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Kai Cieliebak, A. Floer, Helmut Hofer, Konrad Wysocki

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Applications of symplectic homology II: Stability of the action spectrum

K. Cieliebak, A. Floer, H. Hofer, K. Wysocki

Mathematics, ETH Zürich, CH-8092 Zürich, Switzerland
(e-mail: hhofer@math.ethz.ch, Fax 41/1/632 1085)

1 Introduction

Symplectic homology associates to an open subset U of a suitable compact symplectic manifold M and real parameters $0 < a < b$ symplectic homology groups $S^{[a,b)}(M, U)$. These groups measure the symplectic properties of U as a subset of M.

Symplectic homology was introduced by the second and the third author in [11] for open sets in $\mathbb{R}^{2n}$ and it was applied in [12] among other things to the symplectic classification of open ellipsoids and polydisks. In [3] the first three authors extended the construction to certain compact symplectic manifolds M with contact type boundary and their open subsets. For properties of such "convex manifolds" see in particular [7, 15]. We also would like to mention that historically the importance of symplectic convexity was pointed out for the first time by A. Weinstein in [18] for periodic orbit problems. There he doesn't call it convexity but introduces the notion of contact type. For this aspect of convexity we refer the reader to the recent book [14].

In the present paper we are going to use the construction in [3] to study the question what information the interior of a symplectic manifold knows about its boundary. This is a very important question in symplectic topology. It complements another important question. Namely what does the boundary contain about its interior. Very interesting results concerning the two questions can be found in [2, 8, 9, 13].

For the technical background to our results we refer the reader to the papers [3] and [11].

We assume (M, ω) is a compact symplectic manifold with contact type boundary ∂M. This means that there exists an outward pointing transversal vector field η defined on an open neighborhood of ∂M in M such that the Lie derivative of ω

A. Floer died on 15th May 1991

with respect to η satisfies $L_\eta \omega = \omega$. Equivalently (setting $\lambda := i_\eta \omega := \omega(\eta, \cdot)$) there exist a 1-form λ on a neighborhood of ∂M such that $d\lambda = \omega$ and $\lambda \wedge (d\lambda)^{n-1}$ is a volume form on ∂M which determines the orientation of ∂M induced from the orientation ω^n on M. The 2-form ω induces on ∂M a canonical line bundle $\mathscr{L}_{\partial M} \to \partial M$ via the formula

$$\mathscr{L}_\omega := \{(x, v) \in T(\partial M) \mid \omega(v, w) = 0 \text{ for all } w \in T_x(\partial M)\}.$$

The line bundle $\mathscr{L}_{\partial M}$ is naturally oriented as follows. Choose a Hamiltonian H defined near ∂M which is equal to 0 on ∂M and whose outward normal derivative on ∂M is positive. The Hamiltonian vector field X_H defined by

$$i_{X_H} \omega := \omega(X_H, \cdot) = dH$$

gives a nowhere vanishing section of $\mathscr{L}_{\partial M} \to \partial M$ and hence an orientation of $\mathscr{L}_{\partial M}$. We shall restrict our considerations to the following two cases:

A. The symplectic form ω vanishes on $\pi_2(M)$, and the first Chern class c_1 vanishes for pullback bundles $u^*TM \to S^2$ and $[u] \in \pi_2(M)$.
B. The symplectic form ω is exact, i.e. the 1-form $\lambda = i_\eta \omega$ near ∂M extends to a 1-form on M satisfying $d\lambda = \omega$ everywhere.

In case A, we denote by $\mathscr{C}_A$ the collection of all periodic orbits of $\mathscr{L}_\omega$ on ∂M which are contractible in M. Any element $x \in \mathscr{C}_A$ is a T-periodic smooth immersion $x : [0, T] \to \partial M$ which solves the equation $\dot{x} = -X(x)$, where X is the so called Reeb vector field defined by

$$i_X \lambda = 1 \text{ and } i_X d\lambda = 0.$$

Here λ is the 1-form given by $\lambda = i_\eta \omega$. With an $x \in \mathscr{C}_A$ we associate two numerical invariants. We choose a positive parametrisation $x : S^1 \to \partial M$ of $x \in \mathscr{C}_A$ and take an extension $u : D \to M$ of x to a disk D. Now define

$$A(x) := -\int_D u^* \omega$$

In view of the assumption $[\omega] \mid_{\pi_2(M)} = 0$ this integral does not depend on the choice of the extention u and will be called the action of x.
Next we associate with x a well- defined Maslov type index, $\mathrm{ind}_{RS}(x)$. This index, due to Robbin and Salamon, [17], is a generalisation of the well-known Conley-Zehnder index [4]. Assume that $x : [0, T] \to \partial M$ is a T-periodic solution of the Reeb vector field X, $\dot{x} = -X(x)$. We linearize this equation along the orbit x to obtain a path of symplectic maps

$$\Psi(t) : T_{x(0)}M \to T_{x(t)}M .$$

Here we extend the linearisation, which apriori is only defined in $T(\partial M)$ in the normal direction as the identity map. Then, the map $\Psi(t)$ sends $X(x(0))$ onto $X(x(t))$, $\eta(x(0))$ onto $\eta(x(t))$ and

$$\xi_{x(0)} := ker\lambda(x(0))$$

onto $\xi_{x(t)} = ker\lambda(x(t))$. We will call $x \in \mathscr{C}_A$ a nondegenerate periodic orbit if the map

$$\Psi(T) \mid \xi_{x(0)} : \xi_{x(0)} \to \xi_{x(0)}$$

does not have 1 in its spectrum.
We choose a disk map $u : D \to M$ extending x, i.e. $x(t) = u(e^{2\pi it/T})$, for $0 \leqslant t \leqslant T$, and a symplectic trivialisation $\beta : u^*TM \to D \times \mathbb{R}^{2n}$. Consider symplectic maps $\Gamma(t) : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ determined by

$$\Gamma(t) = \beta(e^{2\pi it/T})(\Psi(t))(\beta(1))^{-1}.$$

According to [17] we can associate with a path Γ a Maslov type index $\mu(\Gamma)$ which, as it will be shown in section 3, is an integer if $x \in \mathscr{C}_A$ is nondegenerate in the above sense. Moreover, this index is independent of the choices involved in view of the condition on c_1. Hence $\text{ind}_{RS}(x) = \mu(\Gamma) \in \mathbb{Z}$ is a well defined invariant of a nondegenerate perodic orbit $x \in \mathscr{C}_A$.

We define the action spectrum $\mathscr{A}_A(\partial M)$ by setting

$$\mathscr{A}_A(\partial M) := \{(A(x), \text{ind}_{RS}(x)) \mid x \in \mathscr{C}_A, A(x) \neq 0\} \subset \mathbb{R} \times \mathbb{Z},$$

each pair counted with multiplicity.

The importance of the action spectrum was recognized by Ekeland and Hofer in the construction of symplectic invariants for open sets in the symplectic vector space $\mathbb{C}^n$, [5, 6]. It is now considered an important ingredient in the study of symplectic rigidity.

In case B we fix the 1-form λ and and denote by $\mathscr{C}_B$ the collection of all periodic trajectories of $\mathscr{L}_\omega$ on ∂M. If $x \in \mathscr{C}_B$ then its action is defined by

$$A(x) := -\int_{S^1} x^*\lambda.$$

and the action spectrum in this case is

$$\mathscr{A}_B(\partial M) := \{A(x) \mid x \in \mathscr{C}_B\} \subset \mathbb{R}.$$

Note that in both cases we also count multiple covered periodic trajectories. In section 3 we shall define in the situation of case A symplectic homology groups $S_k^{[a,b)}(M,J)$ with coefficients in $\mathbb{Z}_2$. Here $0 < a < b$ or $a < b < 0$, $k \in \mathbb{Z}$ and J is a suitable almost complex stucture defined in the interior of M. In the case B we get groups $S_*^{[a,b)}(M,J)$ without the $\mathbb{Z}$-grading. We could also define groups with coefficients in arbitrary groups based on the resolution of orientation questions for families of certain linear Fredholm operators in [10].

