

Symplectic homology II: a general construction

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Symplectic homology II

A general construction

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1 Introduction

In [11] the second and third author introduced a symplectic homology theory. By means of a general construction, given real numbers $a < b$ and an integer k they assigned to each open set U of \mathbb{C}^n a group $S_k^{[a,b]}(U)$ and studied its properties. In the present paper which continues the work in [11], we show how this construction can be carried over to more general manifolds. We assume the reader to be familiar with [11], since there are many constructions which we recall here in a more general set up without giving a detailed proof. In fact, the arguments given in [11] work under more general circumstances at least if some topological assumptions are met. Only in the case that there is a considerable difference we give complete details.

For applications we refer the reader to [13] and the forthcoming paper [14]. For motivation of the present construction we refer the reader to [11]. However we recall that the crucial observation made [2] is that periodic orbits for Hamiltonian systems can be used to construct many new symplectic invariants. Our construction in [11] and the present paper precisely exploits the same aspects of this observation. We also would like to point the attention to [3, 4, 5] for other applications of this "philosophy".

* Andreas Floer died on May 15th, 1991.

2 A general construction

In the following we describe the construction under suitable hypotheses on the symplectic manifold (M, ω) . Other assumptions allow similar constructions and we outline some generalizations later on.

We assume (M, ω) is a compact symplectic manifold with or without boundary, such that $[\omega]$ vanishes on $\pi_2(M)$. Moreover we assume that the first Chern class c_1 for pullback bundles $u^*TM \rightarrow \mathbb{S}^2$ and $[u] \in \pi_2(M)$ vanishes, where $TM \rightarrow M$ carries the structure of a complex vectorbundle induced by a ω -calibrated almost complex structure J . (" ω -calibrated" means that $\omega \circ (J \times id)$ is a Riemannian metric. The space of such structures is contractible.)

If $\partial M \neq \emptyset$, we assume that ∂M is of contact type. This means that there exists an outward pointing transversal vector field η defined on an open neighborhood of ∂M in M such that the Lie derivative satisfies $L_\eta \omega = \omega$. Equivalently (set $\lambda = i_\eta \omega$), there exists a 1-form λ on a neighborhood of ∂M such that $d\lambda = \omega$ and $\lambda \wedge (d\lambda)^{n-1}|_{\partial M}$ is a volume form determining the orientation on ∂M induced from the orientation ω^n on M .

We call a smooth Hamiltonian $H : S^1 \times M \rightarrow \mathbb{R}$ **admissible** if

- There exists a nonempty open subset U of M with

$$\text{cl}(U) \subset M \setminus \partial M \quad \text{and} \quad H|(\mathbb{S}^1 \times \text{cl}(U)) < 0.$$
- If $\partial M = \emptyset$, all 1-periodic solutions are nondegenerate .
- If $\partial M \neq \emptyset$, there exists a constant $m = m(H) > 0$,
such that $H(t, x) = m$ for all $(t, x) \in \mathbb{S}^1 \times W$, where W
is a neighborhood of ∂M . Moreover, every 1-periodic
solution $x : \mathbb{S}^1 \rightarrow M$ satisfying $\int_0^1 H(t, x(t))dt < m$ is
nondegenerate. Further, $H(t, x) \leq m$ for all $(t, x) \in$
 $\mathbb{S}^1 \times M$ and the set of 1-periodic solutions x with
 $\int_0^1 H(t, x(t))dt < m$ is finite.

We denote by $C_{\text{cntr}}^\infty(\mathbb{S}^1, M)$ the set of all smooth contractible loops in M . Given $x \in C_{\text{cntr}}^\infty(\mathbb{S}^1, M)$ we denote by $\bar{x} : D \rightarrow M$ a smooth extension of x to the disk. We define the action of x denoted by $A(x)$ via

$$A(x) = - \int_D \bar{x}^* \omega. \quad (2)$$

This is well defined in view of the assumption $[\omega]|_{\pi_2(M)} = 0$. Given a smooth arc in $C_{\text{cntr}}^\infty(\mathbb{S}^1, M)$ say $(\tau, t) \mapsto x_\tau(t)$ is smooth, the map $\tau \mapsto A(x_\tau)$ is of class C^∞ . Let $x_0 = x$, and $\xi(t) = \frac{d}{d\tau} x_\tau(t)|_{\tau=0}$. We compute

$$\begin{aligned}\frac{d}{d\tau} \Big|_{\tau=0} A(x_\tau) &= - \int_0^1 \omega(\xi(t), \dot{\xi}(t)) dt \\ &= \int_0^1 \omega(\dot{\xi}(t), \xi(t)) dt.\end{aligned}$$

We define Φ_H by

$$\Phi_H(x) = A(x) - \int_0^1 H(t, x(t)) dt$$

and observe that

$$d\Phi_H(x)\xi = \int_0^1 \omega(\dot{x}(t) - X_{H_t}(x(t)), \xi(t)) dt$$

for all smooth vector fields ξ along the contractible curve $x \in C_{\text{cntr}}^\infty(\mathbb{S}^1, M)$. Here X_{H_t} is the vector field defined by $i_{X_{H_t}}\omega = dH_t$. Hence

$$d\Phi_H(x) = 0 \iff \dot{x}(t) = X_{H_t}(x(t)), \quad t \in \mathbb{S}^1$$

If H is admissible and x is a contractible 1-periodic solution of the Hamiltonian system associated to H satisfying $\Phi_H(x) > -m(H)$, we infer that $\int_0^1 H(t, x(t)) dt < m(H)$. In fact, if we have equality x has to be constant and $\Phi_H(x) = -m(H)$. Hence x has to be nondegenerate.

We denote for $a \in \mathbb{R}, a > -m(H)$ by $\mathcal{P}_a(H)$ the finite set consisting of all contractible 1-periodic solutions satisfying $\Phi_H(x) \geq a$. We call $\Phi_H(x)$ the energy of x . The energy will be one of the important numerical values attached to a contractible 1-periodic solution. Another invariant is the Conley-Zehnder index, [1]. The version we need is described [19]. Let x be as above and linearize the Hamiltonian vector field along x . The linearized system defines a symplectic loop $X(t)$ satisfying

$$X(t) : T_{x(0)}M \longrightarrow T_{x(t)}M$$

is symplectic and

$$X(0) = \text{Id}, \quad X(1) : T_{x(0)} \longrightarrow T_{x(0)}$$

since $x(0) = x(1)$. We infer that $1 \notin \text{spec}(X(1))$ since x is nondegenerate. Take an extension $\bar{x} : D \rightarrow M$ of x and symplectically trivialize $\bar{x}^*TM \rightarrow D$. In view of the condition on c_1 the trivializations induced for $x^*TM \rightarrow \mathbb{S}^1$ are homotopic and independent of the choice of the extension \bar{x} . Hence take such a trivialization

$$\Psi : x^*TM \xrightarrow{\cong} \mathbb{S}^1 \times \mathbb{C}^n.$$

Let us write $\Psi(t)$ for the induced map

$$\Psi(t) : T_{x(t)}M \longrightarrow \mathbb{C}^n.$$

We consider the arc Γ given by $t \mapsto \Psi(t)X(t)\Psi(0)^{-1}$. Then $\Gamma(0) = \text{Id}$ and $\Gamma(1) = \Psi(0)X(1)\Psi(0)^{-1}$ so that $\text{spec}(\Gamma(1)) = \text{spec}(X(1))$. Hence $1 \notin$

$\text{spec}(\Gamma(1))$. According to [19], Γ has a Conley-Zehnder index $\text{ind}_{CZ} \in \mathbb{Z}$, which is independent of the choice of Ψ (if Ψ is as described above). Hence $\text{ind}_{CZ}(x) := \text{ind}_{CZ}(\Gamma) \in \mathbb{Z}$ is a well defined invariant of a contractible nondegenerate 1-periodic solution.

