Symplectic boundaries: creating and destroying closed characteristics

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SYMPLECTIC BOUNDARIES: CREATING AND DESTROYING CLOSED CHARACTERISTICS

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1 Introduction

A *symplectic manifold* (M, ω) is a smooth manifold M of even dimension $2n$ equipped with a nondegenerate closed 2-form ω. A map ϕ between symplectic manifolds (M, ω) and (M', ω') is called *symplectic* if $\phi^* \omega' = \omega$. A symplectic diffeomorphism is called *symplectomorphism*.

If a symplectic manifold (M, ω) has nonempty boundary ∂M, then

$$\ker(\omega|_{\partial M})_x := \{v \in T_x \partial M \mid \omega(x)(v, w) = 0 \text{ for all } w \in T_x \partial M\}$$

is 1-dimensional for all $x \in \partial M$, by the nondegeneracy of ω. So we get a line bundle

$$\ker(\omega|_{\partial M}) \to \partial M$$

whose integral curves make up the *characteristic foliation* $\mathcal{L}_\omega$. It is uniquely determined by ∂M and ω. Leaves of $\mathcal{L}_\omega$ will be called *characteristics*.

The boundary ∂M has a natural orientation defined by the volume form

$$i_\nu(\omega^n)|_{\partial M}$$

for an outward pointing vector field ν along ∂M. The line bundle $\ker(\omega|_{\partial M})$ also has a natural orientation in which a vector $v \in \ker(\omega|_{\partial M})_x$ is positive if

$$\omega(\nu, v) > 0$$

for some (and hence every) outward pointing vector $\nu \in T_x M$. We will assume all characteristics to be positively oriented.

If two symplectic manifolds with boundaries are symplectomorphic, then the characteristic foliations are *conjugate*, i.e. there exists a diffeomorphism of the boundaries (which is in this case just the restriction of the symplectomorphism) mapping the leaves onto leaves.

Question. *If we assume only that the interiors $(\overset{o}{M}, \omega)$ and $(\overset{o}{M'}, \omega')$ are symplectomorphic, what can we conclude about the characteristic foliations $\mathcal{L}_\omega$ and $\mathcal{L}_{\omega'}$ of their boundaries?*

Y. Eliashberg and H. Hofer have examples which show that under this assumption $\mathcal{L}_\omega$ and $\mathcal{L}_{\omega'}$ need not be conjugate ([EHof2], see also [EHof1]). On the other hand, there is a surprising rigidity result for closed characteristics due to A. Floer, H. Hofer and K. Wysocki usually referred to as 'stability of the action spectrum'. To state this result we first need some definitions.

A 1-form λ on an oriented manifold N of odd dimension $2n-1$ is called *contact form* if

$$\lambda \wedge (d\lambda)^{n-1} > 0$$

with respect to the orientation. Following A. Weinstein ([W2]), we say that the boundary of a symplectic manifold (M, ω) is of *contact type* if there exists a contact form λ on ∂M, where ∂M is equipped with the natural orientation defined above, such that

$$d\lambda = \omega|_{\partial M} .$$

An *exact symplectic manifold* $(M, d\mu)$ is a manifold with a 1-form μ such that $d\mu$ is symplectic. A map ϕ between exact symplectic manifolds $(M, d\mu)$ and $(M', d\mu')$ is called *exact symplectic* if $\phi^*\mu' - \mu$ is an exact 1-form.

In an exact symplectic manifold (M, μ) we define the *action* A_μ of a (positively parametrized) closed characteristic $y : S^1 \to \partial M$ by

$$A_\mu(y) := \int_{S^1} y^*\mu ,$$

and the *action spectrum*

$$\mathcal{A}(\partial M, \mu) := \{k \cdot A_\mu(y) \mid y \text{ closed characteristic on } \partial M, \ k \in \mathbf{N}\} ,$$

where $\mathbf{N}$ denotes the natural numbers without zero.

Finally, we have to define what it means for a closed characteristic x on the boundary ∂M of a symplectic manifold to be nondegenerate. Choose a hypersurface S in ∂M intersecting x transversally in some point x_0. Consider the Poincaré return map $\phi : S \to S$ (which is defined near x_0) and its linearization

$$D\phi(x_0) : T_{x_0}S \to T_{x_0}S .$$

The spectrum of $D\phi(x_0)$ does not depend on the choice of S. We call x *nondegenerate* if the spectrum does not contain 1, and *strongly nondegenerate* if it contains no root of unity. Strong nondegeneracy of x corresponds to the nondegeneracy of all iterates of x.

STABILITY OF THE ACTION SPECTRUM ([CFHW]). *Let $(M, d\mu)$ and $(M', d\mu')$ be compact exact symplectic manifolds with boundaries such that $\mu|_{\partial M}$ and $\mu'|_{\partial M'}$ are contact forms. Assume that all closed characteristics on ∂M and $\partial M'$ are strongly nondegenerate, and that the interiors $(\overset{o}{M}, d\mu)$ and $(\overset{o}{M'}, d\mu')$ are exact symplectomorphic. Then the closed characteristics on ∂M and $\partial M'$ are in 1-1 correspondence, and*

$$\mathcal{A}(\partial M, \mu) = \mathcal{A}(\partial M', \mu') \ .$$

In this paper we shall investigate what happens if one drops the assumptions that both boundaries be of contact type and all closed characteristics be strongly nondegenerate.

The following results are entitled 'corollaries' because they all follow from a single construction which will be described in section 2 (Theorems 1 and 2).

Background material about symplectic geometry can be found in the monographs [HofZ] and [MS]. However, the present article requires no familiarity with symplectic geometry.

a) The first question that arises naturally is the following: If (M, ω) and (M', ω') are symplectic manifolds with boundaries whose interiors are symplectomorphic, and if $(\partial M, \omega)$ is of contact type, does this imply that $(\partial M', \omega')$ is also of contact type?

Let us call a 1-form λ on an oriented $(2n - 1)$-dimensional manifold N a *confoliation form* (this expression is due to Y. Eliashberg) if it has the following properties:

1. $\lambda \neq 0$, i.e. λ is nowhere vanishing.
2. $d\lambda$ is *maximally nondegenerate*, i.e. $\ker(d\lambda)$ is everywhere 1-dimensional.
3.
$$\lambda \wedge (d\lambda)^{n-1} \geq 0$$
   with respect to the orientation on N.

The boundary ∂M of a symplectic manifold (M, ω) is said to be of *confoliation type* if there exists a confoliation form λ on ∂M with

$$d\lambda = \omega|_{\partial M} \ .$$

COROLLARY A. *For any symplectic manifold (M, ω) of dimension ≥ 4 with contact type boundary $(\partial M, \omega)$ there exists a symplectic form ω' on M such that $(\overset{o}{M}, \omega')$ is symplectomorphic to $(\overset{o}{M}, \omega)$, and the boundary $(\partial M, \omega')$ is of confoliation type, but not of contact type.*

REMARK. Often the symplectic manifold (M, ω) comes along with a symplectic embedding

$$: (M, \omega) \hookrightarrow (\hat{M}, \hat{\omega})$$

into some symplectic manifold without boundary. In this case we can choose the symplectic form ω' in Corollary A such that (M, ω') is also symplectically embedded into $(\hat{M}, \hat{\omega})$ (see section 2, Theorem 2 (iii)). This is also true for all the other corollaries below.

EXAMPLE. Let

$$\omega_{2n} := \sum_{i=1}^{n} dq_i \wedge dp_i$$

be the standard symplectic form on $\mathbf{R}^{2n}$ with coordinates $(q_1, p_1, \ldots, q_n, p_n)$, and let B_1^{2n} be the closed unit ball. ∂B_1^{2n} is of contact type with the contact form

$$\lambda_{2n} := \tfrac{1}{2} \sum_{i=1}^{n} (q_i dp_i - p_i dq_i) \ .$$

By Corollary A and the remark following it, for $n \geq 2$ we find a compact subset $M \subset \mathbf{R}^{2n}$ with smooth boundary such that $(\overset{o}{M}, \omega_{2n})$ is symplecto-morphic to $(\overset{o}{B_1^{2n}}, \omega_{2n})$, and the boundary $(\partial M, \omega_{2n})$ is of confoliation type, but not of contact type.

b) In view of Corollary A one may ask whether the new boundary $(\partial M, \omega')$ will at least always be of confoliation type. Again the answer is 'No'.

COROLLARY B. *For any symplectic manifold (M, ω) of dimension ≥ 4 with contact type boundary there exists a symplectic form ω' on M such that $(\overset{o}{M}, \omega')$ is symplectomorphic to $(\overset{o}{M}, \omega)$, and the boundary $(\partial M, \omega')$ is not of confoliation type.*

c) For the closed unit ball B_1^{2n} in $\mathbf{R}^{2n}$ we can make the statement of Corollary B more explicit. Consider the foliation

$$B_1^{2n} \setminus \{0\} = \bigcup_{s \in (0,1]} S^{2n-1}(s)$$

of $B_1^{2n} \setminus \{0\}$ by spheres of radii $s \in (0, 1]$.

COROLLARY C. *For $n \geq 2$ there exists an embedding $\Psi : B_1^{2n} \hookrightarrow \mathbf{R}^{2n}$ such that the interior of $(\Psi(B_1^{2n}), \omega_{2n})$ is symplectomorphic to $(\overset{o}{B_1^{2n}}, \omega_{2n})$, and the foliation $N_s := \Psi(S^{2n-1}(s))$, $s \in (0, 1]$, has the following properties (see Figure 1):*
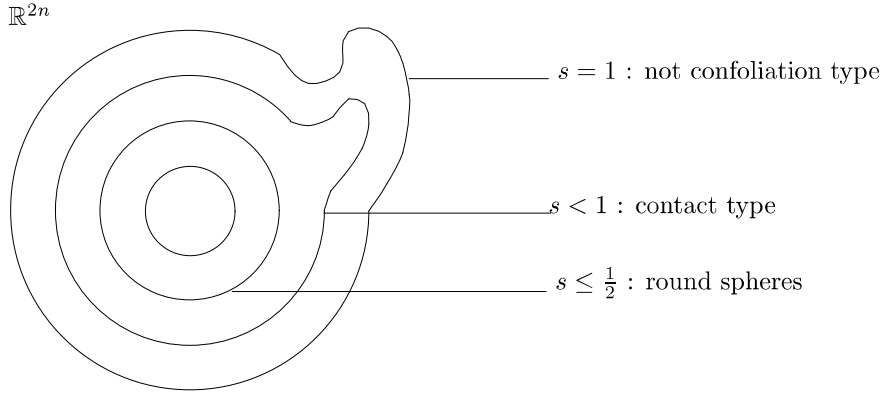
$\mathbb{R}^{2n}$

$s = 1$: not confoliation type

$s < 1$: contact type

$s \leq \frac{1}{2}$: round spheres

Figure 1.

(i) $N_s = S^{2n-1}(s)$ for $s \in (0, \frac{1}{2}]$,

(ii) N_s is of contact type for $s \in (0, 1)$,

(iii) N_1 is not of confoliation type.

In particular this shows that there are hypersurfaces in $\mathbf{R}^{2n}$ which can be smoothly approximated by contact type hypersurfaces but which are not of confoliation type.

d)   Now let us turn to the action spectrum. Recall that 'stability of the action spectrum' holds under the condition that both boundaries are of contact type. Such a result does not hold any more if for one of the boundaries the condition 'contact type' is weakened to 'confoliation type'.

COROLLARY D. *Let $(M, d\mu)$ be an exact symplectic manifold of dimension ≥ 4 with boundary such that $\mu|_{\partial M}$ is a contact form, and assume that ∂M carries only finitely many closed characteristics. Then there exists a 1-form μ' on M such that $(\overset{\circ}{M}, d\mu')$ is exact symplectomorphic to $(\overset{\circ}{M}, d\mu)$, $\mu'|_{\partial M}$ is a confoliation form, and*

$$\mathcal{A}(\partial M, \mu') = \{0\} \ .$$

REMARK.   The proof will show that the characteristic foliation of $(\partial M, d\mu')$ has at least 2 closed characteristics of action zero which are degenerate. We do not know if one can achieve that all closed characteristics of $(\partial M, d\mu')$ are nondegenerate.

EXAMPLE. For $r = (r_1, \ldots, r_n) \in (\mathbf{R}^+)^n$ define the ellipsoid

$$E(r) := \left\{ (q_1, p_1, \ldots, q_n, p_n) \in \mathbf{R}^{2n} \ \Big| \ \sum_{i=1}^n \frac{|q_i|^2 + |p_i|^2}{r_i} \leq 1 \right\} .$$

The boundary $(\partial E(r), \omega_{2n})$ is of contact type with the contact form λ_{2n} from a). If $\frac{r_i}{r_j}$ is irrational for all $i \neq j$, then $\partial E(r)$ carries precisely n closed characteristics, and Corollary D can be applied.

e) What happens to the old action spectrum in Corollary D? It turns out that the set of numbers $\mathcal{A}(\partial M, \mu)$ is still present in $(\partial M, \mu')$, but as the actions of 'heteroclinic chains'.

If (N, λ) is a manifold with a confoliation form, then a *heteroclinic chain* in (N, λ) is an m-tuple $(h_1, \ldots, h_m)$ with the following properties:
$h_1, \ldots, h_m : \mathbf{R} \to N$ are non-closed characteristics of $(N, d\lambda)$ (i.e. integral curves of $\ker(d\lambda)$), and there exist closed characteristics $x_1, \ldots, x_m$ such that

$$h_i(t) \longrightarrow x_i \text{ as } t \to -\infty ,$$
$$h_i(t) \longrightarrow x_{i+1} \text{ as } t \to +\infty$$

for $i = 1, \ldots, m$, where we have set $x_{m+1} := x_1$.

COROLLARY E. *Under the hypotheses of Corollary D, to every action value $a \in \mathcal{A}(\partial M, \mu)$ there corresponds a heteroclinic chain $(h_1, \ldots, h_m)$ in $(\partial M, \mu')$ with action*

$$\sum_{i=1}^m \int_{h_i} \mu' = a .$$

OPEN QUESTION. Can one extend the definition of 'action spectrum' to include, besides closed characteristics, more general invariant subsets such as, for example, the heteroclinic chains above, such that the larger 'action spectrum' is invariant under symplectomorphisms of the interior?

f) In Corollary D we have destroyed a finitely generated action spectrum, creating only one new action value 0. If we drop the condition that the new boundary be of confoliation type, then we can also create a prescribed finitely generated action spectrum. However, since in this case the interior will in general not remain exact symplectomorphic, we shall consider another situation in which we can define an action spectrum.

Let (M, ω) be a symplectic manifold with boundary. We say that ω vanishes on $\pi_2(M)$ if

$$\int_{S^2} f^* \omega = 0$$

for all smooth maps $f : S^2 \to M$. Under this hypothesis we can define the *action* of a closed characteristic $y : S^1 \to \partial M$ which is contractible in M unambiguously as

$$A_\omega(y) := \int_{B_1^2} \bar{y}^* \omega$$

for any smooth map $\bar{y} : B_1^2 \to M$ with $\bar{y}|_{\partial B_1^2} = y$. Define the *contractible action spectrum*

$$\mathcal{A}^{contr}(\partial M, \omega) := \{k \cdot A_\omega(y) \mid y \text{ closed characteristic on } \partial M$$
$$\text{contractible in } M, \; k \in \mathbf{N}\} \; .$$

In this situation we have again 'stability of the action spectrum' if both boundaries are of contact type (see [CFHW]). Dropping the 'contact type' condition for one of the boundaries, we obtain the following result.

COROLLARY F. *Let (M, ω) be a symplectic manifold of dimension ≥ 4 with boundary (which may or may not be of contact type) such that ω vanishes on $\pi_2(M)$. Assume that there are only finitely many closed characteristics on ∂M, and let $a_1, \ldots, a_l$ be given real numbers. Then there exists a symplectic form ω' on M, vanishing on $\pi_2(M)$, such that $(\overset{\mathrm{o}}{M}, \omega')$ is symplectomorphic to $(\overset{\mathrm{o}}{M}, \omega)$, and*

$$\mathcal{A}^{contr}(\partial M, \omega') = \{k \cdot a_i \mid i \in \{1, \ldots, l\}, \; k \in \mathbf{N}\} \cup \{0\} \; .$$

g) Now let us go beyond closed characteristics and study what can happen to invariant tori.

Let $E(r) \subset \mathbf{R}^{2n}$ be an ellipsoid. The characteristic foliation of $(\partial E(r), \omega_{2n})$ can easily be calculated explicitly. It is *completely integrable* in the sense that up to a set of measure zero, $\partial E(r)$ is foliated by smoothly embedded invariant n-dimensional tori on which the characteristic foliation is *linear*. Here we call a 1-dimensional foliation on the n-torus $T^n = \mathbf{R}^n / \mathbf{Z}^n$ linear if it is smoothly conjugate to a foliation by parallel straight lines. Note that this definition of integrability is different from the integrability in the sense of Liouville (via independent integrals of motion).

COROLLARY G. *Let $n = 2$ and $r = (r_1, r_2) \in (\mathbf{R}^+)^2$ with $\frac{r_1}{r_2}$ irrational. Then there exists a compact subset $M \subset \mathbf{R}^4$ with smooth confoliation type boundary such that $(\overset{\mathrm{o}}{M}, \omega_4)$ is symplectomorphic to $(\overset{\mathrm{o}}{E}(r), \omega_4)$, and the characteristic foliation on ∂M has no continuously embedded invariant 2-torus.*

REMARK. The familiar picture of the transition from an integrable to a nonintegrable system, as described by KAM and Aubry-Mather theory (in dimension 2), is that of certain invariant tori breaking up into cantori, others following, until all invariant tori have disappeared. The proof of Corollary G reveals a quite different picture: There exists a smooth family $(N_s)_{0 \le s \le 1}$ of embedded hypersurfaces starting from $N_0 = \partial E(r)$ and ending with $N_1 = \partial M$. All invariant tori persist for $s \in [0, 1)$, breaking up simultaneously at $s = 1$.

h) We may also have the opposite case to g), namely that the boundary remains completely integrable, but the characteristic foliation is changed.