Our main result is the following

Theorem 1.1 *Assume in the situation A or B that all elements of $\mathscr{C}_A$ or $\mathscr{C}_B$ are nondegenerate. Then for $a \in \mathbb{R} \setminus \{0\}$ the symplectic homology groups $S_*^{[a-\varepsilon,a+\varepsilon)}(M,J)$ become independent of ε and J if ε is sufficiently small. We denote them by $S_*^a(M)$. If $(a,k) \notin \mathscr{A}_A$ for all $k \in \mathbb{Z}$ or $a \notin \mathscr{A}_B$, then $S_*^a(M) = \{0\}$. If $(a,k) \in \mathscr{A}_A$ has multiplicity n then it gives rise to n copies of $\mathbb{Z}_2$ in $S_k^a(M)$ and n copies in $S_{k+1}^a(M)$. If $a \in \mathscr{A}_B$ has multiplicity n then $S_*^a(M) = 2n \cdot \mathbb{Z}_2$.*

As a corollary we obtain the so-called "stability of the action spectrum". For a compact manifold M with boundary ∂M we denote by $\dot{M}$ the set $M \setminus \partial M$.

Theorem 1.2 *Let M and N be compact symplectic manifolds with contact type boundary which both satisfy assumption A or B. Assume that all elements of $\mathscr{C}_A(M)$ and $\mathscr{C}_A(N)$, respectively $\mathscr{C}_B(M)$ and $\mathscr{C}_B(N)$, are nondegenerate. Let $\psi: \dot{M} \to \dot{N}$ be a symplectic diffeomorphism of the interiors which in case B we assume to be exact symplectic (i.e. $\psi^*\mu - \lambda$ is exact, where λ, μ are the fixed 1-forms on M and N).*

Then

$$\begin{aligned} \mathscr{A}_A(\partial M) &= \mathscr{A}_A(\partial N) \quad \textit{resp.} \\ \mathscr{A}_B(\partial M) &= \mathscr{A}_B(\partial N). \end{aligned}$$

Proof. Consider case A. It follows from the construction of the symplectic homology groups that ψ induces an isomorphism

$$S_k^{[a,b)}(M,J) \cong S_k^{[a,b)}(N,(D\psi)\circ J \circ (D\psi)^{-1}).$$

This implies that $S_k^a(M) \cong S_k^a(N)$ for all $a \in \mathbb{R}\setminus\{0\}$. But according to Theorem 1.1 the groups S_k^a give us complete knowledge of the action spectra of the boundaries. Hence $\mathscr{A}_A(\partial M) = \mathscr{A}_A(\partial N)$. For case B note that since ψ is exact symplectic, it takes periodic solutions to periodic solutions with the same action. The argument now works as in the previous case.

2 Local Floer homology

In this section we are going to show that a periodic solution of a time-independent Hamiltonian system splits under a small time-dependent perturbation into two solutions, and that the Floer homology of these two solutions is the homology of S^1.

In [4] C. Conley and E. Zehnder introduced an index $\mathrm{ind}_{CZ}(x) \in \mathbb{Z}$ for nondegenerate periodic solutions. We will make use of a generalisation due to J. Robbin and D. Salamon ([17]). They associate with every periodic solution a half integer $\mathrm{ind}_{RS}(x)$ which agrees with $-\mathrm{ind}_{CZ}$ in the nondegenerate case. We briefly recall their construction. Consider $\mathbb{R}^{2n}$ with coordinates $(q_1,\ldots,q_n,p_1,\ldots,p_n)$ equipped with the standard symplectic form $\omega_0 = \Sigma dq_i \wedge dp_i$. Let $\Lambda(t), t \in [a,b]$ be a smooth path of Lagrangian subspaces. For each t_0 we define a quadratic form Q_{t_0} on $\Lambda(t_0)$ as follows. Take a Lagrangian complement W of $\Lambda(t_0)$.

For $v \in \Lambda(t_0)$ and t near t_0 define $w(t) \in W$ by $v + w(t) \in \Lambda(t)$. Then $Q_{t_0}(v) := \frac{d}{dt}\mid_{t=t_0} \omega_0(v, w(t))$ is independent of the choice of W.

Now fix a Lagrangian subspace V and define

$$\begin{aligned} \Sigma_k(V) &:= \{L \mid L = \text{ Lagrangian subspace of } \mathbb{R}^{2n},\ dim(L \cap V) = k\} \\ \Sigma(V) &:= \bigcup_{k=1}^{n} \Sigma_k(V). \end{aligned}$$

Homotope Λ to a path having only regular crossings with $\Sigma(V)$. This means that whenever $\Lambda(t) \in \Sigma(V)$, the quadratic form $\Gamma_t := Q_t \mid_{\Lambda(t)\cap V}$ is nonsingular.

The Maslov index of the path $\Lambda(t)$ is then defined as the half integer

$$\mu(\Lambda, V) := \frac{1}{2}\,\text{sign}\,\Gamma_a + \sum_{\substack{a<t<b \\ \Lambda(t)\in\Sigma(V)}} \text{sign}\,\Gamma_t \; + \frac{1}{2}\,\text{sign}\,\Gamma_b\,,$$

where sign is the signature of a quadratic form, the number of positive minus the number of negative eigenvalues of the quadratic form.

If $\Psi : [a, b] \to Sp(2n)$ is a path of symplectic matrices, then the graphs of $\Psi(t)$, $gr(\Psi(t))$, form a path of Lagrangian subspaces in the symplectic vector space $(\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}, ((-\omega_0) \oplus \omega_0)$. We define the Maslov index of the path Ψ by setting

$$\mu(\Psi) := \mu(gr(\Psi), \Delta),$$

where Δ is the diagonal of $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$. This index μ has the following nice properties, for details see [17]:

1. Two paths of symplectic matrices are homotopic with end points fixed if and only if they have the same index.
2. For each k every path in $Sp_k(2n) := \{\Psi \mid \Psi = \text{symplectic}, gr(\Psi) \in \Sigma_k(\Delta)\}$ has index zero.
3. If $\Psi : [a, b] \to Sp(2n)$ and $a < c < b$ then $\mu(\Psi) = \mu(\Psi \mid_{[a,c]}) + \mu(\Psi \mid_{[c,b]})$.
4. If Ψ, Ψ' are paths in $Sp(2n)$, $Sp(2n')$ respectively, then $\mu(\Psi \oplus \Psi') = \mu(\Psi) + \mu(\Psi')$.

Assume now that (M, ω) is a symplectic manifold such that for every map $u : S^2 \to M$ the number $c_1(u^*TM)[S^2]$ vanishes. Here c_1 is the first Chern number of the complex bundle u^*TM. Let $H(t, x)$ be a Hamiltonian which is 1-periodic in the t-variable and X_{H_t} be a Hamiltonian vector field defined by $i_{X_{H_t}}\omega = dH_t$. Assume that $x: S^1 \to M$ is a contractible 1-periodic solution of $\dot{x} = X_{H_t}(x)$. Denote by ψ_t the flow of X_{H_t}. Then the tangent map $D\psi_t(x(0)): T_{x(0)}M \to T_{x(t)}M$ is symplectic. Choose $u : D \to M$ so that $x(t) = u(e^{2\pi i t})$, where D is a unit disk in $\mathbb{C}$. We take a symplectic trivialization of u^*TM, denoted by $\Psi : u^*TM \to D \times \mathbb{R}^{2n}$, and consider the path of symplectic matrices in $\mathbb{R}^{2n}$, $\Gamma : [0, 1] \to Sp(2n)$, given by

$$\Gamma(t) = \Psi(e^{2\pi i t})(D\psi_t(x(0)))\Psi(1)^{-1}.$$

We define the generalized Conley-Zehnder index of the 1-periodic orbit x by setting

$$\operatorname{ind}_{RS}(x) := \mu(\Gamma).$$

One easily verifies that, in view of our assumption on c_1, this definition is independent of the choices involved.

We say that a 1-periodic orbit x of the Hamiltonian vector field X_{H_t} is nondegenerate if 1 is not an eigenvalue of the map $D\psi_1(x(0)) : T_{x(0)}M \to T_{x(0)}M$. In this situation $\operatorname{ind}_{RS}(x)$ is an integer and is equal to $-\operatorname{ind}_{CZ}(x)$, where $\operatorname{ind}_{CZ}(x)$ stands for the Conley-Zehnder index. If H is time-independent and x a non-constant 1-periodic then it is necessarily degenerate because $x(t-a)$, $a \in \mathbb{R}$, are also 1-periodic solutions. We call such an x transversally nondegenerate if the eigenspace to the eigenvalue 1 of the map $D\psi_1(x(0)): T_{x(0)}M \to T_{x(0)}M$ is one-dimensional. Equivalently, in the above notation, this means that $gr(\Psi(1)) \in \Sigma_1(\Delta)$. From the definition it is clear that if x is transversally nondegenerate then $\operatorname{ind}_{RS}(x) \notin \mathbb{Z}$.