Summing up we have a map $\mathcal{P}_a(H) \rightarrow \mathbb{R} \times \mathbb{Z}$ associating to $x \in \mathcal{P}_a(H)$ the "local invariants" $(\Phi_H(x), \text{ind}_{CZ}(x))$, where $a > -m(H)$ and H is admissible. Then for every $x \in \mathcal{P}_a(H)$ we have $x(\mathbb{S}^1) \subset M \setminus \partial M$.

A smooth time-dependent almost complex structure $J : S^1 \times TM \rightarrow TM$ is called ω -calibrated if $\omega \circ (J_t \times id)$ is a time-dependent Riemannian metric on M , where $J_t = J(t, \cdot)$. If $\partial M \neq \emptyset$ we further assume the following: There exists an outward pointing transversal vector field η near ∂M with $L_\eta \omega = \omega$. The flow $(\psi_t)_{-\varepsilon < t \leq 0}$ of η yields a diffeomorphism $\psi : (-\varepsilon, 0] \times \partial M \rightarrow W$ onto some neighbourhood W of ∂M in M . Let $W_s := \psi_s(\partial M)$. The restriction of $\lambda = i_\eta \omega$ to W_s is a contact form with contact bundle $\xi = \ker(\lambda|_{W_s})$ and Reeb vector field $X \in TW_s$ defined by $i_X d\lambda|_{W_s} = 0$, $i_X \lambda = 1$. We assume that on W , J is time-independent, leaves ξ invariant and maps X onto η . Denote by \mathcal{J} the set of all such J .

We claim that \mathcal{J} is nonempty, and any two elements of \mathcal{J} can be joined by a smooth path $(J_\tau)_{0 \leq \tau \leq 1}$ in \mathcal{J} .

To see this observe that the existence of a W, η, λ, ξ, X as described above follows directly from our assumption that ∂M is of contact type. Now it is well-known that for every symplectic vector bundle (E, ω) the space of almost complex structures on E for which $\omega \circ (J \times id)$ is a bundle metric is nonempty and contractible. So we find an almost complex structure on ξ such that $\omega|_\xi \circ (J \times id)$ is a bundle metric on ξ . Extending it first to the bundle $TM|_W$ via $J(X) = \eta$, $J(\eta) = -X$, and then to the whole bundle $TM \rightarrow M$, we obtain a $J \in \mathcal{J}$. For the second part let $J^0, J^1 \in \mathcal{J}$ be given with associated vectorfields η_0, η_1 near ∂M . Since both η_0 and η_1 are transversal to ∂M and outward pointing, the same is true for all $\eta_\tau := \tau \eta_1 + (1 - \tau) \eta_0$, $0 \leq \tau \leq 1$. Moreover, $L_{\eta_\tau} \omega = \omega$ for all τ . Arguing as above we can associate to every η_τ a $\tilde{J}^\tau \in \mathcal{J}$, depending smoothly on τ . We can also achieve that $\tilde{J}^0 = J^0$, $\tilde{J}^1 = J^1$, and the claim is proved.

Now consider pairs (J, H) , where H is an admissible Hamiltonian and $J \in \mathcal{J}$.

Given $x, y \in \mathcal{P}_a(H)$, $a > -m(H)$, we consider the set $\mathcal{M}(x, y; J, H)$ defined by

$$\mathcal{M}(x, y; J, H) = \{u : Z \rightarrow M \mid u \text{ is smooth and satisfies (3) below}\},$$

where $Z = \mathbb{R} \times \mathbb{S}^1$, $(s, t) \in Z$, and

$$\begin{aligned} u_s + J(t, u)u_t + \nabla_{J_t} H(t, u) &= 0 \\ u(s, *) &\longrightarrow x \text{ in } C^\infty \text{ for } s \longrightarrow -\infty \\ u(s, *) &\longrightarrow y \text{ in } C^\infty \text{ for } s \longrightarrow +\infty. \end{aligned} \tag{3}$$

Here ∇_{J_t} is the gradient taken with respect to the Riemannian metric $\omega \circ (J_t \times id)$. It follows from [18], Lemma 2.4, that if W is a foliated neighborhood of ∂M as described above and $H \equiv m(H)$ on W , then $u(Z) \subset M \setminus W$.

By C^∞ -modifying J (see [12]) we can achieve that the new pair (\tilde{J}, H) still satisfies all the hypotheses above, and for every pair $(x, y), \mathcal{M}(x, y; \tilde{J}, H)$ is a finite dimensional manifold. This is due to the fact that $\mathcal{M}(x, y; \tilde{J}, H)$ can be considered as the zero set of a regular Fredholm section. We call such a pair (\tilde{J}, H) **admissible**.

Let us denote for simplicity by $Ad(M)$ the collection of all admissible pairs (J, H) . Following [11], we define for $(J, H) \in Ad(M)$ and $a > -m(H)$ the graded free Abelian group $C_a(J, H)$:

$$\begin{aligned} C_a(J, H) &= \bigoplus_k C_a^k(J, H) \\ C_a^k(J, H) &= \bigoplus_{x \in \mathcal{P}_a(H)^k} \mathbb{Z}x \\ \mathcal{P}_a(H)^k &= \{x \in \mathcal{P}_a(H) \mid \text{ind}_{CZ}(x) = k\}. \end{aligned} \quad (4)$$

We define $\delta_k : C_a^k(J, H) \rightarrow C_a^{k+1}(J, H)$ by

$$\delta_k(x) = \sum_{y \in \mathcal{P}_a(H)^{k+1}} \left(\sum_{\hat{u} \in \widehat{\mathcal{M}}(x, y; J, H)} \tau(\hat{u}) \right) y ,$$

where $\widetilde{\mathcal{M}}(x, y; J, H) = \mathcal{M}(x, y; J, H)/\mathbb{R}$ and $\tau(\hat{u}) \in \{1, -1\}$ are suitable "orientations" attached to the points in $\widehat{\mathcal{M}}$, see [10]. We recall that $\dim \mathcal{M}(x, y; J, H)$ is given by the difference $\text{ind}_{CZ}(y) - \text{ind}_{CZ}(x)$, see [19]. The crucial point is that $\delta_{k+1} \circ \delta_k = 0$. Hence $(C_a(J, H), \delta)$ is a co-chain complex. Moreover, for $b \geq a > -m(H)$ we have the commutative diagram.

$$\begin{array}{ccc} C_a(J, H) & \xrightarrow{\delta} & C_a(J, H) \\ \uparrow & & \uparrow \\ C_b(J, H) & \xrightarrow{\delta} & C_b(J, H) \end{array}$$

where the vertical arrows are given by inclusion.

