.

After applying this *wrinkling* operation to all intersections with $\mathcal{P}$, we may hence assume that L avoids a closed neighbourhood $U_{\mathcal{P}}$ of $S(1) \cup \cdots \cup S(5)$.

Step 4. After Step 3 and a perturbation of L, the curve L_b intersects the two-dimensional submanifold $S(6)$ transversely in finitely many points in $M \setminus U_{\mathcal{P}}$. Thus, we can repeat Steps 2 and 3 with $S(5)$ replaced by $S(6)$ to make L disjoint from an open neighbourhood U_S of S.

Step 5. Now we are in the position to use the flow Ψ_t of X_t to push L_b into V. According to Lemma 4.3, there exists $\tau > 0$ such that $\Psi_\tau(M \setminus U_S) \subset V$ and thus $\Psi_\tau(L_b) \subset V$. Since Ψ_t preserves the binding and pages of the open book (B, π), the homotopy $(\xi_t, L_t)_{t \in [0,\tau]}$ defined by

$$\xi_t := (\Psi_t)_*\xi, \quad L_t := \Psi_t(L)$$

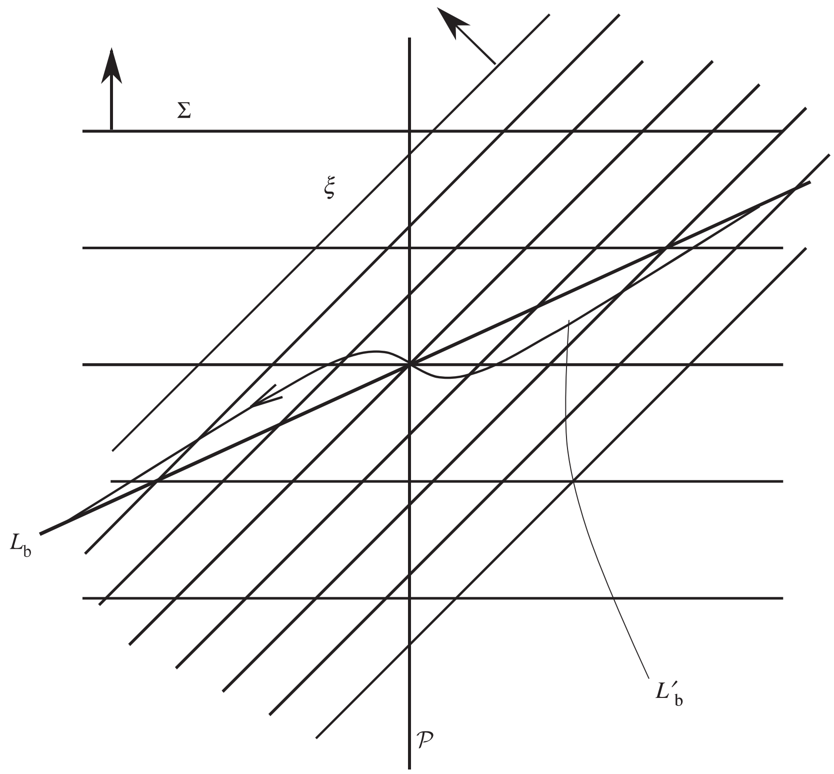

FIGURE 1. *Wrinkling.*

satisfies properties (a) and (b) above. Moreover, the bad (unbraided) part of $\mathcal{L}_\tau$ is given by $\Psi_\tau(L_b)$ and hence contained in V. Note that the new contact form α_τ is given on V by

$$\alpha_\tau = T(1-r^2)(d\phi + (1+\tau)r^2\, d\theta).$$

Step 6. Consider the contact structure $\xi_\tau = \ker \alpha_\tau$ from Step 5 on $V = S^1 \times D^2$. After rescaling in r, we may assume that $\xi_\tau = \xi_{\mathrm{st}} = \ker(d\phi + r^2 d\theta)$. By a result of Bennequin ([**2**], see also [**21**] for a short exposition), we can transversely isotope L_τ relative to ∂V (fixing ξ_τ) to a link which is disjoint from the binding $\{r = 0\}$ and positively transverse to the pages $\{\theta = \text{const}\}$ in V. Since L_τ was already braided outside V, the resulting link $L_A \subset M$ is braided around (B, π).

Finally, let K be the maximum of the intersection numbers of the components of L_A with a page of (B, π). Then for any integer $k_i \geqslant K$ we can further transversely isotope the ith component (pulling it through B) in such a way that at the end it is again braided and its intersection number with a page equals k_i. This concludes the proof of Theorem 4.1. □

5. *Standardization of a contact open book near a transverse knot*

In this section, we prove the following improvement of Theorem 4.1.

COROLLARY 5.1. *Let (M, α) be a closed oriented contact 3-manifold. Let $L = \bigcup_{i=1}^{\ell} L_i \subset M$ be a link (with components L_i) positively transverse to the contact structure $\ker \alpha$. Then there exists a natural number K with the following property. For every collection of numbers $k_i \geqslant K$, $i = 1, \ldots, \ell$, there exists a contact form β with $\ker \beta = \ker \alpha$ and an open book (B, π) supporting β such that L is disjoint from B and positively transverse to the pages, and the intersection number of L_i with a page is k_i. Moreover, near each $L_i = S^1 \times \{0\}$ there exist* *coordinates* $(\theta, r, \phi) \in S^1 \times D^2$ *and a smooth function* f *such that*

$$\beta = \alpha = f(d\theta + r^2\, d\phi) \quad \text{and} \quad \pi(\theta, r, \phi) = k_i \theta.$$

REMARK 5.2. Note that here θ is the coordinate along the knot L_i, so it parametrizes the core circle of the solid torus, whereas in Lemmas 3.8–3.10, θ parametrized the meridian of the solid torus. The reason for this choice of coordinates is that θ parametrizes the target circle of the open book projection in both cases. This new choice of coordinates will be used in Sections 5–7.

Proof. By Theorem 4.1, we find a contact form β and an open book (B, π) having all the properties in the corollary except the last one, that is, β need not agree with α near L and coordinates (θ, r, ϕ) standardizing both π and $\ker \alpha$ need not exist there. Consider a component L_i of L. It has a tubular neighbourhood $V \cong S^1 \times D^2_\delta$ with coordinates $\theta \in S^1$ and polar coordinates (r, ϕ) on the disc D^2_δ in which the contact structure $\xi = \ker \beta$ is given by

$$\xi_{\mathrm{st}} = \ker \alpha_{\mathrm{st}}, \quad \alpha_{\mathrm{st}} = d\theta + r^2\, d\phi.$$

By Lemma 5.3, there exist

(i) an open book (B, π') on M which agrees with (B, π) outside V such that $\pi'(\theta, r, \phi) = k_i \theta$ near L_i and
(ii) a contact form β', supported by (B, π') and defining ξ, which coincides with β outside V and with α_{st} near L_i.

By Lemma 5.4, β' can be modified near L_i to a contact form β'', still supported by (B, π') and defining ξ, such that $\beta'' = \alpha$ near L_i. Then β'' and (B, π') have the desired properties. □

It remains to prove the two lemmas used in the proof of Corollary 5.1. We consider a tubular neighbourhood $V = S^1 \times D^2_\delta$ of the knot $K = \{r = 0\}$ with coordinates (θ, r, ϕ) and the contact structure $\xi_{\mathrm{st}} = \ker \alpha_{\mathrm{st}}$ as above. Note that an open book without binding on V is simply a submersion $\pi : V \to S^1$, and it supports a contact form α if and only if $d\pi \wedge d\alpha > 0$.

LEMMA 5.3. *Let* α *be a contact form on* V *defining* ξ_{st} *and* $\pi : V \to S^1$ *be a submersion such that* $d\pi \wedge d\alpha > 0$ *and* $\pi|_K : K \to S^1$ *is a covering of degree* $n \in \mathbb{N}$. *Then there exist*

(i) *a submersion* $\pi' : V \to S^1$ *which agrees with* π *near* ∂V *such that* $\pi'(\theta, r, \phi) = n\,\theta$ *near* K *and*
(ii) *a contact form* α' *on* V, *defining* ξ_{st} *and satisfying* $d\pi' \wedge d\alpha' > 0$, *which coincides with* α *near* ∂V *and with* α_{st} *near* K.

Note that for the submersion $\pi'(\theta, r, \phi) = n\,\theta$, the condition $d\pi' \wedge d\alpha' > 0$ is just $d\theta \wedge d\alpha' > 0$.

LEMMA 5.4. *Let* α_0, α_1 *be two contact forms on* V *defining* ξ_{st} *such that* $d\theta \wedge d\alpha_i > 0$ *for* $i = 0, 1$. *Then there exists a contact form* α *on* V, *defining* ξ_{st} *and satisfying* $d\theta \wedge d\alpha > 0$, *which coincides with* α_0 *near* ∂V *and with* α_1 *near* K.

Proof. We write $\alpha_i = f_i \alpha_{\mathrm{st}}$ for functions $f_i : V \to \mathbb{R}_+$ and $\alpha = f \alpha_{\mathrm{st}}$, where

$$f := (1 - \rho(r)) f_0 + \rho(r) f_1.$$

Here $\rho : [0, \delta] \to [0, 1]$ is a non-increasing function as in Lemma 3.7 which equals 1 near 0 and 0 near δ and satisfies $|r\rho'(r)| \leqslant \varepsilon$, for arbitrarily small constants $\varepsilon, \delta > 0$ that will be chosen below. Now

$$d\theta \wedge d\alpha = d\theta \wedge \frac{\partial f}{\partial r} dr \wedge r^2 d\phi + f\, d\theta \wedge 2r\, dr \wedge d\phi = \left(2f + r\frac{\partial f}{\partial r}\right) d\theta \wedge r\, dr \wedge d\phi$$

is positive if and only if

$$2f + r\frac{\partial f}{\partial r} > 0. \tag{14}$$

To show (14), we estimate with positive constants c_0, c_1, c_2 depending only on f_0, f_1:

$$2f \geqslant 2 \min\left\{\min_V f_0, \min_V f_1\right\} \geqslant c_0 > 0,$$

$$\left|r\frac{\partial f}{\partial r}\right| \leqslant r\left|(1-\rho)\frac{\partial f_0}{\partial r} + \rho\frac{\partial f}{\partial r}\right| + |r\rho'(r)|\,|f_1 - f_0| \leqslant c_1\delta + c_2\varepsilon.$$

Thus, for ε, δ sufficiently small (14) holds and Lemma 5.4 follows. □

Proof of Lemma 5.3. *Step* 1. We make a coordinate change of V of the form

$$(\theta, r, \phi) \longmapsto (\Theta = \Theta(\theta), r, \phi),$$

where the diffeomorphism $\theta \mapsto \Theta(\theta)$ of $K \cong S^1$ is chosen in such a way that in the new coordinates the covering on K is given by

$$\pi(\Theta, 0, \phi) = n\theta. \tag{15}$$

Note that $d\theta \wedge d\alpha$ is positive at $\{r = 0\}$ by the contact condition, so after shrinking V we may assume it is positive on the whole of V and thus

$$d\Theta \wedge d\alpha = \Theta'(\theta)\, d\theta \wedge d\alpha > 0.$$

Step 2. Since the 1-forms $d\pi$ and $n\, d\Theta$ are cohomologous, we have

$$d\pi = n\, d\Theta + df,$$

for some function $f : V \to \mathbb{R}$, hence (after adding a constant to f if necessary) $\pi = n\theta + f$. Equation (15) then implies that $f|_K = 0$, that is, we have $|f| \leqslant C_f r$ for some constant C_f depending only on f. We take a non-decreasing function $\rho : [0, \delta] \to [0, 1]$ as provided by Lemma 3.7 (replacing ρ by $1 - \rho$) which equals 0 near $r = 0$ and 1 near $r = \delta$ and satisfies $r\rho'(r) \leqslant \varepsilon$. Consider the map

$$\pi_1 := n\theta + \rho(r) f : V \longrightarrow S^1.$$

Claim. For ε sufficiently small, we have $d\pi_1 \wedge d\alpha > 0$.