COROLLARY H. *Let (M, ω) be a symplectic manifold of dimension ≥ 4 with boundary, and suppose that the characteristic foliation $\mathcal{L}_\omega$ on ∂M possesses a smoothly embedded invariant 2-torus T on which the foliation is linear. Then there exists a symplectic form ω' on M such that $(\overset{\mathrm{o}}{M}, \omega')$ and $(\overset{\mathrm{o}}{M}, \omega)$ are symplectomorphic, T is invariant for the new foliation $\mathcal{L}_{\omega'}$, and the following holds:*

(i) *On $\partial M \setminus T$ the foliations $\mathcal{L}_{\omega'}$ and $\mathcal{L}_\omega$ are conjugate.*
(ii) *The foliation $\mathcal{L}_{\omega'}|_T$ has precisely 2 closed leaves, in particular it is not linear.*

The situation of Corollary H occurs, e.g., if $\dim M = 4$ and the characteristic foliation on ∂M is completely integrable in the sense defined in g). In this case the foliation remains completely integrable and is altered on precisely one invariant torus.

i) The next application is related to the Seifert conjecture on S^3. H. Seifert had asked in 1950 whether every nonsingular vector field on S^3 has a periodic orbit. The answer is now known to be 'No' due to the C^1 vector field constructed by P.A. Schweitzer ([Sc]) in 1974 and the smooth vector field constructed by K. Kuperberg ([Ku]) in 1994. But it remains an interesting problem to pose the question for restricted classes of vector fields.

A vector field X on S^3 is called *volume preserving* if there exists a smooth volume form Ω on S^3 such that

$$L_X \Omega = d(i_X \Omega) = 0 .$$

In 1994 G. Kuperberg produced a volume preserving C^1 vector field on S^3 without periodic orbits ([K]). So in order to get a positive answer to Seifert's question we must impose further restrictions on the vector field X.

Since the 2-form $i_X\Omega$ is closed and S^3 has trivial second cohomology, there exists a 1-form λ such that

$$(*) \qquad\qquad d\lambda = i_X\Omega \ .$$

Here in the choice of λ we have the freedom to add the differential of a function. We call X a *Reeb vector field* if we can find a 1-form λ satisfying $(*)$ such that

$$\lambda(X) > 0 \ .$$

If there exists a nowhere vanishing 1-form λ satisfying $(*)$ such that

$$\lambda(X) \geq 0 \ ,$$

we call X a *confoliation vector field*. Note that in these cases the form λ is a contact respectively confoliation form.

H. Hofer has proved in 1993 that every Reeb vector field on S^3 possesses a periodic orbit ([Hof]). Moreover, there exists at least one periodic orbit which is unknotted ([HofWyZ]). In contrast to this, for confoliation vector fields we have the following result.

COROLLARY I. *Let c be any prescribed oriented knot type in S^3. Then there exists a confoliation vector field X on S^3 with precisely two periodic orbits, and they have knot types $\pm c$.*

Moreover, X can be chosen transversal to the standard contact structure on S^3.

Here the standard contact structure on S^3 is the plane distribution

$$TS^3 \supset \ker(\lambda_0) \to S^3 \ ,$$

where λ_0 is the contact form on S^3 obtained by restricting the form λ_4 of a) to the unit sphere in $\mathbf{R}^4$.

REMARKS. 1. By the result of [HofWyZ], for a nontrivial knot type c the flow of the vector field X in Corollary I is not C^0-conjugate to the flow of any Reeb vector field.

2. A *contact vector field* is a nonvanishing vector field Y with $(L_Y\lambda) \wedge \lambda = 0$ for some contact form λ. Combining Corollaries G and I we find a confoliation vector field on S^3 whose flow possesses no invariant 2-torus and no unknotted periodic orbit. The flow of this vector field is not C^0-conjugate to the flow of any contact vector field, which can be seen as follows: Consider the set $N := \{x \in S^3 \mid \lambda(Y)(x) = 0\}$. The condition that Y is a contact vector field implies that $d(\lambda(Y)) \neq 0$ along N, and Y is tangent to N. Hence N is either empty or a union of smoothly embedded

invariant 2-tori. By the choice of Y, N must be empty. But then Y is a Reeb vector field, and the statement follows from Remark 1.

j) A smooth hypersurface $S \subset \mathbf{R}^{2n}$ has a characteristic foliation induced by $\ker(\omega_{2n}|_S)$. It is an old question in Hamiltonian dynamics how many closed characteristics there must exist on a compact hypersurface. In 1994 V. Ginzburg constructed examples of smooth compact hypersurfaces in $\mathbf{R}^{2n}$, $n \geq 4$, without closed characteristics ([G]). The same result has been obtained independently by M. Herman, as well as examples of compact hypersurfaces of class $C^{3-\epsilon}$ in $\mathbf{R}^6$ without closed characteristics ([H]). On the other hand, a result by C. Viterbo from 1987 states that every compact hypersurface of contact type has at least one closed characteristic ([V]). Here a hypersurface S is said to be of contact type (resp. confoliation type) if there exists a contact form (resp. confoliation form) λ on S with $d\lambda = \omega_{2n}|_S$.

Our construction yields the following result.

COROLLARY J. *For any $n \geq 2$ and $k \geq 2$ there exist compact smooth hypersurfaces in $\mathbf{R}^{2n}$ of confoliation type with precisely k closed characteristics.*

For $n = 2$ this contrasts a result due to H. Hofer, K. Wysocki and E. Zehnder that a hypersurface in $\mathbf{R}^4$ bounding a strictly convex domain always possesses precisely 2 or infinitely many closed characteristics. In particular, for $k > 2$ the hypersurfaces in Corollary J cannot bound a strictly convex domain in $\mathbf{R}^4$.

It is an open question whether a compact hypersurface in $\mathbf{R}^4$ of contact type can carry a finite number $k \neq 2$ of closed characteristics.

This paper is organized as follows: In section 2 we describe the construction to create and destroy closed characteristics. In section 3, Corollaries A, D,E,G,I and J are proved. In section 4 we slightly generalize the construction, and in section 5 we prove the remaining Corollaries B,C,F and H.

2   The Construction

Let (N, λ) be an (oriented) manifold of odd dimension $2n - 1 \geq 3$ with a confoliation form. The line bundle $\ker(d\lambda)$ generates a foliation which we shall also call characteristic foliation. A *tangent curve* in (N, λ) is a curve which is everywhere tangent to $\ker(\lambda)$. In dimension 3 such a curve is called 'legendrian curve'. Closed curves will be called loops.

The construction is based on the following 3 results.

LEMMA 1. *Let (N, λ) be a manifold of dimension $2n - 1 \geq 3$ with a contact form. Then every embedded smooth curve $c : [a, b] \to N$ can be deformed into an embedded curve tangent to $\ker(\lambda)$ by a C^0-small smooth isotopy of embedded curves fixing any finite number of points on c.*

Theorem 1. *Let (N, λ) be a manifold of dimension $2n - 1 \geq 3$ with a contact form, and let $L \subset N$ be an embedded oriented loop tangent to $\ker(\lambda)$. Then there exists a 1-form λ' on N, agreeing with λ outside a neighborhood of L, with the following properties:*

(i) *The characteristic foliation generated by $\ker(d\lambda')$ has 2 additional closed orbits $L^\pm$ which are isotopic to $\pm L$. Here $-L$ is the loop L with the opposite orientation.*

(ii) *Every closed characteristic of $\ker(d\lambda)$ intersecting L breaks up into two characteristics of $\ker(d\lambda')$ heteroclinic to $L^\pm$ (see Figure 2).*
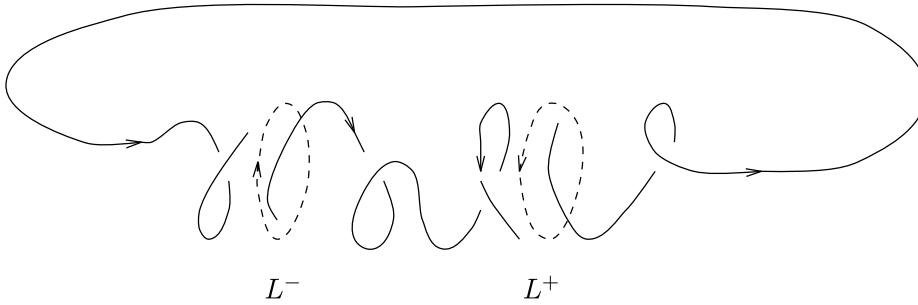

L^- $\qquad\qquad$ L^+

Figure 2. Breaking up a closed characteristic

(iii) *Apart from (i) and (ii) no closed characteristics are created or destroyed.*

(iv) *λ' is a confoliation form, and $\lambda' \wedge (d\lambda')^{n-1} = 0$ precisely on $L^- \cup L^+$.*

Theorem 2. *Let (M, ω) be a symplectic manifold of dimension $2n \geq 4$ with contact type boundary $(\partial M, \lambda)$ such that $\omega|_{\partial M} = \lambda$, and let $L \subset \partial M$ be an*

embedded oriented loop tangent to $\ker(\lambda)$. Then there exists a symplectic form ω' on M, agreeing with ω outside a neighborhood of L in M, with the following properties:

(i) $\omega'|_{\partial M} = d\lambda'$, where for λ' on ∂M the statements of Theorem 1 hold true.

(ii) The interiors $(\overset{o}{M}, \omega)$ and $(\overset{o}{M}, \omega')$ are symplectomorphic.

Moreover, the following holds:

(iii) If (M, ω) is symplectically embedded in some symplectic manifold $(\hat{M}, \hat{\omega})$ without boundary, then we can choose ω' such that (M, ω') is also symplectically embedded in $(\hat{M}, \hat{\omega})$.

(iv) If $\omega = d\mu$ is exact with $\mu|_{\partial M} = \lambda$, then there exists a 1-form μ' on M, agreeing with μ outside a neighborhood of L in M, such that $\mu'|_{\partial M} = \lambda'$, and $(\overset{o}{M}, d\mu)$ and $(\overset{o}{M}, d\mu')$ are exact symplectomorphic.

These results will be applied as follows: Let a symplectic manifold (M, ω) of dimension ≥ 4 with contact type boundary $(\partial M, \lambda)$ be given.

Step 1. Select a finite set $\{y_1, \ldots, y_l\}$ of closed characteristics on ∂M you want to get rid of (this set may also be empty). Choose finitely many isotopy classes $c_1, \ldots, c_k$ of embedded oriented loops in which you want to create closed characteristics, such that every boundary component containing one of the y_j also contains a loop in one of the classes c_i.

Step 2. By Lemma 1 there exist disjoint embedded loops $L_1, \ldots, L_k$ in the classes $c_1, \ldots, c_k$ which are tangent to $\ker(\lambda)$ and such that every y_j intersects one of the L_i.

Step 3. Apply Theorems 1 and 2 to each of the tangent loops $L_1, \ldots, L_k$. We obtain a new symplectic form ω' on M with the properties listed in Theorems 1 and 2, in particular:

- In the new characteristic foliation $\mathcal{L}_{\omega'}$ on ∂M the old closed characteristics $y_1, \ldots, y_l$ have disappeared, and new closed characteristics $L_i^{\pm}$ have been created in the classes $\pm c_i$.
- The interior $(\overset{o}{M}, \omega')$ is symplectomorphic to $(\overset{o}{M}, \omega)$.

Proof of Lemma 1. First we will deform c to make it tangent to $\ker(\lambda)$ near the prescribed points. Taking more points we may assume that two consecutive points are always contained in one Darboux chart. So it suffices to show the following local statement:

Consider $\mathbf{R}^{2n-1}$ with coordinates $(q_1, p_1, \ldots, q_{n-1}, p_{n-1}, z)$ and the standard form $dz + \lambda_{2(n-1)}$, where $\lambda_{2(n-1)} = \frac{1}{2} \sum_{i=1}^{n-1}(q_i dp_i - p_i dq_i)$ as before.

Let

$$c : [-\epsilon, 1 + \epsilon] \to \mathbf{R}^{2n-1}$$

be an embedding which is tangent to $\ker(\lambda)$ *outside* $(0, 1)$. *Then there exists a* C^0-*small smooth isotopy of* c, *fixed outside* $(0, 1)$, *to an embedded curve* L *tangent to*

$$\ker(dz + \lambda_{2(n-1)}) = \left\{ -\lambda_{2(n-1)}(Y)\tfrac{\partial}{\partial z} + Y \mid Y = \sum \left(a_i \tfrac{\partial}{\partial q_i} + b_i \tfrac{\partial}{\partial p_i} \right) \right\} .$$

Without loss of generality assume $c(0) = 0$. Let

$$\pi : \mathbf{R}^{2n-1} \to \mathbf{R}^{2n-2}$$

be the projection onto the $\{z = 0\}$-hyperplane. A curve

$$r = (\pi r, r_z) : [0, 1] \to \mathbf{R}^{2n-1}$$

with $r(0) = 0$ is tangent if and only if

$$(*) \qquad\qquad r_z(t) = -\int_0^t (\pi r)^* \lambda_{2(n-1)} .$$

So setting $\pi r := \pi c$ and defining r_z by $(*)$ we obtain a tangent curve $r : [0, 1] \to \mathbf{R}^{2n-1}$ which matches smoothly with the given curve c at $t = 0$.

Next we want to make r embedded by a C^∞-small perturbation away from the end points $\{0, 1\}$ and keeping it tangent. To this purpose we perturb the projection πr away from the end points until it only has a finite number of transversal self-intersections in $(0,1)$, and define r_z by $(*)$. This yields a new tangent curve which we will still denote by r. Let $0 < t_1 < t_2 < 1$ be times with $\pi r(t_1) = \pi r(t_2)$. Then

$$r_z(t_1) - r_z(t_2) = \int_D d\lambda_{2(n-1)}$$

is the symplectic area of any disk D in $\mathbf{R}^{2(n-1)}$ bounded by $\pi r([t_1, t_2])$. Note that here we have used Stoke's theorem for piecewise smooth boundary. Now we perturb πr to make this area nonzero for all self-intersection points of πr. The resulting r, with r_z again defined by $(*)$, will be tangent and embedded.

The curve r matches with c at $t = 0$, but at $t = 1$ the z-component $r_z(1)$ will in general be different from $c_z(1)$. To compensate for this, let $\pi m : [0, 1] \to \mathbf{R}^{2(n-1)}$ be an embedded closed curve matching smoothly with $\pi r(t + 1)$ at $t = 0$ and with $\pi c(t)$ at $t = 1$. Define the z-component

$$m_z(t) := r_z(1) - \int_0^t (\pi m)^* \lambda_{2(n-1)} .$$

At $t = 1$ we obtain

$$m_z(1) = r_z(1) + \int_D \hat{m}^* d\lambda_{2(n-1)}$$

for any extension $\hat{m}$ of πm to the unit disk D. So we may choose πm in such a way that $m_z(1) = c_z(1)$. The tangent curve $m = (\pi m, m_z)$ then matches smoothly with $r(t + 1)$ at $t = 0$ and with $c(t)$ at $t = 1$. Define the curve $L : [-\epsilon, 1 + \epsilon] \to \mathbf{R}^{2n-1}$ by

$$L(t) := \begin{cases} c(t) & \text{if } t \leq 0 \text{ or } t \geq 1 \ , \\ r(\phi(t)) & \text{if } 0 \leq t \leq \frac{1}{2} \ , \\ m(\phi(t)) & \text{if } \frac{1}{2} \leq t \leq 1 \ , \end{cases}$$

where $\phi : [0, 1] \to [0, 2]$ is a smooth diffeomorphism with $\phi(t) = t$ for t near 0, $\phi(t) = t + 1$ for t near 1, and $\phi(\frac{1}{2}) = 1$.

L is the desired embedded tangent curve. Applying this procedure to a finer partition of c we can get L arbitrarily C^0-close to c, and L is clearly isotopic to c through embedded curves. $\qquad\square$

Changing the foliation on the boundary: Proof of Theorem 1.

Near the tangent loop L the contact form λ has the following normal form: Let B_γ^{2k} be the closed ball around 0 of radius γ in $\mathbf{R}^{2k}$ with coordinates $z = (q_1, p_1, \ldots, q_k, p_k)$. Put

$$k := n - 2$$

and define

$$P := [-\delta, \delta] \times [-\epsilon, \epsilon] \times B_\gamma^{2k} \times S^1$$

with coordinates

$$p = (t, x, z, \theta) \ .$$

Consider the 1-forms $\lambda_{2k} = \frac{1}{2} \sum_{i=1}^k (q_i dp_i - p_i dq_i)$ on B_γ^{2k} and

$$\lambda_0 := dt + x \, d\theta + \lambda_{2k}$$

on P. According to Lemma A1 of the appendix there exists a neighborhood of L in N diffeomorphic to P such that

$$L = \{0\}^{2k+2} \times S^1$$

and

$$\lambda|_P = \lambda_0 \ .$$

Hence Theorem 1 follows from the following proposition, setting $\lambda' := \lambda_1$ on P and $\lambda' := \lambda$ outside.

