J-Holomorphic Curves, Moment Maps, and Invariants of Hamiltonian Group Actions

Kai Cieliebak, Ana Rita Gaio, and Dietmar A. Salamon

Contents

1. Introduction 832
2. Background 833
   2.1. Hamiltonian groupactions 833
   2.2. Connections and curvature 836
   2.3. Equivariant cohomology 837
   2.4. J-holomorphic curves 843
3. Invariants of Hamiltonian groupactions 846
   3.1. An action functional 846
   3.2. Symplectic reduction 848
   3.3. Hamiltonian perturbations 853
   3.4. Moduli spaces 854
   3.5. Fredholm theory 855
   3.6. Compactness 857
   3.7. Invariants 860
   3.8. Adiabatic limits 862
   3.9. Wall crossing and localization 864
4. Floer homology 865
   4.1. Relative fixed points 865
   4.2. Equivariant symplectic action 866
   4.3. Floer homology 867
   4.4. The Arnold conjecture for regular quotients 868
   4.5. Relation with Morse theory 868
   4.6. Equivariant symplectomorphisms 869
   4.7. Boundary value problems 869
   4.8. Adiabatic limits 870
5. Examples 871
   5.1. Vortex equations 871
   5.2. Bradlow pairs 872

5.3. Holomorphic curves in projective space 873
5.4. Toric varieties 873
5.5. The Grassmannian and the Verlinde algebra 874
5.6. Anti-self-dual Yang-Mills equations 875
5.7. Seiberg-Witten equations 876

1 Introduction

This paper outlines the construction of invariants of Hamiltonian group actions on symplectic manifolds. The invariants are derived from the solutions of a nonlinear first-order elliptic partial differential equation involving the Cauchy-Riemann operator, the curvature, and the moment map(see (17)). They are related to the Gromov invariants of the reduced spaces.

Our motivation arises from the proof of the Atiyah-Floer conjecture in [17], [18], and [19], which deals with the relation between holomorphic curves $\Sigma \to \mathcal{M}_S$ in the moduli space $\mathcal{M}_S$ of flat connections over a Riemann surface S and anti-self-dual instantons over the 4-manifold $\Sigma \times S$. In [3] Atiyah and Bott interpret the space $\mathcal{M}_S$ as a symplectic quotient of the space $\mathcal{A}_S$ of connections on S by the action of the group $\mathcal{G}_S$ of gauge transformations. The various terms in the anti-self-duality equations over $\Sigma \times S$ (see (64)) can be interpreted symplectically. Hence they should give rise to meaningful equations in a context where the space $\mathcal{A}_S$ is replaced by a finite-dimensional symplectic manifold M and the gauge group $\mathcal{G}_S$ by a compact Lie group G with a Hamiltonian action on M. In this paper we show how the resulting equations give rise to invariants of Hamiltonian group actions. The same adiabatic limit argument as in [19] then leads to a correspondence between these invariants and the Gromov-Witten invariants of the quotient $M/\!\!/G$ (see Conjecture 3.6). This correspondence is the subject of the Ph.D. thesis [27] of the second author.

In Section 2 we review the relevant background material about Hamiltonian groupactions, gauge theory, equivariant cohomology, and holomorphic curves in symplectic quotients. The heart of this paper is Section 3, where we discuss the equations and their properties, outline the construction of the invariants, and indicate several potential applications. One interesting point is a result about the compactness of the moduli spaces (see Proposition 3.5) which has no analogue for moduli spaces of holomorphic curves. Hence the invariants should lead in many cases to a definition of the Gromov-Witten invariants over the integers. Via the adiabatic limits and wall-crossing arguments, the invariants should also give rise to relations between the Gromov-Witten invariants of symplectic quotients at different values of the moment map. This is reminiscent of the work of Martin [47], [48], [49] about the ordinary cohomology of symplectic quotients. Section 4 deals with the corresponding Floer theory, and Section 5 discusses several examples.

Recently, Ignasi Mundet discovered the equations (17) independently, starting from a different angle, and in his thesis [55] developed a program along similar lines, as outlined in this paper. For $M = \mathbb{C}^n$ the equations also appeared in the physics literature (starting from Witten's work in [80]), where they are known as *gauged sigma models*.

2 Background

2.1 Hamiltonian groupactions

Let (M, ω) be a symplectic manifold, and let G be a compact connected Lie group that acts on M by symplectomorphisms. Let $\mathfrak{g} = \mathrm{Lie}(G)$ denote the Lie algebra of G, and, for every $\xi \in \mathfrak{g}$, denote by $X_\xi : M \to TM$ the vector field whose flow is given by the action of the 1-parameter subgroup generated by ξ. We assume throughout that the Lie algebra $\mathfrak{g}$ carries an invariant inner product $\langle \cdot, \cdot \rangle$. The action of G on M is called *Hamiltonian* if there exists an equivariant function $\mu : M \to \mathfrak{g}$ such that, for every $\xi \in \mathfrak{g}$,

$$d\langle \mu, \xi \rangle = \iota(X_\xi)\omega. \tag{1}$$

This means that X_ξ is the Hamiltonian vector field of the function $\langle \mu, \xi \rangle$. The function μ is called a *moment map*.

Suppose that τ is a regular value of μ and that the isotropy subgroup

$$G_\tau := \left\{ g \in G \mid \tau = g\tau g^{-1} \right\}$$

acts freely on $\mu^{-1}(\tau)$. Then the *Marsden-Weinstein quotient*

$$M/\!\!/G(\tau) := \frac{\mu^{-1}(\tau)}{G_\tau}$$

is a smooth manifold and it inherits the symplectic structure from M. To be more precise, for $x \in \mu^{-1}(\tau)$, there is a chain complex

$$0 \longrightarrow \mathfrak{g}_\tau \xrightarrow{L_x} T_xM \xrightarrow{d\mu(x)} \mathfrak{g} \longrightarrow 0, \tag{2}$$

where $L_x : \mathfrak{g} \to T_xM$ is defined by $L_x\xi = X_\xi(x)$, and

$$\mathfrak{g}_\tau = \left\{ \xi \in \mathfrak{g} \mid [\xi, \tau] = 0 \right\}$$

is the Lie algebra of G_τ. The image of L_x is the symplectic complement of the kernel of $d\mu(x)$. Moreover, the formula

$$d\mu(x)L_x\xi = \big[\xi, \mu(x)\big]$$

shows that $\operatorname{im} L_x \cap \ker d\mu(x) = L_x\mathfrak{g}_\tau$. Thus the quotient $\ker d\mu(x)/L_x\mathfrak{g}_\tau$ inherits the symplectic structure of T_xM, and it can be identified with the tangent space of $M/\!\!/G(\tau)$ at $[x]$.

Remark 2.1. (i) An orbit $\mathcal{O} \subset \mathfrak{g}$ under the adjoint action admits a natural Kähler structure. The tangent space of $\mathcal{O}$ at τ is

$$T_\tau\mathcal{O} = \big\{[\xi, \tau] \mid \xi \in \mathfrak{g}\big\} = \mathfrak{g}_\tau^\perp,$$

and the symplectic form is given by

$$\sigma_\tau\big([\xi, \tau], [\eta, \tau]\big) := \big\langle \tau, [\xi, \eta]\big\rangle.$$

An explicit formula for the complex structure uses the decomposition of $T_\tau\mathcal{O}$ into the eigenspaces

$$V_\alpha = \big\{\xi \in \mathfrak{g} \bigm| \big[\tau, [\tau, \xi]\big] = -\alpha^2\xi\big\}$$

for $\alpha > 0$. Let $\mathfrak{g} \to V_\alpha : \xi \mapsto \xi_\alpha$ denote the orthogonal projection onto V_α, and define $A_\tau : \mathfrak{g} \to \mathfrak{g}$ by

$$A_\tau\xi = \sum_\alpha \alpha\xi_\alpha,$$

where the sum runs over all $\alpha > 0$ such that $-\alpha^2$ is an eigenvalue of $\operatorname{ad}(\tau)^2$. Then the complex structure on $T_\tau\mathcal{O}$ is given by

$$J_\tau\xi = \sum_\alpha \frac{1}{\alpha}[\tau, \xi_\alpha], \qquad J_\tau[\xi, \tau] = A_\tau\xi.$$

The adjoint action of G on $\mathcal{O}$ is Hamiltonian, and one checks easily that the inclusion $\mathcal{O} \hookrightarrow \mathfrak{g}$ is a moment mapfor this action.

(ii) If $\tau \in \mathcal{O}$, then the quotient $M/\!\!/G(\tau)$ can be naturally identified with the symplectic quotient of $M \times \mathcal{O}$ at the zero value of the moment map. Here the product $M \times \mathcal{O}$ is equipped with the symplectic form $\omega - \sigma$, and the moment map $\mu_\mathcal{O} : M \times \mathcal{O} \to \mathfrak{g}$ is given by $\mu_{\mathcal{O}}(x,\tau) = \mu(x) - \tau$. Hence $(M \times \mathcal{O})/\!\!/G \cong \mu^{-1}(\mathcal{O})/G \cong \mu^{-1}(\tau)/G_\tau = M/\!\!/G(\tau)$ for $\tau \in \mathcal{O}$. If $\mu(x) = \tau$, then

$$T_{[x,\tau]}(M \times \mathcal{O})/\!\!/G = \frac{\{(\nu,\eta) \in T_xM \times T_\tau\mathcal{O} \mid d\mu(x)\nu = \eta\}}{\{(L_x\xi, [\xi,\tau]) \mid \xi \in \mathfrak{g}\}}.$$

Equivalently, this tangent space can be identified with the space of all pairs $(\nu,\eta) \in T_xM \times T_\tau\mathcal{O}$ that satisfy

$$d\mu(x)\nu = \eta, \qquad L_x{}^*\nu + J_\tau\eta = 0, \tag{3}$$

where the adjoint operator $L_x{}^*$ is understood with respect to a G-invariant inner product that arises from a G-invariant almost complex structure J on M that is compatible with ω. On the other hand,

$$T_{[x]}M/\!\!/G(\tau) = \frac{\ker d\mu(x)}{\{L_x\xi \mid [\xi,\tau] = 0\}}.$$

The "harmonic" representative of a tangent vector $\nu \in \ker d\mu(x)$ is given by

$$\pi(\nu) = (\nu + L_x\xi, [\xi,\tau]), \qquad L_x{}^*L_x\xi + A_\tau\xi + L_x{}^*\nu = 0. \tag{4}$$

The formula $\omega(\nu,w) = \omega(\nu + L_x\xi, w + L_x\eta) - \langle\tau, [\xi,\eta]\rangle$ for $\nu, w \in \ker d\mu(x)$ shows that the symplectic forms agree.

(iii) Assume that (M,ω,J) is a Kähler manifold and that the action of G preserves the Kähler structure. Then the G-action extends to an action of the complexified group G^c that preserves the complex structure (cf. [33]). The extended action does not preserve the Kähler form. Suppose that $\tau \in \mathfrak{g}$ is a central element, and denote

$$M^\tau = \{x \in M \mid \exists g \in G^c \text{ s.t. } \mu(gx) = \tau\}.$$

Then the complex quotient M^τ/G^c can be naturally identified with $M/\!\!/G(\tau)$. This means that if $\mu(gx) = \mu(x) = \tau$ for $g \in G^c$ and $x \in M$, then gx lies in the G-orbit of x. The proof of this observation relies on the fact that any $g \in G^c$ can be written in the form $g = \exp(i\eta)h$, where $h \in G$ and $\eta \in \mathfrak{g}$. Now consider the path $[0,1] \to M : x(t) = \exp(it\eta)hx$ running from $x(0) = hx$ to $x(1) = gx$. This path satisfies

$$\dot{x}(t) = JX_\eta(x(t)),$$

and hence

$$\frac{d}{dt}\langle \mu(x(t)), \eta\rangle = \langle d\mu(x(t))JX_\eta(x(t)), \eta\rangle = |X_\eta(x(t))|^2.$$

The last identity follows from the definition of the moment map. Since τ is a central element, we have $\mu(x(0)) = \mu(x(1))$; hence $x(t)$ is independent of t, and hence $gx = hx \in Gx$.

(iv) The study of complex quotients of the form M^τ/G^c and their relation to the Marsden-Weinstein quotients is the subject of *geometric invariant theory* (cf. [54], [11], [14], and [15]). In his beautiful recent paper [14], Donaldson treats the infinite-dimensional case, where G is replaced by the group of volume-preserving diffeomorphisms of a manifold S and where M is replaced by the space of maps from S to M. In [15] Donaldson discusses another interesting case, where G is the groupof symplecto-morphisms and M is the manifold of almost complex structures compatible with the given symplectic form.

(v) The condition that G_τ act freely on $\mu^{-1}(\tau)$ is rather strong. In general, if τ is a regular value of μ, then the action of G_τ on $\mu^{-1}(\tau)$ has finite-isotropy subgroups and the quotient $M/\!\!/G(\tau)$ is a symplectic orbifold. Much of the dicussion in this paper extends to that case.

2.2 Connections and curvature

Let X be a compact-oriented smooth manifold, and let $P \to X$ be a principal G-bundle. We think of G as acting on P on the right and denote the infinitesimal action by $p\xi \in T_pP$ for $p \in P$ and $\xi \in \mathfrak{g}$. A *connection* on P is an equivariant horizontal subbundle of TP. Any such subbundle determines an equivariant 1-form $A \in \Omega^1(P, \mathfrak{g})$ whose kernels are the horizontal subspaces and which identifies the vertical subspaces with $\mathfrak{g}$. Thus

$$A_{ph}(vh) = h^{-1}A_p(v)h, \qquad A_p(p\xi) = \xi$$

for $p \in P$, $v \in T_pP$, $h \in G$, and $\xi \in \mathfrak{g}$. A 1-form $A \in \Omega^1(P, \mathfrak{g})$ that satisfies these conditions is called a *connection 1-form*, and the space of connection 1-forms is denoted by $\mathcal{A}(P)$. A *gauge transformation* of P is a smooth function $g : P \to G$ that is equivariant with respect to the adjoint action of G on itself; that is, $g(ph) = h^{-1}g(p)h$ for $p \in P$ and $h \in G$. The groupof gauge transformations is denoted by $\mathcal{G} = \mathcal{G}(P)$. It acts on the left on $\mathcal{A}(P)$ by

$$g_*A = -(dg)g^{-1} + gAg^{-1}.$$

Thus g_*A is the pushforward of A under the automorphism $P \to P : p \mapsto pg(p)$. Let $\Omega^k_{ad}(P, \mathfrak{g})$ denote the space of equivariant and horizontal k-forms on P with values in $\mathfrak{g}$.

Any such form descends to a k-form on X with values in the adjoint bundle $\mathfrak{g}_P := P \times_{\mathrm{ad}} \mathfrak{g}$. Every connection $A \in \mathcal{A}(P)$ gives rise to a covariant derivative operator $d_A : \Omega^k_{\mathrm{ad}}(P, \mathfrak{g}) \to \Omega^{k+1}_{\mathrm{ad}}(P, \mathfrak{g})$ given by

$$d_A \alpha = d\alpha + [A \wedge \alpha].$$

It is interesting to note that $\Omega^0_{\mathrm{ad}}(P, \mathfrak{g})$ is the Lie algebra of the gauge group, $\Omega^1_{\mathrm{ad}}(P, \mathfrak{g})$ is the tangent space of the space of connections, and the infinitesimal action of $\mathrm{Lie}(\mathcal{G}(P))$ on $\mathcal{A}(P)$ is given by minus the covariant derivative.

Now suppose that $X = \Sigma$ is a compact Riemann surface. Then the space $\mathcal{A}(P)$ carries a natural symplectic form

$$\Omega(\alpha, \beta) = \int_\Sigma \langle \alpha \wedge \beta \rangle.$$

Atiyah and Bott [3] noted that the action of $\mathcal{G}(P)$ on $\mathcal{A}(P)$ is Hamiltonian and that a moment mapis given by the curvature

$$F_A = dA + \frac{1}{2}[A \wedge A] \in \Omega^2_{\mathrm{ad}}(P, \mathfrak{g}).$$

Thus the Marsden-Weinstein quotient is the moduli space of gauge equivalence classes of flat connections on P. The analogue of the chain complex (2) for $\tau = 0$ in gauge theory is given by

$$0 \longrightarrow \Omega^0_{\mathrm{ad}}(P, \mathfrak{g}) \xrightarrow{d_A} \Omega^1_{\mathrm{ad}}(P, \mathfrak{g}) \xrightarrow{d_A} \Omega^2_{\mathrm{ad}}(P, \mathfrak{g}) \longrightarrow 0. \tag{5}$$

Here $d_A : \Omega^0_{\mathrm{ad}} \to \Omega^1_{\mathrm{ad}}$ is the infinitesimal action of the gauge group, and $d_A : \Omega^1_{\mathrm{ad}} \to \Omega^2_{\mathrm{ad}}$ is the differential of the function $\mathcal{A}(P) \to \Omega^2_{\mathrm{ad}} : A \mapsto F_A$. The formula $d_A d_A \tau = [F_A \wedge \tau]$ shows that $d_A \circ d_A = 0$ whenever A is flat.

2.3 Equivariant cohomology

Let EG be a contractible space on which the group G acts freely. The *equivariant (co)homology* of a G-space M is defined by

$$H^*_G(M; R) = H^*(M \times_G EG; R),$$
$$H_*^G(M; R) = H_*(M \times_G EG; R).$$

Since there is a natural projection $M \times_G EG \to EG/G =: BG$, $H^*_G(M; R)$ is a module over $H^*(BG; R)$. Explicit representatives of equivariant homology classes can be constructed as follows. Let X be a compact-oriented smooth k-manifold (without boundary), and let $\pi : P \to X$ be a principal G-bundle. An equivariant map $u : P \to M$ determines an equivariant homology class

$$[u] = u_*^G (\pi_*^G)^{-1} [X] \in H_k^G(M;\mathbb{Z}).$$

Here $u_*^G : H_*^G(P;\mathbb{Z}) \to H_*^G(M;\mathbb{Z})$ and $\pi_*^G : H_*^G(P;\mathbb{Z}) \to H_*(X;\mathbb{Z})$ denote the induced homomorphisms on equivariant homology, and $[X] \in H_k(X;\mathbb{Z})$ denotes the fundamental class. Since G acts freely on P, π_*^G is an isomorphism.

Remark 2.2. For every principal bundle $P \to X$, there exists an equivariant map $\phi : P \to EG$. The map $P \to M \times EG : p \mapsto (u(p), \phi(p))$ is equivariant and descends to a function $f : X \to M \times_G EG$ that satisfies

$$f_*[X] = [u].$$

To see this, consider the maps $\phi^G : X \to P \times_G EG$ and $\pi^G : P \times_G EG \to X$, given by $\phi^G(\pi(p)) = [p, \phi(p)]$ and $\pi^G([p, e]) = \pi(p)$. Since EG is contractible, ϕ^G is a homotopy inverse of π^G. Hence ϕ_*^G is the inverse of π_*^G. Moreover, $f = u^G \circ \phi^G$, where $u^G : P \times_G EG \to M \times_G EG$ is given by $u^G([p, e]) = [u(p), e]$. Hence $f_*[X] = u_*^G \phi_*^G [X] = [u]$.