Next we have to investigate the dependence of the co-chain complex on J and H . To do this we define a partial ordering on $Ad(M)$ by

$$(J_1, H_1) \leq (J_2, H_2) \iff H_1(t, z) \leq H_2(t, z) \text{ for all } (t, z). \quad (5)$$

We would like to associate, in analogy to [11], to the above situation a natural chain homotopy class

$$\sigma(J_2, H_2; J_1, H_1) : C_a(J_2, H_2) \rightarrow C_a(J_1, H_1).$$

To this purpose we take a **monotone homotopy**, i.e. a smooth pair $(J, H) = (J(s, t, z), H(s, t, z))$, $(s, t, z) \in \mathbb{R} \times S^1 \times M$ such that

$$\begin{aligned}
(J(s, \cdot), H(s, \cdot)) &= (J_1(\cdot), H_1(\cdot)) \quad \text{for } s \geq s_0 \\
(J(s, \cdot), H(s, \cdot)) &= (J_2(\cdot), H_2(\cdot)) \quad \text{for } s \leq -s_0
\end{aligned} \tag{6}$$

$J(s, \cdot) \in \mathcal{J}$ for all $s \in \mathbb{R}$
 $H(s, t, z) \equiv H(s)$ for z near ∂M

and

$$\partial_s H(s, t, z) \leq 0 \quad \text{for all } (s, t, z) \tag{7}$$

The existence of such a homotopy follows from the discussion after the definition of \mathcal{J} .

For $x_1 \in P_a(H_1)$, $x_2 \in P_a(H_2)$, $a > -m(H_1)$ define $\mathcal{M}(x_2, x_1; J, H)$ as before. In order to get compactness results as in [11] we must ensure that the elements of $\mathcal{M}(x_2, x_1; J, H)$ are bounded away uniformly from ∂M . We can no longer apply Lemma 2.4 from [18] since J may depend on s near ∂M .

Lemma. *Given a monotone homotopy (J, H) and $a > -m(H_1)$, there exist constants $c_0, \delta > 0$ depending on J, H and a such that for all $c \geq c_0$, $(J^c(s, \cdot), H^c(s, \cdot)) = (J(\frac{s}{c}, \cdot), H(\frac{s}{c}, \cdot))$, $x_1 \in P_a(H_1)$, $x_2 \in P_a(H_2)$ and $u \in \mathcal{M}(x_2, x_1; J^c, H^c)$ we have (with respect to some fixed metric on M):*

$$dist(u(Z), \partial M) \geq \delta$$

We call such a (J^c, H^c) a **slow monotone homotopy**.

Proof. Choose an $\varepsilon > 0$ such that for all $s \in \mathbb{R}$ we have diffeomorphisms $\psi(s) : (-2\varepsilon, 0] \times \partial M \xrightarrow{\cong} W(s)$ induced by the vector field $\eta(s, \cdot)$ near ∂M . We may assume that $H(s, \cdot) \equiv H(s)$ on $\bigcup_{s \in \mathbb{R}} W(s)$. Take a smooth function $f : \mathbb{R} \times M \rightarrow [-\frac{3}{2}\varepsilon, 0]$ satisfying $f \equiv -\frac{3}{2}\varepsilon$ on $M \setminus \bigcup W(s)$, $f(s, z) = \tau$ for $x \in \psi(s)(\{\tau\} \times \partial M)$, $\tau \in [-\varepsilon, 0]$, and $|df(s, \cdot)| \leq 1$ where $|\cdot|$ is taken with respect to the metric $\omega \circ (J(s, \cdot) \times id)$. Pick an open neighbourhood W of ∂M contained in $\bigcap_{s \in \mathbb{R}} \psi(s)([-\varepsilon, 0] \times \partial M)$ and having smooth boundary. Since W is compact, there is a $\rho > 0$ such that every closed curve in W with length $< \rho$ is contractible in W . Let $a > -m(H_1)$ be given and define

$$\begin{aligned}
b &:= \sup_{\mathbb{R} \times (M \setminus W)} f < 0 \\
\Delta &:= \max\{\Phi_{H_1}(x_1) - \Phi_{H_2}(x_2) \mid x_1 \in P_a(H_1), x_2 \in P_a(H_2)\}
\end{aligned}$$

Take an $l > 0$ large enough such that

$$\sqrt{\frac{\Delta}{l}} < \min\{\rho, -\frac{b}{2}, a + m(H_1)\}$$

and let

$$\mu := -\frac{2b}{l^2} > 0$$

By passing from (J, H) to (J^c, H^c) for some c large enough we may assume that on W

$$|\partial_s^2 f| + \frac{1}{2} |\partial_s \lambda|^2 + \frac{1}{2} |d(\partial_s f)|^2 < \mu$$

where $\lambda = i_\eta \omega$ and norms are taken with respect to the metric $\omega \circ (J_s \times \text{id})$.

Now let $x_1 \in P_a(H_1)$, $x_2 \in P_a(H_2)$ and $u \in \mathcal{M}(x_2, x_1; J, H)$ be given and consider the function $g(s, t) := f(s, u(s, t))$ on the cylinder $Z = \mathbb{R} \times S^1$. We will show that $g \leq \frac{b}{4} < 0$, from which the lemma follows.

From $\partial_s H \leq 0$ it follows that

$$\begin{aligned} \int_Z |u_s|^2 ds dt &\leq \Phi_{H_1}(x_1) - \Phi_{H_2}(x_2) \\ &\leq \Delta \end{aligned}$$

So there exist $s_k, k \in \mathbb{Z}$, with $|s_{k+1} - s_k| \leq l$ and $\int_0^1 |u_s(s_k, t)|^2 dt \leq \frac{\Delta}{l}$. For $k \in \mathbb{Z}$ let $x_k(t) := u(s_k, t)$.

Case 1: $x_k(t) \in W$ for all $t \in S^1$. Using that on W u satisfies the equation $u_s + J(s, u)u_t = 0$ we calculate

$$\begin{aligned} \text{length}(x_k) &\leq \left(\int_0^1 |\dot{x}_k|^2 dt \right)^{\frac{1}{2}} \\ &= \left(\int_0^1 |u_s(s_k, t)|^2 dt \right)^{\frac{1}{2}} \\ &\leq \sqrt{\frac{\Delta}{l}} \\ &< \rho \end{aligned}$$

By definition of ρ this implies that x_k is contractible in W . Let $\bar{x}_k : D \rightarrow W$ be a smooth extension of x_k to the closed unit disk. Using $\omega = d\lambda(s_k, \cdot)$ on W we get

$$\begin{aligned} |A(x_k)| &\equiv \left| \int_D \bar{x}_k^* w \right| \\ &= \left| \int_{S^1} x_k^* \lambda(s_k, \cdot) \right| \\ &= \left| \int_0^1 \omega(\eta(s_k, x_k), \dot{x}_k) dt \right| \\ &\leq \int_0^1 |\dot{x}_k| dt \\ &\leq \sqrt{\frac{\Delta}{l}} \end{aligned}$$

Hence

$$\begin{aligned} \Phi_{H(s_k, \cdot)}(x_k) &\equiv A(x_k) - \int_0^1 H(s_k, x_k) dt \\ &\leq \sqrt{\frac{\Delta}{l}} - m(H_1) \\ &< a \end{aligned}$$

in contradiction to $a \leq \Phi_{H_2}(x_2) \leq \Phi_{H(s_k, \cdot)}(x_k)$. So case 1 does not occur, and for all $k \in \mathbb{Z}$ we are in

Case 2: There is a $t_k \in S^1$ such that $x_k(t_k) \notin W$. Then for any $t \in S^1$ we find a $t_0 \in S^1$, $t - 1 < t_0 \leq t$, such that $x_k(t_0) \notin W$ and $x_k(t') \in W$ for all $t_0 < t' \leq t$. This yields

$$\begin{aligned} g(s_k, t) &\leq g(s_k, t_0) + \int_{t_0}^t |\partial_t g(s_k, t')| dt' \\ &\leq b + \int_{t_0}^t |df(s_k, x_k) \cdot \dot{x}_k| dt' \\ &\leq b + \int_{t_0}^t |\dot{x}_k| dt' \\ &\leq b + \sqrt{\frac{\Delta}{l}} \\ &\leq \frac{b}{2} \end{aligned}$$

where the last but one inequality follows as in case 1.

