Hölder Index at a Given Point for Density States of Super-α-Stable Motion of Index $1+\beta$

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Abstract A Hölder regularity index at *given points* for density states of $(\alpha, 1, \beta)$-superprocesses with $\alpha > 1+\beta$ is determined. It is shown that this index is strictly greater than the optimal index of *local* Hölder continuity for those density states.

Keywords Hölder continuity at a given point · Optimal exponent · Multifractal spectrum · Hausdorff dimension

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1 Introduction and Statement of Results

For $0 < \alpha \leq 2$ and $1+\beta \in (1,2)$, the so-called (α, d, β)-*superprocess* $X = \{X_t\colon\ t \geq 0\}$ in R^d is a finite measure-valued process related to the log-Laplace equa-

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tion

$$\frac{\mathrm{d}}{\mathrm{d}t}u = \boldsymbol{\Delta}_\alpha u + au - bu^{1+\beta}, \tag{1.1}$$

where $a \in \mathsf{R}$ and $b > 0$ are any fixed constants. Its underlying motion is described by the fractional Laplacian $\boldsymbol{\Delta}_\alpha := -(-\boldsymbol{\Delta})^{\alpha/2}$ determining a symmetric α-stable motion in R^d of index $\alpha \in (0,2]$ (Brownian motion if $\alpha = 2$), whereas its continuous-state branching mechanism

$$v \mapsto -av + bv^{1+\beta}, \quad v \geq 0, \tag{1.2}$$

belongs to the domain of attraction of a stable law of index $1+\beta \in (1,2)$ (the branching is critical if $a = 0$).

From now on we assume that $d < \frac{\alpha}{\beta}$. Then X has a.s. *absolutely continuous states* $X_t(\mathrm{d}x)$ at fixed times $t > 0$ (cf. Fleischmann [3] with the obvious changes for $a \neq 0$). Moreover, as is shown in Fleischmann, Mytnik, and Wachtel [4, Theorem 1.2(a), (c)], there is a *dichotomy* for their density function (also denoted by X_t): There is a continuous version $\tilde{X}_t$ of the density function if $d = 1$ and $\alpha > 1+\beta$, but otherwise the density function X_t is locally unbounded on open sets of positive $X_t(\mathrm{d}x)$-measure. (The case $\alpha = 2$ had been derived earlier in Mytnik and Perkins [12].) In the case of continuity, Hölder regularity properties of $\tilde{X}_t$ had been studied in [4], too.

Let us first recall the notion of an optimal Hölder index at a point (see, e.g., Jaffard [6]). We say that a function f is Hölder continuous with index $\eta \in (0,1]$ *at the point* x if there is an open neighborhood $U(x)$ of x and a constant C such that

$$\bigl|f(y) - f(x)\bigr| \leq C|y-x|^\eta \quad \text{for all } y \in U(x). \tag{1.3}$$

The *optimal Hölder index* $H(x)$ of f at the point x is defined as

$$H(x) := \sup\bigl\{\eta \in (0,1]\colon\ f \text{ is Hölder continuous at } x \text{ with index } \eta\bigr\} \tag{1.4}$$

and set to 0 if f is not Hölder continuous at x.

Going back to the continuous (random) density function $\tilde{X}_t$, in what follows, $H(x)$ will denote the (random) optimal Hölder index of $\tilde{X}_t$ at $x \in \mathsf{R}$. In [4, Theorem 1.2(a), (b)], the so-called *optimal index* for *local* Hölder continuity of $\tilde{X}_t$ had been determined by

$$\eta_\mathrm{c} := \frac{\alpha}{1+\beta} - 1 \in (0,1). \tag{1.5}$$

This means that in any nonempty open set $U \subset \mathsf{R}$ with $X_t(U) > 0$ one can find (random) points x such that $H(x) = \eta_\mathrm{c}$. This however left unsolved the question whether there are points $x \in U$ such that $H(x) > \eta_\mathrm{c}$.

The *purpose* of this note is to verify the following theorem conjectured in [4, Section 1.3]. To formulate it, let $\mathcal{M}_\mathrm{f}$ denote the set of finite measures on R^d, and $B_\epsilon(x)$ the open ball of radius $\epsilon > 0$ around $x \in \mathsf{R}^d$.

Theorem 1.1 (Hölder Continuity at a Given Point) *Fix* $t > 0$, $z \in \mathsf{R}$, *and* $X_0 = \mu \in \mathcal{M}_\mathrm{f}$. *Let* $d = 1$ *and* $\alpha > 1+\beta$.