Proposition 2.1. Let M be a finite-dimensional smooth manifold, and let G be a compact Lie groupthat acts smoothly on M.

(i) For every 2-dimensional equivariant homology class $B \in H_2^G(M;\mathbb{Z})$, there exists a compact-oriented Riemann surface Σ, a principal bundle $P \to \Sigma$, and an equivariant map $u : P \to M$, such that $[u] = B$.

(ii) Suppose that G is connected. Let $P \to \Sigma$ and $P' \to \Sigma$ be principal G-bundles over Σ, and let $u : P \to M$, $u' : P' \to M$ be equivariant maps such that $[u] = [u'] \in H_2^G(M;\mathbb{Z})$. Then P is isomorphic to P'. □

Proof. Given B, choose a compact-oriented Riemann surface Σ and a map $f : \Sigma \to M \times_G EG$ such that $f_*[\Sigma] = B$. Note that $M \times EG$ is a principal G-bundle over $M \times_G EG$, and denote by $P \to \Sigma$ the pullback bundle of f. Thus

$$P = \{(z, x, e) \in \Sigma \times M \times EG \mid [x, e] = f(z)\}.$$

There are two equivariant maps $u : P \to M$ and $\phi : P \to EG$, given by

$$u(z, x, e) = x, \qquad \phi(z, x, e) = e.$$

By definition, the map $(u, \phi) : P \to M \times EG$ descends to f. Hence, by Remark 2.2, $[u] = f_*[\Sigma] = B$. This proves (i).

To prove (ii), choose two equivariant maps $\phi : P \to EG$ and $\phi' : P' \to EG$. Define $f, f' : \Sigma \to M \times_G EG$ as the maps induced by $(u, \phi) : P \to M \times EG$ and $(u', \phi') : P' \to M \times EG$. Then, by Remark 2.2,

$$f_*[\Sigma] = [u] = [u'] = f'_*[\Sigma].$$

Consider the induced maps $\bar{\phi} : \Sigma \to BG$ and $\bar{\phi}' : \Sigma \to BG$. They can be expressed in the form $\bar{\phi} = \pi \circ f$ and $\bar{\phi}' = \pi \circ f'$, where $\pi : M \times_G EG \to BG$ denotes the obvious projection. Hence $\bar{\phi}$ and $\bar{\phi}'$ are homologous; that is, $\bar{\phi}_*[\Sigma] = \bar{\phi}'_*[\Sigma]$. Since G is connected, BG is simply connected. Hence two maps $\Sigma \to BG$ are homologous if and only if they are homotopic. (To see this, note that every mapfrom Σ to a simply connected space factors, up to homotopy, through a map of degree 1 from Σ to S^2.) This shows that our maps $\bar{\phi}$ and $\bar{\phi}'$ are homotopic. Hence P and P' are isomorphic. This proves the proposition. ■

Assertion (ii) in Proposition 2.1 can be restated as follows. An equivariant homology class $B \in H_2^G(M; \mathbb{Z})$ descends to a homology class $b \in H_2(BG; \mathbb{Z})$, and the latter determines an isomorphism class of principal G-bundles $P \to \Sigma$ (over any orientable Riemann surface).

The de Rham model of equivariant cohomology is defined as follows. Let $\Omega_G^*(M)$ denote the space of equivariant polynomials from $\mathfrak{g}$ to $\Omega^*(M)$. To be more explicit, choose a basis $e_1, \dots, e_m$ of $\mathfrak{g}$ and write $\xi = \sum_{i=1}^m \xi^i e_i \in \mathfrak{g}$. Then any $\alpha \in \Omega_G^k(M)$ can be written in the form

$$\alpha(\xi) = \sum_I \xi^I \alpha_I,$$

where $I = (i_1, \dots, i_m)$, $\xi^I = (\xi^1)^{i_1} \cdots (\xi^m)^{i_m}$, and $\alpha_I \in \Omega^{k-2|I|}(M)$. The equivariance of the function $\alpha : \mathfrak{g} \to \Omega^*(M)$ can be expressed in the form

$$D\alpha(\xi)[\xi, \eta] = \mathcal{L}_{X_\eta} \alpha(\xi)$$

for $\xi, \eta \in \mathfrak{g}$. Here the linear operator $D\alpha(\xi) : \mathfrak{g} \to \Omega^*(M)$ denotes the differential of the function $\mathfrak{g} \to \Omega^*(M) : \xi \mapsto \alpha(\xi)$ at the point ξ. The differential $d_G : \Omega_G^k(M) \to \Omega_G^{k+1}(M)$ is given by

$$(d_G \alpha)(\xi) := d(\alpha(\xi)) + \iota(X_\xi)\alpha(\xi) = \sum_I \xi^I \big(d\alpha_I + \iota(X_\xi)\alpha_I \big).$$

Cartan's formula asserts that $d_G \circ d_G = 0$. The equivariant version of de Rham's theorem asserts that, for every smooth manifold M with a smooth G-action, there is a natural isomorphism (see [39]):

$$H_G^k(M;\mathbb{R}) \cong \frac{\ker d_G : \Omega_G^k(M) \longrightarrow \Omega_G^{k+1}(M)}{\operatorname{im} d_G : \Omega_G^{k-1}(M) \longrightarrow \Omega_G^k(M)}.$$

Next we describe the pairing between an equivariant cohomology class, represented by a G-closed k-form $\alpha \in \Omega_G^k(M)$, and an equivariant homology class, represented by an equivariant map $u : P \to M$ defined on the total space of a principal G-bundle $\pi : P \to X$ over a compact-oriented smooth k-manifold X. An explicit formula for this pairing relies on a G-connection $A \in \mathcal{A}(P)$ and on the covariant derivative of u determined by A. This covariant derivative is defined as follows. Think of u as a section of the associated bundle

$$\widetilde{M} = P \times_G M \longrightarrow X$$

with fibres diffeomorphic to M. The connection A on P determines a connection on this bundle. More precisely, the tangent space of $\widetilde{M}$ at $[p, x]$ is the quotient

$$T_{[p,x]}\widetilde{M} = \frac{T_pP \times T_xM}{\left\{(p\xi, -X_\xi(x)) \mid \xi \in \mathfrak{g}\right\}},$$

the vertical space consists of equivalence classes of the form $[0, w]$ with $w \in T_xM$, and the horizontal space consists of those equivalence classes $[v, w]$, where $v \in T_pP$ and $w \in T_xM$ satisfy $w + X_{A_p(v)}(x) = 0$.[1] The covariant derivative of a section $u : P \to M$ with respect to the connection A is the 1-form $d_Au : TP \to u^*TM$ given by

$$d_Au(p)v = du(p)v + X_{A_p(v)}(u(p)) \tag{6}$$

(the vertical part of the vector $[v, du(p)v] \in T_{[p,u(p)]}P \times_G M$). This 1-form is obviously G-equivariant, and it satisfies $d_Au(p)p\xi = 0$ for every $\xi \in \mathfrak{g}$. Hence d_Au descends to a 1-form on X with values in the bundle u^*TM/G.

Given a basis $e_1, \ldots, e_m$ of $\mathfrak{g}$ and an equivariant k-form $\alpha = \sum_I \xi^I \alpha_I \in \Omega_G^k(M)$ as above, we define $\alpha(u, A) \in \Omega^k(P)$ by

$$\alpha(u, A) = \big((d_Au)^*\alpha\big)(F_A) = \sum_I \omega^I \wedge (d_Au)^*\alpha_I.$$

[1] In the terminology of [51 , Chapter 6] the connection form on $\widetilde{M}$ is induced by the 2-form $\omega - d\langle\mu, A\rangle$ on $P \times M$ whenever (M, ω, μ) is a symplectic manifold with a Hamiltonian group action.

Here $F_A = \sum_i \omega^i e_i$ and $\omega^I = (\omega^1)^{i_1} \wedge \cdots \wedge (\omega^m)^{i_m}$. Since $\alpha(u, A) \in \Omega^k(P)$ is equivariant and horizontal (see Proposition 2.2), it descends to a k-form on X, still denoted by $\alpha(u, A)$. The pairing between the equivariant cohomology class of α and the equivariant homology class of u is given by

$$\langle [\alpha], [u] \rangle := \int_X \alpha(u, A).$$

That this number is well defined and depends only on the equivariant cohomology class of α and on the homotopy class of the pair (u, A) is the content of the following proposition.

Proposition 2.2. Let M be a smooth G-manifold, and let $\pi : P \to X$ be a principal G-bundle over a compact smooth manifold X. Let $\alpha \in \Omega_G^\ell(M)$ and $\beta \in \Omega_G^{\ell+1}(M)$.

(i) Let $A \in \mathcal{A}(P)$ and suppose that $u : P \to M$ is an equivariant smooth function. Then $\alpha(u, A) \in \Omega^\ell(P)$ is equivariant and horizontal.

(ii) If $d_G \alpha = \beta$, then $d\alpha(u, A) = \beta(u, A)$.

(iii) Let $A_0, A_1 \in \mathcal{A}(P)$ and suppose that $u_0, u_1 : P \to M$ are equivariantly homotopic. Then $\alpha(u_1, A_1) - \alpha(u_0, A_0)$ is an exact ℓ-form on X. □

Proof. The form $\alpha(u, A)$ is obviously horizontal. We prove that it is equivariant. Denote by c_{ij}^k the structure constants of $\mathfrak{g}$. This means that

$$[e_i, e_j] = \sum_k c_{ij}^k e_k.$$

Then the equivariance of α can be expressed in the form

$$\sum_{i,k} c_{ij}^k \xi^i \alpha_k(\xi) = \mathcal{L}_{X_{e_j}} \alpha(\xi),$$

where $\alpha_k := \partial_k \alpha : \mathfrak{g} \to \Omega^*(M)$. Hence, with $F_A = \omega = \sum_i \omega^i e_i \in \Omega^2(P, \mathfrak{g})$, it follows that

$$\sum_{i,k} c_{ij}^k \omega^i \wedge \alpha_k(u, A) = \left(\mathcal{L}_{X_{e_j}} \alpha\right)(u, A). \tag{7}$$

Moreover, with $A = \sum_i a^i e_i \in \Omega^1(P, \mathfrak{g})$, the Bianchi identity $d_A F_A = 0$ takes the form

$$d\omega^k = \sum_{i,j} c_{ij}^k \omega^i \wedge a^j. \tag{8}$$

For $j = 1, \ldots, m$, denote by $v_j \in \mathrm{Vect}(P)$ the vector field $p \mapsto pe_j$. Then $a^i(v_j) = \delta_j^i$. Hence, by (8),

$$\mathcal{L}_{v_j} \omega^k = \iota(v_j) d\omega^k = \sum_i c_{ij}^k \omega^i,$$

and hence, by (7),

$$\sum_k \big(\mathcal{L}_{v_j}\omega^k\big)\wedge\alpha_k(u,A) = \big(\mathcal{L}_{X_{e_j}}\alpha\big)(u,A) = -\sum_I \omega^I\wedge\mathcal{L}_{v_j}\big((d_Au)^*\alpha_I\big).$$

Here we have used the identity $(d_Au)^*\mathcal{L}_{X_{e_j}}\alpha_I = -\mathcal{L}_{v_j}((d_Au)^*\alpha_I)$. It follows that

$$\mathcal{L}_{v_j}\alpha(u,A) = \sum_k(\mathcal{L}_{v_j}\omega^k)\wedge\alpha_k(u,A) + \sum_I \omega^I\wedge\mathcal{L}_{v_j}\big((d_Au)^*\alpha_I\big) = 0,$$

and hence $\alpha(u,A)$ is equivariant. This proves (i).

The proof of (ii) relies on the following identity, for $\alpha\in\Omega^k(M)$,

$$\begin{aligned} &d\big((d_Au)^*\alpha\big) - (d_Au)^*d\alpha \\ &\qquad = \sum_i \omega^i\wedge(d_Au)^*\iota(X_{e_i})\alpha - \sum_i a^i\wedge(d_Au)^*\mathcal{L}_{X_{e_i}}\alpha. \end{aligned} \tag{9}$$

For $k=0,1$, the proof of (9) is a computation using $\omega^{kk}+\sum_{i<j}c_{ij}^k a^i\wedge a^j$ and

$$\iota(X_{e_j})\mathcal{L}_{X_{e_i}}\alpha - \iota(X_{e_i})\mathcal{L}_{X_{e_j}}\alpha - d\alpha(X_{e_i},X_{e_j}) = \alpha\big([X_{e_i},X_{e_j}]\big).$$

For general k, (9) follows easily by induction. With this understood, we obtain

$$\begin{aligned} d\alpha(u,A) &= \sum_k d\omega^k\wedge\alpha_k(u,A) + \sum_I \omega^I\wedge d\big((d_Au)^*\alpha_I\big) \\ &= \sum_{I,j}\omega^I\wedge a^j\wedge(d_Au)^*\mathcal{L}_{X_{e_j}}\alpha_I + \sum_I \omega^I\wedge d\big((d_Au)^*\alpha_I\big) \\ &= \sum_{I,j}\omega^I\wedge\omega^j\wedge(d_Au)^*\iota(X_{e_j})\alpha_I + \sum_I \omega^I\wedge(d_Au)^*d\alpha_I \\ &= \beta(u,A). \end{aligned}$$

Here the second identity follows from (7) and (8), the third identity from (9), and the last identity from $d_G\alpha=\beta$; that is, $\sum_I \xi^I(d\alpha_I+\iota(X_\xi)\alpha_I) = \sum_I \xi^I\beta_I$. This proves (ii).

We prove (iii). Let $\mathbb{R}\to\mathcal{A}(P): t\mapsto A_t$ be a smooth family of connections, and let $\mathbb{R}\times P\to M:(t,p)\mapsto u_t(p)$ be a smooth family of equivariant functions. Think of the path $t\mapsto A_t$ as a connection $\widetilde{A}$ on the bundle $\widetilde{P}=\mathbb{R}\times P$ over $\widetilde{X}=\mathbb{R}\times X$, and think of the path $t\mapsto u_t$ as a function $\widetilde{u}:\widetilde{P}\to M$. Given a G-closed ℓ-form $\alpha\in\Omega_G^\ell(M)$, write

$$\alpha\big(\widetilde{u},\widetilde{A}\big) = \widetilde{\alpha} = \alpha_t + \beta_t\wedge dt\in\Omega^\ell(\widetilde{P}),$$

where $\alpha_t = \alpha(u_t,A_t)\in\Omega^\ell(P)$ and $\beta_t\in\Omega^{\ell-1}(P)$. By (i), $\widetilde{\alpha}$ is horizontal, equivariant, and closed. Hence α_t and β_t are equivariant and horizontal, α_t is closed, and $\partial_t\alpha_t = d\beta_t$ for every t. Hence

$$\alpha(u_1, A_1) - \alpha(u_0, A_0) = d\int_0^1 \beta_t\, dt.$$

Since β_t descends to X for every t, this proves the proposition. ∎

2.4 J-holomorphic curves

Suppose that (M, ω, μ) is a symplectic manifold with a Hamiltonian group action. Denote by $\mathcal{J}(M, \omega, \mu)$ the space of all almost complex structures J on TM which are invariant under the G-action and compatible with ω; that is, $\omega(\cdot, J\cdot)$ is a Riemannian metric on M. It follows from [51, Proposition 2.50] that the space $\mathcal{J}(M, \omega, \mu)$ is nonempty and contractible. Namely, there is a natural homotopy equivalence from the (contractible) space of G-invariant Riemannian metrics on M to the space $\mathcal{J}(M, \omega, \mu)$.

An almost complex structure $J \in \mathcal{J}(M, \omega, \mu)$ determines an almost complex structure on the quotient

$$M/\!\!/G = \frac{\mu^{-1}(0)}{G}.$$

The tangent space of this quotient is given by

$$T_{[x]} M/\!\!/G = \ker d\mu(x) \cap \ker {L_x}^*,$$

and the identity

$$d\mu(x)J = {L_x}^*$$

shows that this space is invariant under J. Hence a map $u : \mathbb{C} \to \mu^{-1}(0)$ represents a J-holomorphic curve in $M/\!\!/G$ if and only if there exist functions $\Phi, \Psi : \mathbb{C} \to \mathfrak{g}$ such that

$$\partial_s u + X_\Phi(u) + J\big(\partial_t u + X_\Psi(u)\big) = 0. \tag{10}$$

Here we denote by $s + it$ the coordinate on $\mathbb{C}$. This equation implies that the vectors $\partial_s u + X_\Phi(u)$ and $\partial_t u + X_\Psi(u)$ are the unique harmonic representatives of the derivatives with respect to s and t. Thus they are uniquely determined by the equations

$$\begin{aligned} {L_u}^* L_u \Phi + {L_u}^* \partial_s u &= 0, \\ {L_u}^* L_u \Psi + {L_u}^* \partial_t u &= 0. \end{aligned}$$

There is a natural gauge groupof maps $g : \mathbb{C} \to G$. It acts on triples (u, Φ, Ψ) by

$$g^*(u, \Phi, \Psi) = \left(g^{-1}u, g^{-1}\partial_s g + g^{-1}\Phi g, g^{-1}\partial_t g + g^{-1}\Psi g\right).$$

This action preserves the space of solutions of (10). Note that the action on the space of 1-forms $\Phi\, ds + \Psi\, dt$ coincides with the action of the gauge groupon the space of connections.