So for the finite cylinders $Z_k := [s_k, s_{k+1}] \times S^1$ we have shown that $g \leq \frac{b}{2} < 0$ on ∂Z_k . It remains to estimate g in the interior of Z_k . If $(s, t) \in Z_k$ and $g(s, t) > b$ then $u(s, t) \in W$, and we can compute

$$\begin{aligned} \partial_s(\lambda(s, u) \cdot u_t) - \partial_t(\lambda(s, u) \cdot u_s) &= \partial_s \lambda(s, u) \cdot u_t + u^* d\lambda(\partial_s, \partial_t) \\ &= \partial_s \lambda(s, u) \cdot u_t + \omega(u_s, u_t) \\ &= \partial_s \lambda(s, u) \cdot u_t - \frac{1}{2}(|u_s|^2 + |u_t|^2) \end{aligned}$$

Using $df \circ J = \lambda$ on W we obtain

$$\begin{aligned} g_s(s, t) &= \partial_s f(s, u) - \lambda(s, u) \cdot u_t \\ g_t(s, t) &= \lambda(s, u) \cdot u_s \end{aligned}$$

and hence

$$\begin{aligned} \Delta g &= g_{ss} + g_{tt} \\ &= \partial_s^2 f + d(\partial_s f) \cdot u_s - \partial_s \lambda \cdot u_t + \frac{1}{2}(|u_s|^2 + |u_t|^2) \\ &\geq -|\partial_s^2 f| - \frac{1}{2}|d(\partial_s f)|^2 - \frac{1}{2}|\partial_s \lambda|^2 \\ &> -\mu \end{aligned}$$

for all $(s, t) \in Z_k$ with $g(s, t) > b$.

Define

$$\hat{g}(s, t) := g(s, t) + \frac{\mu}{2} \left(s - \frac{s_k + s_{k+1}}{2} \right)^2$$

For $(s, t) \in Z_k$ with $\hat{g}(s, t) > \frac{3}{4}b$ we have $g(s, t) > b$ (remember $\mu = -\frac{2b}{l^2}$ and $|s_{k+1} - s_k| \leq l$) and therefore

$$\Delta \hat{g} = \Delta g + \mu > 0$$

So \hat{g} cannot have an interior maximum $> \frac{3}{4}b$ in Z_k . On the other hand, $g \leq \frac{b}{2}$ on ∂Z_k implies $\hat{g} \leq \frac{b}{4}$ on ∂Z_k , thus $g(s, t) \leq \hat{g}(s, t) \leq \frac{b}{4}$ for all $(s, t) \in Z_k$. Since $k \in \mathbb{Z}$ was arbitrary, the lemma is proved. \blacksquare

This lemma ensures that for slow monotone homotopies (J, H) the solution spaces $\mathcal{M}(x_2, x_1; J, H)$ have the usual compactness properties. Now we can proceed as in [11]: For admissible pairs $(J_1, H_1) \leq (J_2, H_2)$, regular slow monotone homotopies between (J_1, H_1) and (J_2, H_2) induce a unique chain homotopy class

$$\sigma(J_2, H_2; J_1, H_1) : C_a(J_2, H_2) \longrightarrow C_a(J_1, H_1)$$

such that the following properties hold:

$$\sigma(J, H; J, H) = id \quad (8)$$

For $(J_1, H_1) \leq (J_2, H_2) \leq (J_3, H_3)$ we have

$$\sigma(J_2, H_2, J_1, H_1) \circ \sigma(J_3, H_3, J_2, H_2) = \sigma(J_3, H_3, J_1, H_1) \quad (9)$$

For $b \geq a > -m(H_1)$ the following diagram commutes:

$$\begin{array}{ccc} C_a(J_2, H_2) & \xrightarrow{\sigma} & C_a(J_1, H_1) \\ \uparrow & & \uparrow \\ C_b(J_2, H_2) & \xrightarrow{\sigma} & C_b(J_1, H_1). \end{array} \quad (10)$$

Note in (9) that if two slow monotone homotopies between (J_1, H_1) and (J_2, H_2) respectively (J_2, H_2) and (J_3, H_3) are sufficiently slow then their composition is a slow monotone homotopy between (J_1, H_1) and (J_3, H_3) . Then use uniqueness of σ .

Now let R be any commutative ring. We apply the functor $\text{Hom}(*, R)$ to the quotient co-chain complexes C_a/C_b , where $b \geq a > -m(H)$ to obtain the chain complexes

$$D^{[a,b)}(J, H) := \text{Hom}(C_a(J, H)/C_b(J, H); R) \quad (11)$$

with boundary operator ∂ induced by δ . For numbers $b \geq a > -m(H)$ and $b' \geq a' > -m(H)$, such that $b \geq b'$ and $a \geq a'$ we have obviously induced homotopy classes of maps

$$C_a/C_b \longrightarrow C_{a'}/C_{b'}$$

and consequently

$$D^{[a',b']} \longrightarrow D^{[a,b)}(J, H). \quad (12)$$

These maps behave functorially, i.e. for $b_i \geq a_i > -m(H), i = 1, 2, 3$, and $b_3 \geq b_2 \geq b_1, a_3 \geq a_2 \geq a_1$ we have

$$\begin{array}{ccc}
D^{[a_1, b_1)} & \longrightarrow & D^{[a_2, b_2)} \\
& \searrow & \downarrow \\
& & D^{[a_3, b_3)}
\end{array} \tag{13}$$

If $(J, H) \leq (\tilde{J}, \tilde{H})$ then the following diagram is also commutative

$$\begin{array}{ccc}
D^{[a, b)}(J, H) & \xrightarrow{\sigma} & D^{[a, b)}(\tilde{J}, \tilde{H}) \\
\uparrow & & \uparrow \\
D^{[a', b')}(\tilde{J}, \tilde{H}) & \xrightarrow{\sigma} & D^{[a', b')}(\tilde{J}, \tilde{H})
\end{array} \tag{14}$$

where σ is the dual of $\sigma(\tilde{J}, \tilde{H}; J, H)$ and the vertical maps are those in (12). Here a, b, a', b' are as described above.

Now we pass to homology obtaining the symplectic homology groups $S^{[a, b)}(J, H)$ of the admissible pair $(J, H) \in Ad(M)$.