Now we define the local Floer homology group of a transversally nondegenerate 1-periodic orbit $\bar{x}$ of the Hamiltonian vector field X_H. For this we take a sufficiently small neighborhood U of $\bar{x}(S^1)$ in M. In particular U should not contain any other 1-periodic solution except $\bar{x}$ and its translates $\bar{x}(\cdot + t)$. That such a neighborhood U exists follows from the fact that $\bar{x}$ is transversally nondegenerate.

Let J be an almost complex structure on U which is compatible with ω in the sense that $\langle \cdot,\cdot \rangle = \omega \circ (J \times \mathrm{Id})$ defines a Riemannian metric on M. We introduce the space of time-dependent perturbations

$$\mathscr{H} := \{ h \in C^\infty(S^1 \times U, \mathbb{R}) \big| |\nabla h(t,x)| \leq 1 \quad \text{for all } (t,x) \in S^1 \times U \},$$

where ∇ denotes the gradient with respect to the variable x and $|\cdot|$ the norm with respect to $\langle \cdot,\cdot \rangle$. For $h \in \mathscr{H}$ and $\delta > 0$ define $H_\delta := H + \delta h$. For generic $h \in \mathscr{H}$ and small $\delta > 0$ all 1-periodic solutions x will be nondegenerate and hence $\operatorname{ind}_{RS}(x) \in \mathbb{Z}$.

Let $C_k(J, H_\delta)$ be the $\mathbb{Z}_2$ vector space with basis the 1-periodic solutions of $\dot{x} = X_{H_\delta}(t,x)$ in U with $\operatorname{ind}_{RS}(x) = k$. Given periodic solutions x^-, x^+ of indices $k, k-1$ define

$$\begin{aligned} \mathscr{M}(x^-, x^+, J, H_\delta, U) \; := \; & \{ u : \mathbb{R} \times S^1 \to U \mid u_s + J(u)u_t + \nabla H_\delta(u) = 0, \\ & u(s,\cdot) \to x^\pm \quad \text{as } s \to \pm\infty \}. \end{aligned}$$

Note the occurence of U in the definition of $\mathscr{M}(x^-, x^+, J, H_\delta, U)$.

As we shall see below the following is true. For δ small enough and generic h and J we obtain a boundary operator

$$\begin{aligned} \partial_k : C_k \; & \to \; C_{k-1} \quad \text{linear} \\ x \; & \to \; \sum_{\operatorname{ind}(y)=k-1} \langle \partial_k x, y \rangle \cdot y , \end{aligned}$$

where $\langle \partial_k x, y \rangle$ is the number mod 2 of points in $\mathscr{M}(x, y, J, H_\delta)/\mathbb{R}$ ($\mathbb{R}$ acts on $\mathscr{M}$ by translation in s).

The local Floer homology groups of $\bar{x}$ are

$$HF_k^{\mathrm{loc}}(\bar{x}) := \ker(\partial_k)/\operatorname{im}(\partial_{k+1})$$

and, as we shall see below, are independent of the choices involved. We begin with

Lemma 2.1 *Assume that $\bar{x}$ is a transversally nondegenerate 1-periodic orbit of X_H and U is an open neighborhood of $\bar{x}(S^1)$ which does not contain any other 1-periodic orbit of X_H. Then for any open neighborhood V which satisfies $\bar{x}(S^1) \subset V \subset U$ there exists $\delta_0 > 0$ such that for any $0 < \delta < \delta_0$ we have*

1. *All 1-periodic solutions of $\dot{x} = X_{H_\delta}(t,x)$ in U are contained in V .*
2. *All $u \in \mathcal{M}(x^-, x^+, J, H_\delta, U)$ are contained in V .*

Moreover, the local Floer homology groups $HF_k^{\mathrm{loc}}(\bar{x})$ are well-defined and independent of J and $h \in \mathcal{H}$.

Proof. (1) We argue by contradiction. Assume that there exist a neighborhood V such that $\bar{x}(S^1) \subset V \subset \overline{V} \subset U$, sequences $\delta_n \to 0$ and $\{x_n\}$ of 1-periodic solutions of $\dot{x_n}(t) = X_{H_{\delta_n}}(t, x_n)$ such that $x_n(S^1) \subset U$ but $x_n(t_n) \notin V$ for some $t_n \in S^1$. It follows by the compact Sobolev embedding $H^{1,2}(S^1) \to C^0(S^1)$ that there is a subsequence of $\{x_n\}$, still denoted by $\{x_n\}$, and y such that $x_n \to y$ in $C^0(S^1)$ and $\dot{y}(t) = X_H(y(t))$. Since $\bar{x}$ and its translates are the only 1-periodic solutions of X_H contained in U we have that $y(S^1) \subset V$. On the other hand $x_n(t_n) \to y(\bar{t}) \notin V$ for some $\bar{t} \in S^1$. This contradiction proves the first part of the lemma.
(2) From the proof of (1) we know that each 1-periodic solution x of X_{H_δ} is $H^{1,2}$-close to a translate of $\bar{x}$ if δ is small. We define the energy of the periodic orbit x by

$$\Phi(x) = A(x) - \int_0^1 H_\delta(t,x)dt.$$

We span in a small cylinder Z bounded by x and $\bar{x}$. With the proper orientation of Z we have

$$A(x) = A(\bar{x}) - \int_Z \omega,$$

where $A(\bar{x})$ is fixed.

Now assume that for some neighborhood W of $\bar{x}(S^1)$ we would find a sequence $\delta_n \to 0$ and $u_n \in \mathcal{M}(x_n^-, x_n^+, J, H_{\delta_n}, U)$ not contained in W. By shifting in the s-variable we can achieve that $u_n(0,\cdot)$ is not contained in W. We have by (1), $x_n^- \to \bar{x}(a+\cdot)$, $x_n^+ \to \bar{x}(b+\cdot)$ for some $a, b \in S^1$. Hence $A(x_n^+) - A(x_n^-) \to 0$. On the other hand by the well-know compactness results u_n converges in C^∞_{loc} to some u. Since $u_n(0, S^1)$ is not contained in W the same is true for u. But since $A(x_n^+) - A(x_n^-) \to 0$, u must be independent of s, $u(s,t) = x(t)$ for some 1-periodic solution x of $\dot{x} = X_H(x)$. The only such solutions are translates of $\bar{x}$, so u is contained in W, and we have a contradiction which proves (2).

To prove the last part of the lemma note that (2) ensures that all elements $u \in \mathcal{M}(x^-, x^+, J, H_\delta, U)$ are bounded away uniformly from the boundary of U. Since $\pi_2(U) = 0$, no bubbling off of holomorphic spheres can occur. Hence all the usual Floer homology constructions work and the lemma follows.

The lemma allows us to calculate the local Floer homology by choosing nice data (J, h).

Let $\bar{x}$ be an l-fold covered 1-periodic orbit which means that $l \in \mathbb{N}$ is the maximal natural number such that $\bar{x}(t + \frac{1}{l}) = \bar{x}(t)$. We choose a Morse function h_0 on S^1 having exactly two critical points, a minimum at 0 and a maximum at $1/2$. Define the function h on $\bar{x}(S^1)$ by $h(t, \bar{x}(s)) = h_0(ls - lt)$ and extend it to the neighborhood U in a suitable way. We will describe this extension in the proof of the following proposition. This perturbation destroys the critical circle and gives rise to two obvious solutions of $\dot{x} = X_{H_\delta}(t,x)$, $H_\delta = H + \delta h$, namely $x^-(t) = \bar{x}(t)$ and $x^+(t) = \bar{x}(t + \frac{1}{2l})$ which correspond to the minimum and maximum of h_0.

Proposition 2.2 *For h as above and δ sufficiently small, x^+ and x^- are the only 1-periodic solutions of $\dot{x} = X_{H_\delta}(t,x)$ in U. They have indices*

$$\mathrm{ind}_{RS}(x^\pm) = \mathrm{ind}_{RS}(\bar{x}) \pm \frac{1}{2},$$

and the boundary operator is trivial: $\partial(x^+) = \partial(x^-) = 0$.
Thus the local Floer homology $HF_k^{\mathrm{loc}}(\bar{x})$ is $\mathbb{Z}_2$ in dimensions

$$k = \mathrm{ind}_{\mathrm{RS}}(\bar{x}) \pm 1/2$$

and zero otherwise.