(a) (Hölder continuity at a given point) *For each* $\eta > 0$ *satisfying*

$$\eta < \bar{\eta}_c := \min\left\{\frac{1+\alpha}{1+\beta} - 1, 1\right\},$$

with probability one, the continuous version $\tilde{X}_t$ *of the density is Hölder continuous of order* η *at the point* z:

$$\sup_{x \in B_\epsilon(z),\, x \neq z} \frac{|\tilde{X}_t(x) - \tilde{X}_t(z)|}{|x - z|^\eta} < \infty, \quad \epsilon > 0.$$

(b) (Optimality of $\bar{\eta}_c$) *If additionally* $\beta > (\alpha - 1)/2$, *then with probability one for any* $\epsilon > 0$,

$$\sup_{x \in B_\epsilon(z),\ x \neq z} \frac{|\tilde{X}_t(x) - \tilde{X}_t(z)|}{|x - z|^{\bar{\eta}_c}} = \infty \quad \textit{whenever } X_t(z) > 0.$$

Theorem 1.1(b) states the optimality of $\bar{\eta}_c$ in the case $\beta > (\alpha - 1)/2$. But it is easy to see that the opposite case $\beta \leq (\alpha - 1)/2$ implies that $\bar{\eta}_c = 1$. Therefore the optimality of $\bar{\eta}_c$ follows here automatically from the definition of $H(z)$. But opposed to the local unboundedness of the ratio $\frac{|\tilde{X}_t(x) - \tilde{X}_t(z)|}{|x-z|^{\bar{\eta}_c}}$ in the case $\beta > (\alpha - 1)/2$, we *conjecture* that $\tilde{X}_t$ is even Lipschitz continuous at the given z for $\beta < (\alpha - 1)/2$.

Since $\eta_c < \bar{\eta}_c$, at each given point $z \in \mathsf{R}$ the density $\tilde{X}_t$ allows some Hölder exponents η larger than η_c, the optimal Hölder index for local domains. Thus, Theorem 1.1 nicely complements the main result of [4].

The full program however would include proving that for any $\eta \in (\eta_c, \bar{\eta}_c)$, with probability one, there are (random) points $x \in \mathsf{R}$ such that the optimal Hölder index $H(x)$ of $\tilde{X}_t$ at x is exactly η. Moreover, one would like to establish the *Hausdorff dimension*, say $D(\eta)$, of the (random) set $\{x\colon\ H(x) = \eta\}$. The function $\eta \mapsto D(\eta)$ then reveals the so-called *multifractal spectrum* related to the optimal Hölder index at points. As we already mentioned in [4, Conjecture 1.4], we *conjecture* that

$$\lim_{\eta \downarrow \eta_c} D(\eta) = 0 \quad \text{and} \quad \lim_{\eta \uparrow \bar{\eta}_c} D(\eta) = 1 \quad \text{a.s.} \tag{1.6}$$

The investigation of such multifractal spectrum is left for future work.

The multifractal spectrum of random functions and measures has attracted attention for many years and has been studied, for example, in Dembo et al. [1], Durand [2], Hu and Taylor [5], Klenke and Mörters [10], Le Gall and Perkins [11], Mörters and Shieh [13], and Perkins and Taylor [14]. The multifractal spectrum of singularities that describe the Hausdorff dimension of sets of different Hölder exponents of functions was investigated for deterministic and random functions in Jaffard [6–8] and Jaffard and Meyer [9].

Note also that in the case $\alpha = 2$, for the optimal exponents η_c and $\bar{\eta}_c$, we have

$$\eta_c \downarrow 0 \quad \text{and} \quad \bar{\eta}_c \downarrow \frac{1}{2} \quad \text{as } \beta \uparrow 1, \tag{1.7}$$

whereas for continuous super-Brownian motion ($\beta = 1$), one would have $\eta_c = \frac{1}{2} = \bar{\eta}_c$. This discontinuity reflects the essential differences between continuous and discontinuous super-Brownian motion concerning Hölder continuity properties of density states, as discussed already in [4, Sect. 1.3].

After some preparation in the next section, the proof of Theorem 1.1(a), (b) will be given in Sects. 3 and 4, respectively.

2 Some Proof Preparation

Let p^α denote the continuous α-stable transition kernel related to the fractional Laplacian $\boldsymbol{\Delta}_\alpha = -(-\boldsymbol{\Delta})^{\alpha/2}$ in R^d, and S^α the related semigroup. Fix $X_0 = \mu \in \mathcal{M}_\mathrm{f} \backslash \{0\}$.