PROPOSITION 1. *There exist 1-forms $(\lambda_s)_{0 \leq s \leq 1}$ on P, agreeing with λ_0 near ∂P, with the following properties:*

(i) *The cylinder*
$$Z := [-\delta, \delta] \times \{0\}^{2k+1} \times S^1$$
is invariant under the flow of $\ker(d\lambda_s)$ for all $s \in [0,1]$. The circles
$$L^\pm := \left\{\pm\tfrac{\delta}{2}\right\} \times \{0\}^{2k+1} \times S^1$$
are closed orbits of $\ker(d\lambda_1)$ oriented in direction $\pm\frac{\partial}{\partial\theta}$.

(ii) *All orbits of $\ker(d\lambda_1)$ on Z are asymptotic to $L^\pm$ as shown in Figure 3.*
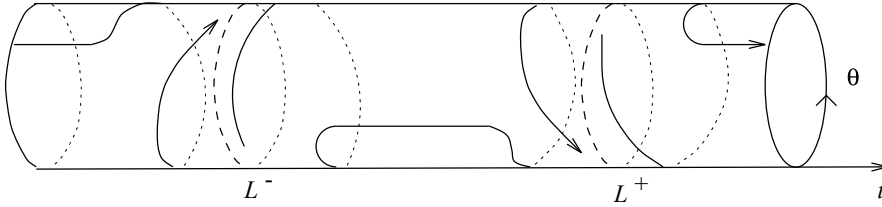

Figure 3. The foliation on Z

(iii) *For $0 \leq s < 1$ all orbits of $\ker(d\lambda_s)$ in P enter P at some point $(-\delta, x, z, \theta)$ and exit P at the opposite point $(+\delta, x, z, \theta)$. The same is true for all orbits of $\ker(d\lambda_1)$ in $P \setminus Z$.*

(iv) *λ_s are contact forms for $0 \leq s < 1$, and λ_1 is a confoliation form. Moreover, $\lambda_1 \wedge (d\lambda_1)^{n-1} = 0$ precisely on $L^- \cup L^+$.*

REMARK. There cannot exist a contact form λ_1 on P, agreeing with λ_0 near ∂P, and having the properties (i)-(iii). This follows from the discussion preceding Corollary I: If such a contact form existed then Corollary I would yield a Reeb vector field on S^3 without unknotted periodic orbits, contradicting Hofer's result.

Proof. The following construction is taken from [G]. It is a symplectic version of F. Wilson's mirror image plugs (cf. [Wi]).

Let H and f be functions on $Q := [-\delta, \delta] \times [-\epsilon, \epsilon] \times B_\gamma^{2k}$ depending only on t, x and $|z|$, and satisfying

(H1) $H = x$ near ∂Q;

(H2) $H_x - |H| - |z| \, |H_z| > 0$ for all $(t, x, z) \neq (\pm\tfrac{\delta}{2}, 0, 0)$;

(H3) $H_x(\pm\tfrac{\delta}{2}, 0, 0) = 0$;

(H4) $H(t, 0, z) = 0$ for all (t, z);

(H5) $H(-t, x, z) = H(t, x, z)$;

(H6) $H(t,x,z) = x\big((t \mp \frac{\delta}{2})^2 + x^2 + |z|^2\big)$ for (t,x,z) near $(\pm\frac{\delta}{2}, 0, 0)$;

and

(f1) $f = 0$ near ∂Q;
(f2) $f_x(\frac{\delta}{2}, 0, 0) > 0$, $f_x(-\frac{\delta}{2}, 0, 0) < 0$;
(f3) f is sufficiently C^2-small;
(f4) $f(-t,x,z) = -f(t,x,z)$
(f5) $f(t,x,z) = \pm ax$ for (t,x,z) near $(\pm\frac{\delta}{2}, 0, 0)$, with some $a > 0$.

Here H_x etc. are partial derivatives, and H_z denotes the gradient of H with respect to the z-variables. The level lines of H on $\{z = 0\}$ are shown in Figure 4. The conditions (H6) and (f5) are not really necessary, but they will simplify some local computations.
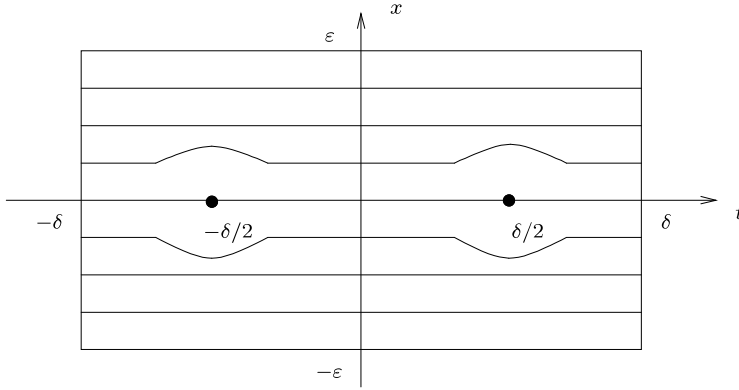

Figure 4. The level lines of H for $z = 0$

A function H satisfying (H1-6) can be constructed as follows (cf. [G]). Choose the following smooth functions:

- $g : [-\delta, \delta] \to [0, 1]$ even such that $g \equiv 1$ near $\pm\delta$, $g = 0$ precisely at $\pm\frac{\delta}{2}$, and $g(t) = (t \mp \frac{\delta}{2})^2$ for t near $\pm\frac{\delta}{2}$.
- $h : [-\epsilon, \epsilon] \to [-\epsilon, \epsilon]$ odd with $h(x) = x$ for x near $\pm\epsilon$, $h(x) = x^3$ near 0, $h'(x) - |h(x)| \geq x^2$, and $|h(x)| \leq |x|$.
- $l : [0, \gamma] \to [0, 1]$ monotone increasing such that $l \equiv 1$ near γ, $l(r) = r^2$ near 0, $l(r) \geq r^2$, and $l'(r) \leq c\,r$ for some constant $c > 0$.

Define

$$H(t,x,z) := (1 - l(|z|))\big[(1 - g(t))h(x) + g(t)x\big] + l(|z|)x \ .$$

Properties (H1) and (H3-5) follow immediately. For (H2) we calculate, choosing $|x| \leq \epsilon$ sufficiently small,

$$
\begin{aligned}
H_x &- |H| - |z|\,|H_z| \\
&= (1-l)[(1-g)h' + g] + l - |(1-l)[(1-g)h + gx] + lx| \\
&\quad - |z|\,|-l'[(1-g)h + gx] + l'x| \\
&\geq (1-l)(1-g)(h' - |h|) + (1-l)g(1 - |x|) + l(1 - |x|) \\
&\quad - |z|\,|l'|(1-g)|h - x| \\
&\geq (1-l)(1-g)x^2 + \tfrac{1}{2}(1-l)g + \tfrac{1}{2}l - 2c|z|^2|x| \\
&\geq (1-l)(1-g)x^2 + \tfrac{1}{2}(1-l)g + \tfrac{1}{2}l[1 - 4c|x|] \\
&\geq 0
\end{aligned}
$$

if $4c\epsilon < 1$, with equality if and only if $l = g = x = 0$, i.e. at $(t, x, z) = (\pm\frac{\delta}{2}, 0, 0)$.

Finally, for $(t, x, z,)$ near $(\pm\frac{\delta}{2}, 0, 0)$ we have

$$
H(t, x, z) = (1 - |z|^2)\Big[\big(1 - (t \mp \tfrac{\delta}{2})^2\big)x^3 + \big(t \mp \tfrac{\delta}{2}\big)^2 x\Big] + |z|^2 x \ ,
$$

which agrees with (H6) up to higher order terms. So we can modify H in a neighborhood of $(\pm\frac{\delta}{2}, 0, 0)$ to fulfill (H6).

Now define

$$
H^s(t, x, z) := sH(t, x, z) + (1 - s)x
$$

and

$$
\lambda_s := \big(1 - s\,f(t, x, z)\big)dt + H^s(t, x, z)d\theta + \lambda_{2k} \ .
$$

Clearly $\lambda_s = \lambda_0$ near ∂P for all $s \in [0, 1]$.

(iv) λ_s is nowhere vanishing because for $|f| < 1$ the dt-term is never zero. Its exterior derivative is

$$
d\lambda_s = sf_x\,dt\,dx + H_x^s\,dx\,d\theta + H_t^s\,dt\,d\theta + s\,dt \wedge d_z f + d_z H^s \wedge d\theta + d\lambda_{2k} \ ,
$$

where d_z denotes the differential with respect to the z-variables. It follows that

$$
\begin{aligned}
(d\lambda_s)^{k+1} &= (k+1)(sf_x\,dt\,dx + H_x^s\,dx\,d\theta + H_t^s\,dt\,d\theta) \wedge (d\lambda_{2k})^k \\
&\quad + k(k+1)(sf_x\,d_z H^s - sH_x^s\,d_z f) \wedge dt\,dx\,d\theta \wedge (d\lambda_{2k})^{k-1} \ ,
\end{aligned}
$$

where the coefficients $(k+1)$ and $k(k+1)$ do not depend on the convention for the wedge product. By (H2) and (H3), the term $(k+1)H_x^s\,dx\,d\theta \wedge (d\lambda_{2k})^k$ vanishes only if $s = 1$ and $(t, x, z) = (\pm\frac{\delta}{2}, 0, 0)$. But then by (f2) the term $(k+1)s\,f_x\,dt\,dx \wedge (d\lambda_{2k})^k$ is nonzero. Hence $d\lambda_s$ is maximally nondegenerate for all s.

To check the confoliation type property, compute

$$\lambda_s \wedge (d\lambda_s)^{k+1} = (k+1)[(1-sf)H_x^s + sH^s f_x]\, dt\, dx\, d\theta \wedge (d\lambda_{2k})^k$$
$$+ k(k+1)\lambda_{2k} \wedge (sf_x\, d_z H^s - sH_x^s\, d_z f) \wedge dt\, dx\, d\theta \wedge (d\lambda_{2k})^{k-1}$$
$$= (k+1)[(1-sf)H_x^s + sH^s f_x + h] \cdot vol \ ,$$

where

$$vol := dt\, dx\, d\theta \wedge (d\lambda_{2k})^k \ ,$$

and h is the function defined by

$$k\, \lambda_{2k} \wedge (sf_x\, d_z H^s - sH_x^s\, d_z f) \wedge dt\, dx\, d\theta \wedge (d\lambda_{2k})^{k-1} = h \cdot vol \ .$$

Since λ_{2k} is of order $|z|$, h can be estimated by

$$|h| \le |k|\, |z|\, (|f_x|\, |H_z^s| + |f_z|\, H_x^s) \ .$$

Assuming $\epsilon, \gamma \le \frac{1}{2}$ and choosing f C^1-small this leads to the following estimate:

$$(1-sf)H_x^s + sH^s f_x + h \ge (1 - |f| - k|z|\, |f_z|)H_x^s - |f_x|(|H^s| + k|z|\, |H_z^s|)$$
$$\ge \tfrac{1}{2}(sH_x + (1-s)) - \tfrac{1}{2}(s|H| + (1-s)|x| + s|z|\, |H_z|)$$
$$(0) \qquad\qquad = \tfrac{1}{2}s(H_x - |H| - |z|\, |H_z|) + \tfrac{1}{2}(1-s)(1-|x|)$$
$$\ge 0$$

for all $0 \le s \le 1$, with equality if and only if $s = 1$ and $(t, x, z) = (\pm\frac{\delta}{2}, 0, 0)$, i.e. on $L^+ \cup L^-$. This proves (iv).

(i) and (ii) $\ker(d\lambda_s)$ is generated by the nonvanishing vector field

$$X^s = sf_x \tfrac{\partial}{\partial\theta} + H_x^s \tfrac{\partial}{\partial t} - H_t^s \tfrac{\partial}{\partial x} + sf_x\, v_{H^s} - sH_x^s\, v_f \ ,$$

where for a function g on P, v_g denotes the vector field having no $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial\theta}$ components and satisfying

$$d_z g = i_{v_g} d\lambda_{2k} \ .$$

Note that since f and H are functions of t, x and $|z|$ only,

$$d_z f(v_{H^s}) = d_z H^s(v_f) = 0 \ .$$

Using this one easily verifies that X^s indeed satisfies $i_{X^s} d\lambda_s = 0$.

Because of (H4) and once again the dependence of H and f on t, x and $|z|$ only, X^s reduces along the cylinder Z to

$$X^s = sf_x \tfrac{\partial}{\partial\theta} + H_x^s \tfrac{\partial}{\partial t} \ .$$

This shows that Z is invariant under X^s for all $s \in [0, 1]$. The flow of X^1 on Z is depicted in Figure 3. It has the two closed orbits $L^\pm$ corresponding to

the points where H_x vanishes. The orientations of L^- and L^+ are opposite to each other because $f_x\left(-\frac{\delta}{2}, 0, 0\right) = -f_x\left(+\frac{\delta}{2}, 0, 0\right)$. Outside $L^\pm$ the $\frac{\partial}{\partial t}$-component of X^1 is strictly positive, so all other orbits on Z are asymptotic to $L^\pm$ as shown in Figure 3.

(iii) Suppose that either $s < 1$, or $s = 1$, and in the latter case only points in $P \setminus Z$ are considered. In these cases the $\frac{\partial}{\partial t}$-component of X^s is strictly positive, and we can normalize X^s to

$$\frac{1}{H_x^s} X^s = \frac{\partial}{\partial t} + \frac{s f_x}{H_x^s} \frac{\partial}{\partial \theta} - \frac{H_t^s}{H_x^s} \frac{\partial}{\partial x} + \frac{s f_x}{H_x^s} v_{H^s} - \frac{s H_x^s}{H_x^s} v_f$$
$$:= \frac{\partial}{\partial t} + Y^s \ .$$

Hence after reparametrisation the integral curves of X^s are curves

$$t \mapsto (t, y(t))$$

(we use the notation $y = (x, z, \theta)$) satisfying

$$\dot{y}(t) = Y^s(t, y(t)) \ .$$

Now by (H5) and (f4) the time-dependent vector field Y^s satisfies

$$Y^s(t, y) = -Y^s(-t, y) \ .$$

Thus for every solution $y(t)$, the curve $t \mapsto y(-t)$ is also a solution, and by uniqueness we obtain

$$y(-t) = y(t) \ .$$

In particular, $y(-\delta) = y(\delta)$, and (iii) follows. This finishes the proof of Proposition 1 and Theorem 1. $\qquad\square$

Keeping the interior symplectomorphic: Proof of Theorem 2.

Let $(\partial M, \lambda)$ be the contact type boundary of (M, ω). In order to find a normal form for ω near the tangent loop $L \subset \partial M$, we need the following fact:

There exists a neighborhood $(0, 1] \times \partial M$ of ∂M in M, ∂M corresponding to $\{1\} \times \partial M$, and a 1-form μ on $(0, 1] \times \partial M$ such that

$$d\mu = \omega|_{(0,1] \times \partial M} \ ,$$
$$\mu|_{\{1\} \times \partial M} = \lambda \ ,$$

and

$$\mu|_{\{s\} \times \partial M}$$

is a contact form for all $s \in (0, 1]$.

Such a neighborhood can be constructed as follows: By the relative Poincaré Lemma (see [W1]) we find a 1-form μ on a neighborhood V of

∂M in M with

$$\omega|_V = d\mu$$

and

$$\mu|\partial M = \lambda \ .$$

Define a vector field Y on V by

$$i_Y \omega = \mu \ .$$

Then Y is transverse to ∂M because $\omega^n(Y, X, Z_1, \ldots, Z_{2n-2}) > 0$ for X the Reeb vector field of λ, and $Z_1, \ldots, Z_{2n-2}$ a positively oriented basis of $\ker(\lambda)$ on ∂M. Let g be a positive smooth function on V such that the flow ϕ_s of $-gY$ starting on ∂M exists until time 1 and stays in V. If g is sufficiently small this leads to an embedding

$$(0, 1] \times \partial M \hookrightarrow M \ ,$$
$$(s, x) \mapsto \phi_{1-s}(x) \ ,$$

onto a neighborhood of ∂M in V. From

$$L_{(gY)}\mu = i_{(gY)}\omega + d(i_{(gY)}\mu)$$
$$= g\,\mu$$

it follows that

$$\phi_{1-s}^*(\mu|_{\phi_{1-s}(\partial M)})$$

is a positive multiple of λ and hence a contact form, and the fact is proved.

Now consider the embedded tangent loop $L \subset \partial M$. Recall from the proof of Theorem 1 that $P = [-\delta, \delta] \times [-\epsilon, \epsilon] \times B_\gamma^{2k} \times S^1$. Define on $(0, 1] \times P$ with coordinates (s, t, x, z, θ) the 1-form

$$\mu_0 := s\,dt + x\,d\theta + \lambda_{2k} \ .$$

By Lemma A5 of the appendix we find, after a change of the foliation $(\{s\} \times \partial M)_{s \in (\rho, 1]}$ near L, a neighborhood $(\rho, 1] \times P$ of L in M such that

$$L = \{1\} \times \{0\}^{2k+2} \times S^1$$

and

$$\mu|_{(\rho, 1] \times P} = \mu_0 \ .$$

Properties (i) and (ii) of Theorem 2 are contained in the following proposition, setting $\omega' := d\mu_1$ on $(\rho, 1] \times P$ and $\omega' := \omega$ outside.

PROPOSITION 2. (i) *There exists a 1-form μ_1 on $(\rho, 1] \times P$ such that*

$$\mu_1 = \mu_0 \ \ \text{near} \ \{\rho\} \times P \cup (\rho, 1] \times \partial P \ ,$$
$$d\mu_1 \ \text{is a symplectic form} \ ,$$

$$\mu_1|_{\{s\}\times P} \text{ is a contact form for } s \in (\rho, 1) ,$$

and

$$\mu_1|_{\{1\}\times P} = \lambda_1 ,$$

where λ_1 is the confoliation form of Proposition 1.