These homology groups are independent of J (by the discussion above), but they depend on H . In order to obtain invariants of the manifold, we have to pass to a direct limit over the Hamiltonians. This is done as follows:

Let $U \subset M$ be open (possibly empty) with $cl(U) \subset M \setminus \partial M$. Let $Ad(M, U)$ be the subset of $Ad(M)$ consisting of those pairs (J, H) for which $H|_{S^1 \times cl(U)} < 0$. The density results on admissible pairs (see [12], [19]) imply that $(Ad(M, U), \leq)$ is a directed set. Thus we may take the direct limit over $Ad(M, U)$ to obtain the symplectic homology groups

$$S^{[a, b)}(M, U) = \varinjlim S^{[a, b)}(J, H). \tag{15}$$

Note that $S^{[a, b)}(J, H)$ is well defined if $b \geq a > -m(H)$; omit in the direct limit all those (J, H) for which this not true. Given numbers $c \geq b \geq a$ the exact sequence of chain complexes

$$0 \longrightarrow D^{[a, b)}(J, H) \longrightarrow D^{[a, c)}(J, H) \longrightarrow D^{[b, c)}(J, H) \longrightarrow 0 \tag{16}$$

for $(J, H) \in Ad(M, U)$; $a > -m(H)$, gives, since direct limits preserve exactness, the long exact homology sequence

$$\begin{array}{ccccc}
S^{[a, b)}(M, U) & \xrightarrow{\quad} & S^{[a, c)}(M, U) & & \\
\uparrow \partial_* & & \swarrow & & \\
& & S^{[b, c)}(M, U) & &
\end{array} \tag{17}$$

where the connecting homomorphism ∂_* has degree -1 . We shall denote the diagram (17) by $\Delta_{a,b,c}(M, U)$ and call it the exact triangle. If $a \leq b \leq c$ and $a' \leq b' \leq c'$ with $a' \leq a, b' \leq b$ and $c' \leq c$, the maps from (12) induce a morphism between triangles

$$\Delta_{a',b',c'}(M, U) \longrightarrow \Delta_{a,b,c}(M, U). \quad (18)$$

which is functorial.

This construction has many nice properties, which we shall study in the next section.

3 Elementary properties of symplectic homology

Let us first recall the properties already outlined in the previous section. Given $b \geq a$ and $b' \geq a'$, such that $a \geq a'$ and $b \geq b'$ we have natural transformations.

$$S^{[a',b')} \longrightarrow S^{[a,b)}. \quad (19)$$

There is another natural transformation ∂_* of degree -1 , which together with (19) yields for $a \leq b \leq c$ the exact triangle $\Delta_{a,b,c}$.

$$\begin{array}{ccc} S^{[a,b)} & \xrightarrow{\hspace{2cm}} & S^{[a,c)} \\ \downarrow \partial_* & & \downarrow \\ S^{[b,c)} & & \end{array} \quad (20)$$

The natural transformations in (19) induce morphisms between exact triangles. In order to be more precise, assume $a \leq b \leq c, a' \leq b' \leq c'$ and $a' \leq a, b' \leq b, c' \leq c$. Then the following diagram is commutative

$$\begin{array}{ccccc} & & \partial_* & & \\ S^{[a,b)} & \xleftarrow{\hspace{2cm}} & & \xrightarrow{\hspace{2cm}} & S^{[b,c)} \\ \uparrow & \searrow & & \swarrow & \uparrow \\ & S^{[a,c)} & & & \\ \uparrow & & & & \uparrow \\ S^{[a',c')} & & & & \\ \uparrow & \nearrow & & \swarrow & \uparrow \\ S^{[a',b')} & \xleftarrow{\hspace{2cm}} & & \xrightarrow{\hspace{2cm}} & S^{[b',c')} \end{array} \quad (21)$$

In short we have $\Delta_{a',b',c'} \rightarrow \Delta_{a,b,c}$ which is functorial in the numbers a, b , e.t.c.

Next assume $U \subset V$ are open with $\text{cl}(V) \subset M \setminus \partial M$. Then $Ad(M, V) \subset Ad(M, U)$, hence we obtain a natural map

$$S^{(a,b)}(M, V) \longrightarrow S^{(a,b)}(M, U) \quad (22)$$

inducing a morphism between exact triangles and being compatible with all other natural transformations.

Next assume $\Psi : M \xrightarrow{\sim} M$ is a symplectic diffeomorphism mapping U into V , where U and V are open subsets of M with their closures contained in $M \setminus \partial M$. We shall write

$$\Psi : (M, U) \xrightarrow{s} (M, V). \quad (23)$$

If $(J, H) \in Ad(M, V)$ then H_Ψ defined by $H_\Psi(t, x) = H(t, \Psi(x))$, satisfies $H_\Psi|_{(\mathbb{S}^1 \times \bar{U})} < 0$. We define further J_Ψ by

$$J_\Psi(t, x)h = T\Psi(x)^{-1}J(t, \Psi(x))T\Psi(x)h.$$

Then $(J_\Psi, H_\Psi) \in Ad(M, U)$.

If $u \in \mathcal{M}(x, y; J, H)$ then $\Psi^{-1}(u)$ is in $\mathcal{M}(\Psi^{-1}(x), \Psi^{-1}(y); J_\Psi, H_\Psi)$. So, the new connecting orbit spaces are obtained by applying Ψ^{-1} . Consequently we obtain a co-chain map $\Psi^\# : C_a(J, H) \rightarrow C_a(J_\Psi, H_\Psi)$ by $x \mapsto \Psi^{-1}(x)$ inducing

$$\Psi_\# : D^{(a,b)}(J_\Psi, H_\Psi) \longrightarrow D^{(a,b)}(J, H) \quad (24)$$

and consequently the isomorphism

$$\Psi_{\#} : S^{(a,b)}(J_\Psi, H_\Psi) \longrightarrow S^{(a,b)}(J, H). \quad (25)$$

Passing to the direct limit induces an isomorphism

$$\widehat{\Psi} : S^{(a,b)}(M, \Psi^{-1}(V)) \xrightarrow{\sim} S^{(a,b)}(M, V). \quad (26)$$

Combining $\widehat{\Psi}^{-1}$ with the natural map (22), namely since $U \subset \Psi^{-1}(V)$:

$$S^{(a,b)}(M, \Psi^{-1}(V)) \longrightarrow S^{(a,b)}(M, U) \quad (27)$$

gives a morphism Ψ^* , i.e

$$\begin{array}{ccc} S^{(a,b)}(M, V) & \xrightarrow{\Psi^*} & S^{(a,b)}(M, U) \\ \downarrow \widehat{\Psi}^{-1} & \nearrow & \\ S^{(a,b)}(M, \Psi^{-1}(V)) & & \end{array} \quad (28)$$

It follows immediately from the construction that Ψ^* gives a morphism between the exact triangles $\Delta_{a,b,c}(M, V)$ and $\Delta_{a,b,c}(M, U)$. Moreover, it is compatible with all the introduced natural transformations. Further, it is clear that $(\Phi\Psi)^* = \Psi^*\Phi^*$ and $\text{Id} : (M, U) \rightarrow (M, U)$ induces Id . (22) is in fact induced by $\text{Id} : (M, U) \rightarrow (M, V)$ for $U \subset V$.

The key point will be that an isotopy of symplectic maps $\Psi_s : (M, U) \rightarrow (M, V)$ with $s \in [0, 1]$ will induce maps $(\Psi_s)^*$ which are independent of $s \in [0, 1]$. This will be shown in the next section.

4 Isotopy invariance

We follow the ideas of [11]. Assume (M, ω) is a symplectic manifold with the properties described before. Let H be an admissible Hamiltonian and fix numbers $b \geq a > -m(H)$. Define

$$g(H, [a, b]) := \min \left\{ \inf_{\Phi_H(x) < a} (a - \Phi_H(x)), \inf_{\Phi_H(x) < b} (b - \Phi_H(x)) \right\}, \quad (29)$$

where the infima are taken over all contractible 1-periodic solutions of $\dot{x} = X_{H_t}(x)$ with $\Phi_H(x) > -m(H)$. Since the set of such solutions is finite, we have $g(H, [a, b]) > 0$. We shall call $g(H, [a, b])$ the "gap".