To prove this, we use $\alpha = h(d\theta + r^2 d\phi)$ to write out

$$\begin{aligned} d\theta_1 \wedge d\alpha &= d(n\theta + \rho f) \wedge d\alpha \\ &= (n\, d\Theta + \rho\, df) \wedge d\alpha + f\, d\rho \wedge d(h(d\theta + r^2\, d\phi)) \\ &= (n\, d\Theta + \rho\, df) \wedge d\alpha + f\rho'(r)[h_\phi - r^2 h_\theta]\, dr \wedge d\phi \wedge d\theta. \end{aligned} \tag{16}$$

Since both expressions

$$S_1 := n\, d\Theta \wedge d\alpha$$

and

$$S_2 := (n\, d\theta + df) \wedge d\alpha = d\pi \wedge d\alpha$$

are positive, their convex combination

$$(1-\rho)S_1 + \rho S_2 = (n\,d\Theta + \rho\,df) \wedge d\alpha \geqslant \min(S_1, S_2)$$

is bounded from below by a constant independent of ρ. So it remains to estimate the last summand in (16). Note that the ϕ-derivative of any function vanishes at $\{r=0\}$ and thus we have an estimate $|h_\phi| \leqslant C_h r$ with a constant C_h depending only on h. Using this as well as $|f| \leqslant C_f r$ and $r\rho'(r) \leqslant \varepsilon$, we estimate

$$|f\rho'(r)[h_\phi - r^2 h_\theta]| \leqslant C_f r \rho'(r) C_h r \leqslant C_f C_h \varepsilon r.$$

Since $r\,dr \wedge d\phi \wedge d\theta$ is a smooth volume form, this shows that the last summand in (16) becomes arbitrarily small for ε small and thus proves the claim.

Thus, $\pi_1 : V \to S^1$ is a submersion, satisfying $d\pi_1 \wedge d\alpha > 0$, which agrees with π near ∂V and with $n\theta$ near K. After shrinking V and renaming π_1 back to π, we may hence assume that $\pi = n\theta$ on V.

Step 3. Note that, after Step 2, α and α_{st} are two contact forms defining ξ_{st} with $d\Theta \wedge \alpha > 0$ and $d\Theta \wedge \alpha_{\mathrm{st}} > 0$. So by Lemma 5.4 (in coordinates (Θ, r, ϕ)), we find a contact form β on V, defining ξ_{st} and satisfying $d\theta \wedge d\beta > 0$, which coincides with α near ∂V and with α_{st} near K. After shrinking V and renaming β back to α, we may hence assume that $\alpha = \alpha_{\mathrm{st}}$ and $\pi = n\theta$ on V.

Step 4. It remains to modify π such that it equals $n\theta$ near K. The argument is similar to Step 2 but simpler: The forms $d\Theta$ and $d\theta$ are cohomologous, so we have $n\,d\Theta = n\,d\theta + df$ for some function f. In other words (after adding a constant to f if necessary), $n\theta = n\theta + f$. We define

$$\pi' := n\,\theta + \rho(r) f : V \longrightarrow S^1,$$

with a cutoff function ρ as in Step 2. Since $d\alpha_{\mathrm{st}} = 2r\,dr \wedge d\phi$, we have $f\,d\rho \wedge d\alpha_{\mathrm{st}} = 0$ and thus

$$d\pi' \wedge d\alpha_{\mathrm{st}} = (n\,d\theta + \rho\,df) \wedge d\alpha_{\mathrm{st}},$$

which is a convex combination of the positive terms $n\,d\theta \wedge d\alpha_{\mathrm{st}}$ and $n\,d\Theta \wedge d\alpha_{\mathrm{st}}$ and hence positive. This concludes the proof of Lemma 5.3. □

6. *Contact open books with boundary*

Consider an oriented contact 3-manifold (N, α) whose boundary $\partial N = T_1 \amalg \cdots \amalg T_\ell$ is a union of 2-tori. We assume that α is *T^2-invariant* near each boundary component T_i in the following sense (by a slight abuse of language as in Subsection 2.2): there exists a collar neighbourhood $K_i \cong [l_i, R_i] \times T^2$ of $T_i = \{r = l_i\} \cong T^2$ with oriented coordinates (r, ϕ, θ) in which α is given by

$$\alpha_h = h_1(r)\,d\phi + h_2(r)\,d\theta, \tag{17}$$

for some immersion $h : [l_i, R_i] \to \mathbb{C}$ satisfying

$$h_1' h_2 - h_2' h_1 > 0. \tag{18}$$

(The immersion h of course depends on i, but we suppress this dependence to keep the notation simple.) We are interested in α that are supported by an open book decomposition (B, π) in the usual sense (that is, $\alpha > 0$ on the binding B and $d\alpha > 0$ on the interior of the pages), where $B \subset \operatorname{int} N$ and the projection $\pi : N \setminus B \to S^1$ is in the above coordinates near each T_i a linear projection $\pi(r, \phi, \theta) = a_1\phi + a_2\theta$. Here $a \in \mathbb{Z}^2$ is some integer vector (again depending on i) and positivity of $d\alpha$ on the pages, $d\pi \wedge d\alpha > 0$, is equivalent to

$$h_1' a_2 - h_2' a_1 > 0.$$

(Such 'relative contact open books' were previously considered in [**1**, **25**].) After a linear coordinate change on T_i, we may then assume that $a = (0, n_i)$ for some $n_i \in \mathbb{N}$, so $\pi(r, \phi, \theta) = n_i\theta$ and the positivity condition $d\theta \wedge d\alpha > 0$ becomes

$$h_1' > 0. \tag{19}$$

Note that the page Σ of the open book has two kinds of boundary components: those that get collapsed to binding components, and others that give rise to the boundary tori T_i.

Recall Giroux's result [**13**] that for any cooriented contact structure on a closed oriented 3-manifold (without boundary) there exists a defining contact form which is supported by an open book. The goal of this section is to prove the following relative version.

PROPOSITION 6.1. *Let (N, α) be a compact oriented contact 3-manifold with toric boundary $\partial N = T_1 \amalg \cdots \amalg T_\ell$ as above, that is, there exist collar neighbourhoods $K_i \cong [l_i, R_i] \times T^2$ of $T_i = \{r = l_i\} \cong T^2$ on which α is given by (17). Assume that on each K_i we have $d\theta \wedge d\alpha > 0$ (that is, $h_1' > 0$).*

Then there exists a natural number K such that for any collection $\{n_i\}_{i=1,\ldots,\ell}$ of natural numbers with $n_i \geqslant K$ there exist a homotopy of contact forms α_t, $t \in [0,1]$, and an open book decomposition (B, π) of N with the following properties .

(i) *The contact form α_1 is supported by the open book decomposition (B, π). Moreover, $d\alpha_1 = d\alpha$ near ∂N.*

(ii) *Near T_i the fibration π is given by $\pi(\theta, r, \phi) = n_i\theta$.*

(iii) *The forms α_t are T^2-invariant near ∂N for all t, and $\alpha_0 = \alpha$.*

(iv) *Writing $\alpha = h_1(r)\, d\phi + h_2(r)\, d\theta$ and $\alpha^t = h_1^t(r)\, d\phi + h_2^t(r)\, d\theta$ near T_i, we have $h_1^1(l_i) > 0$, and if $h_1(l_i) > 0$, then $h_1^1(l_i) < h_1(l_i)$. Moreover, if $h_1(l_i) \leqslant 0$, then $(h^t)'(l_i)$ has rotation number -1 (that is, it makes one full* negative *(that is, clockwise) turn in the (h_2, h_1) plane as t runs from 0 to 1). If $h_1(l_i) > 0$, then we have a choice: we can choose $(h^t)'(l_i)$ to have rotation number -1 or 0.*

(v) *If near some T_i we have $h(r) = ((r - l_i), 1)$, then instead of* (iv) *the T^2-invariant homotopy α_t near T_i can be chosen to be constant.*

Proof. *Special case.* First we treat the special case that $h(r) = ((r - l_i), 1)$ near some T_i. After a shift in the r coordinate by l_i, we may assume that $\alpha = d\theta + r\, d\phi$ and $r \in [0, R_i]$. We remove the torus $\{0\} \times T^2$ and on the remaining $(0, R_i] \times T^2$ we do a coordinate change $r_1 := r^{1/2}$ and then rename r_1 back to r. Now the 1-form α looks like $d\theta + r^2\, d\phi$. So it extends to the solid torus obtained by collapsing the ϕ-direction in the removed torus $\{0\} \times T^2$. This gives rise to a closed contact manifold $(\bar{N}, \bar{\alpha})$ with a transverse link $K_1 \amalg \cdots \amalg K_\ell$ obtained from the T_i. We apply Corollary 5.1 to obtain a contact form $\bar{\beta}$ on $\bar{N}$, defining the same contact structure as $\bar{\alpha}$ and supported by an open book $(\bar{B}, \bar{\pi})$, such that $\bar{\beta} = \bar{\alpha}$ and $\bar{\pi} = n_i\theta$ near K_i. Replacing K_i back by T_i and changing coordinates back from $r^{1/2}$ to r yields a contact homotopy α_t (by linear interpolation from $\bar{\alpha}$ to $\bar{\beta}$) and open book satisfying conditions (i–iii) and (v) in the proposition. This was 'dream situation' in which we did not have to worry how to get back from $\bar{N}$ to N. Now we turn to the following:

General case. Consider one neighbourhood $K_i = [l_i, R_i] \times T^2$. After a shift, we may assume that $l_i > 0$. We consider the solid torus $S^1 \times D^2$ with coordinates $\theta \in S^1$ and polar coordinates (r, ϕ) on D^2, where $r \in [0, l_i]$, so that we can write $S^1 \times D^2 = \{r \in [0, l_i]\}$. We identify the boundary of this solid torus with T_i via the identity map. Gluing in these solid tori for all i gives us a closed manifold $\bar{N}$. Let L_i denote the core circle $\{r = 0\}$ of the solid torus $\{r \in [0, l_i]\}$. We consider the union of this solid torus with the collar neighbourhood of the respective boundary

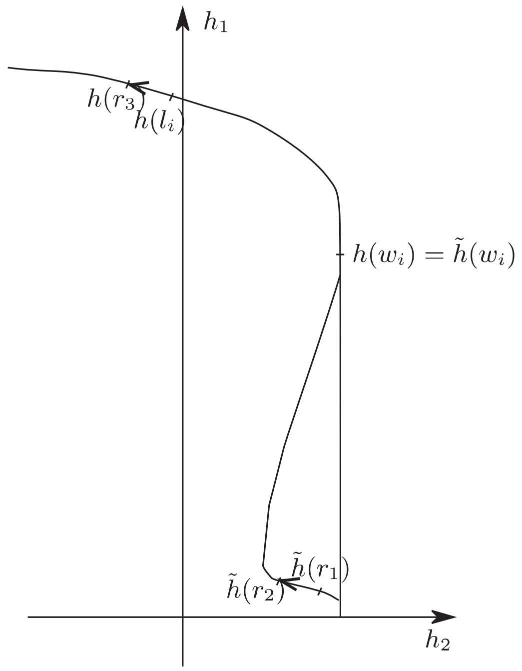

FIGURE 2. *The case* $h_1(l_i) > 0$.

component

$$V_i := K_i \cup \{r \in [0, l_i]\} = \{r \in [0, R_i]\}.$$

We extend the contact form α from N to $\bar{N}$ as follows. Recall that on K_i we have $\alpha = \alpha_h$ for an immersion $h : [l_i, R_i] \to \mathbb{C}$ satisfying (18) and (19). We extend h from $[l_i, R_i]$ to $[0, R_i]$ so that the contact condition (18) holds and $h(r) = (r^2, 1)$ near $r = 0$. Moreover, if $h_1(l_i) > 0$, then we arrange that $h_1' > 0$ on $(0, l_i]$ (see Figure 2), and if $h_1(l_i) \leqslant 0$, then we let h rotate as in Figure 3.

The condition $h(r) = (r^2, 1)$ near $r = 0$ means that near L_i we have $\alpha = \alpha_{\mathrm{st}}$ with

$$\alpha_{\mathrm{st}} := d\theta + r^2\, d\phi,$$

which extends smoothly over $r = 0$. This gives us the desired extension of α_h from K_i to V_i and thus the extension of α from N to $\bar{N}$. To get back to N, we just have to cut out the solid tori $\{r < l_i\}$.

Let $\xi := \ker \alpha$ denote the contact structure defined by α on $\bar{N}$. An application of Corollary 5.1 to $(\bar{N}, L := \bigcup L_i, n_i, \alpha)$ gives us a new defining form β for ξ and an open book $(\bar{B}, \bar{\pi})$ with the following properties. The contact form β is supported by the open book $(\bar{B}, \bar{\pi})$, and both the open book projection and the contact form restrict to a neighbourhood

$$W_i := \{r \leqslant w_i\}$$

of L_i in a standard way: $\bar{\pi}(r, \phi, \theta) = n_i\theta$ and $\beta = \alpha = \alpha_{\mathrm{st}}$.