(ii) There exists a diffeomorphism $\Psi : (\rho, 1) \times P \to (\rho, 1) \times P$ such that

$$\Psi = \text{id} \quad \text{near } \{\rho\} \times P \cup (\rho, 1) \times \partial P$$

and

$$\Psi^*(d\mu_1) = d\mu_0 .$$

Proof. (i) Let $\sigma : (\rho, 1] \to [0, 1]$ be a smooth function satisfying

$$(\sigma) \qquad \begin{aligned} \sigma &\equiv 0 \text{ near } \rho , \\ \sigma(1) &= 1 , \\ \sigma(s) &< 1 \text{ for } s < 1 \end{aligned}$$

(see Figure 5). Define $\Phi_0 : (\rho, 1] \times P \to \mathbf{R} \times P$,

$$\Phi_0(s, t, x, z, \theta) := (s, t, H^{\sigma(s)}(t, x, z), z, \theta) ,$$

where H satisfies (H1-6).

a)
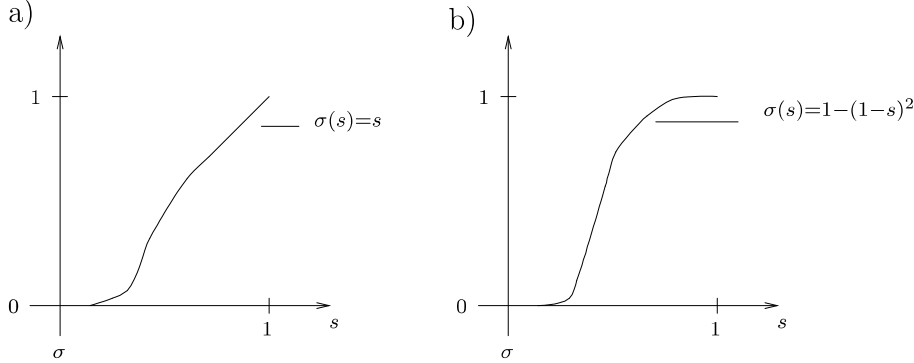
b)

Figure 5. The two choices of σ

We will need the following lemma:

LEMMA 2. (a) Φ_0 is a homeomorphism onto its image, and outside the circles $L^\pm := \{1\} \times \{\pm\frac{\delta}{2}\} \times \{0\}^{2k+1} \times S^1$ it is a diffeomorphism.

(b) For every neighborhood U of $L^- \cup L^+$ there exists a constant $c(U) > 0$ such that if $\Phi : (\rho, 1] \times P \to \mathbf{R} \times P$ satisfies

$$\|\Phi - \Phi_0\|_{C^1} \leq c(U) ,$$

then $D\Phi(X)$ is invertible for all $X \notin U$, and $\Phi(X_1) \neq \Phi(X_2)$ for all $X_1 \neq X_2$ with $X_1 \notin U$.

Proof. (a) First note that a map of the form
$$(X, Y) \mapsto (F(X, Y), Y)$$
is a homeomorphism respectively diffeomorphism if and only if for every fixed Y the map $X \mapsto F(X, Y)$ is a homeomorphism respectively diffeomorphism. So in this case we only have to show that for every fixed $Y = (s, t, z, \theta)$ the map
$$\Phi_0^Y : x \mapsto H^{\sigma(s)}(t, x, z)$$
is a homeomorphism onto its image and a diffeomorphism outside $L^\pm$. Now by (H2), the derivative of Φ_0^Y satisfies
$$\begin{aligned} D\Phi_0^Y(x) &= H_x^{\sigma(s)}(t, x, z) \\ &= (1 - \sigma(s)) + \sigma(s) H_x(t, x, z) \\ &> 0 \end{aligned}$$
for $(s, t, x, z) \neq (1, \pm\frac{\delta}{2}, 0, 0)$. This shows that Φ_0^Y is injective and a diffeomorphism outside $L^\pm$. Since $[-\epsilon, \epsilon]$ is compact, Φ_0^Y is also a homeomorphism onto its image.

(b) By part (a), there exists a constant $b > 0$ such that
$$\det D\Phi_0(X) \geq b \text{ for all } X \notin U \ .$$
Hence
$$\begin{aligned} \det D\Phi(X) &\geq b - \text{const} \|\Phi - \Phi_0\|_{C^1} \\ &> 0 \end{aligned}$$
if $X \notin U$, and $\|\Phi - \Phi_0\|_{C^1}$ is sufficiently small.

By part (a), there also exists a constant $c > 0$ such that
$$|\Phi_0(X_1) - \Phi_0(X_2)| \geq c|X_1 - X_2| \text{ for } X_1 \notin U \ .$$
Hence
$$\begin{aligned} |\Phi(X_1) - \Phi(X_2)| &\geq |\Phi_0(X_1) - \Phi_0(X_2)| - |(\Phi - \Phi_0)(X_1) - (\Phi - \Phi_0)(X_2)| \\ &\geq c|X_1 - X_2| - \|\Phi - \Phi_0\|_{C^1}|X_1 - X_2| \\ &> 0 \end{aligned}$$
if $X_1 \notin U$, and $\|\Phi - \Phi_0\|_{C^1}$ is sufficiently small. $\qquad \square$

Now we are going to modify Φ_0 in order to get a smooth embedding. Let $\Gamma : (\rho, 1] \times P \to \mathbf{R} \times P$,
$$\Gamma(s, t, x, z, \theta) := \left(s - \sigma(s) f(t, x, z), t, H^{\sigma(s)}(t, x, z) - g(s, t, x, z), z, \theta \right) \ ,$$
where H, f and σ satisfy (H1-6), (f1-5) and (σ) respectively, and $g : (\rho, 1] \times Q \to \mathbf{R}$ is a smooth function vanishing near $\{\rho\} \times Q \cup (\rho, 1] \times \partial Q$ with
$$|g| + |g_x| + |g_z| \leq \tfrac{1}{10}(1 - s) \ .$$

Here $Q = [-\delta, \delta] \times [-\epsilon, \epsilon] \times B_\gamma^{2k}$ as in the proof of Proposition 1. Let U be a neighborhood of $L^- \cup L^+$ on which H and f are of the forms (H6) and (f5), and assume that on U we have

$$\sigma(s) = s \ ,$$
$$g(s, t, x, z) = \pm b(1 - s) \ ,$$

with a constant $b > 16a$, where $a > 0$ is the constant in (f5) (see Figure 5a). Moreover, let $\|f\|_{C^1}$ and $\|g\|_{C^1}$ be small enough such that Lemma 2 can be applied to Γ.

LEMMA 3. Γ *is a smooth embedding.*

Proof. 1. In view of Lemma 2, we only have to show that the restriction $\Gamma|_U$ is an embedding. As noted in the proof of Lemma 2, this is equivalent to showing that for every $Y = (t, z, \theta)$ the map

$$\Gamma^Y : (s, x) \mapsto \left(s - \sigma(s)f(t, x, z), H^{\sigma(s)}(t, x, z) - g(s, t, x, z)\right)$$

is an embedding.

To show that Γ^Y is an immersion, let us compute the Jacobi determinant at a point $(s, t, x, z, \theta) \in U$:

$$\det D\Gamma^Y(s, x) = \det \begin{pmatrix} 1 - \sigma' f & -\sigma \, f_x \\ \sigma'(H - x) - g_s & \sigma \, H_x + (1 - \sigma) \end{pmatrix}$$

$$\geq \tfrac{1}{2}(\sigma H_x + (1 - \sigma)) - \sigma g_s f_x - \sigma |\sigma'| \, |f_x| \, |H - x|$$

if $|\sigma'| \, |f| \leq \tfrac{1}{2}$. Now on U we have $H_x \geq 3x^2$, $|H| \leq |x|$, $g_s f_x = -ab$ and $|f_x| = a$. Hence

$$\det D\Gamma^Y(s, x) \geq \tfrac{1}{2}(3\sigma x^2 + (1 - \sigma)) + ab\sigma - 2a\sigma|\sigma'| \, |x|$$
$$= \tfrac{1}{2}\sigma[3x^2 - 4a|\sigma'| \, |x|] + \tfrac{1}{2}(1 - \sigma) + ab\sigma$$
$$= \tfrac{1}{2}\sigma[x^2 + 2(|x| - a|\sigma'|)^2 - 2a^2|\sigma'|^2] + \tfrac{1}{2}(1 - \sigma) + ab\sigma$$
$$(1) \qquad \geq \tfrac{1}{2}\sigma x^2 + \tfrac{1}{2}(1 - \sigma) + ab\sigma - a^2|\sigma'|^2\sigma \ .$$

Inserting $\sigma(s) = s$ and dropping the first 2 terms in (1) we get

$$\det D\Gamma^Y(s, x) \geq abs - a^2 s$$
$$> 0$$

for $b > a$.

2. Next we will show that Γ^Y is injective. Therefore suppose that

$$\Gamma^Y(s, x) = \Gamma^Y(s', x')$$

for some $Y = (t, z, \theta)$ and some (s, x), (s', x') such that both (s, t, x, z, θ) and (s', t, x', z, θ) lie in U. By the definition of Γ^Y this is equivalent to the 2 equations (we omit the arguments (t, z, θ))

$$(s' - s) - \sigma(s')f(x') + \sigma(s)f(x) = 0 \ ,$$
$$\sigma(s')H(x') - \sigma(s)H(x) + (1 - \sigma(s'))x' - (1 - \sigma(s))x - g(s', x') + g(s, x) = 0 \ ,$$

or, after recollecting terms,

(2) $$(s' - s) - (\sigma(s') - \sigma(s))f(x) = \sigma(s')(f(x') - f(x)) \ ,$$

(3)
$$\sigma(s')(H(x') - H(x)) + (1 - \sigma(s'))(x' - x)$$
$$= (\sigma(s') - \sigma(s))(x - H(x)) + g(s', x') - g(s, x)$$

Inserting $\sigma = s$, $f = \pm ax$ and $g = \pm b(1 - s)$, equations (2) and (3) simplify to

(4) $$s' - s = \frac{\pm as'}{1 \mp ax}(x' - x) \ ,$$

(5) $s'(H(x') - H(x)) + (1 - s')(x' - x) + (s' - s)(H(x) - x \pm b) = 0$.

Without loss of generality suppose that $x' \geq x$. Using (H6) we can estimate

$$H(x') - H(x) \geq (x')^3 - x^3$$
$$= (x' - x)((x')^2 + x'x + x^2)$$
(6) $$\geq \tfrac{1}{2}(x' - x)((x')^2 + x^2) \ .$$

Inserting (4) in (5) and then using (6) and $|H| \leq |x|$ yields

$$0 = s'(H(x') - H(x)) + (1 - s')(x' - x) + \tfrac{\pm as'}{1 \mp ax}(x' - x)(H(x) - x \pm b)$$
$$\geq s'(H(x') - H(x)) + \tfrac{\pm as'}{1 \mp ax}(x' - x)(H(x) - x) + \tfrac{abs'}{1 \mp ax}(x' - x)$$
$$\geq s'(H(x') - H(x)) - 4as'|x|\,(x' - x) + \tfrac{1}{2}abs'(x' - x)$$
$$\geq \tfrac{1}{2}s'(x' - x)[x^2 - 8a|x| + ab]$$
$$= \tfrac{1}{2}s'(x' - x)[(|x| - 4a)^2 - 16a^2 + ab]$$
$$\geq \tfrac{1}{2}s'(x' - x)(ab - 16a^2)$$
$$> 0$$

if $x' > x$, since $b > 16a$. This contradition proves $x = x'$ and thus $s = s'$ by equation (4).

This proves that Γ^Y is an injective immersion. Since $\Gamma^Y = \mathrm{id}$ near $\{\rho\} \times [-\epsilon, \epsilon]$, Γ^Y is an embedding. $\qquad \square$

We continue in the proof of (i). Define

$$\mu_1 := \Gamma^* \mu_0 \ .$$

Near $\{\rho\} \times P \cup (\rho, 1] \times \partial P$ we have $\Gamma = \mathrm{id}$ and hence $\mu_1 = \mu_0$. The 2-form $d\mu_1 = \Gamma^* d\mu_0$ is symplectic because $d\mu_0$ is symplectic.

Replacing H^s by $H^{\sigma(s)} - g$ in the proof of Proposition 1 (iv) we obtain

$$\mu_1 \wedge (d\mu_1)^{k+1}\Big|_{\{s\} \times P} = (k+1)G \cdot vol \; ,$$

where the estimate (0) for the function G is modified to

$$G \geq \tfrac{1}{2}\sigma(H_x - |H| - |z|\,|H_z|) + \tfrac{1}{2}(1-\sigma)(1-|x|) - 2|g_x| - \tfrac{1}{2}(|g| + |z|\,|g_z|)$$
$$\geq \tfrac{1}{4}(1-\sigma) - 2(|g_x| + |g| + |g_z|) \; ,$$

where we have used (H2) again in the last step. By assumption, there exists a number $s_0 \in (\rho, 1)$ with $\sigma(s) = s$ for $s \geq s_0$. For $s \leq s_0$ we have $\sigma(s) < 1$, and therefore $G > 0$ if $\|g\|_{C^1}$ is sufficiently small. For $s_0 \leq s < 1$ the hypothesis on g yields

$$h \geq \tfrac{1}{4}(1-s) - \tfrac{2}{10}(1-s)$$
$$> 0 \; .$$

Hence $\mu_1|_{\{s\} \times P}$ is a contact form for $s < 1$. Finally,

$$\mu_1|_{\{1\} \times P} = (1-f)dt + H\,dx + \lambda_{2k}$$
$$= \lambda_1 \; ,$$

and (i) is proved.

(ii) Define $\Phi : (\rho, 1] \times P \to \mathbf{R} \times P$ by

$$\Phi(s, t, x, z, \theta) := \big(s - \sigma(s)f(t, x, z), t, H^{\sigma(s)}(t, x, z), z, \theta\big) \; ,$$

where H, f and σ satisfy (H1-6), (f1-5) and (σ) respectively, and on some neighborhood U of $L^- \cup L^+$ we have (H6), (f5) and

$$\sigma(s) = 1 - (1-s)^2$$

(see Figure 5b). Moreover, suppose that $\|f\|_{C^1}$ is small enough such that Lemma 2 can be applied to Φ.

LEMMA 4. Φ *is a homeomorphism onto its image, and outside the circles* $L^\pm$ *it is a diffeomorphism.*

Proof. 1. In view of Lemma 2, it suffices to prove the statement for the restriction $\Phi|_U$. By the same argument as in (i), it suffices to show that for every fixed $Y = (t, z, \theta)$ the map

$$\Phi^Y : (s, x) \mapsto (s - \sigma(s)f(t, x, z), H^{\sigma(s)}(t, x, z))$$

is a homeomorphism, and $\det D\Phi^Y(s, x) \neq 0$ for $(s, t, x, z) \neq (1, \pm\tfrac{\delta}{2}, 0, 0)$.

Let us first prove the last statement. Putting $b = 0$, $\sigma = 1 - (1 - s)^2$ and $\sigma' = 2(1 - s)$ in inequality (1) we get

$$
\begin{aligned}
\det D\Phi^Y(s, x) &\geq \tfrac{1}{2}\sigma x^2 + \tfrac{1}{2}(1 - \sigma) - a^2|\sigma'|^2\sigma \\
&\geq \tfrac{1}{2}[(1 - s)^2 - 8a^2(1 - s)^2\sigma] \\
&\geq \tfrac{1}{2}(1 - s)^2[1 - 8a^2] \\
&> 0
\end{aligned}
$$

if $s < 1$, and $a^2 < \frac{1}{8}$.

If $s = 1$ we insert $\sigma(1) = 1$ and $\sigma'(1) = 0$ in the first expression for the Jacobi determinant to get

$$
\begin{aligned}
\det D\Phi^Y(1, x) &= H_x(t, x, z) \\
&> 0
\end{aligned}
$$

for $(t, x, z) \neq (\pm\frac{\delta}{2}, 0, 0)$.