Given admissible Hamiltonians H and K we define their distance $d(H, K)$ as follows. Consider a smooth path $\tau \mapsto H_\tau$ of Hamiltonians such that each H_τ is constant close to ∂M , $\tau \in [\tau_1, \tau_2]$ and $H_{\tau_1} = H, H_{\tau_2} = K$. Take the integral $d((H_\tau))$ defined by

$$d((H_\tau)) = \int_{\tau_1}^{\tau_2} \left[\max_{(t,x)} \left| \frac{\partial H_\tau}{\partial \tau}(t, x) \right| \right] d\tau. \quad (30)$$

Then let $d(H, K)$ be the infimum of all numbers $d((H_\tau))$ taken all smooth paths as just described.

Now let (J_1, H_1) and (J_2, H_2) be admissible, a and b fixed such that

$$b \geq a > \max\{-m(H_1), -m(H_2)\}.$$

Suppose we have the estimate

$$d(H_1, H_2) < \min \left\{ g(H_1, [a, b]), g(H_2, [a, b]) \right\}. \quad (31)$$

Then we can take a regular homotopy $(J(s, t, z), H(s, t, z))$ which satisfies (6) and is slow in the sense of the lemma of section 2, but instead of being monotone satisfies the weaker condition

$$d((H_s)) < \min \left\{ g(H_1, [a, b]), g(H_2, [a, b]) \right\}. \quad (32)$$

For $u \in \mathcal{M}(x_2, x_1; J, H)$ we calculate

$$\frac{d}{ds} \Phi_{H_s}(u(s)) = \|\Phi'_{H_s}(u(s))\|_{L^2(s)}^2 - \int_0^1 \frac{\partial H}{\partial s}(s, t, u(s, t)) dt. \quad (33)$$

Here $L^2(s)$ is the L^2 -section space along loops equipped with the L^2 -inner product associated to the ω -calibrated t -dependent almost complex structure $J(s, *, *)$. Integrating (33) we infer via (32)

$$\Phi_{H_1}(x_1) - \Phi_{H_2}(x_2) > -\min \left\{ g(H_1, [a, b)), g(H_2, [a, b)) \right\}. \quad (34)$$

If $x_2 \in P_\mu(H_2)$, $\mu = a, b$, then (34) and the definition of the "gaps" imply

$$\Phi_{H_1}(x_1) \geq \mu.$$

Now we can proceed as in section 2 to obtain unique chain homotopy classes

$$\sigma(J_2, H_2; J_1, H_1) : C_\mu(J_2, H_2) \longrightarrow C_\mu(J_1, H_1) \quad (35)$$

for $\mu = a, b$. Furthermore, if (J_i, H_i) , $i = 1, 2, 3$, are admissible pairs satisfying

$$\begin{aligned} d(H_1, H_2) &< \frac{1}{2} \min \left\{ g(H_1, [a, b)), g(H_2, [a, b)) \right\} \\ d(H_2, H_3) &< \frac{1}{2} \min \left\{ g(H_2, [a, b)), g(H_3, [a, b)) \right\} \end{aligned} \quad (36)$$

then

$$\sigma(J_2, H_2; J_1, H_1) \circ \sigma(J_3, H_3; J_2, H_2) = \sigma(J_3, H_3; J_1, H_1). \quad (37)$$

Taking $(J_3, H_3) = (J_1, H_1)$ we conclude that in this case $\sigma(J_2, H_2; J_1, H_1)$ induces an isomorphism in homology

$$S^{[a,b)}(J_1, H_1) \xrightarrow[sdi]{\cong} S^{[a,b)}(J_2, H_2) \quad (38)$$

with inverse induced by $\sigma(J_1, H_1; J_2, H_2)$. Here "sdi" stands for the small distance isomorphism. This uniquely defined isomorphism always exists if the distance between H_1 and H_2 is small with respect to the "gaps", more precisely if the first equation of (36) holds.

Now we are in the position to prove the isotopy invariance. Assume U, V are open subsets of M such that $\bar{U}, \bar{V} \subset M \setminus \partial M$. Suppose (Ψ_τ) is a smooth isotopy of symplectic maps such that $\Psi_\tau(U) \subset V$, $\Psi_\tau : M \xrightarrow{\cong} M$. Fix $b \geq a$ and choose an admissible pair (J, H) with $H|(\mathbb{S}^1 \times \bar{V}) < 0$ and define $H_\tau : \mathbb{S}^1 \times M \rightarrow \mathbb{R}$ by

$$H_\tau(t, x) = H(t, \Psi_\tau(x))$$

and

$$J_\tau(t, x) = T\Psi_\tau(x)^{-1} J(t, \Psi_\tau(x)) T\Psi_\tau(x).$$

Clearly (J_τ, H_τ) is admissible for all $\tau \in [0, 1]$. Since $U \subset \Psi_\tau^{-1}(V)$ for every $\tau \in [0, 1]$ we can find an admissible $(\widehat{J}, \widehat{K})$ such that

$$\begin{aligned} \widehat{K} \mid (\mathbb{S}^1 \times \bar{U}) &< 0 \\ (\widehat{J}, \widehat{K}) &\geq (J_\tau, H_\tau) \text{ for all } \tau \in [0, 1]. \end{aligned} \quad (39)$$

Taking for fixed τ a slow monotone homotopy we obtain a unique chain homotopy class σ_τ

$$C_\mu(\widehat{J}, \widehat{K}) \longrightarrow C_\mu(J_\tau, H_\tau) \quad (40)$$

for every $\mu > -m(H)$. We observe that the distance $d(H_{\tau_0}, H_{\tau_1})$ for fixed τ_0 and τ_1 close to τ_0 is as small as we wish. Consequently, the map in (40) for $\tau = \tau_0$ and $\mu = a, b$ factors through a monotone homotopy to $C_\mu(J_{\tau_1}, H_{\tau_1})$ and the "smallness map" from $C_\mu(J_{\tau_1}, H_{\tau_1})$ to $C_\mu(J_{\tau_0}, H_{\tau_0})$ provided τ_1 is close enough to τ_0 . Hence for τ_1 close to τ_0 we have the commutative diagram

$$\begin{array}{ccc} S^{(a,b)}(J_{\tau_0}, H_{\tau_0}) & \xrightarrow{\quad} & S^{(a,b)}(\widehat{J}, \widehat{K}) \\ \downarrow sdi & & \nearrow \\ S^{(a,b)}(J_{\tau_1}, H_{\tau_1}) & & \end{array} \quad (41)$$

Here the unmarked arrows are coming from slow montone homotopies. If we can show that the "sdi"-map in the notation of (25) is given by

$$(\Psi_{\tau_1}^{-1}\Psi_{\tau_0})_{\#} : S^{(a,b)}(J_{\tau_0}, H_{\tau_0}) \xrightarrow{\cong} S^{(a,b)}(J_{\tau_1}, H_{\tau_1})$$

we have as an easy corollary the isotopy invariance. In fact, in this case we obtain the commutative diagramm

$$\begin{array}{ccc} S^{(a,b)}(J_0, H_0) & \xrightarrow{\quad} & S^{(a,b)}(\widehat{J}, \widehat{K}) \\ \downarrow (\Psi_1^{-1}\Psi_0)_{\#} & & \nearrow \\ S^{(a,b)}(J_1, H_1) & & \end{array} \quad (42)$$