Remark. The proposition shows that $HF_*^{\mathrm{loc}}(\bar{x})$ is just the ordinary homology of S^1 shifted in dimension by $\mathrm{ind}_{RS}(\bar{x}) - 1/2 \dim(S^1)$. A similar result should also hold for more general critical manifolds in place of S^1.

Proof. The proof of the proposition will be divided into four steps.
1. Since the normal bundle of $\bar{x}(S^1)$ in M is trivial (because it is orientable), there exists a diffeomorphism $\psi : U_0 \to U_1$ of an open neighborhood $U_0 \subset M$ of $\bar{x}(S^1)$ onto an open neighborhood $U_1 \subset S^1 \times \mathbb{R}^{2n-1}$ of $S^1 \times \{0\}$ such that $\psi^*\omega_0 = \omega$. Here $S^1 \times \mathbb{R}^{2n-1}$ is equipped with the form $\omega_0 = \Sigma dq_i \wedge dp_i$ in coordinates $(q_1, p_1, \dots, q_n, p_n)$, $q_1 \in S^1$, $(p_1, q_2, p_2, \dots, p_n) \in \mathbb{R}^{2n-1}$. Under this transformation the periodic orbit $\bar{x}$ corresponds to $\psi(\bar{x}(t)) = (lt, 0) \in S^1 \times \mathbb{R}^{2n-1}$ and the Hamiltonian function H corresponds to $H \circ \psi^{-1}$ which will be still denoted by the same letter H. We will make another change of variables.
Define a Hamiltonian function $K : U_1 \to \mathbb{R}$ by $K(q_1, p_1, \dots) = -l\, p_1$. It generates the flow $\Delta_t(q_1, y) = (q_1 - lt, y)$ on $S^1 \times \mathbb{R}^{2n-1}$. If $x(t)$ is a 1-periodic solution of $\dot{x} = X_{H_\delta}(t,x)$ then $\hat{x}(t) = \Delta_t \circ x(t)$ is a 1-periodic solution of $\dot{x} = X_{\hat{H}_\delta}(t,x)$, where $\hat{H}_\delta(t,x) = H_\delta(t, \Delta_t^{-1}(x)) + K(x)$. Gradient lines in the Floer complex are transformed in a similar way. Hence we obtain an equivalent system in which the unperturbed Hamiltonian $\hat{H}$ has critical points all along $S^1 \times \{0\}$.

Also note that after this transformation the unperturbed Hamiltonian $\hat{H}$ becomes time dependent. The solutions $t \mapsto \bar{x}(t+a)$ correspond to constant solutions $t \mapsto (la, 0)$. In particular, the orbits x^- and x^+ become constant solutions $(0,0)$ and $(1/2, 0)$.

In this new situation we can define the perturbation $\hat{h}$ simply as

$$\hat{h}(q_1, y) := h_0(q_1).$$

The perturbation $\hat{h}$ is time-independent, but it becomes time-dependent when we transform back to the original situation:

$$\begin{aligned} h(t, q_1, y) &= \hat{h}(\Delta_t(q_1, y)) \\ &= h_0(q_1 - lt). \end{aligned}$$

And so we have $h(t, \bar{x}(s)) = h_0(ls - lt)$, in accordance with its previous description.

We shall suppress the $\hat{}$ in $\hat{H}_\delta$, $\hat{h}$ and $\hat{x}$ from our notation, assuming that we have already performed the described transformations.

2. Since $X_{H_t} = 0$ on $S^1 \times \{0\}$, all 1-periodic solutions of $\dot{x} = X_{H_\delta}(t, x)$ in a neighborhood U_1 of $S^1 \times \{0\}$ will be small and therefore contractible in $S^1 \times \mathbb{R}^{2n-1}$. So we can lift them to loops in $\mathbb{R}^{2n}$. Set $E := H^{1,2}(S^1, \mathbb{R}^{2n})$ and $G := L^2(S^1, \mathbb{R}^{2n})$ and consider a smooth nonlinear operator $F : E \to G$ given by

$$F(x) := -J_0\dot{x} - \nabla H(x) = -J_0(\dot{x} - X_H(x)),$$

where ∇H is the gradient with respect to the standard inner product in $\mathbb{R}^{2n}$ and J_0 is the standard symplectic matrix

$$J_0 = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{pmatrix}.$$

Let $N = F^{-1}(0)$ be the subspace of constant solutions $x_a(t) = (la, 0) \in \mathbb{R} \times \mathbb{R}^{2n-1}$, $a \in \mathbb{R}$. We linearize F at x_a to obtain an operator $A := DF(x_a): E \to G$. The operator A is Fredholm of index 0 and its kernel coincides with N. Viewed as an unbounded operator on G, A is self-adjoint. If W denotes the L^2-orthogonal complement of N in G, then the above remarks imply that

$$A\,|_{W \cap E} : W \cap E \xrightarrow{\cong} W$$

is an isomorphism.

In particular, there exists a constant $c > 0$ such that $\|DF(x_a) \cdot y\|_G \geq c\|y\|_E$ for all $a \in \mathbb{R}$ and $y \in W \cap E$.

Now define $f: E \to G$ by $f(x)(t) = h'(x(t))$. Recall that $h(q_1, y) = h_0(q_1)$ for $(q_1, y) \in \mathbb{R} \times \mathbb{R}^{2n-1}$.

Then

$$f(x_a)(t) = h'(x_a(t)) = (h_0'(la), 0)$$

and so $f(x_a) \in N$. We write an arbitrary element of E near N as $x_a + y$ with $y \in W \cap E$, and calculate

$$(F + \delta f)(x_a + y) = F(x_a) + DF(x_a) \cdot y + \delta f(x_a) + \delta Df(x_a) \cdot y + O(\|y\|^2_{H^{1,2}}),$$

where $F(x_a) = 0$, $DF(x_a) \cdot y \in W$ and $f(x_a) \in N$. This yields

$$\begin{aligned} \|(F + \delta f)(x_a + y)\|_{L^2} &\geq c\|y\|_E + \delta\|f(x_a)\|_G - \delta c'\|y\|_E + O(\|y\|^2_E) \\ &\geq c''\|y\|_E + \delta\|f(x_a)\|_{L^2} \end{aligned}$$

if $\delta > 0$ and $\|y\|_E$ are sufficiently small.
Thus $(F+\delta f)(x_a+y) = 0$ if and only if $y = 0$ and $f(x_a) = 0$. On the other hand from Lemma 2.1 we know that for small δ all 1-periodic solutions of $\dot{x} = X_{H_\delta}(t,x)$ are $H^{1,2}$-close to N.

Consequently, the only 1-periodic solutions are $x_a(t)$ with $h_0'(la) = 0$, hence x^+ or x^-.

3. Now we calculate the generalized Conley-Zehnder indices of $x^- = (0,0)$ and $x^+ = (1/2, 0)$. To do this consider the flow ψ_t of $\dot{x} = X_H(t,x)$ and its linearization

$$\Psi(t) := D\psi_t(a,0) \colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}, \ 0 \leqslant t \leqslant 1,$$

where $a = 0$ or $1/2$. It gives a path of symplectic matrices on $(\mathbb{R}^{2n}, \omega_0)$ starting at the identity I and ending in $Sp_1(2n)$. Since

$$X_{H_t} = 0 \ \text{ on } S^1 \times \{0\} \subset S^1 \times \mathbb{R}^{2n-1}$$

and the matrices $\Psi(t)$ are symplectic, it follows that

$$\Psi(t)e_1 = e_1 \ \text{ and } \Psi^*(t)e_2 = e_2,$$

where $\Psi^*(t)$ is the transpose of $\Psi(t)$ and e_1, e_2 are the first and the second vector of the standard basis in $\mathbb{R}^{2n}$. Let $\mu(\Psi)$ be the Maslov index of the path Ψ.
Denote by $\Psi_\delta(t)$ and $\Phi_\delta(t)$ the linearizations of the flows of

$$\dot{x} = X_{H_\delta}(t,x) \ \text{ and } \dot{x} = X_{\delta h}(x).$$

Hence

$$\Phi_\delta(t) = e^{\delta h_0''(a)tJ_0B},$$

where $B = diag(1, 0, \dots, 0)$. The paths $\Psi_\delta(t)$ and $\Psi(t)\Phi_\delta(t)$ are homotopic with endpoints in $Sp_0(2n)$. The homotopy between these two paths is given by

$$L(s,t) = \Psi_{s\delta}(t)\Phi_{(1-s)\delta}(t)$$

for $(s,t) \in [0,1] \times [0,1]$. If $\delta > 0$ is small enough, then the path $s \to L(s,1)$ belongs to $Sp_0(2n)$. To see this note that by the variation of constants formula we have