First, we want to recall the *martingale decomposition* of the (α, d, β)-superprocess X (valid for any α, d, β; see, e.g., [4, Lemma 1.6]): For all sufficiently smooth bounded nonnegative functions φ on R^d and $t \geq 0$,

$$\langle X_t, \varphi \rangle = \langle \mu, \varphi \rangle + \int_0^t \mathrm{d}s \, \langle X_s, \boldsymbol{\Delta}_\alpha \varphi \rangle + M_t(\varphi) + a I_t(\varphi) \tag{2.1}$$

with discontinuous martingale

$$t \mapsto M_t(\varphi) := \int_{(0,t] \times \mathsf{R}^d \times \mathsf{R}_+} \tilde{N}\big(\mathrm{d}(s, x, r)\big) r \varphi(x) \tag{2.2}$$

and increasing process

$$t \mapsto I_t(\varphi) := \int_0^t \mathrm{d}s \, \langle X_s, \varphi \rangle. \tag{2.3}$$

Here $\tilde{N} := N - \hat{N}$, where $N(\mathrm{d}(s, x, r))$ is a random measure on $(0, \infty) \times \mathsf{R}^d \times (0, \infty)$ describing all the jumps $r\delta_x$ of X at times s at sites x of size r (which are the only discontinuities of the process X). Moreover,

$$\hat{N}\big(\mathrm{d}(s, x, r)\big) = \varrho \, \mathrm{d}s \, X_s(\mathrm{d}x) r^{-2-\beta} \, \mathrm{d}r \tag{2.4}$$

is the compensator of N, where $\varrho := b \ (1 + \beta)\beta / \Gamma(1 - \beta)$ with Γ denoting the Gamma function.

Recall that we assumed $d < \frac{\alpha}{\beta}$, and fix $t > 0$. Then the random measure $X_t(\mathrm{d}x)$ is a.s. absolutely continuous. From the Green's function representation related to (2.1) (see, e.g., [4, (1.9)]) we obtain the following representation of a version of the density function of $X_t(\mathrm{d}x)$ (see, e.g., [4, (1.12)]):

$$\begin{aligned} X_t(x) \ &= \ \mu * p_t^\alpha(x) + \int_{(0,t] \times \mathsf{R}^d} M\big(\mathrm{d}(s, y)\big) p_{t-s}^\alpha(y - x) \\ &\quad + a \int_{(0,t] \times \mathsf{R}^d} I\big(\mathrm{d}(s, y)\big) p_{t-s}^\alpha(y - x) \\ &=: \mathsf{Z}_t^1(x) + \mathsf{Z}_t^2(x) + \mathsf{Z}_t^3(x), \quad x \in \mathsf{R}^d \end{aligned} \tag{2.5}$$

(with notation in the obvious correspondence). Here $M(\mathrm{d}(s, y))$ is the martingale measure related to (2.2), and $I(\mathrm{d}(s, y))$ the random measure related to (2.3).

Let $\Delta X_s := X_s - X_{s-}$, $s \in (0, t)$, denote the jumps of the measure-valued process X by time t. Recall that they are of the form $r\delta_x$. By an abuse of notation, we also write $r =: \Delta X_s(x)$. Put

$$f_{s,x} := \log\bigl((t-s)^{-1}\bigr)1_{\{x\neq 0\}} \log\bigl(|x|^{-1}\bigr). \tag{2.6}$$

As a further preparation, we turn to the following lemma. Recall that $t > 0$ is fixed.

Lemma 2.1 (A Jump Mass Estimate) *Fix* $X_0 = \mu \in \mathcal{M}_\mathrm{f}\backslash\{0\}$. *Suppose that* $d = 1$ *and* $\alpha > 1 + \beta$. *Let* $\varepsilon > 0$ *and* $q > 0$. *There exists a constant* $c_{(2.7)} = c_{(2.7)}(\varepsilon, q)$ *such that*

$$\mathbf{P}\Bigl(\Delta X_s(x) > c_{(2.7)}\bigl((t-s)|x|\bigr)^{\frac{1}{1+\beta}}(f_{s,x})^{\ell} \text{ for some } s < t,\ x \in B_{1/\mathrm{e}}(0)\Bigr) \le \varepsilon, \tag{2.7}$$

where

$$\ell := \frac{1}{1+\beta} + q. \tag{2.8}$$

Proof For any $c > 0$ (later to be specialized to some $c_{(2.7)}$), set

$$Y := N\Bigl((s,x,r)\colon\ (s,x) \in (0,t) \times B_{1/\mathrm{e}}(0),\ r \ge c\bigl((t-s)|x|\bigr)^{1/(1+\beta)}(f_{s,x})^{\ell}\Bigr).$$