Here we can pass to the direct limit to obtain

$$\begin{array}{ccc}
S^{[a,b)}(M, \Psi_0^{-1}(V)) & \longrightarrow & S^{[a,b)}(M, U) \\
\downarrow \widehat{\Psi}_1^{-1} \widehat{\Psi}_0 & & \nearrow \\
S^{[a,b)}(M, \Psi_1^{-1}(V)) & &
\end{array} \tag{43}$$

Now the composites $\sigma \circ \widehat{\Psi}_0^{-1}$ and $\sigma \circ \widehat{\Psi}_1^{-1}$, where σ stands for the appropriate natural map induced by inclusion, define Ψ_0^* and Ψ_1^* respectively. Hence we obtain the commutative diagram

$$\begin{array}{ccccc}
& & S^{[a,b)}(M, \Psi_0^{-1}(V)) & & \\
& \swarrow \widehat{\Psi}_0^{-1} & & \searrow & \\
S^{[a,b)}(M, V) & \xrightarrow{\Psi_0^* = \Psi_1^*} & S^{[a,b)}(M, U) & & \\
& \searrow \widehat{\Psi}_1^{-1} & & \swarrow & \\
& & S^{[a,b)}(M, \Psi_1^{-1}(V)) & &
\end{array} \tag{44}$$

So it remains to prove the identity $sdi = (\Psi_{\tau_1}^{-1} \Psi_{\tau_0})_{\#}$ for τ_1 close to τ_0 . In order to simplify notation we note that the above reduces to the following question:

Assume $\tau \rightarrow \Psi_\tau$ is a smooth arc of symplectic maps with $\Psi_0 = Id, (J, H)$ is admissible and, $b \geq a > -m(H)$, is it then true that $(\Psi_\tau)_{\#} = sdi$ for τ close to zero? Let $\beta : \mathbb{R} \rightarrow [0, 1]$ be a smooth map satisfying $\beta(s) = 0$ for $s \leq 0$, $\beta'(s) > 0$ for $s \in (0, 1)$ and $\beta(s) = 1$ for $s \geq 1$. Define

$$\begin{aligned}
H^\varepsilon(s, t, x) &:= H(t, \Psi_{\varepsilon\beta(s)}(x)) \\
J^\varepsilon(s, t, x) &:= T\Psi_{\varepsilon\beta(s)}(x)^{-1} \circ J(t, \Psi_{\varepsilon\beta(s)}(x)) \circ T\Psi_{\varepsilon\beta(s)}(x)
\end{aligned} \tag{45}$$

for all $(s, t, x) \in \mathbb{R} \times \mathbb{S}^1 \times M$ and $\varepsilon > 0$. Let $u \in \mathcal{M}(x, x^\varepsilon; J^\varepsilon, H^\varepsilon)$, i.e. u satisfies

$$u_s + J^\varepsilon(s, t, u)u_t + \nabla_{J^\varepsilon(s,t)}H^\varepsilon(s, t, u) = 0 \tag{46}$$

and converges for $s \rightarrow \pm\infty$ to contractible 1-periodic solutions x, x^ε of H, H^ε respectively. Define $v : Z \rightarrow M$ by

$$u(s, t) = \Psi_{\varepsilon\beta(s)}^{-1}(v(s, t))$$

and insert this in (46) to obtain

$$\begin{aligned}
0 &= \left(\frac{d}{ds} \Psi_{\varepsilon\beta(s)}^{-1} \right)(v(s, t)) + T\Psi_{\varepsilon\beta(s)}^{-1}(v(s, t))v_s(s, t) \\
&\quad + J^\varepsilon(s, t, \Psi_{\varepsilon\beta(s)}^{-1}(v(s, t)))T\Psi_{\varepsilon\beta(s)}^{-1}(v(s, t))v_t(s, t) \\
&\quad + \nabla_{J^\varepsilon(s, t)} H^\varepsilon(s, t, \Psi_{\varepsilon\beta(s)}^{-1}(v(s, t))) \\
&= T\Psi_{\varepsilon\beta(s)}^{-1}(v(s, t)) \left[v_s(s, t) + J(t, v(s, t))v_t(s, t) \right. \\
&\quad \left. + \nabla_{J(t)} H(t, v(s, t)) + T\Psi_{\varepsilon\beta(s)}(v(s, t)) \left(\frac{d}{ds} \Psi_{\varepsilon\beta(s)}^{-1} \right)(v(s, t)) \right]
\end{aligned} \tag{47}$$

Now if $\text{ind}_{CZ}(x) = \text{ind}_{CZ}(x^\varepsilon)$, the only solution of $v_s + J(t, v)v_t + \nabla_{J(t)} H(t, v) = 0$ connecting x and $\Psi_\varepsilon^{-1} \circ x^\varepsilon$ is $v(s, t) \equiv x(t)$.

The expression $T\Psi_{\varepsilon\beta(s)}^{-1}(v(s, t))(\frac{d}{ds} \Psi_{\varepsilon\beta(s)}^{-1})(v(s, t))$ defines a small perturbation (depending on ε) compactly supported in s . By the compactness estimates in [7] and the inverse function theorem, for small ε (47) has a unique solution \tilde{v} close to every $v(s, t) = x(t)$, where x is a contractible 1-periodic solution of $\dot{x} = X_{H_i}(x)$.

Consequently, (46) has for every 1-periodic solution x in $P_a(H) \setminus P_b(H)$ a unique solution u connecting x with $\Psi_\varepsilon^{-1}(x)$ provided ε is small enough. Hence the map from $C_\mu(J, H) \rightarrow C_\mu(J_\Psi, H_\Psi)$ for $\mu = a, b$ induced by the homotopy $(J^\varepsilon, H^\varepsilon)$ is precisely the map $\Psi_\varepsilon^\#$.

On the other hand for ε small, $(J^\varepsilon, H^\varepsilon)$ is a regular slow homotopy satisfying the smallness condition (32).

Hence we have proved

Proposition 1. *Let (M, ω) be as previously described and U, V open subsets of M such that $\bar{U}, \bar{V} \subset M \setminus \partial M$. Given a smooth isotopy of symplectic maps $\Psi_s : (M, U) \hookrightarrow (M, V)$ we have that $(\Psi_s)^*$ is independent of s .*

5 Possible extensions

To define symplectic homology groups $S^{[a,b]}(J, H)$ we have introduced the space \mathcal{J} of ω -calibrated almost complex structures on M behaving nicely at the boundary. One can also consider the bigger space $\tilde{\mathcal{J}}$ of ω -calibrated almost complex structures defined on the **interior** of M (not necessarily extendible to the boundary) and satisfying the analogous condition near ∂M (where η, λ etc. may also not be extendible to ∂M). For $J \in \tilde{\mathcal{J}}$ and H admissible, solutions in $\mathcal{M}(x_1, x_2; J, H)$ still stay away from the boundary by [18], Lemma 2.4. Hence $S^{[a,b]}(J, H)$ can be defined as before. It is however not clear if two elements of $\tilde{\mathcal{J}}$ can be joined by a path in $\tilde{\mathcal{J}}$. Therefore it cannot be excluded as before that $S^{[a,b]}(J, H)$ might depend on the choice of $J \in \tilde{\mathcal{J}}$. But if we fix a $J \in \tilde{\mathcal{J}}$, we can still take the direct limit over H to obtain symplectic homology groups $S^{[a,b]}(M, U, J)$ having the same properties as before. These groups will be used in [14] to prove the so-called “stability of the action spectrum”.