Unfortunately, neither β nor $\bar{\pi}$ is nice on K_i: It may even happen that the binding $\bar{B}$ intersects K_i, and the form β need not be T^2-invariant on K_i. In order to take care of this, we introduce a new contact structure and new contact forms. We begin by picking a subdivision

$$0 < r_1 < r_2 < w_i < l_i < r_3 < R_i, \tag{20}$$

and an immersion $\tilde{h} : [0, w_i] \to \mathbb{C}$ with the following properties (see Figures 2–4):

(i) $r_2 - r_1 = r_3 - l_i$, or equivalently, $r_3 - r_2 = l_i - r_1 =: l$;

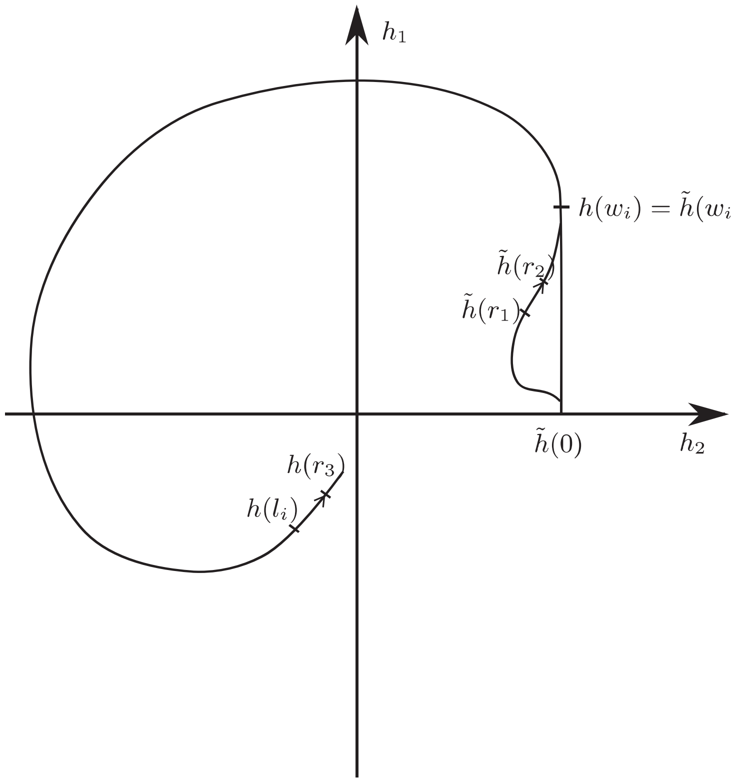

FIGURE 3. *The case $h_1(l_i) \leqslant 0$.*

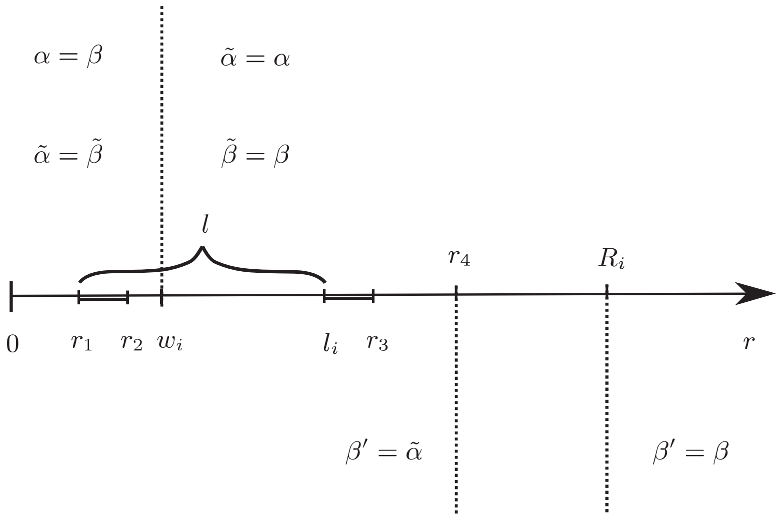

FIGURE 4. *The subdivision.*

(ii) near 0 and w_i we have $\tilde{h}(r) = (r^2, 1)$;
(iii) on $[r_1, r_2]$ we have $\tilde{h}(r) = h(r + l) + (A, B)$ for some constant $(A, B) \in \mathbb{C}$;
(iv) the function $\tilde{h}$ satisfies conditions (18) and (19).

Of course, $r_{+1}, r_2, r_3, \tilde{h}$ depend on i, but we suppress this from the notation. We extend $\tilde{h}$ as h over the interval $[w_i, R_i]$. This defines a contact form $\alpha_{\tilde{h}}$ on each V_i and thus a contact form $\tilde{\alpha}$ on $\bar{N}$ which differs from α only on the neighbourhoods W_i. Recall that α and β coincide on

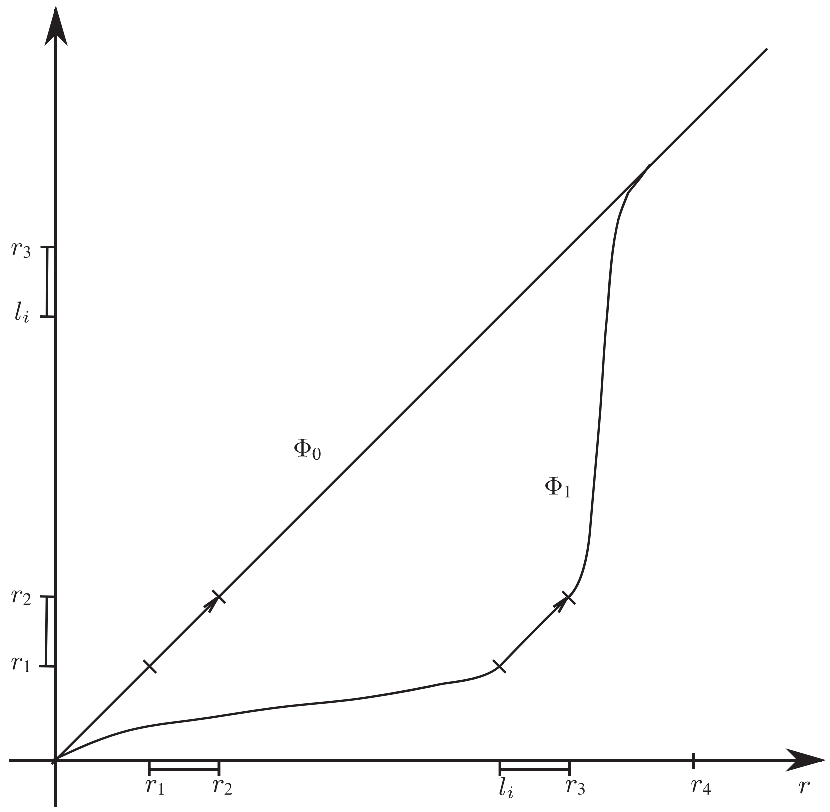

FIGURE 5. *The diffeomorphisms* Φ_t.

W_i, so we can define a contact form $\tilde{\beta}$ on $\bar{N}$ as $\tilde{\alpha}$ on each W_i and as β on the rest of $\bar{N}$. It is crucial to note that the $\tilde{h}_1' > 0$ implies that the contact form $\tilde{\beta}$ is supported by $(\bar{B}, \bar{\pi})$. Note also that the contact forms $\tilde{\alpha}$ and $\tilde{\beta}$ define the same contact structure that we denote by $\tilde{\xi}$.

We introduce a third contact form defining $\tilde{\xi}$. For this, we introduce another subdivision point into subdivision (20), namely r_4:

$$0 < r_1 < r_2 < w_i < l_i < r_3 < r_4 < R_i.$$

Now we pick a contact form β' defining $\tilde{\xi}$ that coincides with $\tilde{\alpha}$ on $\{r \in [0, r_4]\}$ and with $\beta = \tilde{\beta}$ on a neighbourhood of $\bar{N} \setminus (\bigcup V_i)$.

We need one more ingredient: an isotopy of diffeomorphisms $\{\Phi_t\}_{t\in[0,1]}$ of the interval $[0, r_4]$ with the following properties (see Figure 5):

(i) $\Phi_0 = \mathbb{1}$;
(ii) $\Phi_t = \mathbb{1}$ near 0 and r_4 for all $t \in [0, 1]$;
(iii) for $r \in [l_i, r_3]$ we have $\Phi_1(r) = r - l$.

For each Φ_t, we also denote the induced radial diffeomorphism of V_i by Φ_t. This diffeomorphism extends from $\bigcup V_i$ to the rest of $\bar{N}$ by setting it to be the identity on the complement of $\bigcup V_i$. The resulting diffeomorphism will be denoted as Φ_t again. This completes the necessary constructions and we now describe a homotopy of contact forms on $\bar{N}$ in three steps. The desired homotopy α_t will be obtained by restricting this homotopy to N.

Step 1. We start with the contact form $\tilde{\alpha}$ and linearly homotope it to β' (they define the same contact structure $\tilde{\xi}$). Since β' and $\tilde{\alpha}$ coincide on $\{r \leqslant r_4\}$, the homotope is fixed there, in particular on $\{r \in [l_i, r_3]\}$.

Step 2. We deform β' by setting $\beta'_t := \Phi_t^*\beta'$. On $\{r \in [0, r_4]\}$, we have $\beta'_t = \Phi_t^*\tilde{\alpha} = \Phi_t^*\alpha_{\tilde{h}} = \alpha_{\tilde{h}\circ\Phi_t}$. Since $\Phi_t = \mathbb{1}$ near 0 and r_4, the last equality is justified and we see that β'_t is T^2-invariant on $\{r \in [0, r_4]\}$ (so in particular on $\{r \in [l_i, r_3]\}$) for all t.

Step 3. Finally, we deform $\Phi_1^*\beta'$ by deforming β' to $\tilde{\beta}$. Namely, we consider the linear homotopy $(1-t)\beta' + t\tilde{\beta}$ between the two forms $\beta', \tilde{\beta}$ defining the contact structure $\tilde{\xi}$ and set

$$\beta_t := \Phi_1^*((1-t)\beta' + t\tilde{\beta}).$$

Note that on $\{r \in [0, w_i]\}$, we have

$$(1-t)\beta' + t\tilde{\beta} = \alpha_{\tilde{h}},$$

and thus on $\{r \in [l_i, r_3]\}$ we see that

$$\beta_t = \Phi_1^*\alpha_{\tilde{h}} = \alpha_{\tilde{h}\circ\Phi_1} = \alpha_{\tilde{h}(r-l)} = \alpha_{h(r)+(A,B)} = \alpha_h + A\,d\phi + B\,d\theta = \alpha + A\,d\phi + B\,d\theta$$

is fixed.

We define the homotopy α_t by restricting the homotopy constructed in Steps 1–3 above to N, thus $\alpha_0 = \alpha$ and $\alpha_1 = \Phi_1^*\tilde{\beta}|_N$. We define the open book decomposition $(B, \pi) := (\Phi_1^{-1}(\bar{B})|_N, \bar{\pi} \circ \Phi_1)$ on N. Now we check conditions (i–v) in the proposition with '$\{r \in [l_i, r_3]\}$' in place of 'near ∂N'.

(i) The contact form $\Phi_1^*\tilde{\beta}$ is supported by the open book $(\Phi_1^{-1}(\bar{B}), \bar{\pi} \circ \Phi_1)$ because $\tilde{\beta}$ is supported by $(\bar{B}, \bar{\pi})$. The last computation in Step 3 above shows that on $\{r \in [l_i, r_3]\}$ we have $d\alpha_1 = d\alpha_h = d\alpha$.

For (ii), note that $\bar{\pi}(\theta, r, \phi) = n_i\theta$ on $\{r \in [r_1, r_2]\}$ implies that $\bar{\pi} \circ \Phi_1(\theta, r, \phi) = n_i\theta$ on $\{r \in [l_i, r_3]\}$.

(iii) In the construction in Steps 1–3 above, we always checked the T^2-invariance of the contact forms on $\{r \in [l_i, r_3]\}$.

(iv) Recall that the homotopies in Steps 1 and 3 are constant near $T_i = \{r = l_i\}$, so the homotopy α_t is given near T_i by the homotopy $\beta'_t = \alpha_{h^t}$ in Step 2, where $h^t := \tilde{h} \circ \Phi_t$. Thus, for $t = 1$ the first component satisfies $h_1^1(l_i) = \tilde{h}_1(r_1) > 0$, and if $h_1(l_i) > 0$, then $h_1^1(l_i) = \tilde{h}_1(r_1) < \tilde{h}_1(l_i) = h_1(l_i)$ because we chose $\tilde{h}_1$ strictly increasing on $(0, l_i]$ in this case. Since $\Phi_t(l_i)$ decreases from l_i to r_1 as t increases from 0 to 1, Figures 2 and 3 show that the derivatives $(h^t)'(l_i) = \Phi'_t(l_i)\tilde{h}'(\Phi_t(l_i))$ have rotation number 0 if $h_1(l_i) > 0$ and -1 if $h_1(l_i) < 0$. Moreover, by adding positive rotations in the construction of h, we can decrease the rotation numbers by arbitrary integers, in particular we can make it -1 in the case $h_1(l_i) > 0$.