Clearly,

$$\begin{aligned}&\mathbf{P}\Bigl(\Delta X_s(x) > c\bigl((t-s)|x|\bigr)^{1/(1+\beta)}(f_{s,x})^{\ell} \text{ for some } s < t \text{ and } x \in B_{1/\mathrm{e}}(0)\Bigr)\\ &\quad = \mathbf{P}(Y \ge 1) \le \mathbf{E}Y,\end{aligned} \tag{2.9}$$

where in the last step we have used the classical Markov inequality. From (2.4) we have

$$\begin{aligned}\mathbf{E}Y &= \varrho\mathbf{E}\int_0^t \mathrm{d}s \int_{\mathsf{R}} X_s(\mathrm{d}x)1_{B_{1/\mathrm{e}}(0)}(x) \int_{c((t-s)|x|)^{1/(1+\beta)}(f_{s,x})^{\ell}}^{\infty} \mathrm{d}r\, r^{-2-\beta}\\ &= \varrho\frac{c^{-1-\beta}}{1+\beta}\int_0^t \mathrm{d}s\,(t-s)^{-1}\log^{-1-q(1+\beta)}\bigl((t-s)^{-1}\bigr)\\ &\quad \times \int_{\mathsf{R}} \mathbf{E}X_s(\mathrm{d}x)1_{B_{1/\mathrm{e}}(0)}(x)|x|^{-1}\log^{-1-q(1+\beta)}\bigl(|x|^{-1}\bigr).\end{aligned} \tag{2.10}$$

Now, writing C for a generic constant (which may change from place to place),

$$\begin{aligned}&\int_{\mathsf{R}} \mathbf{E}X_s(\mathrm{d}x)1_{B_{1/\mathrm{e}}(0)}(x)|x|^{-1}\log^{-1-q(1+\beta)}\bigl(|x|^{-1}\bigr)\\ &\quad \le \mathrm{e}^{|a|t}\int_{\mathsf{R}} \mu(\mathrm{d}y)\int_{\mathsf{R}} \mathrm{d}x\, p_s^{\alpha}(x-y)1_{B_{1/\mathrm{e}}(0)}(x)|x|^{-1}\log^{-1-q(1+\beta)}\bigl(|x|^{-1}\bigr)\end{aligned}$$

$$\leq C\mu(\mathsf{R})s^{-1/\alpha}\int_{\mathsf{R}}\mathrm{d}x\, 1_{B_{1/\mathrm{e}}(0)}(x)|x|^{-1}\log^{-1-q(1+\beta)}\big(|x|^{-1}\big)$$
$$=: c_{(2.11)}s^{-1/\alpha}, \tag{2.11}$$

where $c_{(2.11)} = c_{(2.11)}(q)$ (recall that t is fixed). Consequently,

$$\mathbf{E}Y \leq \varrho c_{(2.11)}c^{-1-\beta}\int_0^t \mathrm{d}s\, s^{-1/\alpha}(t-s)^{-1}\log^{-1-q(1+\beta)}\big((t-s)^{-1}\big)$$
$$=: c_{(2.12)}c^{-1-\beta} \tag{2.12}$$

with $c_{(2.12)} = c_{(2.12)}(q)$. Choose now c such that the latter expression equals ε and write $c_{(2.7)}$ instead of c. Recalling (2.9), the proof is complete. □

Since $\sup_{0<y<1} y^\gamma \log^\ell \frac{1}{y} < \infty$ for every $\gamma > 0$, we get from Lemma 2.1 the following statement.

Corollary 2.2 (A Jump Mass Estimate) *Fix $X_0 = \mu \in \mathcal{M}_\mathrm{f}\backslash\{0\}$. Suppose that $d = 1$ and $\alpha > 1 + \beta$. Let $\varepsilon > 0$ and $\gamma \in (0, (1+\beta)^{-1})$. There exists a constant $c_{(2.13)} = c_{(2.13)}(\varepsilon, \gamma)$ such that*

$$\mathbf{P}\big(\Delta X_s(x) > c_{(2.13)}\big((t-s)|x|\big)^\lambda \textit{ for some } s < t \textit{ and } x \in B_2(0)\big) \leq \varepsilon, \tag{2.13}$$

where

$$\lambda := \frac{1}{1+\beta} - \gamma. \tag{2.14}$$

Several times we will use the following estimate concerning the α-stable transition kernel p^α taken from [4, Lemma 2.1].

Lemma 2.3 (α-Stable Density Increment) *For every $\delta \in [0, 1]$,*

$$\big|p_t^\alpha(x) - p_t^\alpha(y)\big| \leq C\frac{|x-y|^\delta}{t^{\delta/\alpha}}\big(p_t^\alpha(x/2) + p_t^\alpha(y/2)\big), \quad t > 0,\ x, y \in \mathsf{R}. \tag{2.15}$$

In the proof of our main result we need also a further technical result we quote from [4, Lemma 2.3]. Let $L = \{L_t\colon t \geq 0\}$ denote a spectrally positive stable process of index $\kappa \in (1, 2)$. Per definition, L is an R-valued time-homogeneous process with independent increments and with Laplace transform given by

$$\mathbf{E}\mathrm{e}^{-\lambda L_t} = \mathrm{e}^{t\lambda^\kappa}, \quad \lambda, t \geq 0. \tag{2.16}$$

Note that L is the unique (in law) solution to the following martingale problem:

$$t \mapsto \mathrm{e}^{-\lambda L_t} - \int_0^t \mathrm{d}s\, \mathrm{e}^{-\lambda L_s}\lambda^\kappa \quad \text{is a martingale for any } \lambda > 0. \tag{2.17}$$

Let $\Delta L_s := L_s - L_{s-} > 0$ denote the jumps of L.