$$\Psi_\delta(t) = \Psi(t) + \delta R(t).$$

The term $R(t)$ can be written as $R(t) = -h_0''(a)R_0(t)+R_1(t)$, where $R_0(t)$ is given by

$$R_0(t) = \int_0^t \langle \Psi(s)\cdot, e_1\rangle \Psi(t)\Psi^{-1}(s)e_2 ds$$

and $R_1(t) \longrightarrow 0$ as $\delta \longrightarrow 0$. Assume now that our claim does not hold. Then we find sequences $s_k \longrightarrow s_0$, $\delta_k \longrightarrow 0$ and $X_k \in \mathbb{R}^{2n}$ with $|X_k| = 1$ such that

$$\Psi_{s_k\delta_k}(1)\Phi_{(1-s_k)\delta_k}(1)X_k = X_k.$$

Therefore

$$\begin{aligned}\Psi(1)X_k - X_k \quad &= -s_k\delta_k R(1)X_k - (1-s_k)\delta_k h_0''(a)\Psi(1)J_0BX_k \\ &\quad -s_k(1-s_k)\delta_k^2 h_0''(a)R(1)J_0BX_k.\end{aligned}$$

From this we get that $X_k \longrightarrow \pm e_1$ since $ker(\Psi(1) - I) = \mathbb{R}e_1$. On the other hand, taking an inner product of both sides with e_2 and then passing to the limit we obtain

$$\pm s_0 h_0''(a)\langle R_0(1)e_1, e_2\rangle = \pm(1-s_0)h_0''(a)\langle \Psi(1)J_0Be_1, e_2\rangle.$$

From the formula for R_0 we compute $\langle R_0(1)e_1, e_2\rangle = 1$, and since $\langle \Psi(1)J_0Be_1, e_2\rangle = -1$ we see that $-s_0h_0''(a) = (1-s_0)h_0''(a)$. This gives a contradiction and shows that $L(s, 1) \in Sp_0(2n)$ for all $s \in [0, 1]$.
In particular, taking $s = 1$ we see that x^- and x^+ are nondegenerate solutions of $\dot{x} = X_{H_\delta}(t, x)$ for all $\delta > 0$ small.
By using the homotopy

$$K_0(s,t) = \begin{cases} L(s, \frac{2t}{s+1}) & \text{if } t \leqslant \frac{s+1}{2}, \\ L(2t-1, 1) & \text{if } \frac{s+1}{2} \leqslant t \end{cases}$$

together with the fact that $L(s, 1) \in Sp_0(2n)$, for $s \in [0, 1]$, and properties (1), (2) and (3) of the Maslov index, we conclude that

$$\mu(\Psi_\delta) = \mu(\Psi\Phi_\delta).$$

Another homotopy

$$K_1(s,t) = \begin{cases} \Psi(\frac{2t}{s+1})\Phi_\delta(st) & \text{if } t \leqslant \frac{s+1}{2}, \\ \Psi(1)\Phi_\delta((s+2)t - (s+1)) & \text{if } \frac{s+1}{2} \leqslant t. \end{cases}$$

for $(s, t) \in [0, 1] \times [0, 1]$ with the properties of the Maslov index imply that

$$\mu(\Psi\Phi_\delta) = \mu(\Psi) + \mu(\Psi(1)\Phi_\delta).$$

Hence to show the index formula it is enough to compute $\mu(\Psi(1)\Phi_\delta)$. For this we will use the definition of the index. We abbreviate $M(t) = \Psi(1)\Phi_\delta(t)$ and define

$$\Lambda(t) = \{(X, M(t)X) \mid X \in \mathbb{R}^{2n}\}.$$

Λ defines a path of Lagrangian subspaces of $(\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}, (-\omega_0) \oplus \omega_0)$. One easily sees that $\Lambda(t) \cap \Delta = \{0\}$ for $t \in (0, 1]$ and

$$\Lambda(0) \cap \Delta = \{(X,X) \mid X = (x,0) \in \mathbb{R}^{2n}, x \in \mathbb{R}\}$$

Put $W = \{(X, -\Psi(1)X) \mid X \in \mathbb{R}^{2n}\}$. Then W is a Lagrangian complement of $\Lambda(0)$. For $\hat{X} = (X,X) \in \Lambda(0) \cap \Delta$ we define a vector $\hat{w}(t) = (w(t), -\Psi(1)w(t)) \in W$ with $w(t)$ satisfying

$$X - \Psi(1)w(t) = M(t)(X + w(t)).$$

For such $\hat{w}(t)$ we have $\hat{X} + \hat{w}(t) \in \Lambda(t)$. After differentiating the formula for $w(t)$ at $t = 0$ we obtain that $\dot{w}(0) = \frac{x}{2}\delta h_0''(a)e_2$. Now we have

$$((-\omega_0) \oplus \omega_0)(\hat{X}, \hat{w}(t)) = -2x\langle e_2, w(t)\rangle$$

which after differentiating at $t = 0$ gives

$$Q_0(X) = \frac{d}{dt}\Big|_{t=0} ((-\omega_0) \oplus \omega_0)(\hat{X}, \hat{w}(t)) = -\delta h_0''(a)x^2.$$

Therefore,

$$\mu(\Psi(1)\Phi_\delta) = \begin{cases} \frac{1}{2} & \text{if } h_0''(a) < 0, \\ -\frac{1}{2} & \text{if } h_0''(a) > 0 \end{cases}$$

which shows that $\mathrm{ind}_{RS}(x^\pm) = \mathrm{ind}_{RS}(\bar{x}) \pm \frac{1}{2}$.

4. In this step we prove the statement about the boundary operator.
Using the notation of step 2, we consider $u \in \mathcal{M}(x^-, x^+, J, H_\delta, U)$ as a differentiable curve $u: \mathbb{R} \to E$, $s \mapsto u(s)$, satisfying

$$u'(s) = F\big(u(s)\big) + \delta f\big(u(s)\big).$$

Note that from Lemma 2.1 it follows that $sup_{s\in\mathbb{R}}|a'(s)|$ and $sup_{s\in\mathbb{R}}\|y(s)\|_{H^{1,2}(S^1)}$ converge to zero as $\delta \to 0$. Writing $u(s) = x_{a(s)} + y(s)$ with $x_{a(s)} \in N$ and $y(s) \in W$ the above equation becomes

$$\begin{aligned} x_{a'(s)} + y'(s) &= F(x_{a(s)}) + \delta f(x_{a(s)}) + DF(x_{a(s)}) \cdot y(s) \\ &\quad + \delta Df(x_{a(s)}) \cdot y(s) + O\left(\|y(s)\|^2_{H^{1,2}(S^1)}\right), \end{aligned}$$

where $F(x_{a(s)}) = 0$, $f(x_{a(s)}) \in N$ and $DF(x_{a(s)}) \cdot y(s) \in W$. Collecting together the terms in N and in W and taking the L^2 norm over $\mathbb{R} \times S^1$ we obtain

$$\|x_{a'} - \delta f(x_a)\|^2_{L^2(\mathbb{R}\times S^1)} + \|y' - DF(x_a)\cdot y\|^2_{L^2(\mathbb{R}\times S^1)} \leqslant \epsilon\|y\|^2_{H^{1,2}(\mathbb{R}\times S^1)}$$

with $\epsilon > 0$ becoming arbitrarily small as $\delta \to 0$.

In step 2 we showed that for every $s \in \mathbb{R}$,

$$DF(x_{a(s)})\Big|_{W\cap E} : W \cap E \to W$$

is an isomorphism. Moreover, as $\delta \to 0$, $a'(s)$ becomes small in the supremum norm, which implies that $\frac{d}{ds}\big(DF(x_{a(s)})\big)$ gets small in the operator norm on $\mathcal{L}(W \cap E, W)$ Thus we can apply Proposition 3.14 of [16] to obtain

$$\|y' - DF(x_a)\cdot y\|_{L^2(\mathbb{R}\times S^1)} \geq c\|y\|_{H^{1,2}(\mathbb{R}\times S^1)}$$

with some constant $c > 0$. Combining the last two estimates we get, for $\epsilon < c^2$,

$$\|x_{a'} - \delta f(x_a)\|_{L^2}^2 + (c^2 - \epsilon)\|y\|_{H^{1,2}}^2 \leqslant 0\,.$$

Hence $y = 0$ and $x_{a'} - \delta f(x_a) = 0$.