2. Next we will show that Φ^Y is injective. Therefore suppose that

$$
\Phi^Y(s, x) = \Phi^Y(s', x')
$$

for some $Y = (t, z, \theta)$ and some (s, x), (s', x') such that both (s, t, x, z, θ) and (s', t, x', z, θ) lie in U. This is equivalent to the equations (2) and (3), with $g = 0$ in (3). Since both terms on the left-hand side of equation (3) have the same sign, we deduce, using $|H(x)| \leq |x|$,

$$
(7) \qquad \sigma(s')|H(x') - H(x)| + (1 - \sigma(s'))|x' - x| \leq 2|x|\,|\sigma(s') - \sigma(s)| \ .
$$

From (2) we deduce, using $f = \pm ax$,

$$
\begin{aligned}
a|x' - x| &\geq \sigma(s')|f(x') - f(x)| \\
&= |(s' - s) - (\sigma(s') - \sigma(s))f(x)| \\
&\geq |s' - s| - \max|f|\max|\sigma'|\,|s' - s| \\
&\geq \tfrac{1}{2}|s' - s|
\end{aligned}
$$

if $\max|f|\max|\sigma'| \leq \frac{1}{2}$. Hence

$$
(8) \qquad\qquad\qquad |s' - s| \leq 2a|x' - x| \ .
$$

Without loss of generality assume $\sigma(s') \geq \sigma(s)$. From (8) we get

$$
|1 - s| \leq |1 - s'| + 2a|x' - x| \ .
$$

Taking squares on both sides leads to

$$
(1 - s)^2 \leq (1 - s')^2 + 4a|x' - x|\,|1 - s'| + 4a^2\,|x' - x|^2 \ ,
$$

and therefore
$$|\sigma(s') - \sigma(s)| \leq 4a|x' - x|\,|1 - s'| + 4a^2\,|x' - x|^2 \ .$$

Inserting this in (7) yields
$$\begin{aligned}
\sigma(s')|H(x') - H(x)| &\leq 2|x|\,|\sigma(s') - \sigma(s)| - (1 - \sigma(s'))|x' - x| \\
&\leq 2|x|\,(4a|x' - x|\,|1 - s'| + 4a^2\,|x' - x|^2) \\
&\quad - |1 - s'|^2|x' - x| \\
&= |x' - x|\,\big[-|1 - s'|^2 + 8a|x|\,|1 - s'| \\
&\quad + 8a^2\,|x|\,|x' - x|\big] \\
&= |x' - x|\,\big[-(|1 - s'| - 4a|x|)^2 + 16a^2\,|x|^2 \\
&\quad + 8a^2\,|x|\,|x' - x|\big] \\
&\leq |x' - x|\,a^2\,(16|x|^2 + 8|x|\,|x' - x|) \ .
\end{aligned}$$

For a sufficiently small this implies
$$(9) \qquad |H(x') - H(x)| \leq a|x' - x|(|x|^2 + |x|\,|x' - x|) \ .$$

On the other hand, from inequality (6) we get
$$\begin{aligned}
|H(x') - H(x)| &\geq \tfrac{1}{2}|x' - x|(|x'|^2 + x^2) \\
&\geq \tfrac{1}{4}|x' - x|(|x'| + |x|)^2 \\
&\geq \tfrac{1}{8}|x' - x|(|x|^2 + |x|\,|x' - x|) \ .
\end{aligned}$$

Choosing $a < \tfrac{1}{8}$, it follows from this inequality and (9) that
$$|x' - x|(|x|^2 + |x|\,|x' - x|) = 0 \ ,$$

hence $x = 0$ or $x = x'$. In the latter case, inequality (8) implies $s = s'$, and we are done. If $x = 0$, inequality (7) implies
$$\sigma(s')|H(x') - H(x)| + (1 - \sigma(s'))|x' - x| = 0 \ ,$$

which by (H2) is only possible if $x = x'$. Again it follows that $s = s'$, and the injectivity of Φ^Y is proved.

Since Φ^Y is continuous and injective, and $\Phi^Y = \mathrm{id}$ near $\{\rho\} \times [-\epsilon, \epsilon]$, it follows that Φ^Y is a homeomorphism onto its image. Moreover, by part 1, Φ^Y is a diffeomorphism outside $(s, t, x, z) = (1, \pm\tfrac{\delta}{2}, 0, 0)$, and the lemma is proved. $\qquad\square$

Since $\Gamma = \Phi = \mathrm{id}$ near $\{\rho\} \times P \cup (\rho, 1] \times \partial P$ and $\Gamma = \Phi$ on $\{1\} \times P$, we have
$$\Gamma((\rho, 1) \times P) = \Phi((\rho, 1) \times P) \ .$$

Hence we can define the diffeomorphism
$$\Phi' := \Gamma^{-1} \circ \Phi : (\rho, 1) \times P \to (\rho, 1) \times P$$
and the 1-form
$$\mu'_1 := (\Phi')^* \mu_1 = \Phi^* \mu_0 \ .$$
Clearly $\Phi' = \mathrm{id}$ near $\{\rho\} \times P \cup (\rho, 1) \times \partial P$. The restrictions
$$\begin{aligned} \lambda'_s &:= \mu'_1|_{\{s\} \times P} \\ &= (s - \sigma(s)f)\, dt + H^{\sigma(s)} d\theta + \lambda_{2k} \\ &= (s - 1)\, dt + \lambda_{\sigma(s)} \end{aligned}$$
are contact forms for all $s < 1$, and
$$d\lambda'_s = d\lambda_{\sigma(s)} \ .$$

Now we follow an argument from [EHof2]. The kernel of $d\lambda'_s$ is generated by the vector field $X^{\sigma(s)}$ defined in the proof of Proposition 1. For $s < 1$ we have $H_x^{\sigma(s)} > 0$, so we can normalise $X^{\sigma(s)}$ to
$$Y^s := \frac{1}{H_x^{\sigma(s)}} X^{\sigma(s)} \ .$$
For $t + u \le \delta$ let $\psi^s_u : (t, x, z, \theta) \to (t + u, x', z', \theta')$ be the flow of Y^s. Since
$$\begin{aligned} L_{Y^s} d\lambda'_s &= d(i_{Y^s} d\lambda'_s) \\ &= 0 \ , \end{aligned}$$
the flow ψ^s_u preserves $d\lambda'_s$. Define diffeomorphisms $\psi^s : P \to P$ by
$$^s(t, y) := \psi^s_{t+\delta}(-\delta, y) \ ,$$
where $y = (x, z, \theta)$.

CLAIM. $(\psi^s)^* d\lambda'_s = d\lambda_0$.

Indeed, ψ^s maps each $P_t := \{t\} \times [-\epsilon, \epsilon] \times B_\gamma^{2k} \times S^1$ into itself, and
$$^s|_{P_t} = \psi^s_{t+\delta} \circ \phi_{-t-\delta} \ ,$$
where ϕ_u is the flow generated by $\frac{\partial}{\partial t}$. It follows that
$$\begin{aligned} (\psi^s)^*(d\lambda'_s|_{P_t}) &= (\phi_{-t-\delta})^*(\psi^s_{t+\delta})^*(d\lambda'_s|_{P_t}) \\ &= (\phi_{-t-\delta})^*(d\lambda'_s|_{P_{-\delta}}) \\ &= (\phi_{-t-\delta})^*(d\lambda_0|_{P_{-\delta}}) \\ &= d\lambda_0|_{P_t} \ . \end{aligned}$$
In the direction of $\frac{\partial}{\partial t}$ we have
$$\begin{aligned} D\psi^s(t, y) \cdot \frac{\partial}{\partial t} &= \frac{\partial}{\partial t} \psi^s_{t+\delta}(-\delta, y) \\ &= Y^s(\psi^s_{t+\delta}(-\delta, y)) \\ &= Y^s(\psi^s(t, y)) \ , \end{aligned}$$

which implies

$$i_{\frac{\partial}{\partial t}}((\psi^s)^* d\lambda'_s) = (\psi^s)^*(i_{Y^s} d\lambda'_s)$$
$$= 0$$
$$= i_{\frac{\partial}{\partial t}}(d\lambda_0) \ .$$

Hence $(\psi^s)^* d\lambda'_s$ and $d\lambda_0$ agree on the whole tangent spaces, and the claim is proved.

Note that $\psi^s = \mathrm{id}$ near ∂P. This is trivial except for the boundary part $\{t = +\delta\}$, where it follows from property (iii) of Proposition 1 (see Figure 6).
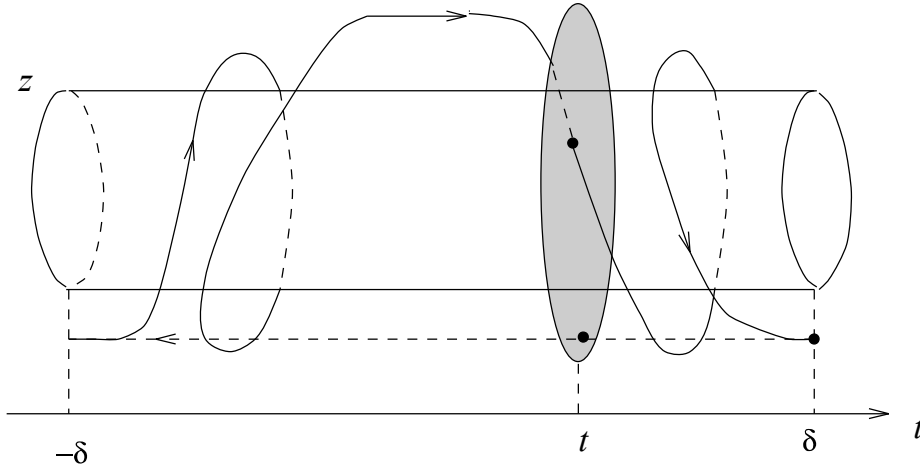
Figure 6. The definition of ψ^s

Define a diffeomorphism $\Psi' : (\rho, 1) \times P \to (\rho, 1) \times P$ by

$$\Psi'(s, p) := (s, \psi^s(p)) \ .$$

Ψ' leaves each $\{s\} \times P$ invariant, and $\Psi' = \mathrm{id}$ near $\{\rho\} \times P \cup (\rho, 1) \times \partial P$. It follows that

$$(\Psi')^* d\mu'_1 = d\mu_0 \quad \text{near} \quad \{\rho\} \times P \cup (\rho, 1) \times \partial P \ .$$

Moreover, by the claim we have just proved,

$$(\Psi')^* d\mu'_1|_{\{s\} \times P} = d\mu_0|_{\{s\} \times P}$$

for $s \in (\rho, 1)$.

The proof of the proposition is now finished by the following general lemma, applied to $N := P$, $\omega_0 := d\mu_0$ and $\omega_1 := (\Psi')^* d\mu'_1$.

LEMMA 4. *Let N be a compact connected manifold of dimension $2n-1$ with boundary such that $H^1(N, \partial N; \mathbf{R}) = 0$. Let ω_0 and ω_1 be symplectic forms on $(\rho, 1) \times N$ with*

$$\omega_0 = \omega_1 \quad \text{near} \quad \{\rho\} \times N \cup (\rho, 1) \times \partial N$$

and

$$\omega_0|_{\{s\} \times N} = \omega_1|_{\{s\} \times N}$$

for all $s \in (\rho, 1)$.

Then there exists a diffeomorphism $\Psi : (\rho, 1) \times N \to (\rho, 1) \times N$, leaving all $\{s\} \times N$ invariant, such that

$$\Psi = \text{id} \quad \text{near} \quad \{\rho\} \times N \cup (\rho, 1) \times \partial N$$

and

$$\Psi^* \omega_1 = \omega_0 \ .$$

Proof. Consider the closed 2-form

$$\hat{\omega} := \omega_1 - \omega_0 \ .$$

Since $\hat{\omega}|_{\{s\} \times N} = 0$ for all s, $\hat{\omega}$ can be written as

$$\hat{\omega} = ds \wedge \gamma$$

with some 1-form γ which is unique if we require $\gamma(\frac{\partial}{\partial s}) = 0$. As $\hat{\omega}$ vanishes near $\{\rho\} \times N \cup (\rho, 1) \times \partial N$, so does γ. From

$$ds \wedge d\gamma = -d\hat{\omega}$$
$$= 0$$

it follows that $\gamma|_{\{s\} \times N}$ is closed for all s. Since $H^1(N, \partial N; \mathbf{R}) = 0$, there exist functions f_s on $\{s\} \times N$ such that

$$\gamma|_{\{s\} \times N} = df_s \ .$$

Moreover, since γ vanishes on $\{s\} \times \partial N$, we can choose $f_s = 0$ near $\{s\} \times \partial N$. Then the f_s are unique and fit together to form a smooth function F on $(\rho, 1) \times N$, vanishing near $\{\rho\} \times N \cup (\rho, 1) \times \partial N$, with

$$\gamma|_{\{s\} \times N} = dF|_{\{s\} \times N}$$

for all s. We conclude that

$$\gamma - dF = G \, ds$$

for some function G, and therefore

$$(*) \qquad\qquad\qquad \hat{\omega} = ds \wedge dF \ .$$

The argument is finished by Moser's trick (see [Mo]). For $0 \leq \tau \leq 1$ define
$$\omega_\tau := \omega_0 + \tau \, ds \wedge dF$$
and compute
$$\omega_\tau^n = \omega_0^n + \tau \, n \, \omega_0^{n-1} \wedge ds \wedge dF \ .$$
Thus $(\omega_\tau^n)_{0 \leq \tau \leq 1}$ provides a linear path from ω_0^n to ω_1^n. Now ω_0^n and ω_1^n are both volume forms, and they define the same orientation because they agree near $\{\rho\} \times N \cup (\rho, 1) \times \partial N$. It follows that all ω_τ are symplectic.

Define a τ-dependent vector field Z_τ by
$$i_{Z_\tau} \omega_\tau = F \, ds \ ,$$
and denote by $\phi_\tau : (\rho, 1) \times N \to (\rho, 1) \times N$ its flow. Z_τ leaves all $\{s\} \times N$ invariant, and $Z_\tau = 0$ near $\{\rho\} \times N \cup (\rho, 1) \times \partial N$. Therefore ϕ_τ exists for all $\tau \in [0, 1]$ (here we use the compactness of N), leaves $\{s\} \times N$ invariant, and $\phi_\tau = \mathrm{id}$ near $\{\rho\} \times N \cup (\rho, 1) \times \partial N$. The simple but famous calculation (using Cartan's formula for the Lie derivative)
$$\begin{aligned}
\tfrac{d}{d\tau}(\phi_\tau^* \omega_\tau) &= \phi_\tau^*\big(L_{Z_\tau} \omega_\tau + \tfrac{d}{d\tau}\omega_\tau\big) \\
&= \phi_\tau^*\big(d(i_{Z_\tau}\omega_\tau) + i_{Z_\tau} d\omega_\tau + ds \wedge dF\big) \\
&= \phi_\tau^* \, d(i_{Z_\tau}\omega_\tau - F \, ds) \\
&= 0
\end{aligned}$$
yields $\phi_1^* \omega_1 = \omega_0$. This concludes the proof of Lemma 4 and of Proposition 2. $\qquad\square$

It remains to show properties (iii) and (iv) of Theorem 1.

(iii) If $\omega = d\mu$ is exact, we may take the 1-form on $(\rho, 1] \times \partial M$ at the beginning of the proof of Theorem 2 to be the given form μ. By Lemma A5 of the appendix we may change μ in a neighborhood of L in $(\rho, 1] \times \partial M$ to achieve
$$\mu|_{(\rho', 1] \times P} = \mu_0$$
for some $\rho' \in (\rho, 1)$. Proceeding as before we find a 1-form μ' on M agreeing with μ outside $(\rho, 1] \times \partial M$, and a diffeomorphism $\psi : \overset{\circ}{M} \to \overset{\circ}{M}$ such that
$$\psi^*(d\mu') = d\mu \ .$$
So $\psi^* \mu' - \mu$ is closed and vanishes on $\overset{\circ}{M} \setminus (\rho, 1) \times \partial M$. Since $H^1(\overset{\circ}{M}, \overset{\circ}{M} \setminus (\rho, 1) \times \partial M; \mathbf{R}) = 0$, $\psi^* \mu' - \mu$ is exact.

(iv) If (M, ω) is symplectically embedded in a symplectic manifold $(\hat{M}, \hat{\omega})$ without boundary, we find a 1-form $\hat{\mu}$ on a tubular neighborhood

$(\rho, 2 - \rho) \times \partial M$ of ∂M in $\hat{M}$ such that

$$d\hat{\mu} = \hat{\omega}|_{(\rho, 2-\rho) \times \partial M} \ ,$$

$$\hat{\mu}|_{\{1\} \times \partial M} = \lambda \ ,$$

and $\hat{\mu}|_{\{s\} \times \partial M}$ is a contact form for all $s \in (\rho, 2 - \rho)$. By Lemma A5 of the appendix we may assume that

$$\hat{\mu}|_{(\rho, 2-\rho) \times P} = \mu_0 \ .$$

In the proof of Proposition 1 we have defined an embedding $\Gamma : (\rho, 1] \times P \hookrightarrow \mathbf{R} \times P$ such that

$$\mu_1 = \Gamma^* \mu_0 \ .$$

By choosing H and f sufficiently small we can achieve that the image of Γ is contained in $(\rho, 1 - \rho) \times P \subset \hat{M}$, and we can extend Γ by the identity to a symplectic embedding

$$(M, \omega') \hookrightarrow (\hat{M}, \hat{\omega}) \ .$$

This finishes the proof of Theorem 2. $\qquad\qquad\square$

3   Applications I

In this section we prove Corollaries A,D,E,G,I and J.

Proof of Corollary A. Apply the construction in Theorems 1 and 2 with one tangent loop $L : S^1 \to \partial M$ which is contractible in ∂M. Property (iv) of Theorem 1 together with $i_{\dot{L}^+} d\lambda' = 0$ implies $\lambda'(\dot{L}^+) \equiv 0$ and therefore

$$\int_{L^+} \lambda' = 0 \ .$$

Arguing by contradiction, suppose that the boundary of the symplectic manifold (M, ω') obtained by means of Theorem 2 is of contact type with a contact form λ''. From $d\lambda' = d\lambda'' = \omega'|_{\partial M}$ and the contractibility of L^+ in ∂M we then get, using Stokes' theorem,

$$\int_{L^+} \lambda'' = \int_D \omega' = \int_{L^+} \lambda' = 0 \ .$$

Here D is a smooth disk in ∂M bounded by L^+. But in view of $i_{\dot{L}^+} d\lambda'' = 0$, the contact condition $\lambda'' \wedge (d\lambda'')^{n-1} > 0$ implies that $\lambda''(\dot{L}^+) > 0$, hence

$$\int_{L^+} \lambda'' > 0 \ ,$$

and we have a contradiction. $\qquad\qquad\square$

Proof of Corollary D. Since there are only finitely many closed characteristics on $(\partial M, d\mu)$, we can choose by Lemma 1 a tangent loop L intersecting all of them. Applying the construction we destroy the closed characteristics. The newly created closed characteristics of $(\partial M, d\mu')$ all have action zero by (iv) of Theorem 1. By (iv) of Theorem 2 the interiors are exact symplectomorphic. □

Proof of Corollary E. We have already shown in the proof of Theorem 1 that each time a closed orbit of $\mathcal{L}_\mu$ intersects one of the L_i it splits up into two heteroclinic orbits. It remains to verify that the sum of the actions of the heteroclinic orbits equals the action of the original orbit.