Lemma 2.4 (Big Values of the Process in the Case of Bounded Jumps) *We have*

$$\mathbf{P}\Big(\sup_{0\le u\le t} L_u \mathbb{1}\Big\{\sup_{0\le v\le u} \Delta L_v \le y\Big\} \ge x\Big) \le \Big(\frac{Ct}{xy^{\kappa-1}}\Big)^{x/y}, \quad t>0,\ x,y>0. \tag{2.18}$$

3 Hölder Continuity at a Given Point: Proof of Theorem 1.1(a)

We will use some ideas from the proofs in Sect. 3 of [4]. However, to be adopted to our case, those proofs require significant changes. Let $d=1$ and fix $t, z, \mu, \alpha, \beta, \eta$ as in the theorem. Consider an $x \in B_1(z)$. Without loss of generality we will assume that $t \le 1$ and, changing μ appropriately, that $z=0$ and $0<x<1$. By definition (2.5) of Z_t^2,

$$\begin{aligned} Z_t^2(x) - Z_t^2(0) = &\int_{(0,t]\times\mathsf{R}} M\big(\mathrm{d}(s,y)\big)\varphi_+(s,y) \\ &- \int_{(0,t]\times\mathsf{R}} M\big(\mathrm{d}(s,y)\big)\varphi_-(s,y), \end{aligned} \tag{3.1}$$

where $\varphi_+(s,y)$ and $\varphi_-(s,y)$ are the positive and negative parts of $p^\alpha_{t-s}(y-x) - p^\alpha_{t-s}(y)$ (for the fixed x). It is easy to check that φ_+ and φ_- satisfy the assumptions in [4, Lemma 2.15]. Thus, there exist spectrally positive stable processes L^+ and L^- such that

$$Z_t^2(x) - Z_t^2(0) = L^+_{T_+} - L^-_{T_-}, \tag{3.2}$$

where $T_\pm := \int_0^t \mathrm{d}s \int_{\mathsf{R}} X_s(\mathrm{d}y)(\varphi_\pm(s,y))^{1+\beta}$. Fix any

$$\varepsilon \in \Big(0, \frac{1}{3}\Big) \quad \text{and} \quad \gamma \in \Big(0, \min\Big\{\frac{\eta_c}{2\alpha}, \frac{\bar{\eta}_c}{2(2\alpha+1)}\Big\}\Big). \tag{3.3}$$

Also fix some $J = J(\gamma)$ and

$$0 =: \rho_0 < \rho_1 < \cdots < \rho_J := 1/\alpha \tag{3.4}$$

such that

$$\rho_\ell(\alpha+1) - \frac{\rho_{\ell+1}}{1+\beta} \ge -\frac{\gamma}{2}, \quad 0 \le \ell \le J-1. \tag{3.5}$$

According to [4, Lemma 2.11], there exists a constant c_ε such that

$$\mathbf{P}(V \le c_\varepsilon) \ge 1-\varepsilon, \tag{3.6}$$

where

$$V := \sup_{0\le s\le t,\ y\in B_2(0)} S_{2^\alpha(t-s)} X_s(y) \tag{3.7}$$

(note that there is no difference in using $B_2(0)$ or its closure for taking the supremum). By Lemma 2.2 we can fix $c_{(2.13)}$ sufficiently large such that the probability of the event

$$A^{\varepsilon,1} := \big\{\Delta X_s(y) \le c_{(2.13)}\big((t-s)|y|\big)^{\lambda} \text{ for all } s<t \text{ and } y \in B_2(0)\big\} \tag{3.8}$$

is larger than $1-\varepsilon$. Moreover, according to [4, Lemma 2.14], there exists a constant $c^* = c^*(\varepsilon,\gamma)$ such that the probability of the event

$$A^{\varepsilon,2} := \big\{\Delta X_s(y) \le c^*(t-s)^{\lambda} \text{ for all } s<t \text{ and } y \in \mathsf{R}\big\} \tag{3.9}$$

is larger than $1-\varepsilon$. Set

$$A^{\varepsilon} := A^{\varepsilon,1} \cap A^{\varepsilon,2} \cap \{V \le c_{\varepsilon}\}. \tag{3.10}$$