The global version of (10) involves principal bundles. Let Σ be a compact-oriented Riemann surface with a fixed complex structure J_Σ. Then a smooth function $\Sigma \to M/\!/G$ need not lift to a smooth function into the ambient space M. However, it does lift to an equivariant function from the total space of a principal G-bundle $\pi : P \to \Sigma$ into M. Hence let (u, A) be a pair consisting of an equivariant smooth function $u : P \to M$ and a connection $A \in \mathcal{A}(P)$. Recall that $d_A u : TP \to u^*TM$ is defined by

$$d_A u = du + L_u A.$$

Think of $d_A u$ as a 1-form on Σ with values in the bundle u^*TM/G. This is a complex vector bundle, and we denote by $\bar{\partial}_{J,A}(u) \in \Omega^{0,1}(\Sigma, u^*TM/G)$ the complex antilinear part of the 1-form $d_A u$. Thus

$$\bar{\partial}_{J,A}(u) = \frac{1}{2}\left(d_A u + J \circ d_A u \circ J_\Sigma\right) \in \Omega^{0,1}\left(\Sigma, u^*TM/G\right).$$

In local coordinates this is the left-hand side of (10). To make sense of this expression in the global form, note that J_Σ acts on the tangent space of Σ but not on that of P. The linear map $d_A u(p) \circ J_\Sigma : T_p P \to T_{u(p)} M$ is defined as follows. Project a vector $v \in T_p P$ onto $T_{\pi(p)}\Sigma$ and then apply J_Σ. Now lift $J_\Sigma d\pi(p)v \in T_{\pi(p)}\Sigma$ to a vector in $T_p P$ and apply $d_A u(p)$. Since $d_A u$ is horizontal, the resulting vector in $T_{u(p)} M$ is independent of the choice of the lift. To understand the $(0,1)$-form $\bar{\partial}_{J,A}(u)$ in a different way, consider the fibre bundle $\widetilde{M} = P \times_G M \to \Sigma$, with fibres diffeomorphic to M, and define $\widetilde{u} : \Sigma \to \widetilde{M}$ by $\widetilde{u}(\pi(p)) = [p, u(p)]$. Now J_Σ, J, and A determine an almost complex structure $\widetilde{J}_A$ on $\widetilde{M}$, and $\widetilde{u}$ is a $\widetilde{J}_A$-holomorphic curve if and only if $\bar{\partial}_{J,A}(u) = 0$. In any case, the global form of (10) is

$$\bar{\partial}_{J,A}(u) = 0, \qquad \mu(u) = 0. \tag{11}$$

It follows that $A \in \mathcal{A}(P)$ is the pullback under u of the connection on the principal bundle $\mu^{-1}(0) \to M/\!/G$ that is determined by the metric $\omega(\cdot, J\cdot)$. Fix an equivariant homology class $B \in H_2^G(M;\mathbb{Z})$, and denote by

$$\mathcal{M}^0_{B,\Sigma}(M,\mu;J) = \frac{\{(u,A) \mid (11),\ [u] = B\}}{\mathcal{G}(P)}$$

the moduli space of gauge equivalence classes of solutions of (11) which represent the class B. If B is the image of a class $\bar{B} \in H_2(M/\!\!/G;\mathbb{Z})$ under the natural homomorphism $H_2(M/\!\!/G;\mathbb{Z}) \to H_2^G(M;\mathbb{Z})$, then the solutions of (11) correspond to J-holomorphic curves in the quotient $M/\!\!/G$ representing the class $\bar{B}$.

Remark 2.3. (i) If $\tau \in \mathfrak{g}$ is a central element, then we can replace the moment map μ by $\mu - \tau$. The solutions of

$$\bar{\partial}_{J,A}(u) = 0, \qquad \mu(u) = \tau \tag{12}$$

correspond to J-holomorphic curves in the quotient $M/\!\!/G(\tau)$.

(ii) Let $\mathcal{O} \subset \mathfrak{g}$ be an orbit under the adjoint action, and consider the product $M \times \mathcal{O}$ with the moment map $\mu_{\mathcal{O}} : M \times \mathcal{O} \to \mathfrak{g}$ given by $\mu_{\mathcal{O}}(x,\tau) = \mu(x) - \tau$. Then (11) takes the form

$$\bar{\partial}_{J,A}(u) = 0, \qquad \bar{\partial}_A(\tau) = 0, \qquad \mu(u) = \tau, \tag{13}$$

where $u : P \to M$ and $\tau : P \to \mathcal{O}$ are equivariant maps and

$$\bar{\partial}_A(\tau) = \frac{1}{2}\big(d_A\tau - J_\tau \circ d_A\tau \circ J_\Sigma\big), \qquad d_A\tau = d\tau + [A,\tau].$$

The solutions of (13) again correspond to J-holomorphic curves in the quotient $M/\!\!/G(\tau)$. Note that (13) is equivalent to (12) whenever $\mathcal{O}$ is a single point (necessarily contained in the centre of $\mathfrak{g}$). In local holomorphic coordinates, (13) has the form

$$\begin{aligned}
&\partial_s u + X_\Phi(u) + J\big(\partial_t u + X_\Psi(u)\big) = 0,\\
&\partial_s \tau + [\Phi,\tau] - J_\tau\big(\partial_t \tau + [\Psi,\tau]\big) = 0,\\
&\mu(u) - \tau = 0.
\end{aligned} \tag{14}$$

As before, these equations imply that the pairs $(\partial_s u + X_\Phi(u), \partial_s\tau + [\Phi,\tau])$ and $(\partial_t u + X_\Psi(u), \partial_t\tau + [\Psi,\tau])$ are the unique harmonic representatives of the derivatives with respect to s and t. Thus, by Remark 2.1(ii), the function $\Phi : \mathbb{C} \to \mathfrak{g}$ is determined by the equation

$$L_u{}^* L_u \Phi + A_\tau \Phi + L_u{}^* \partial_s u + J_\tau \partial_s \tau = 0,$$

and similarly for Ψ. Note that in this local form the quadruple (u, τ, Φ, Ψ) can be gauge-transformed to one where τ is constant. The second equation in (14) then takes the form $[\Phi, \tau] = J_\tau[\Psi, \tau]$ and can be viewed as a constraint on the connection A. The gauge group now consists of maps $\mathbb{C} \to G_\tau$.

(iii) If one removes the assumption that G acts freely on $\mu^{-1}(0)$, then $M/\!\!/G$ is an orbifold and (13) describes J-holomorphic curves in this space. Since every symplectic orbifold can be expressed in this form (cf. [49]), one might be tempted to use (11) to give a rigorous definition of the Gromov-Witten invariants of orbifolds.

3 Invariants of Hamiltonian group actions

3.1 An action functional

Let (M, ω, μ) be a symplectic manifold with a Hamiltonian G-action, and let $\pi : P \to \Sigma$ be a principle G-bundle over a compact Riemann surface (Σ, J_Σ). Denote by $C^\infty_G(P, M)$ the space of equivariant smooth functions $u : P \to M$, and consider the action functional $E : C^\infty_G(P, M) \times \mathcal{A}(P) \to \mathbb{R}$, defined by

$$E(u, A) = \frac{1}{2}\int_\Sigma \left(\left|d_A u\right|^2 + \left|F_A\right|^2 + |\mu(u)|^2\right) \mathrm{dvol}_\Sigma .$$

This functional is invariant under the action of the gauge group $\mathcal{G}(P)$. The Euler equations have the form

$$\nabla_A{}^* d_A u + d\mu(u)^*\mu(u) = 0, \qquad d_A{}^* F_A + L_u{}^* d_A u = 0. \tag{15}$$

Here $\nabla_A : C^\infty(\Sigma, u^*TM/G) \to \Omega^1(\Sigma, u^*TM/G)$ denotes the covariant derivative operator induced by A and by the Levi-Civita connection ∇ of the metric $\omega(\cdot, J\cdot)$ on M. It is defined by

$$\nabla_A \xi = \nabla\xi + \nabla_\xi X_A \tag{16}$$

for $\xi \in C^\infty(\Sigma, u^*TM/G)$. Think of ξ as an equivariant function from P to u^*TM. Then $\nabla_A\xi$ is a 1-form on P with values in u^*TM. This form is obviously equivariant, and, since $\nabla_{p\eta}\xi(p) + \nabla_{\xi(p)} X_\eta(u(p)) = 0$, it is horizontal. Hence it descends to a 1-form on Σ with values in u^*TM/G, still denoted by $\nabla_A\xi$. The symbol $\nabla_A{}^*$ in (15) denotes the L^2-adjoint of ∇_A.

There are first-order equations that give rise to special solutions of (15). They have the form

$$\bar{\partial}_{J,A}(u) = 0, \qquad *F_A + \mu(u) = 0. \tag{17}$$

Here $*$ denotes the Hodge $*$-operator on Σ, and so these equations depend explicitly on the metric on Σ. The study of their solutions is the main purpose of this paper. The next proposition shows that the solutions of (17), if they exist, are the absolute minima of E in a fixed equivariant homology class and hence also solve the Euler equations. The moment mapcondition asserts that the polynomial

$$\mathfrak{g} \longrightarrow \Omega^*(M) : \xi \longmapsto \omega - \langle \mu, \xi \rangle$$

is G-closed and hence defines an equivariant cohomology class that we denote by $[\omega - \mu] \in H^2_G(M; \mathbb{R})$. Hence, by Proposition 2.2, the last term in (18) is a topological invariant.

Proposition 3.1. For every $A \in \mathcal{A}(P)$ and every $u \in C^\infty_G(P, M)$,

$$E(u, A) = \int_\Sigma \left(\left|\bar{\partial}_{J,A}(u)\right|^2 + \frac{1}{2}\left| *F_A + \mu(u)\right|^2 \right) \mathrm{dvol}_\Sigma + \big\langle [\omega - \mu], [u] \big\rangle, \tag{18}$$

where

$$\big\langle [\omega - \mu], [u] \big\rangle = \int_\Sigma \big((d_A u)^* \omega - \langle \mu(u), F_A \rangle \big) = \int_\Sigma \big(u^*\omega - d\langle \mu(u), A \rangle \big). \qquad \square$$

Proof. Choose a holomorphic coordinate chart $\phi : U \to \Sigma$, where $U \subset \mathbb{C}$ is an open set, and let $\widetilde{\phi} : U \to P$ be a lift of ϕ; that is, $\pi \circ \widetilde{\phi} = \phi$. Then the function u and the connection A are in local coordinates given by $u^{loc} = u \circ \widetilde{\phi}$ and $A^{loc} = \widetilde{\phi}^* A = \Phi \, ds + \Psi \, dt$, where $\Phi, \Psi : U \to \mathfrak{g}$. The pullback volume form on U is $\lambda^2 \, ds \wedge dt$ for some function $\lambda : U \to (0, \infty)$, and the metric is $\lambda^2(ds^2 + dt^2)$. Hence

$$\begin{aligned}
\widetilde{\phi}^* F_A &= \big(\partial_s \Psi - \partial_t \Phi + [\Phi, \Psi] \big) \, ds \wedge dt, \\
\widetilde{\phi}^* d_A u &= \big(\partial_s u^{loc} + X_\Phi \big) \, ds + \big(\partial_t u^{loc} + X_\Psi \big) \, dt, \\
\widetilde{\phi}^* \bar{\partial}_{J,A}(u) &= \frac{1}{2}(\xi \, ds - J\xi \, dt), \qquad \xi = \partial_s u^{loc} + X_\Phi + J\big(\partial_t u^{loc} + X_\Psi\big).
\end{aligned}$$

Here X_Φ, X_Ψ, and J are evaluated at u^{loc}. In the following we drop the superscript "loc." Then equations (17) have the form

$$\begin{aligned}
&\partial_s u + X_\Phi(u) + J\big(\partial_t u + X_\Psi(u)\big) = 0, \\
&\partial_s \Psi - \partial_t \Phi + [\Phi, \Psi] + \lambda^2 \mu(u) = 0.
\end{aligned} \tag{19}$$

The pullback of the energy integrand under $\phi : U \to \Sigma$ is given by

$$\begin{aligned}
e :&= \frac{1}{2}\left|\partial_s u + X_\Phi\right|^2 + \frac{1}{2}\left|\partial_t u + X_\Psi\right|^2 + \frac{\lambda^{-2}}{2}\left|\partial_s\Psi - \partial_t\Phi + [\Phi,\Psi]\right|^2 + \frac{\lambda^2}{2}|\mu(u)|^2 \\
&= \frac{1}{2}\left|\partial_s u + X_\Phi + J(\partial_t u + X_\Psi)\right|^2 \\
&\quad + \frac{\lambda^2}{2}\left|\lambda^{-2}\left(\partial_s\Psi - \partial_t\Phi + [\Phi,\Psi]\right) + \mu(u)\right|^2 \\
&\quad + \omega\left(\partial_s u + X_\Phi, \partial_t u + X_\Psi\right) - \left\langle \partial_s\Psi - \partial_t\Phi + [\Phi,\Psi], \mu(u)\right\rangle.
\end{aligned}$$

This proves (18). The identity $(d_A u)^*\omega - \langle \mu(u), F_A\rangle = u^*\omega - d\langle \mu(u), A\rangle$ can, in local coordinates, be expressed in the form

$$\begin{aligned}
&\omega\left(\partial_s u + X_\Phi, \partial_t u + X_\Psi\right) - \left\langle \partial_s\Psi - \partial_t\Phi + [\Phi,\Psi], \mu(u)\right\rangle \\
&\qquad\qquad = \omega(\partial_s u, \partial_t u) - \partial_s\left\langle \mu(u), \Psi\right\rangle + \partial_t\langle \mu(u), \Phi\rangle.
\end{aligned}$$

This follows directly from the definitions and the fact that $\omega(X_\Phi, X_\Psi) = \langle \mu, [\Phi,\Psi]\rangle$. This proves the proposition. ∎

3.2 Symplectic reduction

Denote $C^\infty_G(P, M; B) := \{u \in C^\infty_G(P, M) \mid [u] = B\}$ and consider the space

$$\mathcal{B} := C^\infty_G(P, M; B) \times \mathcal{A}(P).$$

This space carries a natural symplectic form. To see this, note that the tangent space of $\mathcal{B}$ at (u, A) is

$$T_{(u,A)}\mathcal{B} = C^\infty\left(\Sigma, u^*TM/G\right) \oplus \Omega^1(\Sigma, \mathfrak{g}_P),$$

where $C^\infty(\Sigma, u^*TM/G) = C^\infty_G(P, u^*TM)$ is the space of G-equivariant sections of the bundle $u^*TM \to P$ and where $\Omega^1(\Sigma, \mathfrak{g}_P) = \Omega^1_{\mathrm{ad}}(P, \mathfrak{g})$ is the space of equivariant and horizontal Lie algebra-valued 1-forms on P. The symplectic form on $T_{(u,A)}\mathcal{B}$ is given by

$$\Omega\left((\xi, \alpha), (\xi', \alpha')\right) = \int_\Sigma \omega(\xi, \xi')\ \mathrm{dvol}_\Sigma + \int_\Sigma \langle \alpha \wedge \alpha'\rangle \tag{20}$$

for $\xi, \xi' \in C^\infty(\Sigma, u^*TM/G)$ and $\alpha, \alpha' \in \Omega^1(\Sigma, \mathfrak{g}_P)$ (see Section 2.2). Consider the group $\widetilde{\mathcal{G}} = \widetilde{\mathcal{G}}(P)$ of all automorphisms of P that descend to Hamiltonian symplectomorphisms of Σ. There is an exact sequence

$$1 \longrightarrow \mathcal{G} \longrightarrow \widetilde{\mathcal{G}} \longrightarrow \mathrm{Ham}(\Sigma, \mathrm{dvol}_\Sigma) \longrightarrow 1,$$

and the Lie algebra of $\widetilde{\mathcal{G}}$ consists of all equivariant vector fields $v \in \mathrm{Vect}_G(P)$ such that the 1-form $\iota(\pi_* v)\,\mathrm{dvol}_\Sigma \in \Omega^1(\Sigma)$ is exact. Thus, for every $v \in \mathrm{Lie}(\widetilde{\mathcal{G}})$, there exists a unique function $h_v : \Sigma \to \mathbb{R}$ such that

$$\iota(\pi_* v)\,\mathrm{dvol}_\Sigma = dh_v, \qquad \int_\Sigma h_v\,\mathrm{dvol}_\Sigma = 0. \tag{21}$$

The group $\widetilde{\mathcal{G}}$ acts on $\mathcal{B}$ by $(u, A) \mapsto (u \circ f^{-1}, f_* A)$ for $f \in \widetilde{\mathcal{G}}$, and the infinitesimal action is given by the vector fields

$$\mathcal{B} \longrightarrow T\mathcal{B} : (u, A) \longmapsto \big(-du \circ v, -\mathcal{L}_v A\big)$$

for $v \in \mathrm{Lie}(\widetilde{\mathcal{G}})$. It follows from the work of Atiyah and Bott [3] and Donaldson [14] that this action is Hamiltonian. Define $\widetilde{\mu} : \mathcal{B} \to \mathrm{Lie}(\widetilde{\mathcal{G}})^*$ by

$$\langle \widetilde{\mu}(u, A), v \rangle = \int_\Sigma \Big(\big\langle *F_A + \mu(u), A(v) \big\rangle\,\mathrm{dvol}_\Sigma - h_v \big(u^*\omega - d\langle \mu(u), A \rangle \big) \Big). \tag{22}$$

Here $\langle *F_A + \mu(u), A(v) \rangle \in \Omega^0(P)$ and $u^*\omega - d\langle \mu(u), A \rangle \in \Omega^2(P)$. But they both descend to Σ, and we do not distinguish the descendents in notation from the original forms on P.

Proposition 3.2. The function $\widetilde{\mu} : \mathcal{B} \to \mathrm{Lie}(\widetilde{\mathcal{G}})^*$ is a moment mapfor the action of $\widetilde{\mathcal{G}}$ on $\mathcal{B}$. □

Proof. Let $\mathbb{R} \to \mathcal{B} : t \mapsto (u_t, A_t)$ be any smooth path in $\mathcal{B}$, and denote its derivative by $(\xi_t, \alpha_t) \in T_{(u_t, A_t)}\mathcal{B}$. Then, by Cartan's formula,

$$\begin{aligned} &\frac{d}{dt}\big(u_t{}^*\omega - d\langle \mu(u_t), A_t \rangle \big) = d\sigma_t, \\ &\sigma_t = \omega(\xi_t, d_{A_t} u_t) - \langle \mu(u_t), \alpha_t \rangle \in \Omega^1(\Sigma). \end{aligned}$$

Moreover, for every $v \in \mathrm{Lie}(\widetilde{\mathcal{G}})$,

$$\int_\Sigma h_v d\sigma_t = \int_\Sigma \sigma_t \wedge dh_v = \int_\Sigma \sigma_t \wedge \iota(\pi_* v)\,\mathrm{dvol}_\Sigma = -\int_\Sigma \sigma_t(\pi_* v)\,\mathrm{dvol}_\Sigma\,.$$

Hence

$$\begin{aligned} \frac{d}{dt}\langle \widetilde{\mu}(u_t, A_t), v \rangle &= \int_\Sigma \big(\omega(\xi_t, d_{A_t} u_t \circ v) + \big\langle d\mu(u_t)\xi_t, A_t(v) \big\rangle \big)\,\mathrm{dvol}_\Sigma \\ &\quad + \int_\Sigma \big(\big\langle A_t(v), d_{A_t}\alpha_t \big\rangle + \big\langle F_{A_t}, \alpha_t(v) \big\rangle \big) \end{aligned}$$

$$
\begin{aligned}
&= \int_\Sigma \omega(-du_t \circ v, \xi_t)\, \mathrm{dvol}_\Sigma + \int_\Sigma \big\langle \big(- d_{A_t}(A_t \circ v) - \iota(v) F_{A_t} \big) \alpha_t \big\rangle \\
&= \Omega\big(\big(- du_t \circ v, -\mathcal{L}_v A_t \big), (\xi_t, \alpha_t) \big).
\end{aligned}
$$

The last equality uses the formula $\mathcal{L}_v A = d_A(A \circ v) + \iota(v) F_A$. This proves the proposition.