Condition (v) can be arranged as in the special case discussed at the beginning of this proof. □

7. *Existence of supporting open books*

In this section, we prove Theorem 1.3.

7.1. *The structure theorem*

Here, we collect and refine some results from [8] that will be needed in the proof in Subsection 7.2.

PROPOSITION 7.1. *Every stable Hamiltonian structure on a closed* 3*-manifold* M *is stably homotopic to an SHS* (ω, λ) *for which there exists a* (*possibly disconnected and possibly with boundary*) *compact three-dimensional submanifold* $N = N^+ \cup N^- \cup N^0$ *of* M*, invariant under the Reeb flow, and a* (*possibly empty*) *disjoint union* $U = U_1 \cup \cdots \cup U_k$ *of compact integrable regions with the following properties*:

(i) $\operatorname{int} U \cup \operatorname{int} N = M$;

(ii) *the proportionality coefficient* $f := d\lambda/\omega$ *is constant positive, respectively, negative on each connected component of* N^+, *respectively,* N^-;
(iii) *on each* $U_i \cong [0,1] \times T^2$ *the SHS* (ω, λ) *is* T^2*-invariant and* f *is nowhere zero;*
(iv) *on* N^0 *there exists a closed* 1*-form* $\bar{\lambda}$ *representing a primitive integer cohomology class* $\bar{\lambda} \in H^1(N^0; \mathbb{Z})$ *such that* $\bar{\lambda} \wedge \omega > 0$ *and* $\bar{\lambda}$ *is* T^2*-invariant near* ∂N^0.

Remark 7.2. If f is non-negative, then so is the new proportionality coefficient constructed in the proof of Proposition 7.1 and hence $N^- = \emptyset$. Then all binding components of the open book constructed in the proof of Theorem 1.3 in the next section occur in N^+ and thus have positive signs.

Finally, we will need the following simple lemma on immersions. Recall that $\alpha_h = h_1(r)\, d\phi + h_2(r)\, d\theta$ (defined at the beginning of Section 6) is a positive contact form if and only if

$$h_1' h_2 - h_2' h_1 > 0. \tag{21}$$

Lemma 7.3. *Let* $h = (h_1, h_2) : [-\delta, \delta] \to \mathbb{C}$ *be an immersion satisfying the contact condition* (21) *and* $h_1' > 0$. *Then the following conditions are satisfied.*

(a) *For any constant* $B \geqslant 0$, *the immersion* $h + (0, B)$ *satisfies* (21).
(b) *If in addition* $h_2(0) > 0$, *then for every* $\varepsilon > 0$ *there exists a constant* $A \in \mathbb{R}$ *such that for all* $t \in [0,1]$ *the immersion* $h + t(A, 0)$ *satisfies* (21) *near* $r = 0$, *and* $(h_1 + A)(0) \in (0, \varepsilon)$.

Proof. (a) Condition (21), $h_1' > 0$ and $B \geqslant 0$ imply

$$h_1'(h_2 + B) - h_2' h_1 = (h_1' h_2 - h_2' h_1) + h_1' B > 0.$$

(b) At $r > 0$, we have $h_1' h_2 > 0$ by assumption, so for sufficiently small $\delta \in (0, \varepsilon)$

$$0 < h_1' h_2 - h_2' \delta = h_1' h_2 - h_2'(h_1 + A),$$

with $A := \delta - h_1(0)$. Linear interpolation from the right-hand side to $h_1' h_2 - h_2' h_1 > 0$ yields $h_1' h_2 - h_2'(h_1 + tA) > 0$ for all $t \in [0,1]$. □

7.2. *Proof of Theorem* 1.3

Let (ω, λ) be an SHS obtained after application of Proposition 7.1. We will use the following terminology: $N^\pm$ and N^0 are called positive/negative contact parts and the flat parts; connected components of $N^\pm$ and N^0 will be called regions. We will construct the stable homotopy and the supporting open book successively on the different types of regions.

Flat regions. Consider a flat region N with the primitive integer 1-form $\bar{\lambda}$ provided by Proposition 7.1. Integration of $\bar{\lambda}$ over paths from a fixed base point yields a fibration

$$\pi : N \longrightarrow S^1, \tag{22}$$

such that $d\pi = \bar{\lambda}|_N$. Let $k_i \gamma_i$ denote the restriction of the cohomology class $[\bar{\lambda}|_N]$ to the boundary component T_i of N, where $\gamma_i \in H^1(T_i; \mathbb{Z})$ is a primitive integer cohomology class and $k_i \in \mathbb{N}$ is the multiplicity. Since $\bar{\lambda}$ is T^2-invariant near T_i, there exist coordinates near T_i in which the projection is given by $\pi(r, \phi, \theta) = k_i \theta$.

Contact regions. Now let N be a contact region, so $\omega = c\, d\lambda$ on N for some constant $c \neq 0$. After possibly switching the orientation of N, we may assume that λ is a *positive* contact form (but c may still be negative). Let $[-\delta, \delta] \times T^2$ be a tubular neighbourhood of a boundary component $T_i = \{0\} \times T^2$ of N which is contained in an integrable region (such that $[0, \delta] \times$

$T^2 \subset N$). On this neighbourhood the SHS (ω, λ) is given by $(c\, d\alpha_h, \alpha_h)$ for some immersion $h : [-\delta, \delta] \to \mathbb{C}$ satisfying the contact condition (18). After a perturbation of h supported near $r = 0$, we may assume that the cohomology class $[\alpha_h|_{T_i}] \in H^1(T_i; \mathbb{R})$ is rational. Let γ_i be the primitive integer cohomology class in $H^1(T_i; \mathbb{Z})$ positively proportional to $[\alpha_h|_{T_i}]$. We choose linear coordinates (ϕ, θ) on T^2 in which $\gamma_i = [d\theta]$. Then the positive contact condition yields $d\theta \wedge d\alpha_h > 0$ near $r = 0$. Since $\alpha_h = h_1(r)\, d\phi + h_2(r)\, d\theta$, the immersion $h = (h_1, h_2)$ satisfies $h_1(0) = 0$, $h_1'(0) > 0$ and $h_2(0) > 0$. So after a perturbation of h supported near $r = 0$, we may assume that $h = (r, a)$ near $r = 0$ with a constant $a > 0$. By a further deformation (keeping the contact condition) supported near $r = 0$, we can achieve that $h = (r, 1)$ near $r = 0$. After performing these deformations near all boundary components of N, we can apply the easy case of Proposition 6.1 in which condition (v) (with $l_i = 0$) holds near all boundary components. It yields a homotopy of positive contact forms λ_t on N such that $\lambda_t = \lambda$ near ∂N, and λ_1 is supported by an open book (B, π) with $\pi(r, \phi, \theta) = n_i\theta$ near each boundary component T_i. We obtain a corresponding stable homotopy (ω_t, λ_t) by setting $\omega_t := c\, d\lambda_t$ with the constant $c \neq 0$ from above.

Introducing small contact regions. It remains to consider an integrable region $[a, b] \times T^2$. The open book projections on the adjacent contact/flat regions constructed above provide primitive integer cohomology classes $\gamma_a \in H^1(T^2, \mathbb{Z})$ near a and $\gamma_b \in H^1(T^2, \mathbb{Z})$ near b. Note that in general we will have $\gamma_a \neq \gamma_b$, in which case the open book projections π given near the boundary of $[a, b] \times T^2$ do not extend over $[a, b] \times T^2$. To deal with this, we choose a subdivision

$$a = r_0 < r_1 < \cdots < r_n = b, \tag{23}$$

and a sequence $\{\gamma_k\}_{k=1,\ldots,n}$ of primitive integer cohomology classes in $H^1(T^2, \mathbb{Z})$ with the following properties:

(i) $\gamma_1 = \gamma_a$ and $\gamma_n = \gamma_b$;
(ii) on the interval $[r_{k-1}, r_k]$ $(k = 1, \ldots, n)$ we have $\bar{\gamma}_k \wedge \omega > 0$, where $\bar{\gamma}_k = p_k\, d\phi + q_k\, d\theta$ is the T^2-invariant representative of γ_k.

Recall that, according to Proposition 7.1, the proportionality factor $f = d\lambda/\omega$ is nowhere zero on $[a, b] \times T^2$. We apply Corollary 3.2 (with $\{a\} = \{a_k\}$ and followed by averaging) to the level sets $a_k = f(r_k)$, $k = 1, \ldots, n-1$, to find a new T^2-invariant stabilizing 1-form $\tilde{\lambda}$ for ω such that $\tilde{f} := d\tilde{\lambda}/\omega$ satisfies $\tilde{f}(r) \equiv a_k \neq 0$ on some intervals $[r_k - \delta_k, r_k + \delta_k]$. We rename $\tilde{\lambda}, \tilde{f}$ back to λ, f. We will refer to the regions $\{r \in [r_k - \delta_k, r_k + \delta_k]\}$ as *small contact regions* in order to distinguish them from the original (*large*) contact or flat regions constructed above. It is important to note that adjacent small contact regions have the same sign, that is, the contact structures are either both positive or both negative.

We apply Proposition 6.1 to construct contact homotopies and supporting open books on the small contact regions. To extend them over the integrable regions $[r_{k-1}, r_k] \times T^2$, we distinguish two cases: integrable regions $[a, r_1] \times T^2$ and $[r_{n-1}, b] \times T^2$ connect a large contact/flat region to a small contact region, and regions $[r_{k-1}, r_k] \times T^2$, $k = 2, \ldots, n-1$, connecting two small contact regions. Recall that on each such region we have $\bar{\gamma}_k \wedge \omega > 0$ for some T^2-invariant 1-form on T^2 representing a primitive integer cohomology class. After a linear change of coordinates, we may assume that $\bar{\gamma}_k = d\theta$ and thus

$$d\theta \wedge \omega > 0.$$

Integrable regions I. Consider an integrable region $[a, r_1] \times T^2$ connecting a large contact/flat region N with a small contact region $[r_1 - \delta, r_1 + \delta] \times T^2$. (The region $[r_{n-1}, b] \times T^2$ can be treated analogously.)

In both the contact and flat case, the stable homotopy on N constructed above was constant near $\{r = a\}$. So in order to extend the homotopy over $[a, r_1]$, we need to arrange rotation number zero at $r = r_1 - \delta$ in the application of Proposition 6.1 to the small contact region. To achieve this, we prepare the stabilizing 1-form λ before applying Proposition 6.1.

We write $\omega = d\alpha_h$ and $\lambda = \lambda_g$ for functions $h = (h_1, h_2) : [a, b] \to \mathbb{C}$ and $g = (g_1, g_2) : [a, b] \to \mathbb{C}$ satisfying (5). We may assume that λ is a positive contact form on $[r_1 - \delta, r_1 + \delta] \times T^2$. (Otherwise, we make the orientation-reversing coordinate change $\Psi : \phi \mapsto -\phi$ and replace (ω, λ) by $(-\Psi^*\omega, \Psi^*\lambda)$.) So we have that $h(r) - cg(r)$ is a constant complex number on $[r_1 - \delta, r_1 + \delta]$ for some $c > 0$.

Fix some $\varepsilon > 0$ that will be specified later. We choose $B \geqslant 0$ such that $g_2(0) + B > 0$. By Lemma 7.3(a), the homotopy $\{g + t(0, B)\}_{t\in[0,1]}$ satisfies the contact condition on $[r_1 - \delta, r_1 + \delta]$. By Lemma 7.3(b), we find $A \in \mathbb{R}$ and a homotopy $\{g + (0, B) + t(A, 0)\}_{t\in[0,1]}$ of contact immersions on $[r_1 - \delta, r_1 + \delta]$ (for a possibly smaller δ) such that $(g_1 + A)(r) \in (0, \varepsilon)$ for all r. Denote by g^t, $t \in [0, 1]$, the concatenation of these two contact homotopies on $[r_1 - \delta, r_1 + \delta]$ and note that $c(g^t)' = h'$ for all t, so the g^t satisfy (5) for all t. We use Proposition 2.2 to extend this homotopy to $[a, b]$ such that it satisfies (5) and coincides with g outside a neighbourhood of $[r_1 - \delta, r_1 + \delta]$. We rename the new stabilizing function g^1 back to g, so we have achieved that g is a contact immersion on $[r_1 - \delta, r_1 + \delta]$ with first component $g_1 \in (0, \varepsilon)$. Moreover, we still have that $h(r) - cg(r)$ is a constant complex number on $[r_1 - \delta, r_1 + \delta]$ with $c > 0$ as above. (Note that we may have destroyed positivity of $f = d\lambda/\omega$, but we will not need this any more.)