Evidently,

$$\mathbf{P}\big(A^{\varepsilon}\big) \ge 1-3\varepsilon. \tag{3.11}$$

Define $Z_t^{2,\varepsilon} := Z_t^2 \mathsf{1}(A^{\varepsilon})$. We first show that $Z_t^{2,\varepsilon}$ has a version which is locally Hölder continuous of all orders η less than $\bar{\eta}_c$. It follows from (3.2) that, for any $k>0$,

$$\begin{aligned}\mathbf{P}\big(\big|Z_t^{2,\varepsilon}(x) - Z_t^{2,\varepsilon}(0)\big| \ge 2kx^{\eta}\big)\\ \le \mathbf{P}\big(L^+_{T_+} \ge kx^{\eta}, A^{\varepsilon}\big) + \mathbf{P}\big(L^-_{T_-} \ge kx^{\eta}, A^{\varepsilon}\big).\end{aligned} \tag{3.12}$$

Define

$$\tilde{D}_0 := \big\{(s,y) \in [0,t) \times B_2(0)\colon\ y \in \big(-2(t-s)^{1/\alpha-\rho_1},\ x+2(t-s)^{1/\alpha-\rho_1}\big)\big\} \tag{3.13}$$

and, for $1 \le \ell \le J-1$,

$$\tilde{D}_\ell := \big\{(s,y) \in [0,t) \times B_2(0)\colon\ y \in \big(-2(t-s)^{1/\alpha-\rho_{\ell+1}}, x+2(t-s)^{1/\alpha-\rho_{\ell+1}}\big)\big\}.$$

Moreover,

$$D_0 := \tilde{D}_0 \quad \text{and} \quad D_\ell := \tilde{D}_\ell \setminus \tilde{D}_{\ell-1}, \quad 1 \le \ell \le J-1. \tag{3.14}$$

Note that

$$[0,t) \times B_2(0) = \bigcup_{0 \le \ell < J} D_\ell. \tag{3.15}$$

If the jumps of $M(\mathrm{d}(s,y))$ do not exceed $c_{(2.13)}((t-s)|y|)^{\lambda}$ on D_ℓ, then the jumps of the process $u \mapsto \int_{(0,u]\times D_\ell} M(\mathrm{d}(s,y))\varphi_{\pm}(s,y)$ are bounded by

$$c_{(2.13)} \sup_{(s,y)\in D_\ell} \big((t-s)|y|\big)^{\lambda}\varphi_{\pm}(s,y). \tag{3.16}$$

For $0 \le \ell < J$, put

$$\begin{aligned} D_{\ell,1} &:= \big\{(s,y) \in D_\ell\colon\ (t-s)^{1/\alpha-\rho_{\ell+1}} \le x\big\}, \\ D_{\ell,2} &:= \big\{(s,y) \in D_\ell\colon\ (t-s)^{1/\alpha-\rho_{\ell+1}} > x\big\}, \\ D_{\ell,1}(s) &:= \big\{y \in B_2(0)\colon\ (s,y) \in D_{\ell,1}\big\}, \quad s \in [0,t), \\ D_{\ell,2}(s) &:= \big\{y \in B_2(0) : (s,y) \in D_{\ell,2}\big\}, \quad s \in [0,t). \end{aligned} \tag{3.17}$$

Since obviously $D_\ell = D_{\ell,1} \cup D_{\ell,2}$, we get that (3.16) is bounded by

$$\begin{aligned} &c_{(2.13)} \sup_{0<s<t} (t-s)^\lambda \sup_{y \in D_{\ell,1}(s)} |y|^\lambda \varphi_\pm(s,y) \\ &\quad + c_{(2.13)} \sup_{0<s<t} (t-s)^\lambda \sup_{y \in D_{\ell,2}(s)} |y|^\lambda \varphi_\pm(s,y) =: c_{(2.13)}(I_1 + I_2). \end{aligned} \tag{3.18}$$

Clearly,

$$\varphi_\pm(s,y) \le \big|p^\alpha_{t-s}(y-x) - p^\alpha_{t-s}(y)\big|, \quad \text{for all } s, y. \tag{3.19}$$

First, let us bound I_1. Note that for any $(s,y) \in D_{\ell,1}$,

$$|y| \le x + 2(t-s)^{1/\alpha-\rho_{\ell+1}} \le 3x. \tag{3.20}$$

Therefore, we have

$$I_1 \le 3^\lambda x^\lambda \sup_{0<s<t} (t-s)^\lambda \sup_{y \in D_{\ell,1}(s)} \big|p^\alpha_{t-s}(y-x) - p^\alpha_{t-s}(y)\big|. \tag{3.21}$$

Using Lemma 2.3 with $\delta = \eta_c - 2\alpha\gamma$ gives

$$\begin{aligned} &\sup_{y \in D_{\ell,1}(s)} \big|p^\alpha_{t-s}(y-x) - p^\alpha_{t-s}(y)\big| \\ &\le C x^{\eta_c - 2\alpha\gamma} (t-s)^{-\eta_c/\alpha + 2\gamma} \\ &\quad \times \sup_{y \in D_{\ell,1}(s)} \big(p^\alpha_{t-s}\big((y-x)/2\big) + p^\alpha_{t-s}(y/2)\big) \\ &= C x^{\eta_c - 2\alpha\gamma} (t-s)^{-\eta_c/\alpha + 2\gamma - 1/\alpha} \\ &\quad \times \sup_{y \in D_{\ell,1}(s)} \big(p^\alpha_1\big((t-s)^{-1/\alpha}(y-x)/2\big) + p^\alpha_1\big((t-s)^{-1/\alpha} y/2\big)\big). \end{aligned} \tag{3.22}$$