Consequently, the only solutions are the usual gradient lines $u(s,t) = (a(s),0) \in S^1 \times \mathbb{R}^{2n-1}$ with $a'(s) = \delta h_0{}'(a(s))$ and $a(s) \to 0$, respectively $\frac{1}{2}$, as $s \to \pm\infty$. Since there are exactly two gradient lines on S^1 connecting the minimum and maximum of δh_0, the boundary operator between x^- and x^+ is zero (recall that we use $\mathbb{Z}_2$ coefficients), and the proposition is proved.

3 Symplectic homology

We shall give a definition of symplectic homology groups $S^{[a,b)}(M,J)$ which is slightly different from [3] and more adapted to the present situation. We are only going to deal with case A. The only difference in case B is that we do not have well-defined Conley-Zehnder indices. At the end of [3] it has been explained how one can still define relative indices and carry out the same construction.

We call a smooth Hamiltonian $H: S^1 \times M \to (-\infty,0]$ *admissible* if $H(t,x) = 0$ for x in some neigborhood of ∂M, and if every 1-peridodic solution of $\dot{x} = X_H(t,x)$ with

$$\int_0^1 H\left(t,x(t)\right)dt < 0$$

is nondegenerate.

Let $\tilde{\mathcal{J}}$ be the space of almost complex structures defined in the interior of M which has been introduced in [3]. A pair (J,H) with $J \in \tilde{\mathcal{J}}$ and H admissible is called an admissible pair if 0 is a regular value of the Fredholm section

$$u \mapsto u_s + J(t,u)u_t + \nabla H(t,u),$$

for more details see [11]. Fix a $\tilde{J} \in \tilde{\mathcal{J}}$ and let $\mathrm{Ad}(\tilde{J},M)$ be the set of admissible pairs (J,H) where $J = \tilde{J}$ near the boundary ∂M. We introduce a partial ordering on $\mathrm{Ad}(\tilde{J},M)$ by

$$(J_1,H_1) \leqslant (J_2,H_2) \Leftrightarrow H_1(t,z) \leqslant H_2(t,z) \text{ for all } (t,z) \in S^1 \times M\ .$$

Let

$$\begin{aligned} P_a(H)^k \quad := \quad \Big\{x: S^1 \to M \quad & \text{contractible 1-periodic solution of} \\ & \dot{x} = X_H(t,x) \mid \int_0^1 H\left(t,x(t)\right)dt < 0, \\ & \Phi_H(x) \geq a, \quad \mathrm{ind}_{\mathrm{RS}}(x) = k\Big\}, \end{aligned}$$

where the energy of a periodic orbit x, $\Phi_H(x)$, is defined as

$$\Phi_H(x) = -\int_D \bar{x}^* w - \int_0^1 H\left(t, x(t)\right) dt$$

for some extension $\bar{x}$ of x to a disk D. We get vector spaces

$$C_k^a(J, H) = \bigoplus_{x \in P_a(H)^k} \mathbb{Z}_2 x$$

and boundary operators $\partial_k : C_k^a(J, H) \to C_{k-1}^a(J, H)$ defined by

$$\partial_k(x) = \sum_{y \in P_a(H)^{k-1}} \langle \partial_k x, y \rangle \cdot y,$$

where $\langle \partial_k x, y \rangle$ is the number mod 2 of points in $\mathcal{M}(x, y, J, H)/\mathbb{Z}$.

If $a < 0$ the boundary operator will not satisfy $\partial_k \circ \partial_{k+1} = 0$ because of gradient lines connecting elements of $P_a(H)^k$ to solutions with $\int_0^1 H(t, x)\, dt = 0$. But if we take either $0 < a < b$ or $a < b < 0$, in the quotient $C_k^a(J, H)/C_k^b(J, H)$ solutions with $\int_0^1 H(t, x)\, dt = 0$ (and hence $\Phi_H(x) = 0$) will either not appear or be zero, and thus the induced operator

$$C_k^a / C_k^b \to C_{k-1}^a / C_{k-1}^b$$

will satisfy

$$\partial_k \circ \partial_{k+1} = 0.$$

Let

$$S_k^{[a,b)}(J, H) := \ker(\partial_k) / \operatorname{im}(\partial_{k+1})$$

be the associated homology groups. For $(J_1, H_1) \leqslant (J_2, H_2)$ we have induced maps

$$\sigma(J_2, H_2, J_1, H_1) : S_k^{[a,b)}(J_2, H_2) \to S_k^{[a,b)}(J_1, H_1).$$

So we can pass to the direct limit over $\mathrm{Ad}(\tilde{J}, M)$ as $(J, H) \to -\infty$ to obtain the symplectic homology groups

$$S_k^{[a,b)}(M, \tilde{J}) := \varinjlim S_k^{[a,b)}(J, H).$$

These groups are invariant under symplectic diffeomorphisms of the interior in the following sense. Let $(M, \partial M)$ and $(N, \partial N)$ satisfy the conditions of Theorem 1.2 and $\psi : \dot{M} \to \dot{N}$ be a symplectic diffeormorphism. If $\tilde{J} \in \tilde{\mathcal{J}}(N)$, then $\psi^* \tilde{J}$ defined by $\psi^* \tilde{J}(x) = T\psi(x)^{-1} \circ \tilde{J}\left(\psi(x)\right) \circ T\Psi(x)$ is an element of $\tilde{\mathcal{J}}(M)$ and

$$S_k^{[a,b)}(M, \psi^* \tilde{J}) \cong S_k^{[a,b)}(N, \tilde{J}).$$

Now we are going to relate certain symplectic homology groups to periodic trajectories on the boundary.

Recall that there exists an outward pointing vector field η near ∂M satisfying $L_\eta \omega = \omega$. The flow $(\psi_t)_{-r < t \leqslant 0}$ of η yields a diffeomorphism $\psi : (-r, 0] \times \partial M \to$

W onto some neigborhood W of ∂M. Let $W_s := \psi_s(\partial M)$ and $\tau: W \to (-r, 0]$ be the function defined by $\tau_{|W_s} = s$. The restriction of $\lambda = i_\eta \omega$ to W_s is a contact form with contact bundle $\xi = \ker(\lambda|_{W_s})$ and Reeb vector field $X \in TW_s$ defined by

$$i_X \, d\lambda\big|_{W_s} = 0 \text{ and } i_X \lambda = 1.$$

It will be useful to rescale as follows. Let $\hat{\eta} = e^{-\tau}\eta$, $\hat{\lambda} = e^{-\tau}\lambda$, $\hat{X} = e^{\tau}X$. Denote by $\hat{\psi}_t$ the flow of $\hat{\eta}$ and let $\hat{W}_s := \hat{\psi}_s(\partial M)$ and $\hat{\tau}: \hat{W} \to (-\hat{r}, 0]$ be the function defined by $\hat{\tau}|_{\hat{W}_s} = s$. Finally let $\hat{\mathscr{X}}_t$ be the flow of $-\hat{X}$.

One easily verifies that $\hat{\psi}_t$ preserves $d\hat{\tau}$, $\hat{\lambda}$ and $d\hat{\lambda}$. The vector field $\hat{X}$ is uniquely characterized by the conditions $\hat{X} \in \ker(d\hat{\tau})$, $i_{\hat{X}} \, d\hat{\lambda}\big|_{\ker(d\hat{\tau})} = 0$ and $i_{\hat{X}}\hat{\lambda} = 1$, and so it is also preserved by $\hat{\psi}_t$. Hence we obtain

$$\frac{d}{dt}(\hat{\psi}_s \circ \hat{\mathscr{X}}_t) = D\hat{\psi}_s \cdot \left(-\hat{X}(\hat{\mathscr{X}}_t)\right) = -\hat{X}(\hat{\psi}_s \circ \hat{\mathscr{X}}_t),$$

and, by the uniqueness of the flow, this implies that

$$\hat{\psi}_s \circ \hat{\mathscr{X}}_t = \hat{\mathscr{X}}_t \circ \hat{\psi}_s.$$

Since $\hat{\mathscr{X}}_t$ is the Reeb flow for $\hat{\lambda}$, it clearly preserves $\hat{\lambda}|_{\hat{W}_s}$, ξ and $\hat{X}$. The above formula implies that it also preserves $\hat{\eta}$.

Now let $x \in \mathscr{C}_A$ be a positively parametrized periodic trajectory on ∂M of period $T = \int_{S^1} x^*\lambda$, perhaps multiple covered. We fix a parametrization (up to the choice of $x(0)$) by requiring $\dot{x} = -T \cdot X(x)$.