■

Remark 3.1. (i) Consider the action of the gauge group $\mathcal{G} = \mathcal{G}(P)$ on $\mathcal{B}$. The infinitesimal action of $\Omega^0(\Sigma, \mathfrak{g}_P) = \mathrm{Lie}(\mathcal{G})$ is given by the vector fields

$$
(u, A) \longmapsto \big(L_u \eta, -d_A \eta \big)
$$

for $\eta \in \Omega^0(\Sigma, \mathfrak{g}_P)$. By Proposition 3.2 , these vector fields are Hamiltonian, and a moment mapfor the action is given by

$$
\mathcal{B} \longrightarrow \Omega^0(\Sigma, \mathfrak{g}_P) : (u, A) \longmapsto *F_A + \mu(u).
$$

Hence the zero set of the moment mapis the space of solutions of the second equation in (17).

(ii) If M is Kähler, then, under suitable regularity hypotheses, the space

$$
\mathcal{X} = \big\{ (u, A) \in \mathcal{B} \mid \bar{\partial}_{J,A}(u) = 0 \big\}
$$

is a complex, and hence symplectic, submanifold of $\mathcal{B}$. This submanifold is invariant under $\mathcal{G}$, and hence the space of gauge equivalence classes of solutions of (17) can be interpreted as the symplectic quotient $\mathcal{X}/\!/\mathcal{G}$. In this case one can consider the action of the complexified group $\mathcal{G}^c$ on $\mathcal{X}$ and study the quotient $\mathcal{X}^s/\mathcal{G}^c$, where $\mathcal{X}^s \subset \mathcal{X}$ is a suitable subspace of *stable pairs* (u, A). It turns out that, as in the finite-dimensional case, there is a natural correspondence

$$
\frac{\mathcal{X}^s}{\mathcal{G}^c} \cong \mathcal{X}/\!/\mathcal{G}.
$$

This programme was carried out by Mundet in his recent thesis [55].

(iii) The zero set of the moment map $\widetilde{\mu} : \mathcal{B} \to \mathrm{Lie}(\widetilde{\mathcal{G}})^*$ consists of all pairs $(u, A) \in \mathcal{B}$ that satisfy $*F_A + \mu(u) = 0$ and

$$
u^*\omega - d\langle \mu(u), A \rangle = \frac{\big\langle [\omega - \mu], B \big\rangle}{\mathrm{Vol}(\Sigma)}\, \mathrm{dvol}_\Sigma,
$$

where the left-hand side is understood as a 2-form on Σ. However, the action of $\widetilde{\mathcal{G}}$ is not compatible with the condition $\bar{\partial}_{J,A}(u) = 0$.

In the case

$$\Sigma = S^2$$

with the standard metric and complex structure, it is interesting to consider the group $\widehat{\mathcal{G}} \subset \widetilde{\mathcal{G}}$ of all automorphisms of P which descend to isometries of S^2. There is an exact sequence

$$1 \longrightarrow \mathcal{G} \longrightarrow \widehat{\mathcal{G}} \longrightarrow \mathrm{SO}(3) \longrightarrow 1,$$

and the action of $\widehat{\mathcal{G}}$ on $\mathcal{B}$ preserves the submanifold $\mathcal{X}$. The moment map

$$\widehat{\mu} : \mathcal{B} \longrightarrow \mathrm{Lie}\left(\widehat{\mathcal{G}}\right)^*$$

for this action is given by the restriction of $\widetilde{\mu}(u, A)$ to $\mathrm{Lie}(\widehat{\mathcal{G}})$. Hence the zero set of $\widehat{\mu}$ consists of all pairs $(u, A) \in \mathcal{B}$ that satisfy $*F_A + \mu(u) = 0$ and

$$\int_{S^2} h_\xi \left(u^*\omega - d\langle \mu(u), A\rangle\right) = 0 \tag{23}$$

for every $\xi \in \mathfrak{so}(3)$, where $h_\xi : S^2 \to \mathbb{R}$ denotes the Hamiltonian function generating the infinitesimal action of ξ. Thus h_ξ is the restriction of a linear functional on $\mathbb{R}^3$ to S^2. It is interesting to consider all solutions of (17) and (23) and divide by the action of the group $\widehat{\mathcal{G}}$. This is the analogue of the quotient of the space of J-holomorphic spheres $v : S^2 \to M$ by the reparametrization group $\mathrm{PSL}(2, \mathbb{C})$. Namely, $\mathrm{PSL}(2, \mathbb{C})$ is the complexification of $\mathrm{SO}(3)$. It acts on the Riemann sphere $\mathbb{C} \cup \{\infty\}$ by fractional linear transformations of the form

$$\phi(z) = \frac{az + b}{cz + d},$$

where a, b, c, and d are complex numbers such that $ad - bc = 1$. The subgroupof isometries is $\mathrm{SO}(3) \cong \mathrm{SU}(2)/\{\pm 1\!\!1\}$. The next proposition shows that, instead of dividing the space of J-holomorphic curves from the Riemann sphere into a symplectic manifold by $\mathrm{PSL}(2, \mathbb{C})$, one can consider J-holomorphic spheres that satisfy (23) (without the connection term) and only divide by $\mathrm{SO}(3)$. This is another analogue of the relation between the complex quotient M^s/G^c and the Marsden-Weinstein quotient $M/\!\!/G$.

Proposition 3.3. Let $\sigma \in \Omega^2(S^2)$ such that

$$\int_{S^2} \sigma \neq 0.$$

Then there exists a fractional linear transformation $\phi : S^2 \to S^2$ such that

$$\int_{S^2} h_\xi \phi^* \sigma = 0 \tag{24}$$

for every $\xi \in \mathfrak{so}(3)$. If $\phi_0 \in \mathrm{PSL}(2, \mathbb{C})$ and $\phi_1 \in \mathrm{PSL}(2, \mathbb{C})$ both satisfy (24), and if σ is a volume form, then ${\phi_0}^{-1} \circ \phi_1 \in \mathrm{SO}(3)$. □

Proof. We identify the Riemann sphere $\mathbb{C} \cup \{\infty\}$ with the unit sphere $S^2 \subset \mathbb{R}^3$ via stereographic projection. Explicitly, this diffeomorphism is given by

$$S^2 \longrightarrow \mathbb{C} \cup \{\infty\} : x \longmapsto \frac{x_1 + i x_2}{1 - x_3}.$$

Under this correspondence the quotient $\mathrm{PSL}(2, \mathbb{C}) / \mathrm{SO}(3)$ can be identified with the open unit ball $B^3 \subset \mathbb{R}^3$ via the map $B^3 \to \mathrm{PSL}(2, \mathbb{C}) : \eta \mapsto \phi_\eta$ that assigns to $\eta \in B^3$ the diffeomorphism $\phi_\eta : S^2 \to S^2$ given by

$$\phi_\eta(x) = \frac{\sqrt{1 - |\eta|^2}}{1 - \langle x, \eta \rangle} \left(x - \left\langle x, |\eta|^{-1} \eta \right\rangle |\eta|^{-1} \eta \right) + \frac{\left\langle x, |\eta|^{-1} \eta \right\rangle - |\eta|}{1 - \langle x, \eta \rangle} |\eta|^{-1} \eta.$$

If $\eta \in B^3$ converges to $\zeta \in S^2$, then ${\phi_\eta}^{-1}(x)$ converges to ζ uniformly in compact subsets of $S^2 \setminus \{-\zeta\}$. Hence

$$\lim_{\eta \to \zeta} \int_{S^2} \langle \xi, \iota \rangle {\phi_\eta}^* \sigma = \lim_{\eta \to \zeta} \int_{S^2} \left\langle \xi, \iota \circ {\phi_\eta}^{-1} \right\rangle \sigma = \langle \xi, \zeta \rangle \int_{S^2} \sigma$$

for every $\zeta \in S^2$ and every $\xi \in \mathbb{R}^3$, where $\iota : S^2 \to \mathbb{R}^3$ denotes the obvious inclusion and $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^3$. Hence

$$\lim_{\eta \to \zeta} \int_{S^2} \iota {\phi_\eta}^* \sigma = \zeta \int_{S^2} \sigma$$

for every $\zeta \in S^2$, and the convergence is uniform in ζ. It follows from a standard argument in degree theory (see [53]) that there exists an $\eta \in B^3$ such that

$$\int_{S^2} \iota {\phi_\eta}^* \sigma = 0.$$

This identity is equivalent to (24).

Now suppose that $\sigma \in \Omega^2(S^2)$ is a volume form and $\phi \in \mathrm{PSL}(2,\mathbb{C})$ such that

$$\int_{S^2} \iota\phi^*\sigma = \int_{S^2} \iota\sigma = 0. \tag{25}$$

Then the same argument as in Remark 2.1(iii) shows that $\phi \in \mathrm{SO}(3)$. Namely, there exists an $\eta \in B^3$ and a matrix $\Psi \in \mathrm{SO}(3)$ such that

$$\phi(x) = \phi_\eta(\Psi x)$$

for every $x \in S^2$. Hence, for every $\xi \in \mathbb{R}^3$,

$$\int_{S^2} \langle \xi, \iota\rangle {\phi_\eta}^*\sigma = \int_{S^2} \left\langle \Psi^{-1}\xi, \iota\right\rangle \phi^*\sigma = 0.$$

Denote $\lambda(t) = |\eta|^{-1}\tanh(|\eta|t)$, choose $T > 0$ such that $\lambda(T) = 1$, and consider the flow

$$\phi_t(x) := \phi_{\lambda(t)\eta}(x)$$

for $0 \le t \le T$. It satisfies $\phi_0 = \mathrm{id}$, $\phi_T = \phi_\eta$, and, since $\dot\lambda = 1 - |\eta|^2\lambda^2$,

$$\frac{d}{dt}{\phi_t}^{-1}(x) = \eta - \left\langle \eta, {\phi_t}^{-1}(x)\right\rangle {\phi_t}^{-1}(x).$$

Hence

$$\frac{d}{dt}\int_{S^2} \langle \eta, \iota\rangle {\phi_t}^*\sigma = \frac{d}{dt}\int_{S^2} \left\langle \eta, \iota\circ{\phi_t}^{-1}\right\rangle\sigma = \int_{S^2} \left(|\eta|^2 - \left\langle \eta, \iota\circ{\phi_t}^{-1}\right\rangle^2\right)\sigma.$$

The last expression is nonnegative. Integrating from $t = 0$ to $t = T$, we obtain from (25) that it is equal to zero for all t. It follows that $\eta = 0$; hence $\phi_\eta = \mathrm{id}$, and hence $\phi = \Psi \in \mathrm{SO}(3)$, as claimed. This proves the proposition. ∎

3.3 Hamiltonian perturbations

We consider the following perturbations of (17). Let $C^\infty_G(M)$ denote the space of G-invariant smooth functions on M, and let $\mathrm{Vect}_G(M,\omega)$ denote the space of G-invariant Hamiltonian vector fields. Choose a G-invariant horizontal 1-form $\sigma \in \Omega^1(P, C^\infty_G(M))$. One can think of σ either as a 1-form on Σ with values in $C^\infty_G(M)$ or as a 1-form on $P \times M$ which is invariant under the separate action of G on both P and M and which vanishes on all vectors of the form $(p\xi, w) \in T_pP \times T_xM$, where $\xi \in \mathfrak{g}$. Consider the 1-form

$$X_\sigma : TP \longrightarrow \mathrm{Vect}_G(M,\omega), \qquad \iota(X_{\sigma_p(v)})\omega = d(\sigma_p(v)).$$

For every equivariant function $u : P \to M$, the 1-form $X_\sigma(u) \in \Omega^1(P, u^*TM)$, defined by $T_pP \to T_{u(p)}M : v \mapsto X_{\sigma_p(v)}(u(p))$, is equivariant and horizontal. Hence it descends to a 1-form on Σ with values in u^*TM/G which is still denoted by $X_\sigma(u)$. The perturbed equations have the form

$$\bar{\partial}_{J,A}(u) + (X_\sigma(u))^{0,1} = 0, \qquad *F_A + \mu(u) = 0. \tag{26}$$

The space of solutions of (26) is invariant under the action of the gauge group.

Remark 3.2. (i) In local holomorphic coordinates, (26) has the form

$$\begin{aligned} &\partial_s u + X_\Phi(u) + X_F(u) + J\big(\partial_t u + X_\Psi(u) + X_G(u)\big) = 0, \\ &\partial_s \Psi - \partial_t \Phi + [\Phi, \Psi] + \lambda^2 \mu(u) = 0, \end{aligned} \tag{27}$$

where $u : U \to M$, $\Phi, \Psi : U \to \mathfrak{g}$, and $F, G : U \to C^\infty_G(M)$. The local coordinate representatives of A and σ are $A = \Phi\, ds + \Psi\, dt$ and $\sigma = F\, ds + G\, dt$.

(ii) Let $\sigma \in \Omega^1(P \times M)$ be as above. Then σ descends to a 1-form on $P \times_G M$ and $\omega - d\langle \mu, A\rangle - d\sigma$ is a connection 2-form as in [51, Chapter 6]. The covariant derivative of a function $u : P \to M$ with respect to this connection is given by

$$d_{A,\sigma} u = d_A u + X_\sigma(u) \in \Omega^1\big(\Sigma, u^*TM/G\big).$$

The first equation in (26) can now be written in the form $\bar{\partial}_{J,A,\sigma}(u) = 0$, where $\bar{\partial}_{J,A,\sigma}(u)$ is the J-antilinear part of the 1-form $d_{A,\sigma} u$.

(iii) The energy identity (18) continues to hold with $\bar{\partial}_{J,A}(u)$ replaced by $\bar{\partial}_{J,A,\sigma}(u)$ and $d_A u$ replaced by $d_{A,\sigma} u$ (in the definition of $E(u, A)$).

3.4 Moduli spaces

Fix an integer $k \geq 0$, a compact Riemann surface $(\Sigma, J_\Sigma, \mathrm{dvol}_\Sigma)$, and an equivariant homology class $B \in H_2^G(M; \mathbb{Z})$ such that

$$\big\langle [\omega - \mu], B \big\rangle \geq 0. \tag{28}$$

By Proposition 3.1 and Remark 3.2, this condition is necessary for the existence of solutions of (17) or (26). Consider the space

$$\widetilde{\mathcal{M}}_{B,\Sigma,k} = \widetilde{\mathcal{M}}_{B,\Sigma,k}(M, \mu; J, \sigma)$$

of all tuples $(u, A, p_1, \dots, p_k)$, where u and A satisfy (26), u represents the class B, and $p_1, \dots, p_k$ are points in P with distinct base points $\pi(p_i) \in \Sigma$. The gauge group $\mathcal{G}(P)$ acts on this space by

$$g^*(u, A, p_1, \dots, p_k) = \left(g^{-1}u, g^*A, p_1 g(p_1)^{-1}, \dots, p_k g(p_k)^{-1}\right),$$

and the quotient is denoted by

$$\mathcal{M}_{B,\Sigma,k} = \mathcal{M}_{B,\Sigma,k}(M, \mu; J, \sigma) = \frac{\widetilde{\mathcal{M}}_{B,\Sigma,k}(M, \mu; J, \sigma)}{\mathcal{G}(P)}.$$

If $k = 0$, we write $\mathcal{M}_{B,\Sigma} = \mathcal{M}_{B,\Sigma,0}$. Our goal is to use these moduli spaces to define invariants of (M, ω, μ). Note that the symplectic form enters into the definition of the moduli spaces only indirectly through the compatibility condition on the almost complex structure J.

3.5 Fredholm theory

Let $\mathcal{B}$ be as in Section 3.2 , and consider the vector bundle $\mathcal{E} \to \mathcal{B}$ whose fibre over a pair $(u, A) \in \mathcal{B}$ is given by $\mathcal{E}_{u,A} = \mathcal{E}_u = \Omega^{0,1}(\Sigma, u^*TM/G) \oplus \Omega^0(\Sigma, \mathfrak{g}_P)$. The gauge group $\mathcal{G}$ acts on $\mathcal{E}$, the projection $\mathcal{E} \to \mathcal{B}$ is $\mathcal{G}$-equivariant, and the almost complex structure J and the perturbation σ determine a $\mathcal{G}$-equivariant section

$$\mathcal{B} \longrightarrow \mathcal{E} : (u, A) \longmapsto \left(\bar{\partial}_{J,A,\sigma}(u), *F_A + \mu(u)\right). \tag{29}$$

Evidently, the zero set of this section is the space $\widetilde{\mathcal{M}}_{B,\Sigma}$. The vertical differential of the section (29) at a zero (u, A) gives rise to an operator

$$\mathcal{D}_{u,A} : \begin{array}{c} C^\infty(\Sigma, u^*TM/G) \\ \oplus \\ \Omega^1(\Sigma, \mathfrak{g}_P) \end{array} \longrightarrow \begin{array}{c} \Omega^{0,1}(\Sigma, u^*TM/G) \\ \oplus \\ \Omega^0(\Sigma, \mathfrak{g}_P) \\ \oplus \\ \Omega^0(\Sigma, \mathfrak{g}_P) \end{array}$$

given by

$$\mathcal{D}_{u,A}\begin{pmatrix} \xi \\ \alpha \end{pmatrix} = \begin{pmatrix} D\bar{\partial}_{J,A,\sigma}(u)\xi + (L_u\alpha)^{0,1} \\ {L_u}^*\xi - {d_A}^*\alpha \\ d\mu(u)\xi + *d_A\alpha \end{pmatrix}. \tag{30}$$

Here $D\bar{\partial}_{J,A,\sigma}(u) : C^\infty(\Sigma, u^*TM/G) \to \Omega^{0,1}(\Sigma, u^*TM/G)$ is the Cauchy-Riemann operator obtained by differentiating the first equation in (26). In explicit terms this operator is given by

$$D\bar{\partial}_{J,A,\sigma}(u)\xi = (\nabla_{A,\sigma}\xi)^{0,1} - \frac{1}{2}J(\nabla_\xi J)\partial_{J,A,\sigma}(u)$$

for $\xi \in C^\infty(\Sigma, u^*TM/G)$, where ∇ denotes the Levi-Civita connection of the metric $\omega(\cdot, J\cdot)$ on M and $\nabla_{A,\sigma}\xi \in \Omega^1(\Sigma, u^*TM/G)$ is given by

$$\nabla_{A,\sigma}\xi = \nabla\xi + \nabla_\xi X_A + \nabla_\xi X_\sigma. \tag{31}$$

Remark 3.3. A tangent vector $(\xi, \alpha) \in T_{(u,A)}\mathcal{B}$ is L^2-orthogonal to the gauge orbit of (u, A) if and only if

$$L_u{}^*\xi - d_A{}^*\alpha = 0. \tag{32}$$

This is the local slice condition, and the tangent space of the quotient $\mathcal{B}/\mathcal{G}$ at $[u, A]$ can be identified with the space of solutions of (32). Note also that the left-hand side of (32) agrees with the second coordinate of $\mathcal{D}_{u,A}(\xi, \alpha)$ in (30).