Recall the following standard estimate on p^α_1:

$$p^\alpha_1(y) \le c_{(3.23)} |y|^{-(\alpha+1)}, \quad y \in \mathsf{R}, \tag{3.23}$$

for some constant $c_{(3.23)}$. Thus, on $D_{\ell,1}(s)$, we have $|y| \ge 2(t-s)^{1/\alpha-\rho_\ell}$, implying

$$p^\alpha_1\big((t-s)^{-1/\alpha} y/2\big) \le p^\alpha_1\big((t-s)^{-\rho_\ell}\big) \le c_{(3.23)} (t-s)^{\rho_\ell(\alpha+1)}, \tag{3.24}$$

where the last inequality follows by (3.23). A similar estimate holds for the second p_1^α-expression in (3.22). Thus, (3.22) yields

$$\sup_{y\in D_{\ell,1}(s)} \left|p_{t-s}^\alpha(y-x) - p_{t-s}^\alpha(y)\right| \le Cx^{\eta_c - 2\alpha\gamma}(t-s)^{-\eta_c/\alpha+2\gamma-1/\alpha+\rho_\ell(\alpha+1)}. \quad (3.25)$$

Now let us check that

$$\sup_{0<s<t} (t-s)^\lambda (t-s)^{-\eta_c/\alpha+2\gamma-1/\alpha+\rho_\ell(\alpha+1)} \le 1. \quad (3.26)$$

Recall that $\eta_c = \frac{\alpha}{1+\beta} - 1$. Then one can easily get that

$$\lambda - \eta_c/\alpha + 2\gamma - 1/\alpha + \rho_\ell(\alpha+1) = \gamma + \rho_\ell(\alpha+1) \ge \gamma, \quad (3.27)$$

where the last inequality follows by (3.5). Therefore, (3.26) follows immediately. Combining (3.21), (3.25), and (3.26), we see that

$$I_1 \le Cx^{\lambda+\eta_c-2\alpha\gamma} \le Cx^{\bar\eta_c-(2\alpha+1)\gamma}, \quad (3.28)$$

where we used the definitions of η_c and $\bar\eta_c$, given in (1.5) and Theorem 1.1(a), respectively.

Now let us bound I_2. Note that for any $(s, y) \in D_{\ell,2}$,

$$|y| \le x + 2(t-s)^{1/\alpha-\rho_{\ell+1}} \le 3(t-s)^{1/\alpha-\rho_{\ell+1}}. \quad (3.29)$$

Therefore, we have

$$I_2 \le 3^\lambda \sup_{0<s<t}\left((t-s)^{\lambda+(1/\alpha-\rho_{\ell+1})\lambda} \sup_{y\in D_{\ell,2}(s)} \left|p_{t-s}^\alpha(y-x) - p_{t-s}^\alpha(y)\right|\right). \quad (3.30)$$

Using again Lemma 2.3 but this time with $\delta = \bar\eta_c - (2\alpha+1)\gamma$ gives

$$\begin{aligned}
&\sup_{y\in D_{\ell,2}(s)} \left|p_{t-s}^\alpha(y-x) - p_{t-s}^\alpha(y)\right| \\
&\quad\le Cx^{\bar\eta_c-(2\alpha+1)\gamma}(t-s)^{-\bar\eta_c/\alpha+2\gamma+\gamma/\alpha} \\
&\qquad\times \sup_{y\in D_{\ell,2}(s)} \left(p_{t-s}^\alpha\big((y-x)/2\big) + p_{t-s}^\alpha(y/2)\right) \\
&\quad= Cx^{\bar\eta_c-(2\alpha+1)\gamma}(t-s)^{-\bar\eta_c/\alpha+2\gamma+\gamma/\alpha-1/\alpha+\rho_\ell(\alpha+1)}.
\end{aligned} \quad (3.31)$$