Now we apply Proposition 6.1 to the small contact region $(N, \alpha) = ([r_1 - \delta, r_1 + \delta] \times T^2, \lambda_g)$ to find an open book decomposition (B, π) and a contact homotopy λ_t. Write $\lambda_t = \lambda_{g^t}$ near the boundary torus $T_i = \{r_1 - \delta\} \times T^2$. Proposition 6.1(iv) (with $h^t = g^t$ and $l_i = r_1 - \delta$) and $g_1 \in (0, \varepsilon)$ implies that the first component of g^1 satisfies $g_1^1 \in (0, \varepsilon)$. Moreover, we can choose $(g^t)'(r_1 - \delta)$, $t \in [0, 1]$, to have rotation number zero.

We extend λ_t to a stable homotopy (ω_t, λ_t) on $[r_1 - \delta, r_1 + \delta] \times T^2$ by $\omega_t := c\, d\lambda_t$. Near $r = r_1 - \delta$ we have $\omega_t = \omega_{h^t}$ with immersions h^t, $t \in [0, 1]$, defined by $h^t(r) - cg^t(r) \equiv \text{const} \in \mathbb{C}$. Thus, the first components satisfy $|h_1^1 - h_1| = c|g_1^1 - g_1| < c\varepsilon$ near $r = r_1 - \delta$. Since $h_1' > 0$ on $[a, r_1]$, for sufficiently small ε we can extend h^1 from a neighbourhood of $r_1 - \delta$ to an immersion on $[a, r_1 - \delta]$ which coincides with h near $r = a$ and whose first component satisfies $(h_1^1)' > 0$ everywhere. Moreover, since $(h^t)'(r_1 - \delta)$, $t \in [0, 1]$, has rotation number zero, we can extend the h^t from a neighbourhood of $r_1 - \delta$ to immersions on $[a, r_1 - \delta]$ which coincide with h near $r = a$ and satisfy $h^0 = h$. Now we use Proposition 2.2 to extend the g^t from a neighbourhood of $r_1 - \delta$ to functions on $[a, r_1 - \delta]$ which coincide with g near $r = a$ such that $g^0 = g$ and (h^t, g^t) satisfies (5) for all $t \in [0, 1]$. Thus, $(\omega_t, \lambda_t) = (d\alpha_{h^t}, \lambda_{g^t})$ is a stable homotopy on $[a, r_1 - \delta] \times T^2$, which coincides with the previously constructed homotopies near the boundary, from $(\omega_0, \lambda_0) = (\omega, \lambda)$ to (ω_1, λ_1) satisfying $d\theta \wedge \omega_1 > 0$.

Integrable regions II. It remains to consider an integrable region $[r_{k-1}, r_k] \times T^2$ connecting two small contact regions. Recall that the two contact regions have the same sign, so we may assume that they are both positive. (Otherwise, we make the orientation-reversing coordinate change $\Psi : \phi \mapsto -\phi$ and replace (ω, λ) by $(-\Psi^*\omega, \Psi^*\lambda)$.)

Let us ignore for the moment the cohomology classes of ω_t, which we will discuss below. Then we conclude the argument as follows. We apply Proposition 6.1 to both small contact regions, choosing the option 'rotation number -1' in Proposition 6.1(iv) at both boundary tori $T_{k-1} = \{r_{k-1} + \delta\} \times T^2$ and $T_k = \{r_k - \delta\} \times T^2$. Note that the chosen coordinates (r, ϕ, θ) on $[r_{k-1}, r_k] \times T^2$ are related to the coordinates in Proposition 6.1 by the identity map near T_{k-1}, and (for example) by the map $(r, \phi, \theta) \mapsto (-r, -\phi, \theta)$ near T_k. Thus, in the integrable region coordinates (r, ϕ, θ) the rotation number equals -1 at T_{k-1} and $+1$ at T_k. (Here, it is crucial that no positive and negative contact regions are connected by an integrable region.) Thus, the homotopy of HSs ω_t given on the small contact regions extends over the region $[r_{k-1}, r_k] \times T^2$ (by extending the corresponding immersions h^t) such that $\omega_0 = \omega_1 = \omega$. We use Proposition 2.2 to extend the λ_t from the contact regions to stabilizing 1-forms over $[r_{k-1}, r_k] \times T^2$. Note that $d\theta \wedge \omega_1 = d\theta \wedge \omega_0 > 0$ on $[r_{k-1}, r_k] \times T^2$.

Defining the open book. The preceding step completes the construction of the homotopy of SHS (ω_t, λ_t), $t \in [0,1]$, on M. It remains to define the open book (B, π) supporting (ω_1, λ_1). For this, recall that in Proposition 6.1(ii) we have the freedom to prescribe the multiplicities $n_i \geqslant K$ of the open book projection near each boundary component T_i of a contact region, for some constant K depending on this region. Let K_0 be the maximum of these constants K over all (large and small) contact regions. We set the multiplicities at all boundary tori of contact regions equal to K_0, except for those adjacent to flat regions. For the latter the choice can be made as follows. Let $[a,b] \times T^2$ be an integrable region with $T_a = \{a\} \times T^2$ contained in a flat region N and $\{b\} \times T^2$ in a (small) contact region N^c (the opposite case is analogous). By the discussion following equation (22), there is a primitive cohomology class $\gamma_a \in H^1(T_a; \mathbb{Z})$ and a multiplicity $k_a \in \mathbb{N}$ such that $k_a l_a = [d\pi|_{T_a}]$ for the projection $\pi : N \to S^1$. We choose the multiplicity for T_b in Proposition 6.1 to be the product $k_a K_0$. Then the fibration $K_0 \pi : N \to S^1 = \mathbb{R}/\mathbb{Z}$ extends over the integrable region and coincides with the one produced by Proposition 6.1 near T_b. With these choices, the open book projections on the contact regions and the projections on the flat regions extend over all the integrable regions to an open book structure on M supporting (ω_1, λ_1).

Preserving the cohomology class. The discussion so far completes the proof of Theorem 1.3 except for vanishing of the cohomology classes $[\omega_t - \omega] \in H^2(M; \mathbb{R})$. Let us analyse how we can ensure exactness in the previous constructions. On a flat region, we have $\omega_t = \omega$. On a (large or small) contact region N, we have $\omega = c\, d\lambda$ for some constant $c \neq 0$. We change λ by a contact homotopy λ_t and define ω_t by $\omega_t = c\, d\lambda_t$. Next consider an integrable region $U = [a,b] \times T^2$ connecting two contact/flat regions N_0, N_ℓ on which $\omega = c_i d\lambda$ for constants $c_0, c_\ell \neq 0$. Moreover, suppose that we have several small integrable regions $N_i = [r_i - \delta, r_i + \delta] \times T^2$ in U on which $\omega = c_i d\lambda$ for constants $c_i \neq 0$, $i = 1, \dots, \ell - 1$. We change λ by a contact homotopy λ_t and define ω_t by $\omega_t = c_i\, d\lambda_t$ on each N_i, $i = 0, \dots, \ell$. Denote by $\partial^\pm N_i = \{r_i \pm \delta\} \times T^2$ the left, respectively, right boundary components for $i = 1, \dots, \ell - 1$ and set $\partial^+ N_0 = \{a\} \times T^2$, $\partial^- N_\ell = \{b\} \times T^2$. On U, we can write $\lambda = \lambda_g$ and $\omega = d\alpha_h$ for functions $g, h : [a,b] \to \mathbb{C}$ satisfying (5). So near each boundary component $\partial^\pm N_i$, we have a relation

$$h - c_i g = k_i^\pm, \tag{24}$$

for some constants $k_i^\pm \in \mathbb{C}$. Moreover, near $\partial^\pm N_i$ we can write $\lambda_t = \lambda_{g^t}$ for locally defined functions g^t. Now suppose that we find a family of immersions $h^t : [a,b] \to \mathbb{C}$ and a family of constants $k^t \in \mathbb{C}$ such that $h^0 = h$, $k^0 = 0$ and

$$h^t = \begin{cases} c_i g^t + k_i^\pm & \text{near } r = r_i \pm \delta, \\ c_0 g^t + k_0^+ + k^t & \text{near } r = a, \\ c_\ell g^t + k_\ell^- + k^t & \text{near } r = b. \end{cases} \tag{25}$$

Then we obtain an *exact* homotopy of HS $\omega_t = \omega + d\beta_t$ on U, extending the given one on $\bigcup_i N_i$ and T^2-invariant on $\bigcup_i N_i$, by setting

$$\beta_t := \begin{cases} c_0(\lambda_t - \lambda) & \text{on } N_0, \\ c_\ell(\lambda_t - \lambda) & \text{on } N_\ell, \\ c_i(\lambda_t - \lambda) - \alpha_{k^t} & \text{on } N_i,\ i = 1, \dots, \ell - 1, \\ \alpha_{h^t} - \alpha_h - \alpha_{k^t} & \text{on } U \setminus \bigcup_i N_i. \end{cases}$$

Finally, Proposition 2.2 provides a homotopy of stabilizing 1-forms λ_t for ω_t on U, extending the given one on $\bigcup_i N_i$ and T^2-invariant on $\bigcup_i N_i$. Note that in 'Integrable regions I' above we arranged condition (25) with constant $k^t = 0$, but in 'Integrable regions II' this will in general fail.

The $\theta - \phi - \theta$ trick. Consider again an integrable region $[a,b] \times T^2$ connecting two (small) positive contact regions N_a, N_b with an SHS $(\omega, \lambda) = (d\alpha_h, \lambda_g)$ satisfying $d\theta \wedge \omega > 0$, that is, $h_1' > 0$. Near $r = a, b$ we are given a contact homotopy λ_t and define ω_t by $\omega_t = c_a d\lambda_t$, respectively, $\omega_t = c_b d\lambda_t$, that is,

$$h(r) - c_a g(r) = k_a, \quad \text{respectively, } h(r) - c_b g(r) = k_b,$$

for constant $c_a, c_b > 0$ and $k_a, k_b \in \mathbb{C}$. We face the following problem: In order to ensure exactness, we wish to define the homotopy $h^t : [a, b] \to \mathbb{C}$ satisfying (25) near $r = a, b$, but then the difference of the first components of $h^1(b) - h^1(a) = c_b g^1(b) + k_b - c_a g^1(a) - k_a$ may be negative and we cannot achieve $(h_1^1)' > 0$. To overcome this problem, we introduce two new small contact regions $N_x = [x - \delta, x + \delta] \times T^2$ and $N_y = [x - \delta, y + \delta] \times T^2$ around new subdivision points x and y:

$$a < x < y < b.$$

Now we repeat the construction using the following open book projections:

$$\pi(r, \phi, \theta) := \begin{cases} \theta & \text{on } [a, x - \delta], \\ \phi & \text{on } [x + \delta, y - \delta], \\ \theta & \text{on } [y + \delta, b]. \end{cases} \tag{26}$$

It will turn out that, with this trick, we can use the freedom in Proposition 6.1 and Lemma 7.3 to achieve exactness of $[\omega_t - \omega]$ as well as positivity of $d\theta \wedge \omega_1$, respectively, $d\phi \wedge \omega_1$ on the respective regions. This will be carried out in the remaining two steps.

Preparation for the exact homotopy. We first homotop the immersion $h : [a, b] \to \mathbb{C}$ through immersions rel $\partial[a, b]$ to one (still denoted by h) which satisfies

$$h(r) = e^{-i(r - x + \pi/4)}$$

on some small subinterval $[x - \delta, y + \delta] \subset (a, b)$, see Figure 6.

Hence, $h|_{[x-\delta, y+\delta]}$ is contact and satisfies the conditions

$$h_1' > 0, \quad h_2' < 0, \quad ih' = h.$$

Note that we may have lost the condition $h_1' > 0$ outside the interval $[x - \delta, y + \delta]$. The 1-form $h_1(r)\, d\phi + h_2(r)\, d\theta$ stabilizes $d\alpha_h$ on $[x - \delta, y + \delta]$. So by Proposition 2.2, we find a stabilizing form λ_g for $\omega = d\alpha_h$ on $[a, b] \times T^2$ which agrees with the original λ near the boundary and such that $g = h$ on $[x - \delta, y + \delta]$.