Our construction was carried out under the standing assumption that $[\omega]$ and c_1 vanish on $\pi_2(M)$. The assumption on c_1 guarantees that a contractible periodic trajectory has a well defined Morse index: the Conley-Zehnder index. The assumption on $[\omega]$ implies that for a contractible loop x we have a well defined action $\Phi_H(x)$.

It makes of course also sense to look for noncontractible 1-periodic solutions in a given class $\alpha \in [\mathbb{S}^1, M]$. We fix in that case a reference loop $x_0 \in \alpha$. Given another loop $x \in \alpha$ we would like to define a number $\Phi_H(x)$, so that the derivative of Φ_H with respect to x vanishes if and only if x is a 1-periodic solution of the Hamiltonian system $\dot{x} = X_{H_t}(x)$ in the class of α . The only way is the following:

Take a smooth map $\bar{x} : [0, 1] \times \mathbb{S}^1 \rightarrow M$ such that $\bar{x}(0, t) = x_0(t)$ and $\bar{x}(1, t) = x(t)$ for $t \in \mathbb{S}^1$ and define $\Phi_H(x) := \int \bar{x}^* \omega - \int_0^1 H(t, x(t)) dt$. The difficulty here is that $\Phi_H(x)$ is not well defined. Clearly by Stokes's Theorem we obtain the same value if we replace \bar{x} by any \hat{x} which is homotopic to \bar{x} with boundary fixed. So the right objects to study are not the 1-periodic solutions in the class α , but the 1-periodic solutions together with their "history", i.e. a homotopy of a given reference curve x_0 to the loop x . Let $C_\alpha^\infty(\mathbb{S}^1, M)$ be the collection of all loops x of class α , and denote by $\tilde{C}_{\alpha, x_0}^\infty(\mathbb{S}^1, M)$ the universal covering with reference curve x_0 . Then finding 1-periodic solutions of class α turns out to be a variational problem on this space. Problems of this kind can be handled at the expense of some nontrivial algebraic machinery, see [16].

All these difficulties can be of course avoided if the symplectic form ω is exact, i.e. $\omega = d\lambda$ for some 1-form on M . Then we define

$$\Phi_H(x) = - \int_0^1 x^* \lambda - \int_0^1 H(t, x(t)) dt$$

for every loop.

The previous discussion solves the problem of finding a real filtration provided the Morse complex can be handled analytically. (The bubbling off of holomorphic spheres has to be controlled.) The other problem is the fixing of a grading. For this one takes the reference curve x_0 and considers the pullback bundle $x_0^* TM \rightarrow \mathbb{S}^1$, which is a symplectic vectorbundle. Of interest are first order linear differential operators $L_0 : L^2(x_0^* TM) \supset H^{1,2}(x_0^* TM) \rightarrow L^2(x_0^* TM)$, which in a suitable symplectic trivialization $\bar{x}^* TM \simeq \mathbb{S}^1 \times \mathbb{C}^n$ look like

$$(\tilde{L}_0 h)(t) = -ih(t) - A(t)h(t),$$

where $t \rightarrow A(t)$ is a smooth map into the real linear maps of \mathbb{C}^n such that $A(t)$ is symmetric for the real inner product $\langle *, * \rangle = \text{Re}(*, *)$ on \mathbb{C}^n . We assume $\text{Kern}(\tilde{L}_0) = \{0\}$ and call \tilde{L}_0 and L_0 in this case nondegenerate. L_0 is what we call in [10] an asymptotic operator.

Now we consider x_0 and L_0 as some kind of normalizing data. Given any element of $\tilde{C}_{\alpha, x_0}^\infty(\mathbb{S}^1, M)$, i.e. a homotopy class of maps $\bar{x} : [0, 1] \times \mathbb{S}^1 \rightarrow M$ with

$\bar{x}(0, t) = x_0(t)$ and $\bar{x}(1, t) = x(t)$ we define a Morse index for x provided it is a nondegenerate 1-periodic solution of a Hamiltonian system $\dot{x} = X_H(x)$. Namely having fixed a calibrated perhaps t -dependent almost complex structure J we can write $\dot{x} = X_H(x)$ as $-J(t, x(t)) \frac{d}{dt}x(t) - \nabla_{J_t}H(t, x(t)) = 0$. Linearizing the latter expression by varying x in the function space we obtain a second asymptotic operator $L = L(H, x)$. (It only depends on x , not on \bar{x} .)

Now one trivialises $\bar{x}^*TM \rightarrow [0, 1] \times \mathbb{S}^1$. This determines trivializations for L_0 and $L(H, x)$ which depend up to homotopy only on the class of \bar{x} in $\widetilde{\mathcal{C}}_{\alpha, x_0}(\mathbb{S}^1, M)$. Now one has obtained asymptotic operators \widetilde{L}_0 and $\widetilde{L}(H, x)$. One takes now a Fredholm operator A of the type described in [10] having asymptotic operators $A^- = \widetilde{L}_0$ and $A^+ = \widetilde{L}(H, x)$ and defines the generalized Conley-Zehnder index of $[\bar{x}]$ with respect to the normalizing data (x_0, L_0) to be precisely the Fredholm index of A .

It follows from results in [10] that this definition is independent of the constructions involved. Let us write for it $\text{ind}([\bar{x}], H) := \text{ind}_{(x_0, L_0)}([\bar{x}], H)$. If we use this construction for contractible loops under the assumption that c_1 vanishes on $\pi_2(M)$ we see that this index map differs from the Conley-Zehnder index only by a fixed constant. Given an admissible pair (J, H) , consider the manifold of connecting orbits between two nondegenerate 1-periodic solutions in the class α , i.e.

$$\begin{aligned} \mathcal{M}(x^-, x^+; J, H) = \{u : Z \rightarrow M \mid & u_s + J(t, u)u_t + \nabla_{J_t}H(t, u) = 0, \\ & u(s, *) \rightarrow x^\pm \text{ as } s \rightarrow \pm\infty\}. \end{aligned}$$

It turns out as the consequence of the discussions in [10] that the local dimension $\dim_u \mathcal{M}$ close to u satisfies

$$\dim_u \mathcal{M} = \text{ind}([b], H) - \text{ind}([a], H),$$

where $a : [0, 1] \times \mathbb{S}^1 \rightarrow M$ is any fixed homotopy between x_0 and x^- , and b is the homotopy obtained by first carrying out a followed by the homotopy described by u . Let us write $[b] = [a] \& [u]$. Then the above formula reads

$$\dim_u \mathcal{M} = \text{ind}([a] \& [u], H) - \text{ind}([a], H).$$

We note finally that for manifolds M with contact type boundary which are equidimensional submanifolds of a cotangent bundle we have $c_1|\pi_2(M) = 0$ and $\omega|_M = d\lambda$. In that case $\dim_u \mathcal{M}$ is independent of the choice of u , and only depends on x^- and x^+ . One can carry out our homology construction for any class α since in that case no holomorphic spheres exist. The integer grading is fixed by prescribing a pair (x_0, L_0) as described above. The filtration is given naturally through the choice of $\lambda = pdq$. Another case where the construction can be carried out are monotone manifolds in the sense of [9].

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