Consider the forms λ_0, λ_1 on the neighborhood P of one tangent loop L_i as in Proposition 1. The orbits of $\ker(d\lambda_0)$ in P intersecting $L_i = \{0\}^{2k+2} \times S^1$ are

$$y = [-\delta, \delta] \times \{0\}^{2k+1} \times \{\theta\}$$

with fixed $\theta \in S^1$. Their action is

$$\int_y \lambda_0 = 2\delta .$$

As shown in Proposition 1, each such orbit splits into 3 orbits y_-, y_0, y_+ of $\ker(d\lambda_1)$ lying on the invariant cylinder Z (see Figure 3). Hence the result follows from the following lemma.

LEMMA 5. *In the above notation,*

$$\int_{y_-} \lambda_1 + \int_{y_0} \lambda_1 + \int_{y_+} \lambda_1 = 2\delta .$$

Proof. The restriction of λ_1 to the cylinder Z is

$$\lambda_1|_Z = (1 - f)dt .$$

Therefore the orbit

$$y_-(s) = (t(s), 0, 0, \theta(s)), \;\; s \in (-\infty, 0]$$

of X^1 has the action

$$\begin{aligned}
\int_{y_-} \lambda_1 &= \int_{-\infty}^0 (y_-)^* \lambda_1 \\
&= \int_{-\infty}^0 (1 - f(t(s), 0, 0))) \tfrac{dt}{ds} ds \\
&= \int_{-\delta}^{-\frac{\delta}{2}} (1 - f(t, 0, 0)) dt .
\end{aligned}$$

Analogous formulae hold for y_0 and y_+. Adding up we obtain

$$\int_{y_-} \lambda_1 + \int_{y_0} \lambda_1 + \int_{y_=} \lambda_1 = \int_{-\delta}^{\delta} (1 - f(t,0,0))dt$$
$$= 2\delta \ ,$$

because f is odd in t. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Corollary G. Let $n = 2$ and $P = [-\delta,\delta] \times [-\epsilon,\epsilon] \times S^1$, Z, λ_1 and X^1 be as in the proof of Proposition 1. For $n = 2$ the vector field X^1 simplifies to

$$X^1 = f_x \frac{\partial}{\partial \theta} + H_x \frac{\partial}{\partial t} - H_t \frac{\partial}{\partial x} \ .$$

The proof of Corollary G is based on the following lemma about the flow of X^1 near $L^\pm$.

LEMMA 6. *Let $F \subset P$ be a continuously embedded closed surface invariant under the flow of X^1. If F contains one point of Z, then it contains the whole cylinder Z.*

Proof. Under the flow of X^1 the point in $F \cap Z$ tends in forward or backward time to L^- or L^+, say L^+. By invariance and compactness of F this implies

$$L^+ \subset F \ .$$

Take a small closed disk

$$D' \subset [-\delta,\delta] \times [-\epsilon,\epsilon] \times \{\theta_0\}$$

around $p_0 := (\frac{\delta}{2},0,\theta_0) \in L^+$ which is transversal to X^1. Let $D \subset \overset{\mathrm{o}}{D'}$ be a smaller disk around p_0 such that the Poincaré return maps $\phi, \phi^{-1} : D \to D'$ of the flows of X^1 respectively $-X^1$ are defined (see Figure 7). Note that $\phi(p_0) = \phi^{-1}(p_0) = p_0$.

CLAIM 1. $F \cap \overset{\mathrm{o}}{D'}$ *is locally path-connected.*

Proof. Let U be an open neighborhood of $\overset{\mathrm{o}}{D'}$ in P on which the projection

$$\pi : U \to \overset{\mathrm{o}}{D'}$$

along flow lines of X^1 is defined. Around a given point $p \in F \cap \overset{\mathrm{o}}{D'}$ choose an open neighborhood W in U such that $V := W \cap F$ is path connected. Since F is invariant, $\pi(V) = \pi(W) \cap F$ is an open neighborhood of p in $F \cap \overset{\mathrm{o}}{D'}$.

Any point q in $\pi(V)$ can be connected to p by a path in U, first following the flow line from q to a point in V and then connecting this point to p
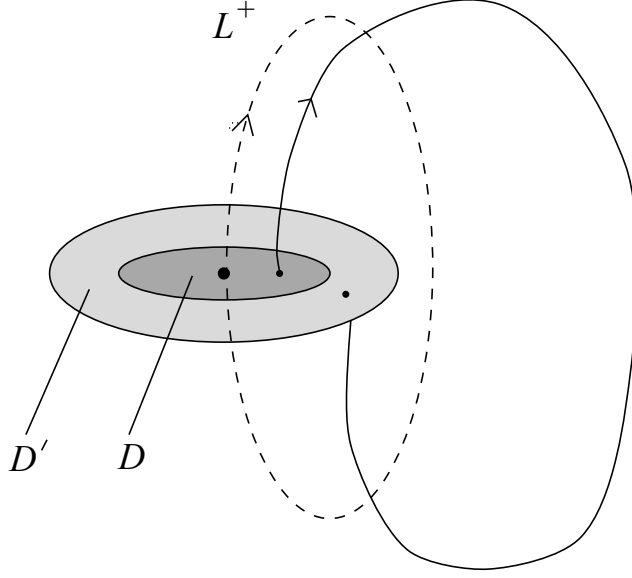
Figure 7. The Poincaré return map on D

in V. The projection of this path onto $\overset{\circ}{D}'$ joins q to p in $\pi(V)$. Hence $\pi(V)$ is path-connected, and the claim follows.

Consider the space

$$T := ([-1, 1] \times \{0\}) \cup (\{0\} \times [-1, 0]) \subset \mathbf{R}^2$$

which has the shape of the letter T, with the topology induced from $\mathbf{R}^2$. We say that a topological space X *contains a T* if there exists a continuous embedding of T into X.

CLAIM 2. $F \cap \overset{\circ}{D}'$ does not contain a T.

Proof. Suppose that we have an embedding $f : T \hookrightarrow F \cap \overset{\circ}{D}'$. Using the flow of X^1 we can extend it to an embedding

$$\bar{f} : T \times (-\rho, \rho) \hookrightarrow F \ ,$$

for some small $\rho > 0$. By invariance of the domain the image

$$\bar{f}((-1, 1) \times \{0\} \times (-\rho, \rho))$$

is open in F. In particular it contains a neighborhood of $f(0, 0)$ in F. But then $\bar{f}$ cannot be injective, and we have a contradiction.

Now let K be the connected component of p_0 in $F \cap \overset{\mathrm{o}}{D}$. K is locally compact, connected and locally path connected by Claim 1. Now each two points in a locally compact, connected and locally connected metric space can be joined by an arc, where an arc is the homeomorphic image of a closed interval ([HoY], Theorem 3.15 and the remark following it). So K has the property that any two points can be joined by an arc.

CLAIM 3. $K \subset [-\delta, \delta] \times \{0\} \times \{\theta_0\} \subset P$.

Proof. Arguing by contradiction, suppose that there exists a point $p_1 \in K$ whose x-component is nonzero. The idea why this should lead to a contradiction is the following: Connect p_1 to p_0 by an arc c_1 in K, and consider the arcs $\phi(c_1)$ and $\phi^{-1}(c_1)$. Since p_0 is a fixed point, $\phi(c_1)$ and $\phi^{-1}(c_1)$ both emanate from p_0. On the other hand, the flow of X^1, and hence also the return map ϕ, strictly increases the t-component outside the set $\{x = 0\}$. Thus the three arcs c_1, $\phi(c_1)$ and $\phi^{-1}(c_1)$ are pairwise different, and we have a situation as shown in Figure 8a. But then the three arcs form a T at p_0, contradicting Claim 2.
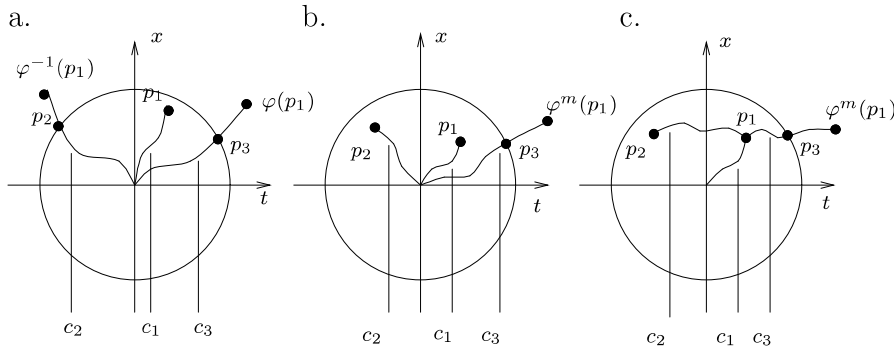
Figure 8.

Now we will make this argument precise. Since the t-component strictly increases along the X^1-orbit of p_1, the iterated images of p_1 under ϕ and ϕ^{-1} eventually leave D. In particular, after replacing p_1 by some iterated image, we may assume that $\phi(p_1) \notin K$.

Let c_1 be an arc in K joining p_0 to p_1. Let $m \geq 0$ be the maximal integer such that $\phi^{-m}(p_1) \in K$.

Case A: $m = 0$. Let c_2 be the maximal subarc of $\phi^{-1}(c_1)$ starting from p_0 and contained in D, and let p_2 be its end point on ∂D. Define c_3 and p_3 in the analogous way for $\phi(c_1)$ (see Figure 8a).

Let q_2 be the first point in which c_2, starting from p_2, hits c_1. The point q_2 cannot be an interior point of c_1 because then K would contain a T. The point q_2 cannot be p_1 because then $\phi(p_1) \in c_1 \subset K$, contradicting the choice of p_1. Hence $q_2 = p_0$.

Let q_3 be the first point in which c_3, starting from p_3, hits the arc $c_1 \cup c_2$. By the previous argument, q_3 cannot be an interior point of c_1 or c_2. It cannot be p_1 because then $\phi^{-1}(p_1) \in c_1 \subset K$. If it were p_2, then $\phi(p_2)$ and $\phi^{-1}(p_2)$ would both lie in $K \subset D$. But since ϕ strictly increases the t-component outside p_0, we can choose the disk D in such a way that every boundary point on ∂D leaves D under either ϕ or ϕ^{-1}. (This is achieved by choosing D such that ∂D has only 2 isolated tangencies with the level lines of H shown in Figure 4). Hence $q_3 = p_0$, and again we find a T in K. This shows that Case A leads to a contradiction.

Next suppose $m > 0$. If $p_2 := \phi^{-m}(p_1)$ lies on c_1, then we shorten the arc c_1, replacing p_1 by $p_1' := p_2$, p_2 by $p_2' := p_1$, ϕ by ϕ^{-1} and c_1 by the subarc c_1' of c_1 from p_0 to p_1'. This yields an equivalent situation, but with p_2' not lying on c_1'. So without loss of generality we may assume that p_2 does not lie on c_1. Let c_2 be an arc in K joining p_2 to p_1. Let q_2 be the first point in which c_2 hits c_1, and replace c_2 by the arc from p_2 to q_2. Arguing as in Case A, q_2 cannot be an interior point of c_1.

Case B: $q_2 = p_0$. Let c_3 be the maximal subarc of $\phi^m(c_1)$ contained in D, and let p_3 be its end point on ∂D (see Figure 8b). Let q_3 be the first point, starting from p_3, in which c_3 hits the arc $c_1 \cup c_2$. In view of Claim 2, q_3 can only be one of the end points p_1, p_2 of $c_1 \cup c_2$. It cannot be p_1, since then $\phi^{-m}(p_1) = p_2$ would lie on c_1, which we have excluded above. Neither can q_3 be p_2 because then $\phi^{-m}(p_2)$ would belong to $c_1 \subset K$, contradicting the choice of m for p_2. Hence $q_3 = p_0$, and again K contains a T.

Case C: $q_2 = p_1$. Let c_3 be the maximal subarc of $\phi^m(c_2)$ starting from p_1 and contained in D, and let p_3 be its end point on ∂D (see Figure 8c). Let q_3 be the first point in which c_3, starting from p_3, hits $c_1 \cup c_2$. Since p_0 is a fixed point of ϕ and is not contained in c_2, p_0 is not contained in $\phi^m(c_2)$ either, so q_3 cannot be p_0. The point q_3 is not p_2 because then $\phi^{-m}(p_2)$ would lie in $c_2 \subset K$, contradicting the choice of m for p_2. Hence $q_3 = p_1$, and once again we have found a T in K.

In all cases we have arrived at a contradiction, so Claim 3 is proved.

The set K does not consist only of the point p_0 because then the loop L^+ would be an open subset of the surface F, which is impossible by invariance

of the domain. Thus K must contain an interval in $[-\delta, \delta] \times \{0\} \times \{\theta_0\}$ containing p_0, say $[\frac{\delta}{2}, \frac{\delta}{2} + \rho] \times \{0\} \times \{\theta_0\}$. By invariance, F contains the cylinder

$$\left[\tfrac{\delta}{2}, \delta\right] \times \{0\} \times S^1 \ .$$

Since F has no boundary, there must be other points of F in any neighborhood of L^+. Arguing as in Claim 3, we conclude that these points can only lie on Z. It follows that F contains $\left(-\frac{\delta}{2}, \delta\right] \times \{0\} \times S^1$. By compactness of F we get $L^- \subset F$, and the same argument as above applied to L^- yields $Z \subset F$. $\qquad \square$

We proceed to prove Corollary G.

In complex coordinates $z_j = q_j + ip_j$ on $\mathbf{R}^4 = \mathbf{C}^2$, $|z_1|$ and $|z_2|$ are integrals of the characteristic flow on $\partial E(r)$. Thus the invariant tori are

$$T_{a_1, a_2} := \left\{(z_1, z_2) \mid |z_1| = a_1, |z_2| = a_2\right\}$$

with $a_1, a_2 \geq 0$ and $\frac{a_1^2}{r_1} + \frac{a_2^2}{r_2} = 1$. $(a_1, a_2) = (0, \sqrt{r_2})$ and $(\sqrt{r_1}, 0)$ correspond to the two closed characteristics. Perturb the loop $S^1 \to \partial E(r)$,

$$t \mapsto (\sqrt{r_1} \cos t, \sqrt{r_2} \sin t) , \qquad t \in [0, 2\pi]$$

to a loop

$$L : t \mapsto (z_1(t), z_2(t))$$

in $\partial E(r)$ tangent to $\ker(\lambda_4|_{\partial E(r)})$, keeping it fixed at $t = 0$ and $t = \frac{\pi}{2}$ (see Lemma 1). It follows that

$$t \mapsto (|z_1(t)|, |z_2(t)|) , \qquad t \in \left[0, \tfrac{\pi}{2}\right]$$

covers the ellipse segment

$$\left\{(a_1, a_2) \mid a_1, a_2 \geq 0 , \ \tfrac{a_1^2}{r_1} + \tfrac{a_2^2}{r_2} = 1\right\} \ .$$

So L intersects not only the two closed characteristics, but also all invariant tori. Applying the construction in Theorems 1 and 2 to a neighborhood P of L as before we obtain a compact subset $M \subset \mathbf{R}^4$ such that on $(\partial M, \omega_4)$ the two old closed characteristics have disappeared and two new ones $L^{\pm}$ been created.

Arguing by contradiction, suppose that $F \subset \partial M$ is a continuously embedded invariant torus for the new characteristic foliation.

Claim. $F \cap Z \neq \emptyset$.

Since the maximal invariant set in P is the cylinder $\left[-\frac{\delta}{2}, \frac{\delta}{2}\right] \times \{0\} \times S^1$ between L^- and L^+, F cannot lie entirely in P. Recall that by construction

$$\partial M \setminus P = \partial E(r) \setminus P \ ,$$

and thus the characteristic foliations are equal in this region. Let $p \in F \setminus P$ be a point which does not belong to one of the two closed characteristics of the old foliation. Denote by $\mathcal{O}_{old}^+(p)$, $\mathcal{O}_{new}^+(p)$ its forward orbits in the old respectively in the new foliation.

If $\mathcal{O}_{new}^+(p)$ tends to $L^+ \cup L^-$ we are done. So suppose this is not the case. Then, by property (iii) of Proposition 1,

$$\mathcal{O}_{new}^+(p) \setminus P = \mathcal{O}_{old}^+(p) \setminus P .$$

Now remember that p lies on some invariant torus T of the old foliation, and $\mathcal{O}_{old}^+(p)$ is dense in T because $\frac{r_1}{r_2}$ is irrational. It follows from the discussion above that $\mathcal{O}_{new}^+(p) \setminus P$ is dense in $T \setminus P$, and invariance and compactness of F imply

$$T \setminus P \subset F \setminus P .$$

Since the tangent loop L intersects all invariant tori, in particular T, there exists an orbit on T which breaks up into a heteroclinic chain to $L^- \cup L^+$ in the new foliation. This means that there exists a point $q \in T \setminus P$ such that $\mathcal{O}_{new}^+(q)$ tends to L^- and therefore lies eventually on Z. By invariance of F, the claim follows.