Next we use Lemma 7.3 and Proposition 2.2 to modify the stabilizing 1-form λ_g on the small contact regions $N_x = [x - \delta, x + \delta] \times T^2$ and $N_y = [y - \delta, y + \delta] \times T^2$ (possibly shrinking δ in the process). For this, we make the following orientation-preserving coordinate change on $[x - \delta, y + \delta] \times T^2$:

$$\hat{\theta} := \phi, \quad \hat{\phi} := -\theta.$$

We rewrite λ_g in the new coordinates,

$$\lambda_g = g_1\, d\phi + g_2\, d\theta = \hat{g}_1\, d\hat{\phi} + \hat{g}_2\, d\hat{\theta}, \quad \hat{g}_1 = -g_2, \quad \hat{g}_2 = g_1,$$

and note that the contact condition $\langle ig', g \rangle > 0$ is preserved by this coordinate change. Moreover, by construction we have $\hat{g}_1' = -g_2' = -h_2' > 0$. Thus, by Lemma 7.3 (applied in the coordinates $(\hat{\phi}, \hat{\theta})$ near x and y) and Proposition 2.2, we can add complex constants to g near x, y to obtain a new stabilizing form, still denoted by λ_g, which coincides with the previous one outside a neighbourhood of $N_x \cup N_y$ and satisfies

$$\begin{cases} 0 < \hat{g}_1 < \varepsilon \text{ and } g_1 = \hat{g}_2 > 0 & \text{on } [x - \delta, x + \delta], \\ 0 < \hat{g}_1 < \varepsilon \text{ and } g_1 = \hat{g}_2 \geqslant A > 0 & \text{on } [y - \delta, y + \delta], \end{cases} \tag{27}$$

for an arbitrarily small constant ε and an arbitrarily large constant A that will be specified later.

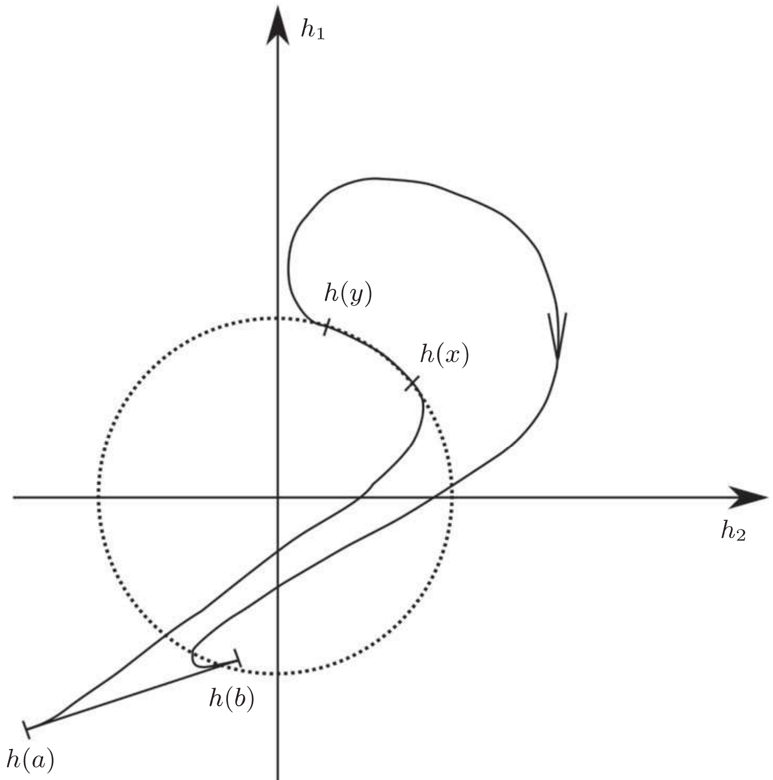

FIGURE 6. *The deformation of* h.

Constructing the exact homotopy. Now we apply Proposition 6.1 to the contact regions N_a, N_b, N_x and N_y, with the open book projections on the connecting integrable regions given by θ, respectively, $\hat{\theta} = \phi$ as in (26). We choose the option 'rotation number -1' in Proposition 6.1(iv) at all boundary tori. Recall that near the boundary component $\partial^{\pm} N_x, \partial^{\pm} N_y, N_a, N_b$ we have the respective relations

$$h - g = k_x^{\pm}, \quad h - g = k_y^{\pm}, \quad h - c_a g = k_a, \quad h - c_b g = k_b,$$

for some constants $c_a, c_b > 0$ and $k_x^{\pm}, k_y^{\pm}, k_a, k_b \in \mathbb{C}$. Moreover, near each boundary torus the homotopy of contact forms λ_t from Proposition 6.1 can be written as λ_{g^t} for locally defined functions g^t. We define a family of immersions h^t near the boundary tori as in (25) with $k^t = -t(K, 0)$, that is,

$$h^t = \begin{cases} g^t + k_x^{\pm} & \text{near } r = x \pm \delta, \\ g^t + k_y^{\pm} & \text{near } r = y \pm \delta, \\ c_a g^t + k_a - t(K,0) & \text{near } r = a, \\ c_b g^t + k_b - t(K,0) & \text{near } r = b, \end{cases} \tag{28}$$

where $K \geqslant 0$ is a real constant that will be chosen below. We need to extend h^t to a family of immersions $[a, b] \to \mathbb{C}$ such that $h^0 = h$ and h^1 is supported by the respective open books, that is, the $d\phi$-component h_1^1 increases on $[a, x - \delta] \cup [y + \delta, b]$ and the $d\hat{\phi}$-component $\hat{h}_1^1 = -h_2^1$ increases on $[x + \delta, y - \delta]$. Note that, once we have constructed h^1 with these properties, the extension of the homotopy h^t follows as in 'Integrable regions II' from our choice of rotation numbers.

First, we consider the interval $[x+\delta, y-\delta]$ in coordinates $(\hat{\phi}, \hat{\theta})$. Near $x+\delta$ and $y-\delta$, we have $\hat{g}_1 \in (0, \varepsilon)$ according to (27), hence $\hat{g}_1^1 \in (0, \varepsilon)$ by Proposition 6.1 and $\hat{h}_1^1 - \hat{h}_1 \in (0, \varepsilon)$ by (28). Since $\hat{h}_1 = -h_2$ strictly increases on $[x, y]$, it follows that $\hat{h}_1^1(y-\delta) > \hat{h}_1^1(x+\delta)$, so we can extend the immersion h^1 over $[x+\delta, y-\delta]$ so that $\hat{h}_1^1$ is strictly increasing.

Next, we consider the interval $[a, x-\delta]$ in coordinates (ϕ, θ). According to (28), the $d\phi$-component of h^1 changes over this interval by the amount

$$h_1^1(x-\delta) - h_1^1(a) = C + K \in \mathbb{R},$$

where

$$C := g_1^1(x-\delta) + (k_x^-)_1 - c_a g_1^1(a) - (k_a)_1 \in \mathbb{R}$$

depends on the functions h, g constructed above near a and $x-\delta$ (but not on the constant A in (27)). We choose the constant $K \geqslant 0$ in (28) large enough such that $C + K > 0$, so we can extend the immersion h^1 over $[a, x-\delta]$ with h_1^1 is strictly increasing.

Finally, we consider the interval $[y+\delta, b]$ in coordinates (ϕ, θ). Recall that near the boundary torus $\{r = y+\delta\}$ the coordinates (r, ϕ, θ) on the integrable region $[a, b] \times T^2$ are related to the coordinates of Proposition 6.1 by the coordinate change $(r, \phi, \theta) \mapsto (-r, -\phi, \theta)$. In the coordinates of Proposition 6.1, the $d\phi$- component of the stabilizing form λ at $y+\delta$ changes from the (very negative) value $-g_1(y+\delta) \leqslant -A$ (here we use (27)!) to a positive value $-g_1^1(y+\delta) > 0$, so the value increases by at least A. Switching back to the integrable region coordinates (r, ϕ, θ), we see a decrease by at least A:

$$g_1^1(y+\delta) - g_1(y+\delta) < -A. \tag{29}$$

The $d\phi$-component of h^1 changes over the interval $[y+\delta, b]$ by the amount

$$h_1^1(b) - h_1^1(y+\delta) = [h_1^1(b) - h_1(b)] + [h_1(b) - h_1(y+\delta)] + [h_1(y+\delta) - h_1^1(y+\delta)].$$

By (28), the first term in [ ] on the right-hand side equals

$$h_1^1(b) - h_1(b) = c_b(g_1^1(b) - g_1(b)) - K,$$

which depends on the functions h, g constructed above near b and the constant K chosen above, but not on the constant A. The second term depends on the function h near $y+\delta$ and b, but not on g and hence not on the constant A. The third term is estimated using (28) and (29) by

$$h_1(y+\delta) - h_1^1(y+\delta) = g_1(y+\delta) - g_1^1(y+\delta) > A.$$

Therefore, by choosing the constant $A \geqslant 0$ large enough we can achieve that $h_1^1(b) - h_1^1(y+\delta) > 0$, so we can extend the immersion h^1 over $[y+\delta, b]$ with h_1^1 strictly increasing. This concludes the proof of Theorem 1.3.

8. *Examples and discussion*

In this section, we consider various examples of SHS supported by open books on simple manifolds: S^3 and $S^1 \times S^2$. We begin with some general remarks on the behaviour of our structures under switching the orientation of the ambient manifold. Let (B, π) be an open book on M. Let $\bar{M}$ denote the manifold M with the orientation reversed. Then $(B, -\pi)$ is an open book on $\bar{M}$, where the orientation of the page and the binding remains the same. Alternatively, we could have switched these orientations instead of introducing a minus in front of π. If (ω, λ) is an SHS on M, then $(\omega, -\lambda)$ and $(-\omega, \lambda)$ are SHS on $\bar{M}$. Now if (ω, λ) is an SHS on M supported by an open book (B, π) with signs $s_{(\omega,\lambda)} : \pi_0(B) \to \{+, -\}$, then $(\omega, -\lambda)$ is supported by $(B, -\pi)$ (on $\bar{M}$!) with reversed signs: $s_{(\omega,-\lambda)} = -s_{(\omega,\lambda)}$.

If a positive contact form α is supported by an open book (B, π) on M, then the *negative* contact form α on $\bar{M}$ is supported by $(B, -\pi)$.

Note that if α is a *negative* contact form on M, then the Reeb vector field of α and that of the SHS $(d\alpha, -\alpha)$ on M differ by sign.

For each SHS (ω, λ), the corresponding homotopy class of oriented plane fields (formal class) will be denoted by $[(\omega, \lambda)]$. Note that this formal class is determined by ω, or equivalently by the Reeb vector field R. So when convenient we will write $[\omega]$ or $[R]$ for that class.

Note also that the supportedness of an SHS (ω, λ) by an open book (B, π) is really a property of ω only.

8.1. S^3

We consider the 3-sphere

$$S^3 = \{(s_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}.$$

Note first the following simple topological lemma.

LEMMA 8.1. *Every nowhere vanishing vector field R on S^3 is homotopic to $-R$ through nowhere vanishing vector fields.*

Proof. Fixing a trivialization of TS^3 and normalizing, nowhere vanishing vector fields on S^3 correspond to maps $F : S^3 \to S^2$. Homotopy classes of such maps are classified by their Hopf invariant $H(F)$ (see, for example, [3]) which can be defined as follows: Pick a normalized positive area form σ on S^2 and a 1-form α on S^3 such that $F^*\sigma = d\alpha$; then $H(F) = \int_{S^3} \alpha \wedge d\alpha$. Now $(-F)^*\sigma = d(-\alpha)$ and $H(-F) = \int_{S^3}(-\alpha) \wedge d(-\alpha) = H(F)$, so F and $-F$ are homotopic. □

We introduce the radial coordinate $r \in [0, 1]$ and two angular coordinates ϕ and θ on S^3 by

$$z_1 = r\, e^{i\theta}, \quad z_2 = \sqrt{1 - r^2}\, e^{i\phi},$$

so $dr \wedge d\theta \wedge d\phi$ is a positive volume form. Let T^2 act on S^3 by rotations in the z_1 and z_2 planes, that is, by shifts along θ and ϕ. Let (B, π) be the open book on S^3 with binding $B = B_0 \cup B_1$, where $B_0 = \{r = 0\}$ and $B_1 = \{r = 1\}$, and projection $\pi := \theta + \phi$. The binding component B_0 is oriented by $d\phi$ and B_1 by $d\theta$. We decorate this open book in four different ways $(+, +)$, $(+, -)$, $(-, +)$ and $(-, -)$ (the first sign is assigned to B_0, the second one to B_1) and denote the respective SHS by $(\omega_{(i,j)}, \lambda_{(i,j)})$ with $i, j \in \{+, -\}$. We will consider T^2-invariant SHS on S^3 defined by immersions $h = h_1 + ih_2 : [0, 1] \to \mathbb{C}$ satisfying the following conditions near $r = 0$ and 1:

(i) all odd derivatives of h vanish at 0 and 1;
(ii) $h_1(0) = h_2(1) = 0$;
(iii) $h_1''(0) \neq 0$ and $h_2''(1) \neq 0$.