The operator $\mathcal{D}_{u,A}$ (between suitable Sobolev completions) is a compact perturbation of the direct sum of the first-order operators $D\bar{\partial}_{J,A,\sigma}(u)$ and

$$\Omega^1(\Sigma, \mathfrak{g}_P) \longrightarrow \Omega^0(\Sigma, \mathfrak{g}_P) \oplus \Omega^0(\Sigma, \mathfrak{g}_P) : \alpha \longmapsto \big(-d_A{}^*\alpha, *d_A\alpha\big).$$

Hence it is Fredholm and, by the Riemann-Roch theorem,

$$\operatorname{index}\mathcal{D}_{u,A} = (2-2g)(n - \dim G) + 2\langle c_1^G, [u]\rangle. \tag{33}$$

Here g is the genus of Σ and $c_1^G = c_1^G(TM, J) \in H_G^2(M; \mathbb{Z})$ denotes the equivariant first Chern class of the tangent bundle. It is defined as the first Chern class of the vector bundle $TM \times_G EG \to M \times_G EG$ with the complex structure given by $J \in \mathcal{J}(M, \omega, \mu)$. If $\mathcal{D}_{u,A}$ is surjective for every $(u, A) \in \widetilde{\mathcal{M}}_{B,\Sigma}$, then it follows from the implicit function theorem (in an infinite-dimensional setting) that $\mathcal{M}_{B,\Sigma,k}$ is a smooth manifold of dimension

$$\dim\mathcal{M}_{B,\Sigma,k} = (2-2g)(n - \dim G) + 2\langle c_1^G, B\rangle + k(2 + \dim G). \tag{34}$$

To obtain smooth moduli spaces it remains to prove that, for a suitable perturbation, the Fredholm operator $\mathcal{D}_{u,A}$ is surjective for every solution (u, A) of (26). This means that the section (29) of the vector bundle $\mathcal{E} \to \mathcal{B}$ is transverse to the zero section. In some cases transversality can be expected to hold for a generic volume form on Σ or a generic almost complex structure on M. In other cases more general perturbations of the equations may be required. In [55] Mundet established transversality for the so-called simple (i.e., not multiply covered) solutions in the case $G = S^1$ by choosing a generic almost complex structure $J \in \mathcal{J}(M, \omega, \mu)$. Alternatively, denote by $\mathcal{M}^*_{B,\Sigma} \subset \mathcal{M}_{B,\Sigma}$ the subset of all those gauge equivalence classes of solutions of (26) that satisfy

$$g \cdot u(p) = u(p) \Longrightarrow g = \mathbb{1} \tag{35}$$

for every p in a dense open subset of P. The next proposition asserts that, for a generic perturbation σ, this subset is a manifold of the predicted dimension. The proof appears elsewhere.

Proposition 3.4. Let $\mathcal{S} = \Omega^1(\Sigma, C^\infty_G(M))$, and denote by $\mathcal{S}_{\mathrm{reg}} \subset \mathcal{S}$ the subset of all perturbations $\sigma \in \mathcal{S}$ such that the operator $\mathcal{D}_{u,A}$ is surjective for every solution (u, A) of (26) that satisfies (35). Then $\mathcal{S}_{\mathrm{reg}}$ is a countable intersection of dense open subsets of $\mathcal{S}$. □

3.6 Compactness

The moduli space $\mathcal{M}_{B,\Sigma}$ is, in general, not compact. The energy identity (18) asserts that

$$E(u, A) = \langle [\omega - \mu], B \rangle$$

for every pair $(u, A) \in \widetilde{\mathcal{M}}_{B,\Sigma}$. Hence the L^2-norms of $d_{A,\sigma}u$ and $\mu(u)$ are uniformly bounded. As in the case of J-holomorphic curves and anti-self-dual Yang-Mills instantons, this is a Sobolev borderline case. Combining the techniques for J-holomorphic curves (Gromov compactness [32]) with those for connections (Uhlenbeck compactness [76]), one can show that, for every sequence $(u^\nu, A^\nu) \in \widetilde{\mathcal{M}}_{B,\Sigma}$ that satisfies a uniform L^p-bound of the form

$$\sup_\nu \int_\Sigma \left(|d_{A^\nu,\sigma} u^\nu|^p + |\mu(u^\nu)|^p \right) \mathrm{dvol}_\Sigma < \infty \tag{36}$$

for some constant $p > 2$, there exists a sequence of gauge transformations $g^\nu \in \mathcal{G}(P)$ such that $((g^\nu)^{-1} u^\nu, (g^\nu)^* A^\nu)$ has a C^∞-convergent subsequence. However, the energy identity only guarantees (36) for $p = 2$, and, in general, this does not suffice to prove compactness of the quotient space $\mathcal{M}_{B,\Sigma}$.

If (36) does not hold, then there must be a sequence of points $p^\nu \in P$ such that either $|d_{A^\nu,\sigma} u^\nu(p^\nu)|$ or $|\mu(u^\nu(p^\nu))|$ diverges to infinity. If the sequence $\mu \circ u^\nu$ is uniformly bounded, one can use the standard rescaling argument in Gromov compactness (cf. [50, Section 4.3]) to prove that, for some sequence of maps

$$\phi^\nu : \left\{ z \in \mathbb{C} \mid |z| < \frac{1}{\varepsilon^\nu} \right\} \longrightarrow P$$

with holomorphic projections $\pi \circ \phi^\nu$, the sequence $u^\nu \circ \phi^\nu$ converges to a nonconstant J-holomorphic curve $v : \mathbb{C} \to M$ that has finite energy. The removable singularity theorem for J-holomorphic curves (cf. [50, Theorem 4.2.1]) then asserts that v extends to a nonconstant J-holomorphic 2-sphere in M. Any such 2-sphere must be topologically nontrivial since

$$E(v) = \int_{S^2} v^*\omega = \int_{S^2} |dv|^2 > 0.$$

Thus, if there are no J-holomorphic spheres in M (e.g., if $\pi_2(M) = 0$), the only obstruction to compactness is the divergence of $\mu \circ u^\nu$. Now there are some interesting cases where the manifold M is noncompact, but all solutions of (17) satisfy a uniform bound on u.

Proposition 3.5. Assume the following.

(i) (M, ω, J) is a Hermitian vector space.

(ii) The group G acts on M by unitary automorphisms.

(iii) The moment map $\mu : M \to \mathfrak{g}$ is proper.

Then there exists a constant $c > 0$ such that every solution (u, A) of (17) (over any compact Riemann surface Σ) satisfies

$$\|u\|_{L^\infty} \leq c. \tag{37}$$

In particular, the moduli space $\mathcal{M}_{B,\Sigma}(M, \mu; J, \sigma)$ is compact for every compact Riemann surface $(\Sigma, J_\Sigma, \mathrm{dvol}_\Sigma)$, every equivariant homology class $B \in H_2^G(M; \mathbb{Z})$, and every compactly supported perturbation σ. □

Proof. Write $V := M$, denote by $\langle \cdot, \cdot \rangle = \omega(\cdot, J\cdot)$ the real inner product on V, denote by U(V) the groupof unitary automorphisms of V, and denote by $\mathfrak{u}(V)$ its Lie algebra. By assumption, the action of G on V is given by a homomorphism $\rho : G \to U(V)$, and we denote by $\dot\rho : \mathfrak{g} \to \mathfrak{u}(V)$ the corresponding Lie algebra homomorphism. We prove that there exists a central element $\tau \in \mathfrak{g}$ such that

$$\big\langle x, \dot\rho(\mu(x))Jx \big\rangle = 2\big\langle \mu(x), \mu(x) - \tau \big\rangle \tag{38}$$

for $x \in V$. To see this, suppose without loss of generality that $M = \mathbb{C}^n$ with its standard

Hermitian structure, and consider the inner product

$$\langle A, B\rangle = \mathrm{tr}(A^*B)$$

on the Lie algebra $\mathfrak{u}(n)$ of skew-symmetric matrices. The moment mapis given by

$$\mu(z) = \pi\Big(-\frac{i}{2}zz^*\Big) + \tau$$

for $z \in \mathbb{C}^n$, where $\tau \in \mathfrak{g}$ is a central element and $\pi : \mathfrak{u}(n) \to \mathfrak{g}$ denotes the adjoint of the Lie algebra homomorphism $\dot\rho : \mathfrak{g} \to \mathfrak{u}(n)$. Hence

$$\big\langle z, \dot\rho(\mu(z))iz\big\rangle = \mathrm{tr}\left(\dot\rho(\mu(z))izz^*\right) = \big\langle \mu(z), \pi(-izz^*)\big\rangle = 2\big\langle \mu(z), \mu(z) - \tau\big\rangle.$$

This proves (38).

Now fix a Riemann surface $(\Sigma, J_\Sigma, \mathrm{dvol}_\Sigma)$ and suppose that (u, A) is a solution of (17). Consider equations (19) in local holomorphic coordinates, where the metric has the form $\lambda^2(ds^2 + dt^2)$. In our situation

$$X_\xi(u) = \dot\rho(\xi)u,$$

and we abbreviate $\nabla_s u := \partial_s u + \dot\rho(\Phi)u$ and $\nabla_t u := \partial_t u + \dot\rho(\Psi)u$. Since

$$\nabla_s\nabla_t u - \nabla_t\nabla_s u = \dot\rho\big(\partial_s\Psi - \partial_t\Phi + [\Phi, \Psi]\big)u$$

and, by (19),

$$\nabla_s u + J\nabla_t u = 0, \qquad \partial_s\Psi - \partial_t\Phi + [\Phi, \Psi] + \lambda^2\mu(u) = 0,$$

we obtain

$$\nabla_s\nabla_s u + \nabla_t\nabla_t u = \lambda^2\dot\rho(\mu(u))Ju.$$

Hence, with $\Delta = \partial_s{}^2 + \partial_t{}^2$,

$$\begin{aligned}
\frac{\Delta|u|^2}{2} &= \partial_s\langle u, \nabla_s u\rangle + \partial_t\langle u, \nabla_t u\rangle \\
&= \left|\nabla_s u\right|^2 + \left|\nabla_t u\right|^2 + \big\langle u, \nabla_s\nabla_s u + \nabla_t\nabla_t u\big\rangle \\
&= \left|\nabla_s u\right|^2 + \left|\nabla_t u\right|^2 + \lambda^2\big\langle u, \dot\rho(\mu(u))Ju\big\rangle \\
&= \left|\nabla_s u\right|^2 + \left|\nabla_t u\right|^2 + 2\lambda^2\big\langle \mu(u), \mu(u) - \tau\big\rangle \\
&\geq 2\lambda^2|\mu(u)|\big(|\mu(u)| - |\tau|\big).
\end{aligned}$$

Now let (s_0, t_0) be a point at which the function $(s,t) \mapsto |u(s,t)|$ attains its maximum. Since Σ is compact, such a point exists in some coordinate chart, and we have $\Delta|u|^2 \leq 0$ at (s_0, t_0). Hence

$$\left|\mu(u(s_0, t_0))\right| \leq |\tau|.$$

Since μ is proper, there exists a constant $c > 0$ such that

$$|\mu(x)| \leq |\tau| \Longrightarrow |x| \leq c.$$

Hence $|u(s_0, t_0)| \leq c$, and it follows that $\sup_{p \in P} |u(p)| \leq c$ for every solution (u, A) of (17). To prove the last assertion just note that the same estimate holds for solutions of the perturbed equation (26) whenever the support of the perturbation is contained in the ball $\{|x| < c\}$. This proves the proposition. ■

Remark 3.4. (i) The proof of Proposition 3.5 is reminiscent of the compactness proof for the Seiberg-Witten equations in Kronheimer and Mrowka [42].

(ii) In Proposition 3.5 the assumption that the moment map be proper is essential. But one would expect that conditions (i) and (ii) can be removed or replaced by weaker assumptions.

(iii) If $\pi_2(M) \neq 0$, then, in general, there may be J-holomorphic spheres in M. In this case the compactification of the moduli space $\mathcal{M}_{B,\Sigma}(M, \mu; J, \sigma)$ should include stable maps, as introduced by Kontsevich [40]. To see this think of the solutions of the first equation in (26) as $\widetilde{J}_{A,\sigma}$-holomorphic curves from Σ to $\widetilde{M} = P \times_G M$ (see Section 2.4 for the case $\sigma = 0$). In the stable maps that appear in the limit, the main component is a solution of (26), and all other components are J-holomorphic spheres in the fibres.

(iv) It is often interesting to allow the complex structure on Σ to vary. Then one has to deal with a suitable compactification of Teichmüller space. This again leads to Kontsevich's stable maps.

(v) Similar techniques, as in the proof of Proposition 3.5, can be used to prove the following unique continuation theorem.

Continuation theorem. Let (u, A) be a solution of (26). If the pair $(d_{A,\sigma}u, \mu \circ u)$ vanishes to infinite order at some point $p \in P$, then $d_{A,\sigma}u \equiv 0$ and $\mu(u) \equiv 0$.

3.7 Invariants

The moduli space $\widetilde{\mathcal{M}}_{B,\Sigma,k}(M, \mu; J, \sigma)$ carries a natural right action of $G^k = G \times \cdots \times G$ on the k-marked points. This action commutes with the action of the gauge group and hence descends to an action on the quotient space $\mathcal{M}_{B,\Sigma,k} = \mathcal{M}_{B,\Sigma,k}(M, \mu; J, \sigma)$. Now there is an evaluation mapev $= (\mathrm{ev}_1, \dots, \mathrm{ev}_k) : \mathcal{M}_{B,\Sigma,k} \to M^k$ given by $\mathrm{ev}_i([u, A, p_1, \dots, p_k]) = u(p_i)$ and a projection $\pi : \mathcal{M}_{B,\Sigma,k} \to \mathcal{A}(P)/\mathcal{G}(P)$ given by $\pi([u, A, p_1, \dots, p_k]) = [A]$. The evaluation mapis G^k-equivariant, and the projection π is G^k-invariant:

$$
\begin{array}{ccc}
\mathcal{M}_{B,\Sigma,k} & \xrightarrow{\mathrm{ev}} & M^k \\
{\scriptstyle \pi}\big\downarrow & & \\
\mathcal{A}/\mathcal{G}. & &
\end{array}
$$

One can use these maps to produce certain natural G^k-equivariant cohomology classes on the moduli space $\mathcal{M}_{B,\Sigma,k}$. Integrating these over the quotient $\mathcal{M}_{B,\Sigma,k}/G^k$ gives rise to the invariants.

To be more precise choose equivariant cohomology classes $\alpha_i \in H^*_G(M)$ for $i = 1, \dots, k$ and a cohomology class $\beta \in H^*(\mathcal{A}/\mathcal{G})$ such that

$$\deg(\beta) + \sum_{i=1}^{k} \deg(\alpha_i) = \dim \mathcal{M}_{B,\Sigma,k} - k \dim G. \tag{39}$$

Then the pullback $\pi^*\beta \smile \mathrm{ev}_1{}^*\alpha_1 \smile \cdots \smile \mathrm{ev}_k{}^*\alpha_k$ is an equivariant cohomology class on $\mathcal{M}_{B,\Sigma,k}$. Let us pretend, for a moment, that $\mathcal{M}_{B,\Sigma,k}$ is a compact smooth manifold of the predicted dimension and that G^k acts freely on this space. Then our equivariant cohomology class on $\mathcal{M}_{B,\Sigma,k}$ descends to a top-dimensional cohomology class on the quotient $\mathcal{M}_{B,\Sigma,k}/G^k$ that we can evaluate on the fundamental cycle. This gives rise to an integer

$$\Phi^{M,\mu}_{B,\Sigma,k}(\beta, \alpha_1, \dots, \alpha_k) := \int_{\mathcal{M}_{B,\Sigma,k}/G^k} \pi^*\beta \smile \mathrm{ev}_1{}^*\alpha_1 \smile \cdots \smile \mathrm{ev}_k{}^*\alpha_k. \tag{40}$$

In general, only the subspace $\mathcal{M}^*_{B,\Sigma,k}$ of all solutions that satisfy (35) for almost every $p \in P$ and for $p = p_i$ is a smooth manifold for a generic perturbation σ and carries a free action of G^k. Even under the hypotheses of Proposition 3.5 this space will not be compact. However, in many cases we expect that this space can be compactified by adding strata of strictly lower dimensions and that (40) can be defined by integrating differential forms whose pullbacks are supported in $\mathcal{M}^*_{B,\Sigma,k}$. Alternatively, one can consider intersection numbers of cycles in $M \times_G EG$. This requires the choice of an equivariant function $\mathcal{M}^*_{B,\Sigma,k} \to EG$, and the easiest way to get such a function is by composition of the evaluation mapwith an equivariant function $\phi : M^* \to EG$, where

$$M^* = \{x \in M \mid gx = x \Longrightarrow g = \mathbb{1}\}.$$

Here we assume that EG has been replaced by a suitable finite-dimensional approximation. Now represent the Poincaré duals of α_i and β by submanifolds $Y_i \subset M \times_G EG$ and $Z \subset \mathcal{A}/\mathcal{G}$. Integrating the differential form then corresponds to counting the solutions $[u, A, p_1, \dots, p_k] \in \mathcal{M}^*_{B,\Sigma,k}$ that satisfy

$$[u(p_i), \phi(u(p_i))] \in Y_i, \qquad [A] \in Z. \tag{41}$$

Since ϕ is defined only on M^*, one has to check that the reducible solutions of (26), if they exist, do not obstruct compactness. With standard cobordism techniques, similar to the ones used in the definition of the Donaldson invariants [12], the Gromov-Witten invariants in [50] and [61], or the Seiberg-Witten invariants in [67], one should then be able to prove that the invariants (40) are independent of the choice of the perturbation σ and the almost complex structure J used to define them. To work this out in detail requires a considerable amount of analysis, which will be carried out elsewhere. Some cases were treated by Mundet [55].

Remark 3.5. In the above discussion the complex structure on the Riemann surface Σ is fixed. Even in this case there is an interesting moduli space $\mathcal{M}_{\Sigma,k}$ of stable Riemann surfaces with k-marked points, where one of the components of the stable surface is Σ itself. Correspondingly, one might wish to extend the definition of the invariants to include, as a base for the bundle P, stable Riemann surfaces, where the main component is Σ and all other components are spheres. With this modification in place there is a projection

$$\mathcal{M}_{B,\Sigma,k} \longrightarrow \mathcal{M}_{\Sigma,k},$$

and one could consider pullbacks of cohomology classes from $\mathcal{M}_{\Sigma,k}$ to get further invariants. Similar observations apply to the case where the complex structure on Σ is allowed to vary.