The claim and Lemma 6 yield

$$Z \subset F .$$

Now let T' be any invariant torus of the old foliation. As above, since L intersects T', there exists a point $q' \in T' \setminus P$ such that $\mathcal{O}_{new}^+(q')$ tends to L^-. So $\mathcal{O}_{new}^+(q')$ meets Z and hence F, and from invariance of F we conclude $q' \in F$. The same argument as above yields

$$T' \setminus P \subset F \setminus P ,$$

for every invariant torus T' of the old foliation. But the invariant tori completely foliate the complement of the two closed characteristics in $\partial E(r)$, so we get

$$\partial M \setminus P \subset F .$$

This in turn implies, in view of the invariance of F,

$$F = \partial M ,$$

which is absurd. Hence F cannot exist, and Corollary G is proved. $\qquad \square$

Proof of Corollary I. Let $E(r) \subset \mathbf{R}^4$ be an ellipsoid with $\frac{r_1}{r_2}$ irrational. Apply the construction to destroy the two closed characteristics, creating two new ones of knot types $\pm c$. The new characteristic foliation is defined

by a confoliation form λ on $\partial E(r) = S^3$. Let X be a positive section of the oriented line bundle $\ker(d\lambda) \to S^3$. Then the condition

$$\lambda \wedge d\lambda \geq 0$$

implies that $\lambda(X) \geq 0$. Choose a 1-form μ on S^3 such that $\mu(X) \equiv 1$, and define the volume form $\Omega := \mu \wedge d\lambda$. Then

$$i_X \Omega = \mu(X) \, d\lambda = d\lambda \ .$$

The proof of the last statement is based on the following local statement on P, where we use again the terminology of the proof of Proposition 1.

LEMMA 7. *There exists a contact form λ_c on P, $\lambda_c = \lambda_0$ near ∂P, such that*

$$\lambda_c(X^1) > 0 \ .$$

Moreover, λ_c can be connected to λ_0 by a path of contact forms fixed near ∂P.

Proof. Define

$$\lambda_c := \lambda_1 + g(t, x, z) d\theta \ ,$$

where $g : Q := [-\delta, \delta] \times [-\epsilon, \epsilon] \times B_\gamma^{2k} \to \mathbf{R}$ is a smooth function satisfying

(g1) $g = 0$ near ∂Q;

(g2) $g \cdot f_x \geq 0$, and $g \cdot f_x > 0$ near $\left(\pm \frac{\delta}{2}, 0, 0 \right)$;

(g3) $|g_x| + |z| \, |g_z| < H_x - |H| - |z| \, |H_z|$ for all $(t, x, z) \neq (\pm \frac{\delta}{2}, 0, 0)$.

(g1) and (g2) can be easily achieved if we choose g to be supported in the region near $\left(\pm \frac{\delta}{2}, 0, 0 \right)$ where $f_x \neq 0$. To satisfy (g3) it suffices in view of (H2) to choose g constant near $\left(\pm \frac{\delta}{2}, 0, 0 \right)$ and sufficiently small.

In

$$\lambda_c(X^1) = \lambda_1(X^1) + g \, f_x$$

both terms are greater or equal to 0. Outside $\left(\pm \frac{\delta}{2}, 0, 0 \right)$, λ_1 is contact and therefore $\lambda_1(X^1) > 0$, and at $\left(\pm \frac{\delta}{2}, 0, 0 \right)$ we have $g \, f_x > 0$. This shows that $\lambda_c(X^1) > 0$.

To prove that λ_c is a contact form, repeat the calculation in the proof of Proposition 1 with H replaced by $H + g$, but keeping the term $g \, f_x$ with the correct sign:

$$\begin{aligned}
\lambda_c \wedge (d\lambda_c)^{k+1} &\geq \big[(1 - |f| - k|f_z|)(H_x + g_x) \\
&\quad - k \, |f_x| \, |z| \, (|H_z| + |g_z|) + f_x(H + g) \big] \cdot vol \\
&\geq \big[\tfrac{1}{2}(H_x - |H| - |z| \, |H_z| - |g_x| - |z| \, |g_z|) + g \, f_x \big] \cdot vol \\
&> 0
\end{aligned}$$

by (g2) and (g3).

For the last statement observe that (g1) and (g3) are still satisfied if we replace H by H^s and f by sf for $s < 1$. Hence we can connect λ_c to $\lambda_0 + g\,d\theta$ by the path of contact forms

$$(\lambda_s + g\,d\theta)_{0 \leq s \leq 1} \ .$$

Then connecting $\lambda_0 + g\,d\theta$ to λ_0 by the path

$$(\lambda_0 + s\,g\,d\theta)_{0 \leq s \leq 1}$$

concludes the proof. □

Now the last statement of Corollary I follows easily: Extend the contact form λ_c by λ_0 to a contact form on S^3 which we will still denote by λ_c. By the last statement of Lemma 7, λ_c can be connected to the standard contact form λ_0 on S^3 by a path of contact forms. Hence by Gray's stability theorem there exists a diffeomorphism ϕ of S^3, isotopic to the identity, such that

$$\phi^* \lambda_0 = h\,\lambda_c$$

for some positive function h on S^3. From Lemma 7 we get

$$\lambda_0(\phi_* X) = h\,\lambda_c(X) > 0 \ .$$

Replacing X by $\phi_* X$ finishes the proof of Corollary I. □

Proof of Corollary J. Apply the construction to an ellipsoid $E(r) \subset \mathbf{R}^{2n}$ with $\frac{r_i}{r_j} \neq \mathbf{Q}$ for $i \neq j$, destroying all (if k is even) respectively all but one (if k is odd) closed characteristics, creating k respectively $k-1$ new ones. □

4   The Generalized Construction

The proof of Theorems 1 and 2 shows that we actually need the contact form λ only in a neighborhood of the tangent loop L. So we can modify the construction as follows:

Let (N, ω) be a manifold of dimension $2n-1 \geq 3$ with a maximally non-degenerate 2-form, and let $L \subset N$ be an embedded loop which is nowhere tangent to $\ker(\omega)$.

By Lemma A2 of the appendix there exists a neighborhood of L diffeomorphic to P such that

$$\omega|_P = d\lambda_0 \ ,$$

and L corresponds to $\{0\}^{2n-2} \times S^1$, where P and λ_0 are as in Proposition 1. Hence Proposition 1 yields

Theorem 1′. *Let (N, ω) and $L \subset N$ be as above. Then there exists a maximally nondegenerate 2-form ω' on N, agreeing with ω outside a neighborhood of L, with the following properties:*

(i) *The characteristic foliation generated by $\ker(\omega')$ has 2 new closed orbits $L^{\pm}$ isotopic to $\pm L$.*

(ii) *Every closed characteristic of $\ker(\omega)$ intersecting L breaks up into two characteristics of $\ker(\omega')$ heteroclinic to $L^{\pm}$.*

(iii) *Apart from (i) and (ii) no closed characteristics are created nor destroyed.* $\square$

Next let (M, ω) be a symplectic manifold of dimension $2n \geq 4$, and $L \subset \partial M$ an embedded loop which is nowhere tangent to $\ker(\omega|_{\partial M})$.

By Lemma A4 of the appendix there exists a foliated neighborhood $(\rho, 1] \times P$ of $L = \{1\} \times \{0\}^{2n-2} \times S^1$ in M such that

$$\omega|_{(\rho, 1] \times P} = d\mu_0 \ ,$$

where μ_0 is the 1-form as in Proposition 2. So Proposition 2 implies

Theorem 2′. *Let (M, ω) and $L \subset \partial M$ be as above. Then there exists a symplectic form ω' on M, agreeing with ω outside a neighborhood of L in M, with the following properties:*

(i) *$\omega'|_{\partial M}$ is as in Theorem 1′.*

(ii) *The interiors $(\overset{\circ}{M}, \omega)$ and $(\overset{\circ}{M}, \omega')$ are symplectomorphic.*

(iii) *If (M, ω) is symplectically embedded in some symplectic manifold $(\hat{M}, \hat{\omega})$ without boundary, then we can choose ω' such that (M, ω') is also symplectically embedded in $(\hat{M}, \hat{\omega})$.* $\square$

5   Applications II

In this section we prove Corollaries B,C,F and H.

Proof of Corollary B. Let $(\partial M, \lambda)$ be the contact type boundary of the symplectic manifold (M, ω). Take an embedded loop $L \subset \partial M$ which is nowhere tangent to $\ker(\omega|_{\partial M})$ and contractible in ∂M, and such that

$$\int_L \lambda \neq 0 \ .$$

More explicitly, take a Darboux chart on ∂M in which λ looks like $dz + \frac{1}{2} \sum_{i=1}^{n-1} (q_i dp_i - p_i dq_i)$ on $\mathbf{R}^{2n-1}$, and let L be a small circle in the (q_1, p_1)-plane.

Let $(\rho, 1] \times P$ be a neighborhood of L as above on which $\omega = d\mu_0$. Apply the generalized construction in Theorems $1'$ and $2'$ to obtain a symplectic form ω' on M such that $(\overset{\text{o}}{M}, \omega')$ is symplectomorphic to $(\overset{\text{o}}{M}, \omega)$, $\omega' = \omega$ outside $(\rho, 1] \times P$, and

$$\omega'|_{\{1\} \times P} = d\lambda_1$$

where P and λ_1 are as in Proposition 1.

We claim that $(\partial M, \omega')$ is not of confoliation type. To see this, suppose that τ is a 1-form on ∂M with

$$d\tau = \omega'|_{\partial M} .$$

By Proposition 1 the line bundles $\ker(\omega|_P)$ and $\ker(\omega'|_P)$ are both tangent to the cylinder

$$Z = [-\delta, \delta] \times \{0\}^{2n-3} \times S^1 ,$$

thus

$$\omega|_Z = \omega'|_Z = 0 .$$

It follows from Stokes' theorem that

$$\int_{L^-} \tau = \int_{L^+} \tau = \int_K \tau$$

and

$$\int_K \lambda = \int_L \lambda \neq 0 ,$$

where $K := \{+\delta\} \times \{0\}^{2n-3} \times S^1$, and all loops are given the orientation $+\frac{\partial}{\partial\theta}$. On the other hand, if L is constructed as described above and P chosen to lie entirely in the Darboux chart, then K is contractible in $\partial M \setminus \overset{\text{o}}{P}$. Since in $\partial M \setminus \overset{\text{o}}{P}$ the forms $\omega = d\lambda$ and $\omega' = d\tau$ agree, Stokes' theorem yields

$$\int_K \tau = \int_K \lambda .$$

Altogether we obtain

$$\int_{L^-} \tau = \int_{L^+} \tau \neq 0 .$$

Now recall that ω' induces on L^- and L^+ opposite orientations, so with these orientations one of the two integrals is positive and the other one negative. This contradiction shows that τ cannot be a confoliation form. □

Proof of Corollary C. The proof is similar to that of Corollary B. However, we must argue a little bit more carefully in order to get property (i). Choose a contractible embedded loop $L \subset S^{2n-1}(1)$ which is nowhere tangent to $\ker(\omega_{2n}|_{S^{2n-1}(1)})$ and such that

$$\int_L \lambda \neq 0$$

for any 1-form λ on $\mathbf{R}^{2n}$ with $d\lambda = \omega_{2n}$. By Lemma A4 of the appendix we find a diffeomorphism $\Phi : B_1^{2n} \to B_1^{2n}$ of the closed ball of radius 1 with

$$\Phi = \mathrm{id} \quad \mathrm{on} \quad B_{\frac{1}{2}}^{2n} \cup S^{2n-1}(1) \ ,$$

and a neighborhood P of $L = \{0\}^{2n-2} \times S^1$ in $S^{2n-1}(1)$ such that

$$\omega_{2n}\big|_{\Phi\big((\rho,1]\times P\big)} = d\mu_0$$

for some $\rho \in (\frac{1}{2}, 1)$. Now apply the generalized construction as in the proof of Corollary B. Using (iii) of Theorem $2'$ we obtain an embedding

$$\Psi' : \Phi(B_1^{2n}) \hookrightarrow \mathbf{R}^{2n}$$

such that

$$\Psi' = \mathrm{id} \quad \mathrm{outside} \quad \Phi((\rho,1] \times P) \ ,$$

and $\Psi := \Psi' \circ \Phi$ has the desired properties. $\qquad\square$

Proof of Corollary F. Let $L \subset \partial M$ be an embedded loop which is contractible in M and nowhere tangent to $\ker(\omega|_{\partial M})$. Let $(\rho, 1] \times P$ be a neighborhood of L in M on which $\omega = d\mu_0$, and

$$K := \{1\} \times \{+\delta\} \times \{0\}^{2n-3} \times S^1 \subset \{1\} \times P \ .$$

The loop K is contractible in $M \setminus (\rho, 1] \times \overset{\mathrm{o}}{P}$: Deform it via $\{\rho\} \times \{\delta\} \times \{0\}^{2n-3} \times S^1$ to $\{\rho\} \times \{0\}^{2n-2} \times S^1$, which is contractible in $M \setminus (\rho, 1] \times \partial M$ by the contractibility of L.

By the construction in Theorems $1'$ and $2'$ we obtain a symplectic form ω' on M with two new closed characteristics $L^\pm$. Using the contractibility of K we conclude, arguing as in the proof of Corollary B, that they have actions

$$A_{\omega'}(L^\pm) = \pm A_\omega(L)$$

or

$$A_{\omega'}(L^\pm) = \mp A_\omega(L) \ ,$$

if $L^\pm$ are given the orientations induced by ω'.

Now the proof can be easily finished. Choose contractible embedded nowhere tangent loops $L_i \subset \partial M$ with

$$A_\omega(L_i) = a_i \ .$$

This can be achieved by simply adding small loops in a Darboux chart until the actions equal a_i. Performing the generalized construction we obtain new closed characteristics with actions $\pm a_i$. In a second step, choose a contractible embedded nowhere tangent loop of action 0 which intersects all the old closed characteristics and the new ones with action $-a_i$, and apply the construction again. This yields the desired action spectrum. $\quad\square$

Proof of Corollary H. Since the foliation on T is linear we find an embedded loop L_1 in T transversal to the foliation. Apply the generalized construction to this loop. It follows from (i) and (iii) of Proposition 1 that T is invariant for the new foliation, and outside T the two foliations are conjugate, i.e. there exists a diffeomorphism mapping leaves onto leaves. On T all orbits of $\mathcal{L}_\omega$ break up, and 2 new orbits $L_1^\pm$ are created. $\qquad\square$

6 Appendix: Normal Forms Near Embedded Loops

In this appendix we derive the normal forms for contact and symplectic forms near embedded loops which are used in this paper. Most of them are special cases of general results due to A. Weinstein. Thus Lemma A3 is an immediate consequence of Theorem 4.1 of [W1], and Lemmas A1 and A5 are essentially contained in Proposition 4.2 of [W3]. For the sake of completeness, we give self-contained proofs here.

Throughout the appendix N will be a smooth manifold of dimension $2k + 3$, $k \geq 0$.

Recall the following definitions from section 3: B_γ^{2k} is the closed ball around 0 of radius γ in $\mathbf{R}^{2k}$ with coordinates $z = (q_1, p_1, \ldots, q_k, p_k)$, and

$$P := [-\delta, \delta] \times [-\epsilon, \epsilon] \times B_\gamma^{2k} \times S^1$$

with coordinates $p = (t, x, z, \theta)$. We have 1-forms

$$\bar{\lambda}_{2k} := \sum_{i=1}^k p_i \, dq_i$$

on B_γ^{2k} and

$$\lambda_0 := dt + x \, d\theta + \bar{\lambda}_{2k}$$

on P.

Our first normal form concerns a contact form near a tangent loop. Here an embedded loop L is called *tangent* for a contact form λ if L is everywhere tangent to $\ker(\lambda)$.

LEMMA A1. *Let λ be a contact form on N and $L \subset N$ an embedded tangent loop. Then (for some δ, ϵ and γ) there exists an embedding $\phi : P \hookrightarrow N$ onto a neighborhood of L with $\phi(\{0\}^{2k+2} \times S^1) = L$ and*

$$\phi^* \lambda = \lambda_0 \ .$$

Proof. The following proof has been pointed out to me by C. Abbas. After pulling back λ by any embedding mapping $\{0\}^{2k+2} \times S^1$ to L we have

two 1-forms λ, λ_0 on P. We will transform λ into λ_0 by a succession of diffeomorphisms fixing L, possibly shrinking P in each step.

1. Since $L = \{0\}^{2k+2} \times S^1$ is tangent, the Reeb vector field X of λ does not point in the ∂_θ-direction near L. Applying a diffeo of P of the form $(t, x, z, \theta) \mapsto (A(\theta) \cdot (t, x, z), \theta)$ we can arrange that the ∂_t-component of X is positive on all of (a possibly smaller) P.

2. Next we want to make $X \equiv \partial_t$ on all of P. This requires finding a diffeomorphism ϕ of P with $\phi_* \partial_t = X$, or explicitly

$$\tfrac{\partial \phi}{\partial t}(t, q) = X(\phi(t, q)) \ ,$$

where we have abbreviated $q = (x, z, \theta)$. Since the ∂_t-component X_1 of X is strictly positive, we can perform the change of variables

$$\begin{cases} \tau(0, q) = 0 \ , \\ d\tau = X_1(\phi(t, q))dt \ . \end{cases}$$

Set $\psi(\tau, q) := \phi(t, q)$. The equation to solve becomes

$$\tfrac{\partial}{\partial \tau}(\tau, q) = \tfrac{1}{X_1(\psi(\tau, q))} X(\psi(\tau, q))$$
$$=: Y(\psi(\tau, q)) \ .$$

Since the ∂_τ-component of Y is identically 1, the first component of ψ is $_1(\tau, q) = \tau$, and the other components ψ_2 must satisfy

$$\tfrac{d\psi_2}{d\tau}(\tau, q) = Y_2(\tau, \psi_2(\tau, q)) \ .$$

This is an ordinary differential equation which has a local solution ψ_2 with $_2(0, q) = q$. Transforming back yields the diffeomorphism ϕ.