These conditions ensure that

$$\alpha_h = h_1(r)\, d\theta + h_2(r)\, d\phi$$

defines a smooth 1-form on S^3 such that

$$\omega = d\alpha_h = h_1'\, dr \wedge d\theta + h_2'\, dr \wedge d\phi$$

is nowhere vanishing. The condition $d\pi \wedge \omega > 0$ for supportedness of ω by (B, π) becomes

$$(d\theta + d\phi) \wedge (h_1'\, dr \wedge d\theta + h_2'\, dr \wedge d\phi) = (h_1' - h_2')\, dr \wedge d\theta \wedge d\phi > 0.$$

Since $dr \wedge d\theta \wedge d\phi$ is a positive volume form on S^3, the last inequality becomes $(h_1 - h_2)' > 0$. With h_1 drawn vertically and h_2 horizontally (as in our pictures), this means that the curve h crosses the lines $h_1 = h_2 +$ constant transversely in the direction from the lower right to the

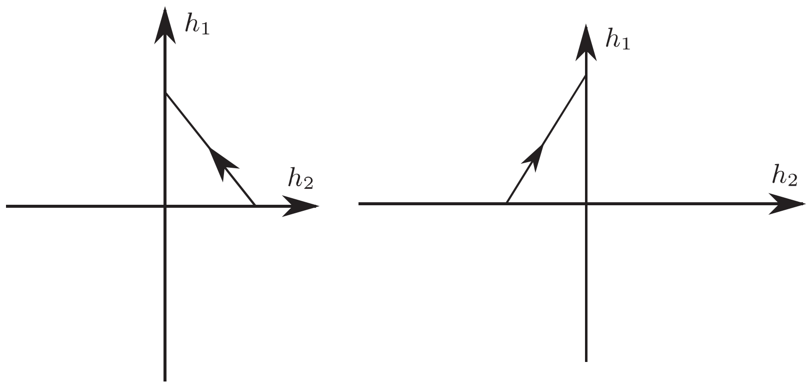

FIGURE 7. *The $(+,+)$-supported HS and the $(+,-)$-supported HS on S^3.*

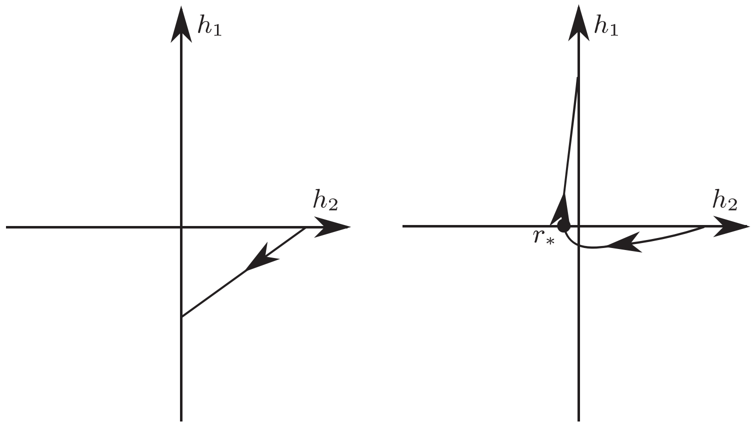

FIGURE 8. *The $(-,+)$-supported and the $(-,-)$-supported HS on S^3.*

upper left quadrant. Let us now discuss the corresponding signs at the binding components. The Reeb vector field is positively proportional to $-h_2'\partial_\theta + h_1'\partial_\phi$. Thus,

$$s(B_0) = \lim_{r\to 0} \operatorname{sign}(h_1'(r)), \quad s(B_1) = -\lim_{r\to 1} \operatorname{sign}(h_2'(r)).$$

With this Figures 7 and 8 should be self-explanatory.

To stabilize $\omega_{(+,-)}$ and $\omega_{(-,+)}$ note that both are exterior derivatives of negative contact forms on S^3. For $\omega_{(+,+)}$ and $\omega_{(-,-)}$, note that we can choose the corresponding primitive 1-forms α_h to be λ_{st}, respectively, λ_{ot}^-, where $\lambda_{\mathrm{st}} = r^2 d\theta + (1-r^2)d\phi$ is the standard positive tight contact form on S^3 and λ_{ot}^- is a negative (!) overtwisted contact form on S^3. See the first picture in Figure 7 and the second one in Figure 8; a family of overtwisted discs $\{r \leqslant r_*, \phi = \phi_*\}$, $\phi_* \in S^1$, occurs at the value r_* in the second picture in Figure 8.

PROPOSITION 8.2. *The formal classes above satisfy*

$$[\omega_{(+,+)}] = [\omega_{(-,-)}] \neq [\omega_{(+,-)}] = [\omega_{(-,+)}].$$

Moreover, the T^2-invariant SHS corresponding to $\omega_{(+,-)}$ and $\omega_{(-,+)}$ are stably homotopic.

The proof uses the following simple lemma.

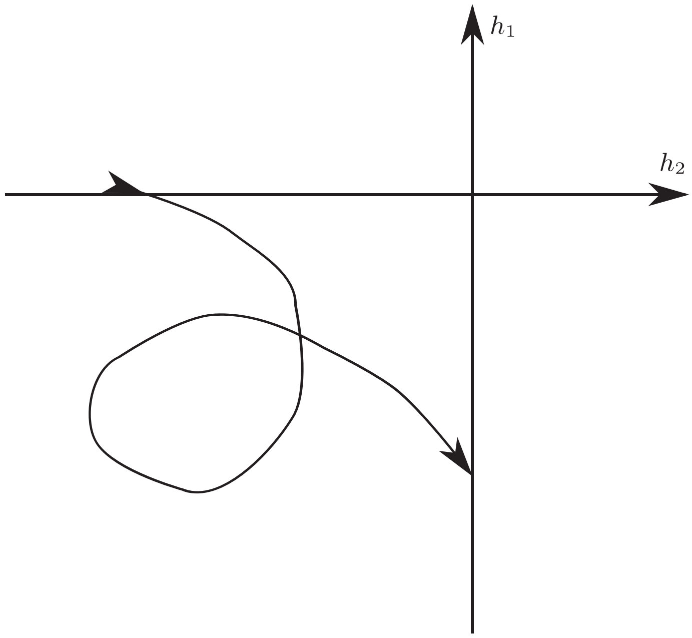

FIGURE 9. $\tilde{\omega}$.

LEMMA 8.3. *Let $h, \tilde{h} : [a, b] \to \mathbb{C}$ be two immersions that coincide on $[a, a + \varepsilon] \cup [b - \varepsilon, b]$. Then the HS $d\alpha_h$ and $d\alpha_{\tilde{h}}$ on $[a, b] \times T^2$ define the same formal class rel $\partial[a, b] \times T^2$.*

Proof. Pick nowhere vanishing vector fields $R, \tilde{R}$ positively generating the kernels of $d\alpha_h$ and $d\alpha_{\tilde{h}}$. These vector fields are tangent to the surfaces $\{r\} \times T^2$ and can be chosen to coincide on $([a, a + \varepsilon] \cup [b - \varepsilon, b]) \times T^2$. Let χ be a cutoff function on $[a, b]$ which equals zero near a and b and 1 on a neighbourhood of $[a + \varepsilon, b - \varepsilon]$. Then

$$(1 - t)R + t\tilde{R} + t(1 - t)\chi(r)\partial_r$$

is a homotopy rel $\partial[a, b] \times T^2$ of nowhere vanishing vector fields connecting R and $\tilde{R}$. □

Proof of Proposition 8.2. For the equality $[\omega_{(+,+)}] = [\omega_{(-,-)}]$, we deform the curve defining $\omega_{(-,-)}$ to the one defining $\tilde{\omega}$ in Figure 9.

Let $\Psi : S^3 \to S^3$ denote the orientation-preserving diffeomorphism of S^3 defined by $\Psi(z_1, z_2) := (\bar{z}_1, \bar{z}_2)$, that is,

$$\Psi(r, \theta, \phi) := (r, -\theta, -\phi).$$

As Ψ is simply a 180° rotation in the $(\mathrm{Im}\, z_1, \mathrm{Im}\, z_2)$ plane, it is isotopic to the identity. Thus, $\tilde{\omega}$ is homotopic to $\Psi^*\tilde{\omega}$ and the latter is defined by a curve h which is the rotation of the one in Figure 9 by 180° . Note that h restricted to $(0, \varepsilon) \cup (1 - \varepsilon, 1)$ coincides with the curve $h_{\mathrm{st}} = (r^2, 1 - r^2)$ defining the standard contact form λ_{st}. Therefore, Lemma 8.3 yields $[\omega_{(+,+)}] = [\Psi^*\tilde{\omega}] = [d\lambda_{\mathrm{st}}] = [\omega_{(-,-)}]$. For the inequality $[\omega_{(-,-)}] \neq [\omega_{(+,-)}]$, consider the orientation-reversing diffeomorphism $\Phi(r, \theta, \phi) = (r, \theta, -\phi)$ of S^3. The first picture in Figure 7 shows that we can choose $\omega_{(+,-)} = d\bar{\lambda}$, where $\bar{\lambda}$ is C^∞-close to $\Phi^*\lambda_{\mathrm{st}}$. By Cieliebak and Volkov [**8**, Lemma 3.44], the formal classes defined by $d\bar{\lambda}_{\mathrm{st}}$ and $d\lambda_{\mathrm{st}}$ are different, from which we conclude that $[\omega_{(+,-)}] \neq [\omega_{(+,+)}]$.

Finally, let us construct a stable homotopy between $\omega_{(+,-)}$ and $\omega_{(-,+)}$. After the obvious T^2-invariant stable homotopies applied to both $\omega_{(+,-)}$ and $\omega_{(-,+)}$, we may assume $\omega_{(+,-)} =$

$-\omega_{(-,+)}$. In other words, $\omega_{(+,-)} = \Psi^*\omega_{(-,+)}$, where $\Psi(r,\theta,\phi) = (r,-\theta,-\phi)$ is the orientation-preserving diffeomorphism $S^3 \to S^3$ defined above. As Ψ is isotopic to the identity, this shows that $\omega_{(+,-)}$ and $\omega_{(-,+)}$ are stably homotopic. $\square$

Recall that $\omega_{(+,+)} = d\lambda_{\rm st}$ and $\omega_{(-,-)} = d\lambda^-_{\rm ot}$, where $\lambda_{\rm st}$ is the standard positive tight contact form and $\lambda^-_{\rm ot}$ is a negative overtwisted contact form defining the same homotopy class of oriented plane fields. This homotopy class of oriented plane fields also contains a positive overtwisted contact form $\lambda^+_{\rm ot}$ (which is not supported by the open book above).

Conjecture 8.4. The SHS corresponding to $\omega_{(+,+)} = d\lambda_{\rm st}$ is not stably homotopic to $\omega_{(-,-)} = d\lambda^-_{\rm ot}$ or $d\lambda^+_{\rm ot}$.

Question 8.5. Are the SHS corresponding to $\omega_{(-,-)} = d\lambda^-_{\rm ot}$ and $d\lambda^+_{\rm ot}$ stably homotopic?