3.8 Adiabatic limits

In her Ph.D. thesis [27], the second author studied the adiabatic limit $\varepsilon \to 0$ in the equations

$$\bar{\partial}_{J,A}(u) = 0, \qquad *F_A + \varepsilon^{-2}\mu(u) = 0. \tag{42}$$

For $\varepsilon = 0$, these equations degenerate into (11), and the solutions of those equations correspond to J-holomorphic curves in the Marsden-Weinstein quotient $M/\!\!/G$ (see Section

2.4). Under suitable conditions on M and for sufficiently small $\varepsilon > 0$, there should be a one-to-one correspondence between the solutions of (42) and those of (11). The arguments are reminiscent of the proof of the Atiyah-Floer conjecture in [19] and [66]. Gaio proves that regular solutions of (11) give rise to solutions of (42) for ε sufficiently small and makes substantial progress towards establishing that, in many cases, all solutions of (42) can be obtained in this way. When completed, this work should lead to a proof of the following conjecture, at least in the case where the quotient $M/\!\!/G$ is *semipositive* (or *weakly monotone* in the terminology of [36] and [50]).

Conjecture 3.6. Suppose that $\mu : M \to \mathfrak{g}$ is proper, that zero is a regular value of μ, that $\mu^{-1}(0)$ is nonempty, and that G acts freely on $\mu^{-1}(0)$. Then, for $\bar{B} \in H_2(M/\!\!/G;\mathbb{Z})$ and $\alpha_1, \ldots, \alpha_k \in H^*_G(M;\mathbb{Z})$,

$$\Phi^{M,\mu}_{B,\Sigma,k}(1, \alpha_1, \ldots, \alpha_k) = \mathrm{GW}^{M/\!\!/G}_{\bar{B},\Sigma,k}(\bar{\alpha}_1, \ldots, \bar{\alpha}_k). \tag{43}$$

Here $1 \in H^0(\mathcal{A}/\mathcal{G})$, $B \in H^G_2(M;\mathbb{Z})$ is the image of $\bar{B}$ under the homomorphism $H_2(M/\!\!/G;\mathbb{Z}) \to H^G_2(M;\mathbb{Z})$ induced by the inclusion $\mu^{-1}(0) \hookrightarrow M$, and $\bar{\alpha}_i \in H^*(M/\!\!/G;\mathbb{Z})$ is the image of α_i under the homomorphism $H^*_G(M;\mathbb{Z}) \to H^*(M/\!\!/G;\mathbb{Z})$ induced by the same inclusion. □

Remark 3.6. (i) Kirwan [54] proved that the homomorphism $H^*_G(M;\mathbb{Z}) \to H^*(M/\!\!/G;\mathbb{Z})$ is surjective.

(ii) Consider the case

$$M/\!\!/G = \{\mathrm{pt}\}.$$

Then $n = \dim G = \dim M/2$, and the only class in the image of the homomorphism $H_2(M/\!\!/G;\mathbb{Z}) \to H^G_2(M;\mathbb{Z})$ is $B = 0$. Moreover, the invariant $\Phi^{M,\mu}_{B,\Sigma,k}(1, \alpha_1, \ldots, \alpha_k)$ can only be nonzero if $\dim \mathcal{M}_{B,\Sigma,k} = 0$. Hence assume

$$n = \dim G, \qquad B = 0, \qquad k = 0.$$

Then Conjecture 3.6 asserts that, if zero is a regular value of μ and G acts freely on $\mu^{-1}(0)$, then

$$\Phi^{M,\mu}_{0,\Sigma,0} = 1.$$

In this case the bundle P is trivial, and, for any $\varepsilon > 0$, there are obvious solutions of (42) that satisfy $\mu \circ u = 0$, $d_A u = 0$, and $F_A = 0$. They are all gauge-equivalent. The energy identity (18) shows that there is no other solution of (42).

(iii) Conjecture 3.6 does not allow for the pullback of classes in $\mathcal{M}_{\Sigma,k}$ or for variations of the complex structure on Σ (see Remark 3.5). But there should be analogous results for those cases.

(iv) The examples in Sections 5.1 and 5.2 show that the invariants $\Phi^{M,\mu}_{B,\Sigma,k}$ can be nontrivial in cases where the symplectic quotient $M/\!/G$ is a point or the empty set (and B does not descend to a homology class in the quotient).

(v) If G does not act freely on $\mu^{-1}(0)$, then Conjecture 3.6 suggests that the solutions of (26) can be used to define the Gromov-Witten invariants of symplectic orbifolds.

3.9 Wall crossing and localization

One should be able to use the formula (43) to find relations between the Gromov-Witten invariants of the quotients $M/\!/G(\tau)$ for different values of τ. Namely, choose a generic path $[0,1] \to \mathfrak{g} : s \mapsto \tau_s$ in the center of $\mathfrak{g}$, and consider the cobordism

$$\mathcal{W}_{B,\Sigma} = \bigcup_{0 \le s \le 1} \{s\} \times \mathcal{M}_{B,\Sigma}(\mu - \tau_s)$$

with boundary

$$\partial \mathcal{W}_{B,\Sigma} = \mathcal{M}_{B,\Sigma}(\mu - \tau_0) \cup \mathcal{M}_{B,\Sigma}(\mu - \tau_1).$$

In some cases the critical parameters should be the singular values of the moment map (e.g., when $G = S^1$). However, the examples in Sections 5.2 and 5.5 show that the moduli space $\mathcal{M}_{B,\Sigma}(\mu - \tau)$ may also have singularities when τ is a regular value of the moment map, and the effect of these on the definition of the invariants remains yet to be fully understood. If the path $s \mapsto \tau_s$ passes through such critical parameters, then the difference of the invariants for τ_0 and τ_1 should be computable in terms of the reducible solutions of (26).

Remark 3.7. (i) For the ordinary cohomology of symplectic quotients, wall-crossing formulae were discovered by Martin [46], [47], [48]. These should correspond to the present case when $B = 0$. In [47] Martin developed techniques for computing the cohomology of symplectic quotients via a reduction argument to the action of the maximal torus. We expect that his ideas can be adapted to our situation and lead to formulae for the computation of the invariants $\Phi_{B,\Sigma,k}$.

(ii) Guillemin and Sternberg [34] showed that passing through a critical value of the moment mapcorresponds to blowing upand down. The resulting formulae should thus lead to an alternative proof of Ruan's results in [63].

(iii) We expect that the wall-crossing relations correspond, under suitable assumptions, to the fixed-point localization formulae of Kontsevich [40] and Givental [29]. In [30] and [31] Givental used localization to compute Gromov-Witten invariants for many examples and, in particular, to prove the mirror conjecture for the quintic in $\mathbb{C}P^4$.

(iv) It should be interesting to relate the wall-crossing formulae for the invariants $\Phi_{B,\Sigma,k}$ to the gluing formulae in contact homology (cf. Eliashberg [20]).

4 Floer homology

4.1 Relative fixed points

Let (M, ω, μ) be a symplectic manifold with a Hamiltonian G-action. Fix a time-dependent Hamiltonian function $\mathbb{R} \times M \to \mathbb{R} : (t, x) \mapsto H_t(x)$ such that $H_t = H_{t+1}$ and $H_t : M \to \mathbb{R}$ is G-invariant for every t. Consider the Hamiltonian differential equation

$$\dot{x}(t) = X_{H_t}(x(t)), \tag{44}$$

and denote by $f : M \to M$ the time-1 map. It is defined by $f(x(0)) = x(1)$ for all solutions of (44). Note that

$$\mu \circ f = \mu,$$

A pair $(x_0, g_0) \in M \times G$ is called a *relative fixed point* of f if

$$f(x_0) = g_0 x_0.$$

Equivalently, the unique solution $x : \mathbb{R} \to M$ of (44) with initial condition $x(0) = x_0$ satisfies $x(t + 1) = g_0 x(t)$ for every $t \in \mathbb{R}$. Note that the set of relative fixed points is invariant under the action of G on $M \times G$ by $(x_0, g_0) \mapsto (gx_0, gg_0g^{-1})$. A relative fixed point (x_0, g_0) is called *regular* if $gx_0 = x_0$ implies $g = \mathbb{1}$. It is called *nondegenerate* if the linear map $df(x_0) - g_0 : T_{x_0}M \to T_{g_0x_0}M$ induces an isomorphism from the quotient $\ker d\mu(x_0)/L_{x_0}\mathfrak{g}_\tau$ to $\ker d\mu(g_0x_0)/L_{g_0x_0}\mathfrak{g}_\tau$, where $\tau = \mu(x_0)$. This means that

$$d\mu(x_0)v = 0, \qquad df(x_0)v - g_0v \in \operatorname{im} L_{g_0x_0} \Longrightarrow v \in \operatorname{im} L_{x_0} \tag{45}$$

for every $v \in T_{x_0}M$. Relative fixed points $(x_0, g_0) \in \mu^{-1}(0) \times G$ appear as the critical points of an equivariant symplectic action functional.

4.2 Equivariant symplectic action

Denote by $D \subset \mathbb{C}$ the closed unit disc, and denote by $\mathcal{L} = \mathcal{L}(M \times \mathfrak{g})$ the space of contractible loops in $M \times \mathfrak{g}$. The universal cover of this space consists of all equivalence classes of triples (x, η, v), where $x : \mathbb{R}/\mathbb{Z} \to M$, $\eta : \mathbb{R}/\mathbb{Z} \to \mathfrak{g}$, and $v : D \to M$ satisfy $v(e^{2\pi i t}) = x(t)$. Two such triples (x_1, η_1, v_1) and (x_2, η_2, v_2) are equivalent if and only if $x_1 = x_2$, $\eta_1 = \eta_2$, and v_1 is homotopic to v_2 with fixed boundary. The space of equivalence classes is denoted by

$$\widetilde{\mathcal{L}} = \widetilde{\mathcal{L}}(M \times \mathfrak{g}).$$

This space carries an action of the group $\widetilde{\mathcal{G}} = \mathrm{Map}(D, G)$ by

$$g^*[x, \eta, v] = \left[g^{-1}x, g^{-1}\partial_t g + g^{-1}\eta g, g^{-1}v\right],$$

where $\partial_t g = \partial/\partial t g(e^{2\pi i t})$. There is a $\widetilde{\mathcal{G}}$-invariant action functional

$$\mathcal{A}_{\mu,H} : \widetilde{\mathcal{L}}(M \times \mathfrak{g}) \longrightarrow \mathbb{R}$$

given by

$$\mathcal{A}_{\mu,H}(x, \eta, v) = -\int_D v^*\omega + \int_0^1 \big(\langle \mu(x(t)), \eta(t)\rangle - H_t(x(t))\big)\, dt.$$

A 1-periodic family of almost complex structures $J_t \in \mathcal{J}(M, \omega, \mu)$ determines an L^2-inner product on the tangent space

$$T_{(x,\eta)}\mathcal{L} = C^\infty(S^1, x^*TM) \times C^\infty(S^1, \mathfrak{g}),$$

and the gradient of $\mathcal{A}_{\mu,H}$ with respect to this inner product is given by

$$\operatorname{grad} \mathcal{A}_{\mu,H}(x, \eta) = \begin{pmatrix} J_t(\dot{x} + X_\eta(x) - X_{H_t}(x)) \\ \mu(x) \end{pmatrix}. \tag{46}$$

Hence the critical points of $\mathcal{A}_{\mu,H}$ are the loops $(x, \eta) : \mathbb{R}/\mathbb{Z} \to M \times \mathfrak{g}$ that satisfy

$$\dot{x} + X_\eta(x) = X_{H_t}(x), \qquad \mu(x) = 0. \tag{47}$$

Let us denote by $\widetilde{\mathrm{Per}}(\mu, H)$ the set of solutions of (47). The loopgroup

$$LG = \mathrm{Map}(S^1, G)$$

acts on this space and the quotient is denoted by

$$\mathrm{Per}(\mu, H) = \frac{\widetilde{\mathrm{Per}}(\mu, H)}{\mathrm{L}G}.$$

This quotient space can be naturally identified with the set of G-orbits of relative fixed points of f in $\mu^{-1}(0) \times G$. Moreover, a relative fixed point (x_0, g_0) is nondegenerate if and only if the corresponding critical point of $\mathcal{A}_{\mu,H}$ is nondegenerate. The proof of this observation is a precise analogue of [18 , proof of Proposition 4.4].

Remark 4.1. A closer look at the equivariant symplectic action should reveal interesting relations to the geometry of the loopgroup(cf. [60] and [10]).

4.3 Floer homology

One can construct Floer homology groups $HF^*(M, \omega, \mu; J, H)$, as in the standard case, by considering the gradient flow lines of the action functional $\mathcal{A}_{\mu,H}$ with respect to the L^2-metric determined by J_t. The formula (46) shows that the gradient flow lines are pairs (u, Ψ), where $u : \mathbb{R}^2 \to M$ and $\Psi : \mathbb{R}^2 \to \mathfrak{g}$ satisfy

$$\partial_s u + J_t\big(\partial_t u + X_\Psi(u) - X_{H_t}(u)\big) = 0, \qquad \partial_s \Psi + \mu(u) = 0, \tag{48}$$

and

$$u(s, t+1) = u(s, t), \qquad \Psi(s, t+1) = \Psi(s, t). \tag{49}$$

The energy of such a flow line is defined by

$$E(u, \Psi) = \int_0^1 \int_{-\infty}^{\infty} \left(\left|\partial_s u\right|^2 + \left|\partial_s \Psi\right|^2\right) ds\, dt.$$

If this energy is finite and the critical points of $\mathcal{A}_{\mu,H}$ are all nondegenerate, then one can show with standard techniques in gauge theory that the limits

$$x^{\pm}(t) = \lim_{s \to \pm\infty} u(s, t), \qquad \eta^{\pm}(t) = \lim_{s \to \pm\infty} \Psi(s, t) \tag{50}$$

exist and are critical points of $\mathcal{A}_{\mu,H}$. The strategy now is to proceed as in the standard case and define a chain complex generated by the critical points of $\mathcal{A}_{\mu,H}$ and define a boundary operator by counting the solutions of (48) and (49) with given limits (50) in the case where the Floer relative Morse index is 1. To carry this out in detail one has to deal with the usual transversality and compactness questions. Additional difficulties arise from the presence of nontrivial isotropy subgroups of critical points of $\mathcal{A}_{\mu,H}$, and this requires an equivariant version of Floer homology (cf. Viterbo [78]). The resulting Floer homology theory is related to the solutions of (17) in the same way that instanton Floer homology [21] is related to the Donaldson invariants, symplectic Floer homology [24] is related to the Gromov-Witten invariants, and Seiberg-Witten Floer homology is related to the Seiberg-Witten invariants. To see this compare (48) with (27). In particular, one should get relative invariants, for Riemann surfaces with cylindrical ends, with values in the Floer homology groups (cf. [59] for the standard case).

4.4 The Arnold conjecture for regular quotients

Suppose that μ is proper, zero is a regular value of μ, and G acts freely on $\mu^{-1}(0)$. Then one hopes to obtain transversality for the solutions of (48) by choosing a generic G-invariant Hamiltonian H. At first glance one might not expect to get anything new because the critical points are the periodic solutions of a Hamiltonian system in $M/\!\!/G$, and one could get an equivalent theory from Floer homology in the reduced space. However, the compactness result of Proposition 3.5 suggests that in many cases the present approach might be simpler than the standard theory and lead to a proof of the Arnold conjecture over the integers.[2] The key point is that the presence of holomorphic spheres with negative Chern number in the quotient $M/\!\!/G$ leads to complications in the standard theory, but not in our approach, provided that they do not lift to holomorphic spheres in M.

4.5 Relation with Morse theory

If zero is not a regular value of μ or G does not act freely on $\mu^{-1}(0)$, then the Floer homology theory outlined in Section 4.3 should lead to new existence theorems for relative fixed points of G-equivariant Hamiltonian symplectomorphisms. In the standard theory one can, in many cases, identify the Floer homology groups with Morse homology by considering the case where H is independent of t (and sufficiently small). The analogue of this argument in the present case leads to equivariant Morse homology on $M \times \mathfrak{g}$ for the function

$$M \times \mathfrak{g} \longrightarrow \mathbb{R} : (x, \eta) \longmapsto \langle \mu(x), \eta \rangle - H(x),$$

[2] A quite different approach to Floer homology over the integers for general symplectic manifolds has recently been proposed by Fukaya [25]. Other approaches to Floer homology for general symplectic manifolds (cf. Fukaya and Ono [26] and Liu and Tian [45]) have so far been established only over the rationals.

where $H : M \to \mathbb{R}$ is G-invariant. The critical points of this function satisfy

$$\nabla H(x) = JX_\eta(x), \qquad \mu(x) = 0$$

and so correspond to critical points of the induced function $\bar{H} : M/\!\!/G \to \mathbb{R}$ whenever the quotient is smooth. The gradient flow equations have the form

$$\dot{u} + JX_\Psi(u) - \nabla H(u) = 0, \qquad \dot{\Psi} + \mu(u) = 0. \tag{51}$$

They are equivalent to (48) whenever H, J, u, and Ψ are independent of t.

4.6 Equivariant symplectomorphisms

One might wish to define the Floer homology groups of general equivariant symplectomorphisms, not just Hamiltonian ones. For this theory one would consider symplectomorphisms $f : M \to M$ that satisfy

$$f(gx) = \rho(g)x, \qquad \mu(f(x)) = \dot{\rho}\big(\mu(x)\big) \tag{52}$$

for all $x \in M$ and some isomorphism $\rho : G \to G$. Here $\dot{\rho} : \mathfrak{g} \to \mathfrak{g}$ denotes the corresponding Lie algebra isomorphism. The Hamiltonian perturbation $H_t \in C^\infty_G(M)$ and the almost complex structures $J_t \in \mathcal{J}(M, \omega, \mu)$ should satisfy the periodicity condition

$$H_t = H_{t+1} \circ f, \qquad J_t = f^* J_{t+1},$$

and (49) should be replaced by

$$u(s, t+1) = f(u(s,t)), \qquad \Psi(s, t+1) = \dot{\rho}(\Psi(s,t)). \tag{53}$$

The resulting solutions of (48) and (53) should give rise to Floer homology groups $HF^*(M, \omega, \mu, f)$ that are independent of H and J. One might hope that these invariants can be used to distinguish equivariant Hamiltonian isotopy classes. In the standard case such results were established by Seidel [69].

4.7 Boundary value problems

It is interesting to consider boundary value problems for the equations (26) or (48). The relevant boundary data would then involve G-invariant Lagrangian submanifolds. In particular, this should lead to Floer homology groups $HF^*(M, \omega, \mu, L_0, L_1)$, where L_0 and L_1 are G-invariant Lagrangian submanifolds of $\mu^{-1}(0)$, the critical points are G-orbits of intersections of L_0 and L_1, and the connecting orbits are solutions of (48) on the strip $\mathbb{R} \times [0, 1]$ that satisfy the boundary condition

$$u(s, 0) \in L_0, \qquad u(s, 1) \in L_1. \tag{54}$$

(See Floer [22], [23], Oh [57], and Lazzarini [43] for the standard case.)