With x as above, the linearized system along x defines a path of symplectic maps $D\hat{\mathscr{X}}_t(x(0))$, $0 \leqslant t \leqslant 1$, satisfing

$$D\hat{\mathscr{X}}_t\big(x(0)\big): T_{x(0)}M \to T_{x(t)}M$$

and mapping $\hat{X}\big(x(0)\big)$ to $\hat{X}\big(x(t)\big)$, $\hat{\eta}\big(x(0)\big)$ to $\hat{\eta}\big(x(t)\big)$ and $\xi_{x(0)}$ to $\xi_{x(t)}$. The orbit x is called nondegenerate if the restriction

$$D\hat{\mathscr{X}}_1\big(x(0)\big): \xi_{x(0)} \to \xi_{x(0)}$$

does not have 1 in its spectrum. Take an extension $u : D \to M$ of x and symplectically trivialize $u^*TM \to D$. In view of our condition on c_1 the induced trivializations for $x^*TM \to S^1$ are homotopic and independent of the choice of the extension u. Hence take such a trivialization

$$\Psi : x^*TM \to S^1 \times \mathbb{R}^{2n}$$

and write $\Psi(t)$ for the map

$$\Psi(t) : T_{x(t)}M \to \mathbb{R}^{2n}.$$

Now define

$$\mathrm{ind}_{RS}(x) = \mathrm{ind}_{RS}(\Psi(t) \circ D\hat{\mathscr{X}}_t(x(0)) \circ \Psi^{-1}(0)).$$

Then by our previous discussion $\mathrm{ind}_{RS}(x)$ is a well-defined invariant of the 1-periodic orbit x.

Let $\phi: (-\hat{r}, 0] \to \mathbb{R}$ be a smooth function and consider the Hamiltonian

$$H = \phi \circ \hat{r}$$

defined on $\hat{W}$. It generates the Hamiltonian vector field X_H which is given by the formula $X_H = -\phi'(\hat{r})\hat{X}$. It follows that if $\phi'(\hat{s}) = T$ for some $\hat{s} \in (-\hat{r}, 0]$, then $\hat{x}(t) = \hat{\psi}_{\hat{s}} \circ x(t)$ is a 1-periodic solution of the equation $\dot{x} = X_H(x)$.

The vector field X_H generates a flow which is given by $\hat{\mathscr{X}}_t^H = \hat{\mathscr{X}}_{\phi'(\hat{r})t}$. Like the flow $\hat{\mathscr{X}}_t$, it preserves ξ and $\hat{X}$. To calculate $D\hat{\mathscr{X}}_t^H \cdot \hat{\eta}$ we use once again the formula

$$\hat{\psi}_s \circ \hat{\mathscr{X}}_t^H = \hat{\mathscr{X}}_t^H \circ \hat{\psi}_s .$$

Differentiating the above formula with respect to s at $s = 0$ we get

$$D\hat{\mathscr{X}}_t^H \cdot \hat{\eta} = \hat{\eta}(\hat{\mathscr{X}}_t^H) + \phi''(\hat{r})t\big(-\hat{X}(\hat{\mathscr{X}}_t^H)\big).$$

Hence with respect to the splitting

$$T_{\hat{x}(t)}M = \mathbb{R}\big(-\hat{X}\big(\hat{x}(t)\big)\big) \oplus \mathbb{R}\hat{\eta}\big(\hat{x}(t)\big) \oplus \xi_{\hat{x}(t)},$$

the mapping $D\hat{\mathscr{X}}_t^H\big(\hat{x}(0)\big) : T_{\hat{x}(0)}M :\longrightarrow T_{\hat{x}(t)}M$ has the following matrix form

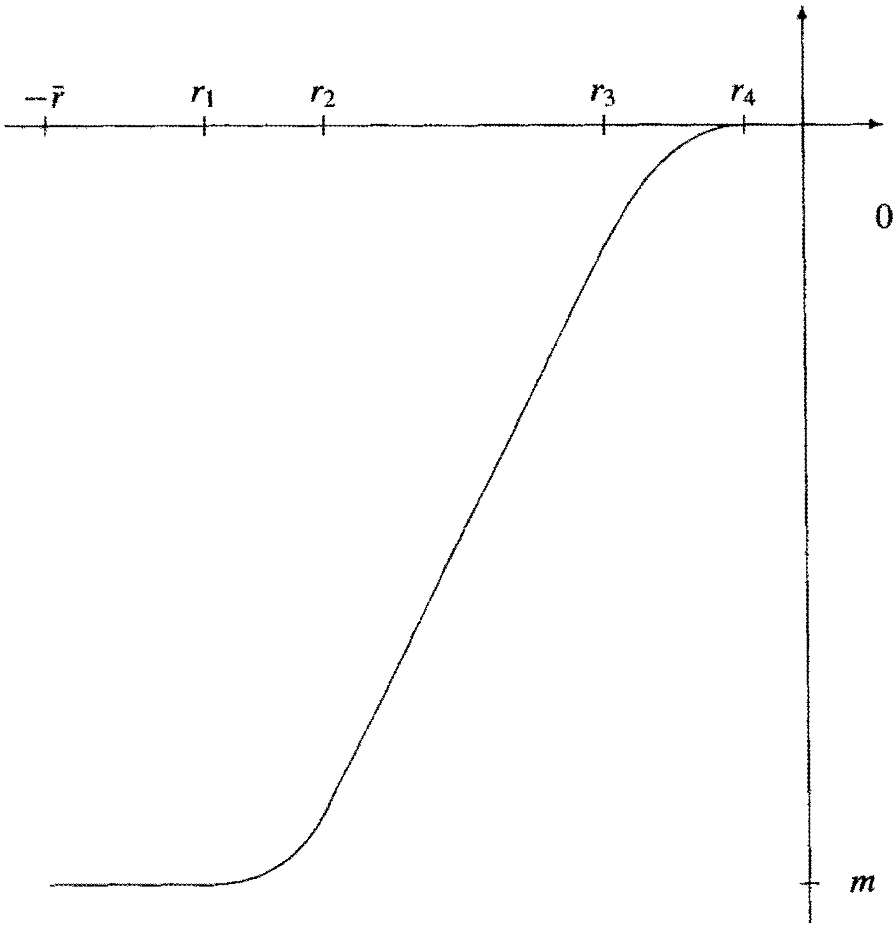
$$\left(\begin{array}{cc|c} 1 & \phi''(\bar{s})t & \\ 0 & 1 & \smash{\raisebox{0.5ex}{0}} \\ \hline \multicolumn{2}{c|}{0} & * \end{array}\right).$$

This shows that $\hat{x}$ is transversally nondegenerate (in the sense of section 2) if and only if x is nondegenerate and $\phi''(\bar{s}) \neq 0$. A similar calculation like in section 2 gives the following index formula

$$\mathrm{ind}_{\mathrm{RS}}(\hat{x}) = \mathrm{ind}_{\mathrm{RS}}(x) + \begin{cases} \frac{1}{2} & \text{if } \phi''(\hat{s}) < 0, \\ -\frac{1}{2} & \text{if } \phi''(\hat{s}) > 0. \end{cases}$$

Now let us consider smooth functions $\phi: (-\hat{r}, 0] \to \mathbb{R}$ of the following form: there exist numbers $-\hat{r} < r_1 < r_2 < r_3 < r_4 < 0$ such that:

- $\phi \equiv m$ for some negative constant m on $(-\hat{r}, r_1]$
- $\phi'' > 0$ on (r_1, r_2)
- $\phi(s) = ps + q$ for some constants p, $q > 0$ on $[r_2, r_3]$ where p is not the period $\int_{S^1} x^*\lambda$ of any periodic trajectory on ∂M
- $\phi'' < 0$ on (r_3, r_4)
- $\phi \equiv 0$ on $[r_4, 0]$.



We extend the Hamiltonian $H = \phi \circ \hat{r}$ on $\hat{W}$ to all of M by putting $H \equiv m$ on $M \setminus \bar{W}$.

The Hamiltonian vector field X_H has of course plenty of constant solutions with energy $-m$ and 0. Any periodic trajectory x on ∂M with period T between 0 and p gives rise to two 1-periodic solutions x_1, x_2 of X_H corresponding to the two values $s_1 \in (r_1, r_2)$ and $s_2 \in (r_3, r_4)$, where ϕ has slope T. By the assumption on ϕ'' these solutions are transversally nondegenerate and have indices $\mathrm{ind}_{\mathrm{RS}}(x) \pm \frac{1}{2}$. Moreover X_H does not have any other 1-periodic solutions except those just described.