3. Consider the bilinear forms $d\lambda(0, \theta)$ and $d\lambda_0(0, \theta)$ on $T_{(0,\theta)}P$, $\theta \in S^1$. Their kernels both equal $\mathbf{R} \cdot \partial_t$, and their restrictions to the hyperplane $\{t = 0\}$ are linear symplectic forms. After applying a diffeomorphism $(t, x, z, \theta) \mapsto (t, A(\theta) \cdot (x, z), \theta)$, $A(\theta)$ linear, we may assume that $d\lambda(0, \theta) = d\lambda_0(0, \theta)$ for all $\theta \in S^1$.

4. Now we are in the situation to use Moser's method (see [Mo]). Let

$$\lambda_s := (1 - s)\lambda_0 + s\lambda \ ,$$

and calculate for a time-dependent vector field Y_s and its flow ϕ_s,

$$\tfrac{d}{ds}(\phi_s^* \lambda_s) = \phi_s^*\big(L_{Y_s}\lambda_s + \tfrac{d}{ds}\lambda_s\big)$$
$$= \phi_s^*(d(i_{Y_s}\lambda_s) + i_{Y_s}d\lambda_s + (\lambda - \lambda_0)) \ .$$

If this equals 0 for all s, ϕ_1 is the desired diffeomorphism with $\phi_1^* \lambda = \lambda_0$.

So it suffices to find a vector field Y_s satisfying
$$\begin{cases} i_{Y_s}\lambda_s = 0 \ , \\ i_{Y_s}d\lambda_s = \lambda_0 - \lambda \ . \end{cases}$$
We infer from 2. and 3. that near L all λ_s are contact forms with Reeb vector field ∂_t. Since the form $d\lambda_s$ is nondegenerate on the hyperplane $\{t{=}0\}$ in each tangent space, we find a unique vector field Y_s with vanishing t-component such that $i_{Y_s}d\lambda_s$ and $\lambda_0 - \lambda$ agree on all these hyperplanes. But as we also have
$$i_{\partial_t}d\lambda_s = 0 \ ,$$
$$i_{\partial_t}(\lambda_0 - \lambda) = 0 \ ,$$
the equation $i_{Y_s}d\lambda_s = \lambda_0 - \lambda$ holds on the whole tangent spaces. Now add a ∂_t-component to Y_s to satisfy the first equation $i_{Y_s}\lambda_s = 0$, which is possible because $\lambda_s(\partial_t) = 1$. Since this does not affect the second equation, the vector field is found and the proof complete. $\square$

Recall that a 2-form ω on N is called maximally nondegenerate if its kernel is one-dimensional. Near a loop which is nowhere tangent to $\ker(\omega)$ the 2-form ω has the following normal form:

LEMMA A2. *Let ω be a closed maximally nondegenerate 2-form on N and $L \subset N$ an embedded loop nowhere tangent to $\ker(\omega)$. Then there exists an embedding $\phi : P \hookrightarrow N$ onto a neighborhood of L with $\phi(\{0\}^{2k+2} \times S^1) = L$ and*
$$\phi^*\omega = d\lambda_0 \ .$$

Proof. 1. Set $\omega_0 := d\lambda_0$, and pull back ω anyhow to P. As in the first three steps in the proof of Lemma A1 we can arrange that $\ker(\omega) = \mathbf{R} \cdot \partial_t$ on all of P, and $\omega = \omega_0$ along $L = \{0\}^{2k+2} \times S^1$.

2. The 2-form $\omega_0 - \omega$ is closed and vanishes along L. Hence by the relative Poincaré Lemma there exists a 1-form α, vanishing along L, with
$$d\alpha = \omega_0 - \omega \ .$$
We claim that moreover $\alpha(\partial_t) = 0$. To see this let us recall the definition of the form α. Abbreviate $q = (t, x, z)$, define maps $\phi_s : P \to P$,
$$\phi_s(q, \theta) := (sq, \theta)$$
and an s-dependent vector field Y_s, $s > 0$, by
$$Y_s(\phi_s) := \tfrac{d}{ds}\phi_s \ .$$
Set $\bar{\omega} := \omega_0 - \omega$ and define
$$\alpha := \int_0^1 \phi_s^*(i_{Y_s}\bar{\omega})ds \ .$$

The integral is well-defined although Y_s is singular at $s = 0$. To see this introduce the nonsingular vector field

$$\tfrac{d}{ds}|_{s=0}\phi_s = (q,0) \ .$$

Then $Y_s = \tfrac{1}{s}(q,0)$, and the integrand can be written as

$$\phi_s^*(i_{Y_s}\bar{\omega}) = \bar{\omega}((\phi_s)_* Y_s, (\phi_s)_* \cdot)$$
$$= \bar{\omega}((q,0), (\phi_s)_* \cdot) \ ,$$

which is nonsingular. We have $\alpha(\partial_t) = 0$ because $i_{\partial_t}\bar{\omega} = 0$, and obviously $\alpha = 0$ along L. Moreover,

$$\begin{aligned}
d\alpha &= \int_0^1 \phi_s^* d(i_{Y_s}\bar{\omega})ds \\
&= \int_0^1 \tfrac{d}{ds}\phi_s^*\bar{\omega} \ ds \\
&= \phi_1^*\bar{\omega} - \phi_0^*\bar{\omega} \\
&= \bar{\omega} \ ,
\end{aligned}$$

and the claim is proved.

3. To apply Moser's method define

$$\omega_s := (1-s)\omega_0 + s\omega \ ,$$

and calculate for a vector field X_s and its flow ψ_s,

$$\begin{aligned}
\tfrac{d}{ds}(\psi_s^*\omega_s) &= \psi_s^*(d(i_{X_s}\omega_s) + (\omega - \omega_0)) \\
&= \psi_s^* d(i_{X_s}\omega_s - \alpha) \ .
\end{aligned}$$

So we are done if we can find a vector field X_s with

$$i_{X_s}\omega_s = \alpha \ .$$

Near L the forms ω_s are all nondegenerate on the hyperplanes $\{t = 0\}$. Hence there exists a unique vector fields X_s with vanishing ∂_t-component such that the equation holds on these hyperplanes. But on the other hand, $i_{\partial_t}\omega_s = 0$ and $\alpha(\partial_t) = 0$. So the equation holds on the whole tangent spaces, and the proof is finished. $\square$

We proceed to derive normal forms in the even-dimensional manifold $\mathbf{R} \times N$, starting with the following technical lemma.

LEMMA A3. *Let $A \subset N$ be compact. Let ω_0 and ω_1 be two symplectic forms on a neighborhood U of $\{1\} \times A$ in $\mathbf{R} \times N$ such that ω_0^{k+2} and ω_1^{k+2} define the same orientation, and $\omega_0|_{(\{1\}\times N)\cap U} = \omega_1|_{(\{1\}\times N)\cap U}$.*

Then there exists a diffeomorphism $\Psi : \mathbf{R} \times N \to \mathbf{R} \times N$ *with*

(i) $\qquad\qquad \Psi = \mathrm{id} \quad on \quad \{1\} \times N \quad and\ outside \quad U$,

(ii) $\qquad \Psi^* \omega_1 = \omega_0 \quad on \quad [1 - \sigma, 1 + \sigma] \times A \quad for\ some \quad \sigma > 0$.

The statement remains valid if we replace $\mathbf{R}$ *by* $(-\infty, 1]$ *and* $[1 - \sigma, 1 + \sigma]$ *by* $[1 - \sigma, 1]$.

Proof. 1. Let us first show that we may assume $\omega_0 = \omega_1$ along $(\{1\} \times N) \cap U$. Define $\Psi : \mathbf{R} \times N \to \mathbf{R} \times N$ by

$$\Psi(s, x) := (h(s, x), x) ,$$

where $h(\cdot, x) : \mathbf{R} \to \mathbf{R}$ are diffeomorphisms with $h(1, x) = 1$, $h(s, x) = s$ for (s, x) outside U, and

$$\left(\tfrac{\partial}{\partial s} h(1, x)\right) \cdot \omega_1^{k+2} = \omega_0^{k+2}$$

for x in a neighborhood V of A in N. Clearly $\Psi = \mathrm{id}$ on $\{1\} \times N$ and outside U. The last condition on h ensures that $(\Psi^* \omega_1)^{k+2} = \omega_0^{k+2}$ along $\{1\} \times V$. Together with the agreement of ω_0 and $\Psi^* \omega_1$ on $\{1\} \times V$ this implies that $\Psi^* \omega_1 = \omega_0$ along $\{1\} \times V$. Replacing U by a smaller neighborhood the statement follows.

2. The form $\omega_1 - \omega_0$ is closed, and by 1. we may assume that it vanishes along $(\{1\} \times N) \cap U$. By the relative Poincaré Lemma there exists a 1-form α on (a possibly smaller) U, $\alpha = 0$ along $(\{1\} \times N) \cap U$, such that

$$d\alpha = \omega_1 - \omega_0 \quad on\ [1 - \sigma, 1 + \sigma] \times A$$

for some $\sigma > 0$. Extend α to $\mathbf{R} \times N$ such that $\alpha = 0$ on $\{1\} \times N$ and outside U.

To apply Moser's method, define

$$\omega_s := (1 - s)\omega_0 + s\omega_1 .$$

Since $\omega_0 = \omega_1$ along $(\{1\} \times N) \cap U$, for U sufficiently small ω_s are symplectic forms on U. Define a vector field X_s on $\mathbf{R} \times N$ by

$$i_{X_s} \omega_s = -\alpha$$

on U and $X_s = 0$ outside U, and let Ψ_s be its flow. Note that X_s vanishes along $\{1\} \times N$ and therefore $\Psi_s|_{\{1\} \times N} = \mathrm{id}$. On $[1 - \sigma, 1 + \sigma] \times A$ we get

$$\tfrac{d}{ds}(\Psi_s^* \omega_s) = \Psi_s^*(d(i_{X_s} \omega_s) + d\alpha)$$
$$= 0 ,$$

and $\Psi_1 : \mathbf{R} \times N \to \mathbf{R} \times N$ is the desired diffeomorphism. Since Ψ_1 maps $(-\infty, 1] \times N$ onto itself, the same proof also works for $(-\infty, 1]$ instead of $\mathbf{R}$. $\qquad\square$

The following lemma shows that we can put a symplectic form on $\mathbf{R} \times N$ and its restriction to $\{1\} \times N$ simultaneously into normal forms. Define the 1-form

$$\mu_0 := s\,dt + x\,d\theta + \bar{\lambda}_{2k}$$

on $\mathbf{R} \times P$, where s is the coordinate on $\mathbf{R}$.

Lemma A4. *Let ω be a symplectic form on a neighborhood U of $\{1\} \times N$ in $\mathbf{R} \times N$, and $L \subset \{1\} \times N$ an embedded loop nowhere tangent to $\ker(\omega|_{\{1\}\times N})$. Then there exists a diffeomorphism $\Psi : \mathbf{R} \times N \to \mathbf{R} \times N$ with*

(i) $$\Psi = \mathrm{id} \quad on \quad \{1\} \times N \quad and \ outside \quad U\,,$$

and an embedding $\Phi : [1-\sigma, 1+\sigma] \times P \hookrightarrow U$ of the form

(ii) $$\Phi(s,p) = (s, \phi(p))$$

with $\phi(\{0\}^{2k+2} \times S^1) = L$, such that

(iii) $$\Phi^*\Psi^*\omega = d\mu_0\,.$$

The statement remains true with $(-\infty, 1]$ instead of $\mathbf{R}$ and $[1-\sigma, 1]$ instead of $[1-\sigma, 1+\sigma]$.

Proof. Let $\phi : P \hookrightarrow (\{1\} \times N) \cap U$ be an embedding as provided by Lemma A2 with $\phi(\{0\}^{2k+2} \times S^1) = L$ and

$$\phi^*(\omega|_{\{1\}\times N}) = d\lambda_0$$
$$= d\mu_0|_{\{1\}\times P}\,.$$

Define $\Phi : [1-\sigma, 1+\sigma] \times P \hookrightarrow U$ by

(ii) $$\Phi(s,p) := (s, \phi(p))\,,$$

and consider the 2-form ω_1 on $[1-\sigma, 1+\sigma] \times \phi(P)$ defined by

$$\Phi^*\omega_1 = d\mu_0\,.$$

After perhaps composing ϕ with the orientation reversing diffeomorphism $(t, x, z, \theta) \mapsto (-t, x, z, \theta)$ of P we may assume that ω_1^{k+2} and ω^{k+2} define the same orientation. Since ω_1 and ω also agree on $\{1\} \times \phi(P)$, Lemma A3 yields a diffeomorphism Ψ of $\mathbf{R} \times N$ satisfying (*i*) and

$$\Psi^*\omega = \omega_1 \quad on \ [1-\sigma, 1+\sigma] \times \phi(P)\,,$$

which together with the definition of ω_1 gives (iii). $\qquad\square$

Finally we want to put a symplectic form near a contact type hypersurface into normal form simultaneously with the contact form on the hypersurface.

LEMMA A5. *Let μ be a 1-form on $(1 - \sigma', 1 + \sigma') \times N$ such that $d\mu$ is symplectic and $\mu \wedge (d\mu)^{k+1}|_{\{s\} \times N} > 0$ with respect to $i_{\partial_s}((d\mu)^{k+2})|_{\{s\} \times N}$ for all $s \in (1 - \sigma', 1 + \sigma')$. Let $L \subset \{1\} \times N$ be an embedded tangent loop and U a neighborhood of L in $(1 - \sigma', 1 + \sigma') \times N$.*

Then there exists a diffeomorphism $\Psi : \mathbf{R} \times N \to \mathbf{R} \times N$ with

(i) $$\Psi = \mathrm{id} \quad on \quad \{1\} \times N \quad and \ outside \quad U \ ,$$

a 1-form μ' on $(1 - \sigma', 1 + \sigma') \times N$ with

(ii) $$\mu' - \Psi^*\mu = dh \quad for\ some\ function \quad h \quad on \quad (1 - \sigma', 1 + \sigma') \times N$$
$$vanishing\ on \quad \{1\} \times N \quad and\ outside \quad U \ ,$$

(iii) *$\mu'|_{\{s\} \times N}$ is a contact form for $s \in [1 - \sigma, 1 + \sigma]$ and some $\sigma \in (0, \sigma')$,*

and an embedding $\Phi : [1 - \sigma, 1 + \sigma] \times P \hookrightarrow U$ of the form

(iv) $$\Phi(s, p) = (s, \phi(p))$$

with $\phi(\{0\}^{2k+2} \times S^1) = L$, such that

(v) $$\Phi^*\mu' = \mu_0 \ .$$

The statement remains valid if $\mathbf{R}$, $(1 - \sigma', 1 + \sigma')$ and $[1 - \sigma, 1 + \sigma]$ are replaced by $(-\infty, 1]$, $(1 - \sigma', 1]$ and $[1 - \sigma, 1]$ respectively.

REMARK. Observe that
$$\Phi^*\Psi^* d\mu = d\mu_0$$
and
$$\Phi^*\Psi^*(\mu|_{\{1\} \times N}) = \lambda_0 \ ,$$
i.e. the lemma really puts $d\mu$ and $\mu|_{\{1\} \times N}$ into normal forms around L.

Proof. Let $\phi : P \hookrightarrow (\{1\} \times N) \cap U$ be the embedding from Lemma A1 with $\phi(\{0\}^{2k+2} \times S^1) = L$ and
$$\phi^*(\mu|_{\{1\} \times N}) = \lambda_0 \ .$$
For small σ define $\Phi : [1 - \sigma, 1 + \sigma] \times P \hookrightarrow U$ by

(iv) $$\Phi(s, p) := (s, \phi(p)) \ ,$$

and consider the 1-form μ_1 on $[1 - \sigma, 1 + \sigma] \times \phi(P)$ defined by
$$\Phi^*\mu_1 = \mu_0 \ .$$
It follows from the hypothesis on the orientation of $\mu \wedge (d\mu)^{k+1}|_{\{1\} \times N}$ that $(d\mu)^{k+2}$ and $(d\mu_1)^{k+2}$ define the same orientation. Since μ and μ_1 agree on $\{1\} \times \phi(P)$, we can apply Lemma A3 to get a diffeomorphism Ψ of $\mathbf{R} \times N$ satisfying (i) and
$$\Psi^* d\mu = d\mu_1 \quad on \quad [1 - \sigma, 1 + \sigma] \times \phi(P) \ .$$

Hence $\mu_1 - \Phi^*\mu$ is a closed 1-form on $[1-\sigma, 1+\sigma] \times \phi(P)$ whose restriction to $\{1\} \times \phi(P)$ vanishes. By the relative Poincaré Lemma there exists a function h on $[1-\sigma, 1+\sigma] \times \phi(P)$ (for possibly smaller P and σ) with $h|_{\{1\} \times \phi(P)} \equiv 0$ and

$$\mu_1 - \Psi^*\mu = dh \quad \text{on} \quad [1-\sigma, 1+\sigma] \times \phi(P) \ .$$

Extend h to $(1-\sigma', 1+\sigma') \times N$ such that it vanishes on $\{1\} \times N$ and outside U, and define

(ii) $$\mu' := \Psi^*\mu + dh \quad \text{on} \quad (1-\sigma', 1+\sigma') \times N \ .$$

Then μ' satisfies (v), and $\mu'|_{\{s\} \times N}$ is a contact form for s sufficiently close to 1. This concludes the proof. $\qquad\square$

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