Lemma 4.1. Let $L \subset M$ be a connected G-invariant Lagrangian submanifold. Then there exists a central element $\tau \in \mathfrak{g}$ such that $L \subset \mu^{-1}(\tau)$. □

Proof. For every $x \in L$, we have

$$\operatorname{im} L_x \subset T_x L \subset \ker d\mu(x).$$

The last inclusion follows from the fact that $T_x L$ is a Lagrangian subspace of $T_x M$ and the kernel of $d\mu(x)$ is the symplectic complement of the image of L_x. The inclusion $T_x L \subset \ker d\mu(x)$ shows that μ is constant on L. The inclusion $\operatorname{im} L_x \subset \ker d\mu(x)$ shows that

$$[\xi, \mu(x)] = d\mu(x) L_x \xi = 0$$

for $\xi \in \mathfrak{g}$ and $x \in L$. Hence $\mu(x)$ is in the center of $\mathfrak{g}$ for every $x \in L$. ■

If zero is a regular value of μ, then the *equivariant diagonal*

$$\Delta^\mu = \{(x, gx) \mid x \in M,\ g \in G,\ \mu(x) = 0\}$$

is a Lagrangian submanifold of $\widehat{M} = M \times M$, with the symplectic form $\widehat{\omega} = (-\omega) \times \omega$, and is invariant under the action of $\widehat{G} = G \times G$. The Floer homology groups of a symplectomorphism $f : M \to M$ that satisfies (52) should be isomorphic to the Floer homology groups of the Lagrangian pair $(\Delta^\mu, \Gamma^\mu(f))$ in $\widehat{M}$, where $\Gamma^\mu(f)$ is the *equivariant graph* of f, that is, the image of Δ^μ under $\mathrm{id} \times f$.

4.8 Adiabatic limits

That the present theory is, for regular quotients, equivalent to the standard theory in $M/\!/G$ follows from an adiabatic limit argument involving the equations

$$\partial_s u + J(\partial_t u + X_\Psi(u) - X_{H_t}(u)) = 0, \qquad \partial_s \Psi + \varepsilon^{-2} \mu(u) = 0. \tag{55}$$

In the limit $\varepsilon \to 0$, the solutions of (55) degenerate to Floer gradient lines in the quotient $M/\!/G$. The details are analogous to the proof of the Atiyah-Floer conjecture in [18], [19], and [66], and to [27, proof of Conjecture 3.6]. The resulting theorem should be the existence of a natural isomorphism

$$\mathrm{HF}^*(M, \omega, \mu, f) \cong \mathrm{HF}^*\left(M/\!/G, \bar{\omega}, \bar{f}\right)$$

whenever zero is a regular value of μ and G acts freely on $\mu^{-1}(0)$. Here $\bar{\omega}$ denotes the induced symplectic form, and $\bar{f}$ denotes the induced symplectomorphism on $M/\!/G$. In the Lagrangian case there should be a natural isomorphism

$$\mathrm{HF}^*\left(M, \omega, \mu, L_0, L_1\right) \cong \mathrm{HF}^*\left(M/\!/G, \bar{\omega}, \bar{L}_0, \bar{L}_1\right),$$

where $\bar{L}_i = L_i/G \subset M/\!/G$ for $i = 0, 1$.

5 Examples

5.1 Vortex equations

Consider the standard action of $G = S^1$ on $M = \mathbb{C}$. Then a moment mapis given by

$$\mu(z) = -\frac{i}{2}|z|^2, \tag{56}$$

and the quotient space is a point. Nevertheless, the space of solutions of (17) is interesting. Let $(\Sigma, J_\Sigma, \mathrm{dvol}_\Sigma)$ be a compact Riemann surface, and let $P \to \Sigma$ be a circle bundle of degree d. An equivariant function $\Theta : P \to \mathbb{C}$ can then be interpreted as a section of the line bundle $E = P \times_{S^1} \mathbb{C} \to \Sigma$, a connection $A \in \mathcal{A}(P)$ determines a Cauchy-Riemann operator

$$\bar{\partial}_A : C^\infty(\Sigma, E) \longrightarrow \Omega^{0,1}(\Sigma, E),$$

and the equations (17), with μ replaced by $\mu + i\tau$ for some $\tau \in \mathbb{R}$, have the form

$$\bar{\partial}_A \Theta = 0, \qquad *iF_A + \frac{|\Theta|^2}{2} = \tau. \tag{57}$$

These are the *vortexequations* . The necessary condition (28) for the existence of solutions has the form

$$\tau > \frac{2\pi d}{\mathrm{Vol}(\Sigma)}.$$

In this case the moduli space is smooth and, by Proposition 3.5, it is compact. These observations are well known (see [28]), as is the fact that the moduli space

$$\mathcal{M}_d(\Sigma) = \frac{\{(\Theta, A) \mid (57)\}}{\mathrm{Map}(\Sigma, S^1)}$$

can be identified with the symmetric product $S^d\Sigma = \Sigma \times \cdots \times \Sigma / S_d$ (via the zeros of Θ). Hence the invariants (40) should be expressible in terms of the cohomology of $S^d\Sigma$. Note that the adiabatic limit argument of Section 3.8 can, in this case, be rephrased in the form $\tau \to \infty$ (by rescaling Θ), and this limit corresponds precisely to the argument of Taubes in [71] for the Seiberg-Witten equations.

5.2 Bradlow pairs

Another example with a trivial quotient is the action of $G = \mathrm{U}(2)$ on $M = \mathbb{C}^2$. A moment mapis given by $\mu(z) = -izz^*/2$. Hence the quotient at any nonzero central element of $\mathfrak{u}(2)$ is the empty set. Let $P \to \Sigma$ be a principal $\mathrm{U}(2)$-bundle of degree $d = \langle c_1(E), [\Sigma] \rangle$, and consider the Hermitian rank-2 bundle $E = P \times_{\mathrm{U}(2)} \mathbb{C}^2 \to \Sigma$. Fix a constant $\tau > \pi d/\mathrm{Vol}(\Sigma)$, and replace the moment map by $\mu + i\tau\mathbb{1}$. Then (17) takes the form

$$\bar{\partial}_A \Theta = 0, \qquad *iF_A + \frac{1}{2}\Theta\Theta^* = \tau\mathbb{1}, \tag{58}$$

where $A \in \mathcal{A}(E)$ and $\Theta \in C^\infty(\Sigma, E)$. The moduli spaces

$$\mathcal{M}_\tau = \frac{\{(\Theta, A) \mid (58)\}}{\mathcal{G}(E)}$$

were studied in detail by Bradlow and others in [7], [8], and [74]. The invariants (40) and the wall-crossing numbers should, in this case, be related to the work of Thaddeus [74]. He studied the cohomology of the moduli space of flat $\mathrm{U}(2)$-connections over Σ via Bradlow pairs. For τ close to $\pi d/\mathrm{Vol}(\Sigma)$ and for large d, $\mathcal{M}_\tau$ is a bundle over the moduli space of flat $\mathrm{U}(2)$-connections with projective spaces as fibres, $\mathcal{M}_\tau = \emptyset$ for $\tau > 2\pi d/\mathrm{Vol}(\Sigma)$, and $\mathcal{M}_\tau$ can be identified with a complex projective space for $\tau = 2\pi d/\mathrm{Vol}(\Sigma) - \varepsilon$ whenever $\varepsilon > 0$ is sufficiently small (see [7]). The critical parameters are

$$\tau_k = \frac{2\pi k}{\mathrm{Vol}(\Sigma)}, \quad \frac{d}{2} < k \le d.$$

For $\tau = \tau_k$, there are reducible solutions of (58); that is, $\mathcal{M}_\tau$ is not smooth and its singular part can be identified with the symmetric product $S^{d-k}\Sigma$. In the context of this paper it is useful to recall the following construction (see [9]). Fix a point $z_0 \in \Sigma$, and denote by $\mathcal{G}_0 \subset \mathrm{Map}(\Sigma, S^1)$ the codimension-1 subgroupof all maps of the form $g = g_0 \exp(\xi)$, where $g_0 : \Sigma \to S^1$ satisfies $d^*({g_0}^{-1} dg_0) = 0$, $g_0(z_0) = 1$, and $\xi : \Sigma \to i\mathbb{R}$ has mean value zero. Then the quotient space

$$\mathcal{M} = \frac{\left\{(\Theta, A) \mid \exists \tau > \dfrac{\pi d}{\mathrm{Vol}(\Sigma)} \text{ s.t. (58) holds}\right\}}{\left\{g \in \mathcal{G}(E) \mid \det \circ g \in \mathcal{G}_0\right\}} \tag{59}$$

is a smooth manifold. It carries a Hamiltonian S^1-action with moment map

$$\mathcal{M} \longrightarrow i\mathbb{R} : (\Theta, A) \longmapsto -\frac{i}{2}\int_\Sigma |\Theta|^2 \, d\mathrm{vol}_\Sigma \,. \tag{60}$$

Hence $\mathcal{M}_\tau$ can be identified with the quotient $\mathcal{M}/\!/S^1(i(2\pi d - 2\tau \mathrm{Vol}(\Sigma)))$.

5.3 Holomorphic curves in projective space

Consider the standard action of $G = S^1$ on $\mathbb{C}^{n+1}$. Then a moment mapis again given by (56), and (17) has the form

$$\bar{\partial}_A \Theta_\nu = 0, \qquad *iF_A + \sum_{\nu=0}^{n} \frac{|\Theta_\nu|^2}{2} = \tau, \tag{61}$$

where $E \to \Sigma$ is a Hermitian line bundle, $A \in \mathcal{A}(E)$, and $\Theta_0, \ldots, \Theta_n \in C^\infty(\Sigma, E)$. By Proposition 3.5 , the moduli space of solutions of (61) is compact and transversality can be easily achieved. By Conjecture 3.6 , the resulting invariants (40) agree with the Gromov-Witten invariants of $\mathbb{C}P^n$. However, in contrast to those, they are defined in terms of compact smooth moduli spaces.

5.4 Toric varieties

The situation is similar for Kähler manifolds that arise as quotients of $\mathbb{C}^N$ by a subgroup $G \subset U(N)$. Here a moment mapis given by

$$\mu(z) = \pi\Big(-\frac{i}{2} z z^*\Big),$$

where $\pi : \mathfrak{u}(N) \to \mathfrak{g}$ denotes the adjoint of the inclusion $\mathfrak{g} \hookrightarrow \mathfrak{u}(N)$. Let $P \to \Sigma$ be a principal

G-bundle, and denote by

$$E = P \times_G \mathbb{C}^N$$

the associated vector bundle. Then the equations (17) can be interpreted as equations for a p air$(\Theta, A) \in C^\infty(\Sigma, E) \times \mathcal{A}(E)$, and they have the form

$$\bar{\partial}_A \Theta = 0, \qquad *F_A + \pi\Big(-\frac{i}{2}\Theta\Theta^*\Big) = \tau \tag{62}$$

for some central element $\tau \in \mathfrak{g}$. Again, Proposition 3.5 guarantees that the moduli space is compact whenever the moment map is proper. One gets integer invariants that should correspond to the Gromov-Witten invariants of the quotient $X = \mathbb{C}^N /\!\!/ G(\tau)$. This is interesting because there are many examples where X contains holomorphic spheres with negative Chern number, and in these cases the direct definition of the Gromov-Witten invariants of X has so far been established only over the rationals (see [26], [44], and [62]).

5.5 The Grassmannian and the Verlinde algebra

In [79] Witten conjectured a relation between the Gromov-Witten invariants of the Grassmannian (see [5]) and the Verlinde algebra (see [77] and [6]). For the quantum cohomology (3-punctured spheres) this conjecture was confirmed by Agnihotri [1]. The Grassmannian can be expressed as a symplectic quotient

$$\mathrm{Gr}(k, n) \cong \mathbb{C}^{k\times n} /\!\!/ \mathrm{U}(k).$$

Think of $\Theta \in \mathbb{C}^{k\times n}$ as a k-frame in $\mathbb{C}^n$. If Θ has rank-k, then the orthogonal complement of its kernel is a k-dimensional subspace of $\mathbb{C}^n$. The groupU (k) acts on $\mathbb{C}^{k\times n}$ on the left, and the function $\mu : \mathbb{C}^{k\times n} \to \mathfrak{u}(k)$ given by

$$\mu(\Theta) = -\frac{i}{2}\Theta\Theta^*$$

is a moment map. Thus $\mu^{-1}(-i/2)$ is the space of unitary k-frames in $\mathbb{C}^n$ and its quotient by $\mathrm{U}(k)$ is the Grassmannian. Now let $P \to \Sigma$ be a principal $\mathrm{U}(k)$-bundle of degree d, and denote by

$$E = P \times_{\mathrm{U}(k)} \mathbb{C}^k \longrightarrow \Sigma$$

the associated complex rank-k bundle. Fix a real number $\tau > 2\pi d/k\mathrm{Vol}(\Sigma)$, and replace μ by $\mu + i\tau\mathbb{1}$. Then (17) takes the form

$$\bar{\partial}_A\Theta_\nu = 0, \qquad *iF_A + \frac{1}{2}\sum_{\nu=1}^{n}\Theta_\nu{\Theta_\nu}^* = \tau\mathbb{1}, \tag{63}$$

where $A \in \mathcal{A}(E)$ and $\Theta_1, \ldots, \Theta_n \in C^\infty(\Sigma, E)$. In this case (63) has reducible solutions whenever

$$\tau = \frac{2\pi d_0}{k_0\,\mathrm{Vol}(\Sigma)}, \quad 0 < k_0 < k, \ \frac{k_0}{k}d < d_0 \leq d.$$

Thus the moduli space is regular for $\tau > 2\pi d/\mathrm{Vol}(\Sigma)$ and, for these values of τ, Conjecture 3.6 asserts that the invariants obtained from the solutions of (63) can be identified with the Gromov-Witten invariants of the Grassmanian. On the other hand, the opposite adiabatic limit $\varepsilon \to \infty$ in (42) should give rise to an identification with the invariants of moduli spaces of flat connections that appear as the structure constants in the Verlinde algebra (cf. Thaddeus [74]). Thus the solutions of (63) might give rise to a geometric approach for the proof of Witten's conjecture.

5.6 Anti-self-dual Yang-Mills equations

There are interesting cases where the solutions of (17) give rise to finite-dimensional moduli spaces even though the symplectic manifold (M, ω) is infinite-dimensional. As an example consider the case of a principal bundle $Q \to S$ over a compact-oriented Riemann surface S with structure groupSU (2) or SO(3). In Section 2.2 we have seen that the space $M = \mathcal{A}(Q)$ of connections on Q carries a natural symplectic structure and that the action of the identity component of the gauge group $G = \mathcal{G}_0(Q) \subset \mathcal{G}(Q)$ is Hamiltonian with moment map $\mathcal{A}(Q) \to \mathrm{Lie}(\mathcal{G}_0(Q)) : A \mapsto *F_A$. Hence equations (19), in local holomorphic coordinates on Σ, have the form

$$\begin{aligned} &\partial_s A - d_A\Phi + *(\partial_t A - d_A\Psi) = 0, \\ &\partial_s\Psi - \partial_t\Phi + [\Phi, \Psi] + \lambda^2 * F_A = 0, \end{aligned} \tag{64}$$

where $A(s,t) \in \mathcal{A}(Q)$ and $\Phi(s,t), \Psi(s,t) \in C^\infty(S, \mathrm{ad}(Q))$, and the metric on Σ is $\lambda^2(ds^2 + dt^2)$. These are the anti-self-dual Yang-Mills equations over the product $\Sigma \times S$. The function $\mathbb{C} \to \mathcal{A}(Q) : s{+}it \mapsto A(s,t)$ plays the role of the map $u : \mathbb{C} \to M$ in (19). The symplectic quotient

$$\mathcal{M}_Q := \frac{\mathcal{A}^{\mathrm{flat}}(Q)}{\mathcal{G}_0(Q)} = M/\!/G$$

is the moduli space of flat connections on Q. It is a symplectic manifold of dimension $6g-6$, where g is the genus of S. If Q is an $SO(3)$-bundle with nonzero second Stiefel-Whitney class, then the moduli space $\mathcal{M}_Q$ is smooth. The adiabatic limit argument of Conjecture 3.6 here gives rise to a correspondence between anti-self-dual instantons over $\Sigma \times S$ and holomorphic curves $\Sigma \to \mathcal{M}_Q$. This is the basic idea of the proof of the Atiyah-Floer conjecture in [17], [18], [19], and [65]. Another reference for this adiabatic limit is the recent thesis by Handfield [35].

Remark 5.1. (i) An automorphism $f : Q \to Q$ (that descends to a diffeomorphism of S) determines an equivariant symplectomorphism

$$\mathcal{A}(Q) \longrightarrow \mathcal{A}(Q) : A \longmapsto f^*A.$$

The corresponding isomorphism of the gauge group is given by

$$\mathcal{G}_0(Q) \longrightarrow \mathcal{G}_0(Q) : g \longmapsto g \circ f.$$

That the symplectic Floer homology groups of the induced symplectomorphism of $\mathcal{M}_Q$ are isomorphic to the instanton Floer homology groups of the corresponding 3-dimensional mapping torus was proved in [19].

(ii) There are interesting Lagrangian submanifolds of $\mathcal{A}(Q)$ whenever S is the boundary of a compact 3-manifold Y and Q admits a trivialization. Then the bundle extends over Y and the flat connections on Y determine a $\mathcal{G}(Q)$-invariant Lagrangian submanifold of $\mathcal{A}(Q)$ (that is contained in the subset of flat connections). The general Atiyah-Floer conjecture [2] relates the Floer homology groups of Lagrangian intersections in $\mathcal{M}_Q$, corresponding to two bordisms Y_0 and Y_1, to the instanton Floer homology groups of the closed 3-manifold $Y = Y_0 \cup_S Y_1$ whenever the latter is a homology 3-sphere.

5.7 Seiberg-Witten equations

Another infinite-dimensional example is the space

$$M = \left\{(\Theta, A) \in C^\infty(S, E) \times \mathcal{A}(E) \mid \bar{\partial}_A \Theta = 0\right\},$$

where $E \to S$ is a Hermitian line bundle of degree $d = \langle c_1(E), [S] \rangle$ over a compact-oriented Riemann surface S. The symplectic form is given by (20), and Proposition 3.2 asserts that the action of the gauge group $G = \mathrm{Map}(S, S^1)$ on this space is Hamiltonian with moment map

$$(\Theta, A) \longmapsto *F_A - \frac{i}{2}|\Theta|^2.$$

The symplectic quotient $M/\!\!/G(-i\tau)$ is the moduli space $\mathcal{M}_d(S)$ of solutions to the vortex equations (57) and hence can be identified with the d-fold symmetric product of S. The equations (19) have the form

$$\begin{aligned}
&\bar\partial_A\Theta = 0,\\
&\partial_s\Theta + \Phi\Theta + i(\partial_t\Theta + \Psi\Theta) = 0,\\
&\partial_s A - d\Phi + *(\partial_t A - d\Psi) = 0,\\
&\partial_s\Psi - \partial_t\Phi + \lambda^2\Big(*F_A - \frac{i}{2}|\Theta|^2 + i\tau\Big) = 0,
\end{aligned} \tag{65}$$

where $A(s,t) \in \mathcal{A}(E)$, $\Theta(s,t) \in C^\infty(S,E)$, and $\Phi(s,t), \Psi(s,t) \in C^\infty(S, i\mathbb{R})$. These are the Seiberg-Witten equations over the product $\Sigma \times S$, so long as the complex structure on S is independent of s and t (the *integrable case*). More precisely, the first two equations in (65) correspond to the Dirac equation and the last two to the curvature equation. The spinor bundle is a rank-2 bundle over $\Sigma \times S$ which naturally splits into a direct sum of two line bundles. In the integrable case one of the two components of the spinor vanishes (see [79]), and this leads to the simpler form of the Seiberg-Witten equations stated above.