Near any nonconstant solution x_i we perform a small time-dependent perturbation of H as in section 2. The solution x_i splits into two nondegenerate 1-periodic solutions $x_i^{\pm}$ corresponding to the maximum and minimum of a Morse function h_0 on S^1. They have indices $\mathrm{ind}_{\mathrm{RS}}(x_i) \pm \frac{1}{2}$ and energy $\Phi_H(x_i) - \delta \max(h_0)$, respectively $\Phi_H(x_i) - \delta \min(h_0)$, and they generate the local Floer homology $H_*^{\mathrm{loc}}(x_i) = \mathbb{Z}_2 x_i^- \oplus \mathbb{Z}_2 x_i^+$.

By another small perturbation we remove the degenerate constant solutions in $M \setminus (r_1, 0] \times \partial M$, leaving a finite number of nondegenerate 1-periodic solutions with energy near $-m$ (and carrying the homology of M).

The resulting Hamiltonian is admissible. We denote by $\overrightarrow{\mathrm{Ad}}(\tilde{J}, M)$ the collection of all admissible pairs $(J, H) \in \mathrm{Ad}(\tilde{J}, M)$ with H as above. Here $\tilde{J} \in \tilde{\mathcal{J}}$ is a fixed almost complex structure.

$\overrightarrow{\mathrm{Ad}}(\tilde{J},M)$ is a cofinal subset of the partially ordered set $\mathrm{Ad}(\tilde{J},M)$. This means that for any $(J,H) \in \mathrm{Ad}(\tilde{J},M)$ there exists a $(J_1,H_1) \in \overrightarrow{\mathrm{Ad}}(\tilde{J},M)$ with $(J_1,H_1) \leqslant (J,H)$. In view of this property we can calculate the symplectic homology by taking the direct limit only over the subset $\overrightarrow{\mathrm{Ad}}(\tilde{J},M)$.

Let $(a,k) \in \mathscr{A}_A(\partial M)$ be of multiplicity n. Thus there are n distinct nondegenerate periodic trajectories $x^{(1)},\dots,x^{(n)}$ on ∂M with $A(x^{(i)}) = a$ and $\mathrm{ind}_{\mathrm{RS}}(x^{(i)}) = k$. Assume first that there are no $(a,l) \in \mathscr{A}_A(\partial M)$ with $l \neq k$.

Choose $\epsilon > 0$ small enough such that there are no other elements of $\mathscr{A}_A(\partial M)$ with action in $[a-2\epsilon, a+2\epsilon]$.

If $(H,J) \in \overrightarrow{\mathrm{Ad}}(\tilde{J},M)$ is small, the maximal slope p of H will be larger than the periods of $x^{(1)},\dots,x^{(n)}$.

As described above, each periodic trajectory x on ∂M with period strictly less than p gives rise to four 1-periodic solutions $x_1^{\pm}$, $x_2^{\pm}$ with actions

$$\begin{aligned} \Phi_H(x_i^+) &= A(x) - \phi(s_i) - \delta\max(h_0) \\ \Phi_H(x_i^-) &= A(x) - \phi(s_i) - \delta\min(h_0) \end{aligned}$$

and indices

$$\begin{aligned} \mathrm{ind}_{\mathrm{RS}}(x_1^-) &= \mathrm{ind}_{\mathrm{RS}}(x) - 1 \\ \mathrm{ind}_{\mathrm{RS}}(x_1^+) &= \mathrm{ind}_{\mathrm{RS}}(x_2^-) = \mathrm{ind}_{\mathrm{RS}}(x) \\ \mathrm{ind}_{\mathrm{RS}}(x_2^+) &= \mathrm{ind}_{\mathrm{RS}}(x) + 1. \end{aligned}$$

As H decreases, $\Phi_H(x_1^{\pm}) \to +\infty$ and $\Phi_H(x_2^{\pm}) \to A(x)$. The actions of the solutions in $M \setminus (r_1,0] \times \partial M$ also tend to $+\infty$. So eventually the 1-periodic solutions of X_H with action in $[a-\epsilon, a+\epsilon)$ will be exactly $x_2^{(1)\pm},\dots,x_2^{(n)\pm}$. In other words,

$$C_*^{a-\epsilon}(J,H)/C_*^{a+\epsilon}(J,H) = \bigoplus_{i=1}^{n}\left(\mathbb{Z}_2 x_2^{(i)-} \oplus \mathbb{Z}_2 x_2^{(i)+}\right).$$

If for arbitrarily small δ there would exists gradient lines between some $x_2^{(i)\pm}$ and $x_2^{(j)\pm}$, $i \neq j$, then in the limit we would obtain a nonconstant gradient line having the same action at both ends, a contradiction. Hence for small δ there are no gradient lines between $x_2^{(i)\pm}$ and $x_2^{(j)\pm}$, $i \neq j$. Therefore

$$S_*^{[a-\epsilon,a+\epsilon)}(J,H) = \bigoplus_{i=1}^{n} HF_*^{\mathrm{loc}}(x_2^{(i)}) = \bigoplus_{i=1}^{n}\left(\mathbb{Z}_2 x_2^{(i)-} \oplus \mathbb{Z}_2 x_2^{(i)+}\right)$$

by Proposition 2.2, for $(J,H) \in \overrightarrow{\mathrm{Ad}}(\tilde{J},M)$ and $\delta > 0$ sufficiently small. If $(J_i,H_i) \in \overrightarrow{\mathrm{Ad}}(\tilde{J},M)$ with $(J_1,H_1) \leqslant (J_2,H_2)$ then they can be connected by a monotone homotopy (J_s,H_s) $1 \leqslant s \leqslant 2$, such that no X_{H_s} has a 1-periodic solution with action in $[a-3\epsilon/2, a-\epsilon/2] \cup [a+\epsilon/2, a+3\epsilon/2]$. In the language of [3], the "gap" $g\big(H_s,[a-\epsilon,a+\epsilon)\big) \geq \epsilon/2$ for all s. We take a sequence $1 = s_1 < s_2 < \dots < s_l = 2$ with $d(H_{s_i},H_{s_{i+1}}) < \epsilon/2$ for $i = 1,\dots,l-1$, where d is the distance defined in [3]. It follows from the results in [3] that the homomophism $\sigma(J_2, H_2, J_1, H_1)$ is equal to the composition of the small distance isomorphisms $\sigma(J_{s_{i+1}}, H_{s_{i+1}}, J_{s_i}, H_{s_i})$.

An indirect argument as above shows that if $|s_{i+1} - s_i|$ is small, then there cannot exist a flow line between solutions $x_2^{(j)\pm}$ of H_{s_i} and $x_2^{(k)\pm}$ of $H_{s_{i+1}}$ with $j \neq k$. Thus $\sigma(J_{s_{i+1}}, H_{s_{i+1}}, J_{s_i}, H_{s_i})$ must map each $x_2^{(j)+}$ to $x_2^{(j)+}$ and $x_2^{(j)-}$ to $x_2^{(j)-}$. Hence the same is true for $\sigma(J_2, H_2, J_1, H_1)$.

This knowledge of the homomorphisms $\sigma(J_2, H_2, J_1, H_1)$ implies that the system $x_2^{(i)\pm}$ for $(J, H) \in \overrightarrow{\mathrm{Ad}}(\tilde{J}, M)$ give rise to basis elements $\bar{x}_2^{(i)\pm}$ of the directed limit with indices $\mathrm{ind}_{\mathrm{RS}}(\bar{x}_2^{(i)-}) = k$ and $\mathrm{ind}_{\mathrm{RS}}(\bar{x}_2^{(i)+}) = k + 1$. Hence

$$S_j^{[a-\epsilon, a+\epsilon)}(M, \tilde{J}) = \begin{cases} \mathbb{Z}_2^n & \text{if j=k} \\ \mathbb{Z}_2^n & \text{if j = k+1} \\ 0 & \text{otherwise}. \end{cases}$$

Finally, if there are $(a, k_j) \in \mathscr{A}_A(\partial M)$ for various $k_j \in \mathbb{Z}$, one shows again that no flow lines exist between solutions corresponding to different (a, k_j). So each (a, k_j) gives a contribution as above to the symplectic homology $S_*^{[a-\epsilon, a+\epsilon)}(M, \tilde{J})$, and Theorem 1.1 is proved.

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