8.2. $S^1 \times S^2$

We view S^2 as the round sphere in $\mathbb{R}^3$,

$$S^2 := \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}.$$

We introduce the angular coordinate θ by

$$x = \sqrt{1-z^2}\cos\theta, \quad y = \sqrt{1-z^2}\sin\theta,$$

so $d\theta \wedge dz$ is a positive volume form on S^2. Let $N := (0,0,1)$ be the north pole and $S := (0,0,-1)$ be the south pole. Let ϕ be the angular coordinate parametrizing S^1. Let (B,π) be the open book on $S^1 \times S^2$ with binding $B = B_0 \cup B_1$, where $B_0 = S^1 \times \{N\}$ and $B_1 = S^1 \times \{S\}$, and projection $\pi := \theta$. The binding component B_0 is oriented by $d\phi$ and B_1 by $-d\phi$. We decorate this open book in four different ways $(+,+)$, $(+,-)$, $(-,+)$ and $(-,-)$ (the first sign is assigned to B_0, the second one to B_1) and consider the respective SHS. We will consider HSs on $S^1 \times S^2$ of the form

$$\omega := d\alpha_l = dz \wedge d\phi + l'(z)\, dz \wedge d\theta,$$

where

$$\alpha_l := z\, d\phi + l(z)\, d\theta,$$

for different choices of the function $l : [-1,1] \to \mathbb{R}$. Its Reeb vector field is a positive multiple of

$$\partial_\theta - l'(z)\partial_\phi.$$

Note that regardless of the choice of l, we have $d\theta \wedge \omega = d\theta \wedge dz \wedge d\phi$, so ω is supported by the open book (B,π). Note also that all such HS are T^2-invariant with respect to the T^2-action given by shifts in θ and ϕ. Recall that the standard tight contact form on $S^1 \times S^2$ can be written as

$$\lambda_{\rm st} = z\, d\phi + (x\, dy - y\, dx) = z\, d\phi + (x^2+y^2)\, d\theta = z\, d\phi + (1-z^2)\, d\theta,$$

that is, $\lambda_{\rm st} = \alpha_l$ for $l(z) = 1 - z^2$. Note that along the binding $d\lambda_{\rm st}$ equals $2dx \wedge dy$, so it restricts positively to S^2 at the north pole and negatively at the south pole. Thus, the SHS $(\omega_{(+,+)}, \lambda_{(+,+)}) := (d\lambda_{\rm st}, \lambda_{\rm st})$ is $(+,+)$-supported by (B,π). Similarly, *negative* tight contact form

$$\bar\lambda_{\rm st} := z\, d\phi - (1-z^2)\, d\theta$$

(obtained by pulling back $\lambda_{\rm st}$ by the reflection $\theta \mapsto -\theta$) gives rise to a $(-,-)$-supported SHS $(\omega_{(-,-)}, \lambda_{(-,-)}) := (d\bar\lambda_{\rm st}, -\bar\lambda_{\rm st})$. Let l_{+-} be an odd function on $[-1,1]$ that equals $(1-z^2)$ near

$z = 1$ and $-(1 - z^2)$ near $z = -1$. Set $l_{-+} := -l_{+-}$. The HS $\omega_{(+,-)} := d\alpha_{l_{+-}}$ can be stabilized by λ_{st} near the north pole and by $\bar{\lambda}_{\mathrm{st}}$ near the south pole, so according to [**8**, Section 3.5] $\omega_{(+,-)}$ is stabilizable. It is $(+,-)$-supported by (B, π) because of the shape of l_{+-} near -1 and 1. Similarly, $\omega_{(-,+)} := d\alpha_{l_{-+}}$ is stabilizable and $(-,+)$-supported by (B, π). The Reeb vector field of the SHS $(\omega_{(i,j)}, \lambda_{(i,j)})$, $i, j \in \{+, -\}$, will be denoted by $R_{(i,j)}$.

Proposition 8.6. *There are three distinct formal classes* $[\omega_{(+,+)}] = [\omega_{(-,-)}]$, $[\omega_{(+,-)}]$ *and* $[\omega_{(-,+)}]$.

Proof. To show the equality $[\omega_{(+,+)}] = [\omega_{(-,-)}]$, we choose a frame for $T(S^1 \times S^2)$ as follows. Recall that we consider S^2 as the unit sphere in $\mathbb{R}^3$ and denote by $\mathrm{pr} : S^1 \times S^2 \to S^2$ the projection. Then

$$T(S^1 \times S^2) \cong \mathbb{R} \oplus \mathrm{pr}^* TS^2 \cong \mathrm{pr}^*(T\mathbb{R}^3|_{S^2}),$$

where we identify $\mathbb{R}$ with the normal bundle to $S^2 \subset \mathbb{R}^3$. Let (e_x, e_y, e_z) be the standard basis of $\mathbb{R}^3$ (do not confuse e_z with the vector field ∂_z tangent to S^2 corresponding to the coordinate z). In this frame, the vector fields corresponding to the coordinates ϕ and θ are given by

$$\partial_\theta = -y e_x + x e_y, \quad \partial_\phi = x e_x + y e_y + z e_z.$$

Note that the Reeb vector fields R_{st} and $\bar{R}_{\mathrm{st}}$ of the contact forms λ_{st} and $\bar{\lambda}_{\mathrm{st}}$, respectively, write out as

$$\begin{aligned} R_{\mathrm{st}} &= (1 + z^2)^{-1}(2z\partial_\phi + \partial_\theta) = (1 + z^2)^{-1}((2zx - y)e_x + (2zy + x)e_y + 2z^2 e_z), \\ R_{\mathrm{st}} &= (1 + z^2)^{-1}(2z\partial_\phi - \partial_\theta) = (1 + z^2)^{-1}((2zx + y)e_x + (2zy - x)e_y + 2z^2 e_z). \end{aligned}$$

We see that the coefficient in front of e_z is always non-negative for both R_{st} and $\bar{R}_{\mathrm{st}}$, so the corresponding maps to S^2 are not surjective and thus homotopic to a constant map. Hence, $[R_{\mathrm{st}}] = [\bar{R}_{\mathrm{st}}]$. Note that $R_{(+,+)} = R_{\mathrm{st}}$ and $R_{(-,-)} = -\bar{R}_{\mathrm{st}}$, thus $[R_{(+,+)}] = [-R_{(-,-)}]$. The orientation-preserving diffeomorphism of $S^2 \subset R^3$ given by $(x, y, z) \mapsto (x, -y, -z)$ is just $180°$ rotation in the (y, z) plane and hence isotopic to the identity. Thus, λ_{st} is contact isotopic to its pullback under this diffeomorphism, which is $-\lambda_{\mathrm{st}}$. In particular, $[R_{(+,+)}] = [-R_{(+,+)}]$, so we conclude that $[R_{(+,+)}] = [R_{(-,-)}]$, that is, $[\omega_{(+,+)}] = [\omega_{(-,-)}]$.

The three classes $[\omega_{(i,j)}]$, $(i, j) = (+,+), (+,-), (-,+)$ will be distinguished by their Euler numbers $E(\omega_{(i,j)})$, that is, the values of the Euler class of the corresponding oriented plane field on $\{\mathrm{pt}\} \times S^2$. By the preceding discussion, we have $[\omega_{(+,+)}] = [-\omega_{(+,+)}]$ and hence $E(\omega_{(+,+)}) = -E(\omega_{(+,+)}) = 0$. Next we compute the Euler numbers of $[\omega_{(+,-)}]$ and $[\omega_{(-,+)}]$. Note that in a neighbourhood of the binding the Reeb vector field $R_{(+,-)}$ of $\omega_{(+,-)}$ has positive ∂_ϕ component. Since the ∂_θ component is always positive away from the binding, we can deform $R_{(+,-)}$ to ∂_ϕ through nowhere vanishing vector fields. Similarly, we can deform the Reeb vector field $R_{(-,+)}$ of $\omega_{(-,+)}$ to $-\partial_\phi$. Now

$$E(\partial_\phi) = -E(-\partial_\phi) = c_1(TS^2) \neq 0$$

concludes the proof. □

Question 8.7. Are the SHS corresponding to $\omega_{(+,+)}$ and $\omega_{(-,-)}$ stably homotopic?

8.3. *Mixed adaptedness for contact structures*

We have seen that in dimension 3 every SHS is homotopic to one supported by an open book (with some signs). If this open book always supported a (positive or negative) contact structure with the same signs, then the homotopy classification of SHS would reduce to a question about

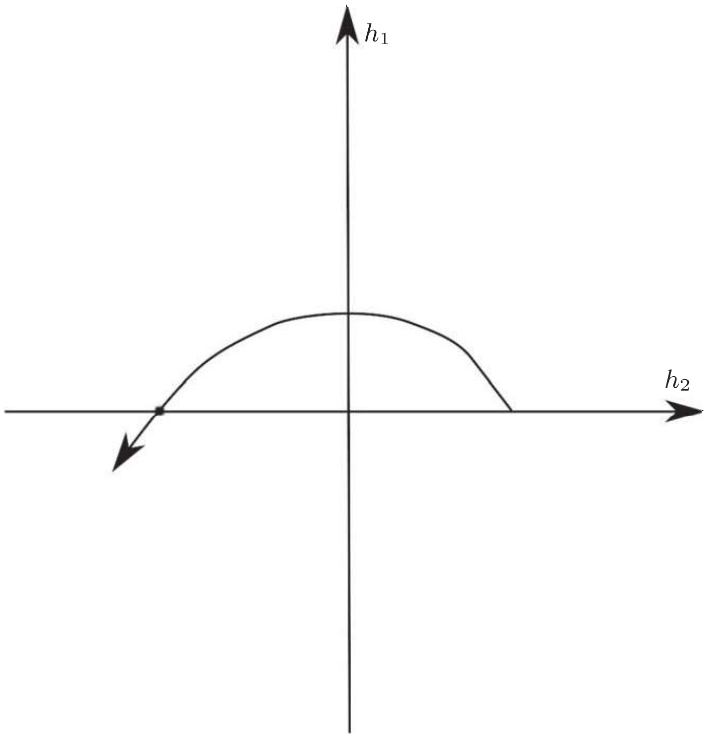

FIGURE 10. *Lutz twist.*

contact structures. Unfortunately (or fortunately?), this is not the case. Moreover, the following question in contact topology appears to be widely open:

Given a decorated open book (B, π) on a closed oriented 3-manifold, does there exist a positive/negative contact form α such that the SHS $(d\alpha, \pm\alpha)$ is supported by the open book with the given signs?

Let us just mention the following easy necessary criterion. It is not sufficient, but we are not giving the corresponding example here.

LEMMA 8.8. *Suppose that the decorated open book (B, π) supports an SHS $(\omega, \lambda) = (d\alpha, \alpha)$ with the given signs. Then at least one binding component must carry a plus sign. Moreover, if at least one binding component carries a minus sign, then the positive contact structure supported by (B, π) (which is not $\ker \alpha$!) is overtwisted.*

Proof. The first statement follows from Stokes' theorem applied to a page P:

$$\int_B \alpha = \int_P d\alpha > 0.$$

For the second statement, let B_0 be a binding component carrying a minus sign. According to [**8**, Lemma 4.10], we can homotop α through contact forms supported by (B, π) to one which near B_0 equals $-r^2 d\theta - (1 + r^2)d\phi$. (Here θ is the open book projection and ϕ the coordinate along B_0.) Then we can perform a Lutz twist around B_0 to obtain a new contact form supported by (B, π) with $+$ sign at B_0 and with a Lutz tube around B_0, see Figure 10. Performing Lutz twist at all negative binding components, we obtain an overtwisted contact form positively supported by (B, π). □

The negative case is discussed in the following corollary.

Corollary 8.9. *Suppose the decorated open book (B, π) supports an SHS $(\omega, \lambda) = (d\alpha, -\alpha)$ with the given signs. Then at least one binding component must carry a minus sign. Moreover, if at least one binding component carries a plus sign, then the negative contact structure supported by (B, π) (which is not $\ker \alpha$!) is overtwisted.*

Proof. In the proof, we will repeatedly use the orientation remarks from the beginning of this section. For the first statement, assume that $(d\alpha, -\alpha)$ is supported by (B, π) with all pluses, then the restriction of α to all binding components is negative and we derive a contradiction by Stokes' theorem as in the proof of Lemma 8.8. For the second statement, note that the open book $(B, -\pi)$ supports the SHS $(d\alpha, \alpha)$ on $\bar{M}$ with at least one minus sign. Therefore, by Lemma 8.8, $(B, -\pi)$ supports a positive overtwisted contact structure on $\bar{M}$. Hence, the open book (B, π) supports a negative overtwisted contact structure on M. □

Example 8.10. Consider the open book (B, π) on $S^1 \times S^2$ from Subsection 8.2. It supports *positive and negative* tight contact forms (with signs $(+, +)$). Hence, by Lemma 8.8 and Corollary 8.9 we can say the following. If (B, π) supports an SHS $(d\alpha, \alpha)$, then the signs must be $(+, +)$. If (B, π) supports an SHS $(d\alpha, -\alpha)$, then the signs must be $(-, -)$. One can put both cases in one sentence: If a contact form α (positive or negative) on $S^1 \times S^2$ is such that $d\alpha$ restricts positively on the interiors of the pages of the open book (B, π), then α must restrict positively to both binding components.

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