The adiabatic limit argument of Conjecture 3.6 now gives rise to a correspondence between the Seiberg-Witten equations over the product $\Sigma \times S$ and holomorphic curves from Σ into the d-fold symmetric product of S (see [66]). There is a somewhat more complicated version of this argument which also applies to the case where the complex structure on S depends on s and t. Then the moduli spaces of solutions of the vortex equations form a bundle over the Teichmüller space of S, this bundle carries a natural connection, and this connection is related in an interesting way to the full version of the Seiberg-Witten equations in the nonintegrable case whenever the 4-manifold in question is a fibration with fibre S. This is discussed in detail in [66]. The correspondence between holomorphic curves and Seiberg-Witten equations indicated here is different from the one in the work of Taubes [70], [71], [72], [73] where he directly compares the Seiberg-Witten monopoles over a general symplectic 4-manifold X with holomorphic curves in X. It is likely that the two approaches are related via the work of Donaldson [13] on symplectic Lefschetz fibrations (see also Auroux [4]). Donaldson proved that every symplectic 4-manifold, after blow-up, admits the structure of a symplectic Lefschetz fibration

$$X \longrightarrow S^2$$

with generic fibre S. Cutting out the singular fibres one obtains a 4-manifold *W*, fibred over the punctured sphere, with cylindrical ends corresponding to the mapping tori of

Dehn twists. The adiabatic limit argument of Conjecture 3.6 now relates the Seiberg-Witten monopoles over X to holomorphic sections of the bundle $X^{(d)}$, where the fibres are replaced by the d-fold symmetric products of S. The latter correspond to multivalued sections of the bundle $X \to S^2$. That these in turn should correspond to holomorphic curves in X itself is the subject of a current research project by Donaldson and Ivan Smith. The adiabatic limit argument for $W^{(d)}$ is the Seiberg-Witten analogue of the Atiyah-Floer conjecture (see [65] and [66]). In the 3-dimensional case this is related to the work of Meng and Taubes [52], Hutchings and Lee [37], [38], Turaev [75], and Donaldson [16].

Remark 5.2. Since there is a correspondence between Donaldson invariants and holomorphic curves in the moduli space $\mathcal{M}^{\text{flat}}(S)$ of flat SO(3)-connections over S on the one hand, and between the Seiberg-Witten invariants and holomorphic curves in the symmetric product $\mathcal{M}_d(S)$ on the other hand, it would be interesting to compare the Gromov-Witten invariants of $\mathcal{M}_d(S)$ with those of $\mathcal{M}^{\text{flat}}(S)$. Such a comparison should be related to the picture of Thaddeus [74] for the ordinary cohomology of these spaces and hence to the study of holomorphic curves in the moduli spaces of Bradlow pairs. Results in this direction might provide an alternative approach (to the one by Pidstrigach and Tyurin [58]) for the comparison of the Donaldson and the Seiberg-Witten invariants in the symplectic case. The discussion of Section 5.2 shows that this fits into the framework of the invariants (40). To be more precise, equations (19) with target space $\mathcal{M}$ given by (59) and moment map(60) take the form

$$
\begin{aligned}
&\bar{\partial}_A \Theta = 0, \\
&\partial_s \Theta + \Phi\Theta + i(\partial_t \Theta + \Psi\Theta) = 0, \\
&\partial_s A - d_A \Phi + *(\partial_t A - d_A \Psi) = 0, \\
&(2\,\mathrm{Vol}(S))^{-1} \int_S \mathrm{tr}(\partial_s \Psi - \partial_t \Phi)\ \mathrm{dvol}_S + \lambda^2 \left(\frac{*F_A - i\Theta\Theta^*}{2} + i\tau \right) = 0,
\end{aligned}
\tag{66}
$$

where $E \to S$ is a Hermitian rank-2 bundle, $A(s,t) \in \mathcal{A}(E)$, $\Theta(s,t) \in C^\infty(S,E)$, and $\Phi(s,t)$, $\Psi(s,t) \in C^\infty(S,\mathrm{End}(E))$. Working with (66), instead of holomorphic curves in $\mathcal{M}_\tau$ (see Section 5.2), eliminates the problems arising from holomorphic spheres with negative Chern number, which exist in $\mathcal{M}_\tau$ but not in $\mathcal{M}$. On the other hand, care must be taken with the solutions of (66) that satisfy $\Theta = 0$.

Acknowledgments

We would like to thank Paul Biran, Simon Donaldson, Shaun Martin, and Paul Seidel for enlightening discussions.

References

[1] S. Agnihotri, *Quantum cohomology and the Verlinde algebra*, Ph.D. dissertation, Oxford Univ., Oxford, 1995.

[2] M. F. Atiyah, "New invariants of 3- and 4-dimensional manifolds" in *The Mathematical Heritage of Hermann Weyl (Durham, N.C., 1987)*, Proc. Sympos. Pure Math. **48**, Amer. Math. Soc., Providence, 1988, 285–299.

[3] M. F. Atiyah and R. Bott, *The Yang-Mills equations over Riemann surfaces*, Philos. Trans. Roy. Soc. London Ser. A **308** (1983), 523–615.

[4] D. Auroux, *Asymptotically holomorphic families of symplectic submanifolds*, Geom. Funct. Anal. **7** (1997), 971–995.

[5] A. Bertram, G. Daskalopoulos, and R. Wentworth, *Gromov invariants for holomorphic maps from Riemann surfaces to Grassmannians*, J. Amer. Math. Soc. **9** (1996), 529–571.

[6] J.-M. Bismut and F. Labourie, *Symplectic geometry and the Verlinde formula*, preprint, Univ. de Paris Sud, Orsay, 1999.

[7] S. B. Bradlow, *Special metrics and stability for holomorphic bundles with global sections*, J. Differential Geom. **33** (1991), 169–213.

[8] S. B. Bradlow and G. Daskalopoulos, *Moduli of stable pairs for holomorphic bundles over Riemann surfaces*, Internat. J. Math. **2** (1991), 477–513.

[9] S. B. Bradlow, G. Daskalopoulos, and R. Wentworth, *Birational equivalences of vortexmoduli*, Topology **35** (1996), 731–748.

[10] T. Davies, *The Yang-Mills functional over Riemann surfaces and the loop group*, Ph.D. dissertation, Univ. of Warwick, Warwick, 1996.

[11] S. K. Donaldson, *Instantons and geometric invariant theory*, Comm. Math. Phys. **93** (1984), 453–460.

[12] ———, *Polynomial invariants of smooth four-manifolds*, Topology **29** (1990), 257–315.

[13] ———, *Symplectic submanifolds and almost-complexgeometry*, J. Differential Geom. **44** (1996), 666–705.

[14] ———, *Moment maps and diffeomorphisms*, Asian J. Math. **3** (1999), 1–15.

[15] ———, "Symmetric spaces, Kähler geometry, and Hamiltonian dynamics" in *Northern California Symplectic Geometry Seminar*, Amer. Math. Soc. Transl. Ser. 2 **196**, Amer. Math. Soc., Providence, 1999, 13–33.

[16] ———, "Topological field theories and formulae of Casson and Meng-Taubes" in *Proceedings of the Kirbyfest (Berkeley, Calif., 1998)*, Geom. Topol. Monogr. **2**, Geom. Topol., Inst. Maths., Coventry, 1999, 87–102.

[17] S. Dostoglou and D. A. Salamon, "Instanton homology and symplectic fixed points" in *Symplectic Geometry*, London Math. Soc. Lecture Note Ser. **192**, Cambridge Univ. Press, Cambridge, 1993, 57–94.

[18] ———, "Cauchy-Riemann operators, self-duality, and the spectral flow" in *First European Congress of Mathematics Vol. 1: Invited Lectures, Part 1*, Progr. Math. **119**, Birkhäuser, Basel, 1994, 511–545.

[19] ———, *Self-dual instantons and holomorphic curves*, Ann. of Math. (2) **139** (1994), 581–640.

[20] Y. Eliashberg, "Invariants in contact topology" in *Proceedings of the International Congress of Mathematicians (Berlin, 1998), Vol. 2*, Doc. Math. **1998**, Extra Vol. 2, Doc. Math., Bielefeld, 327–338, http://www.mathematik.uni-bielefeld.de/documenta/.

[21] A. Floer, *An instanton-invariant for* 3*-manifolds*, Comm. Math. Phys. **118** (1988), 215–240.

[22] ———, *Morse theory for Lagrangian intersections*, J. Differential Geom. **28** (1988), 513–547.

[23] ———, *The unregularized gradient flow of the symplectic action*, Comm. Pure Appl. Math. **41** (1988), 775–813.

[24] ———, *Symplectic fixed points and holomorphic spheres*, Comm. Math. Phys. **120** (1989), 575–611.

[25] K. Fukaya, lectures in Cortona and Kyoto, June/July 1999.

[26] K. Fukaya and K. Ono, *Arnold conjecture and Gromov-Witten invariants for general symplectic manifolds*, preprint, 1996.

[27] A. R. Gaio, J*-holomorphic curves and moment maps*, Ph.D. dissertation, Univ. of Warwick, Warwick, 1999.

[28] O. García-Prada, *A direct existence proof for the vortex equations over a compact Riemann surface*, Bull. London Math. Soc. **26** (1994), 88–96.

[29] A. B. Givental, *Equivariant Gromov-Witten invariants*, Internat. Math. Res. Notices **1996**, 613–663.

[30] ———, "A tutorial on quantum cohomology" in *Symplectic Geometry and Topology (Park City, Utah, 1997)*, IAS/Park City Math. Ser. **7**, Amer. Math. Soc., Providence, 1999, 231–264.

[31] ———, *Stationary phase integrals, quantum Toda lattices, flag manifolds, and the mirror conjecture*, preprint, Univ. of Calif., Berkeley, 1996.

[32] M. Gromov, *Pseudoholomorphic curves in symplectic manifolds*, Invent. Math. **82** (1985), 307–347.

[33] V. Guillemin and S. Sternberg, *Geometric quantization and multiplicities of group representations*, Invent. Math. **67** (1982), 515–538.

[34] ———, *Birational equivalence in the symplectic category*, Invent. Math. **97** (1989), 485–522.

[35] F. G. Handfield, *Adiabatic limits of the anti-self-dual equation*, Ph.D. dissertation, Univ. of Texas, Austin, 1998.

[36] H. Hofer and D. A. Salamon, "Floer homology and Novikov rings" in *The Floer Memorial Volume*, Progr. Math. **133**, Birkhäuser, Basel, 1995, 483–524.

[37] M. Hutchings and Y.-J. Lee, *Circle-valued Morse theory and Reidemeister torsion*, Geom. Topol. **3** (1999), 369–396.

[38] ———, *Circle-valued Morse theory, Reidemeister torsion, and Seiberg-Witten invariants of* 3*-manifolds*, Topology **38** (1999), 861–888.

[39] F. Kirwan, *Cohomology of Quotients in Symplectic and Algebraic Geometry*, Math. Notes **31**, Princeton Univ. Press, Princeton, 1984.

[40] M. Kontsevich, *Enumeration of rational curves via torus actions*, preprint, 1994.

[41] M. Kontsevich and Yu. Manin, *Gromov-Witten classes, quantum cohomology, and enumerative geometry*, Comm. Math. Phys. **164** (1994), 525–562.

[42] P. Kronheimer and T. S. Mrowka, *The genus of embedded surfaces in the projective plane*, Math. Res. Lett. **1** (1994), 797–808.

[43] L. Lazzarini, *Existence of somewhere injective pseudoolomorphic discs*, preprint, 1999.
[44] J. Li and G. Tian, *Virtual moduli cycles and GW invariant*, preprint, 1996.
[45] G. Liu and G. Tian, *Floer homology and Arnold conjecture*, J. Differential Geom. **49** (1998), 1–74.
[46] S. K. Martin, *Symplectic geometry and gauge theory*, Ph.D. dissertation, Oxford Univ., Oxford, 1997.
[47] ———, *Symplectic quotients by a nonabelian group and by its maximal torus*, preprint, Institute for Advanced Study, Princeton, N.J., 1999.
[48] ———, *Transversality theory, cobordisms, and invariants of symplectic quotients*, preprint, Institute for Advanced Study, Princeton, N.J., 1999.
[49] ———, private communication, 1999.
[50] D. McDuff and D. Salamon, *J-holomorphic Curves and Quantum Cohomology*, Univ. Lecture Ser. **6**, Amer. Math. Soc., Providence, 1994.
[51] ———, *Introduction to Symplectic Topology*, 2d ed., Oxford Univ. Press, New York, 1998.
[52] G. Meng and C. H. Taubes, $\underline{\text{SW}}$ = *Milnor torsion*, Math. Res. Lett. **3** (1996), 661–674.
[53] J. Milnor, *Topology from the Differentiable Viewpoint*, Univ. Press of Virginia, Charlottesville, Virginia, 1965.
[54] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric Invariant Theory*, 3rd ed., Ergeb. Math. Grenzgeb. (2) **34**, Springer, Berlin, 1994.
[55] I. Mundet, *Yang-Mills-Higgs theory for symplectic fibrations*, Ph.D. dissertation, Univ. of Madrid, 1999.
[56] M. S. Narasimhan and C. S. Seshadri, *Stable and unitary vector bundles over compact Riemann surfaces*, Ann. of Math. (2) **82** (1965), 540–564.
[57] Y.-G. Oh, *Floer cohomology of Lagrangian intersections and pseudo-holomorphic discs, I*, Comm. Pure Appl. Math. **46** (1993), 949–993.
[58] V. Pidstrigach and A. Tyurin, Localization of Donaldson polynomials along Seiberg-Witten classes, preprint no. 75, Univ. Bielefeld, 1995.
[59] S. Piunikhin, D. Salamon, and M. Schwarz, "Symplectic Floer-Donaldson theory and quantum cohomology" in *Contact and Symplectic Geometry (Cambridge, 1994)*, Publ. Newton Inst. **8**, Cambridge Univ. Press, Cambridge, 1996, 171–200.
[60] A. Pressley and G. Segal, *Loop Groups*, Oxford Math. Monogr., Oxford Univ. Press, New York, 1986.
[61] Y. Ruan, *Topological sigma model and Donaldson-type invariants in Gromov theory*, Duke Math. J. **83** (1996), 461–500.
[62] ———, "Virtual neighborhoods and the monopole equations" in *Topics in Symplectic 4-manifolds (Irvine, Calif., 1996)*, First Int. Press Lect. Ser. **1**, Internat. Press, Cambridge, Mass., 1998, 101–116.
[63] ———, "Surgery, quantum cohomology and birational geometry" in *Northern California Symplectic Geometry Seminar*, Amer. Math. Soc. Transl. Ser. 2 **196**, Amer. Math. Soc., Providence, 1999, 183–198.
[64] D. A. Salamon, "Lagrangian intersections, 3-manifolds with boundary, and the Atiyah-Floer conjecture" in *Proceedings of the International Congress of Mathematicians (Zürich, 1994), Vol. 1*, Birkhäuser, Basel, 1995, 526–536.

[65] ———, "Quantum products for mapping tori and the Atiyah-Floer conjecture" in *Northern California Symplectic Geometry Seminar*, Amer. Math. Soc. Transl. Ser. 2 **196**, Amer. Math. Soc., Providence, 1999, 199–235.

[66] ———, *Seiberg-Witten invariants of mapping tori, symplectic fixed points, and Lefschetz numbers*, Turkish J. Math. **23** (1999), 117–143.

[67] ———, *Spin geometry and Seiberg-Witten invariants*, to be published by Birkhäuser Verlag.

[68] P. Seidel, π_1 *of symplectic automorphism groups and invertibles in quantum cohomology rings*, Geom. Funct. Anal. **7** (1997), 1046–1095.

[69] ———, *Floer homology and the symplectic isotopy problem*, Ph.D. dissertation, Oxford Univ., Oxford, 1997.

[70] C. H. Taubes, *The Seiberg-Witten and the Gromov invariants*, Math. Res. Lett. **2** (1995), 221–238.

[71] ———, SW $\Rightarrow$ Gr*: From the Seiberg-Witten equations to pseudo-holomorphic curves*, J. Amer. Math. Soc. **9** (1996), 845–918.

[72] ———, *Counting pseudo-holomorphic submanifolds in dimension* 4, preprint, Harvard Univ., 1996.

[73] ———, Gr $\Rightarrow$ SW*: From pseudo-holomorphic curves to the Seiberg-Witten invariants*, preprint, Harvard Univ., 1996.

[74] M. Thaddeus, *Stable pairs, linear systems and the Verlinde algebra*, Invent. Math. **117** (1994), 317–353.

[75] V. Turaev, *Torsion invariants of spin*^{c}*-structures on* 3*-manifolds*, Math. Res. Lett. **4** (1997), 679–695.

[76] K. K. Uhlenbeck, *Connections with* L^p *bounds on the curvature*, Comm. Math. Phys. **83** (1982), 31–42.

[77] E. Verlinde, *Fusion rules and modular transformations in* 2*D conformal field theory*, Nuclear Phys. B **300** (1988), 360–376.

[78] C. Viterbo, *Functors and computations in Floer homology with applications*, *I*, Geom. Funct. Anal. **9** (1999), 985–1033.

[79] E. Witten, *Monopoles and four-manifolds*, Math. Res. Lett. **1** (1994), 769–796.

[80] ———, "The Verlinde algebra and the quantum cohomology of the Grassmannian" in *Geometry, Topology, and Physics*, Conf. Proc. Lecture Notes Geom. Topology **4**, Internat. Press, Cambridge, Mass., 1995, 357–422.

Cieliebak: Department of Mathematics, Stanford University, Stanford, California 94305-2125 USA; kai@math.stanford.edu

Gaio: Mathematics Institute, University of Warwick, Coventry CV4 7AL, United Kingdom; rita@maths.warwick.ac.uk; Current: Department of Mathematics, Eidgenössische Technische Hochschule-Zentrum, Ramistrasse 101, 8092 Zürich, Switzerland; rita@math.ethz.ch

Salamon: Department of Mathematics, Eidgenössische Technische Hochschule-Zentrum, Ramistrasse 101, 8092 Zürich, Switzerland; dietmar.salamon@math.ethz.ch