By definition (2.14) of λ,

$$\begin{aligned}
&\lambda + \left(\frac{1}{\alpha} - \rho_{\ell+1}\right)\lambda - \frac{\bar\eta_c}{\alpha} + 2\gamma + \frac{\gamma}{\alpha} - \frac{1}{\alpha} + \rho_\ell(\alpha+1) \\
&\quad= \frac{1}{\alpha}\left(\frac{1+\alpha}{1+\beta} - 1 - \bar\eta_c\right) + \gamma + \gamma\rho_{\ell+1} - \frac{\rho_{\ell+1}}{1+\beta} + \rho_\ell(\alpha+1) \\
&\quad\ge \gamma/2,
\end{aligned} \quad (3.32)$$

where in the last step we used the definition of $\bar{\eta}_c$, given in Theorem 1.1(a), and (3.5). Thus,

$$\sup_{0<s<t} (t-s)^{\lambda+(1/\alpha-\rho_{\ell+1})\lambda-\bar{\eta}_c/\alpha+2\gamma+\gamma/\alpha-1/\alpha+\rho_\ell(\alpha+1)} \le 1. \tag{3.33}$$

Combining estimates (3.30), (3.31), and (3.33), we obtain

$$I_2 \le C x^{\bar{\eta}_c-(2\alpha+1)\gamma}. \tag{3.34}$$

If the jumps of $M(\mathrm{d}(s,y))$ are smaller than $c^*(t-s)^\lambda$ on $\mathsf{R} \setminus B_2(0)$ (where c^* is from (3.9)), then the jumps of the process $u \mapsto \int_{(0,u]\times(\mathsf{R}\setminus B_2(0))} M(\mathrm{d}(s,y))\varphi_\pm(s,y)$ are bounded by

$$c^*(t-s)^\lambda \sup_{y\in\mathsf{R}\setminus B_2(0)} \varphi_\pm(s,y). \tag{3.35}$$

Using Lemma 2.3 once again but this time with $\delta = \bar{\eta}_c - 2\alpha\gamma$, we have

$$\begin{aligned}\left|p^\alpha_{t-s}(y-x) - p^\alpha_{t-s}(y)\right| &\le C x^{\bar{\eta}_c-2\alpha\gamma}(t-s)^{-\bar{\eta}_c/\alpha+2\gamma} \\ &\qquad\times \left(p^\alpha_{t-s}\big((y-x)/2\big) + p^\alpha_{t-s}(y/2)\right).\end{aligned} \tag{3.36}$$

Since $0 < x < 1$,

$$\begin{aligned}&\sup_{y\in\mathsf{R}\setminus B_2(0)} \left(p^\alpha_{t-s}\big((y-x)/2\big) + p^\alpha_{t-s}(y/2)\right) \\ &\quad\le C(t-s)^{-1/\alpha} p^\alpha_1\big((t-s)^{-1/\alpha}/2\big) \le C(t-s).\end{aligned} \tag{3.37}$$

Therefore, (3.19), (3.36), and (3.37) imply

$$\begin{aligned}c^*(t-s)^\lambda \sup_{y\in\mathsf{R}\setminus B_2(0)} \varphi_\pm(s,y) &\le C x^{\bar{\eta}_c-2\alpha\gamma}(t-s)^{\lambda-\bar{\eta}_c/\alpha+2\gamma+1} \\ &\le c_{(3.38)} x^{\bar{\eta}_c-2\alpha\gamma}\end{aligned} \tag{3.38}$$

for some constant $c_{(3.38)} = c_{(3.38)}(\varepsilon)$. Here we have used that $\bar{\eta}_c \le (1+\alpha)/(1+\beta)-1$ implies $\lambda - \bar{\eta}_c/\alpha + 2\gamma + 1 \ge 1$.

Combining (3.16), (3.18), (3.28), (3.34), and (3.38), we see that all jumps of the process $u \mapsto \int_{(0,u]\times\mathsf{R}} M(\mathrm{d}(s,y))\varphi_\pm(s,y)$ on the set A^ε are bounded by

$$c_{(3.39)} x^{\bar{\eta}_c-(2\alpha+1)\gamma} \tag{3.39}$$

for some constant $c_{(3.39)} = c_{(3.39)}(\varepsilon)$. Therefore, by an abuse of notation writing L for L^+ and L^-,

$$\begin{aligned}&\mathbf{P}\left(L_{T_\pm} \ge kx^\eta, A^\varepsilon\right) \\ &\quad= \mathbf{P}\Big(L_{T_\pm} \ge kx^\eta, \sup_{0<u<T_\pm} \Delta L_u \le c_{(3.39)} x^{\bar{\eta}_c-(2\alpha+1)\gamma}, A^\varepsilon\Big) \\ &\quad\le \mathbf{P}\Big(\sup_{0<v\le T_\pm} L_v \mathsf{1}\Big\{\sup_{0<u<v} \Delta L_u \le c_{(3.39)} x^{\bar{\eta}_c-(2\alpha+1)\gamma}\Big\} \ge kx^\eta, A^\varepsilon\Big).\end{aligned} \tag{3.40}$$

Since

$$T_{\pm} \le \int_0^t \mathrm{d}s \int_{\mathsf{R}} X_s(\mathrm{d}y) \big| p^{\alpha}_{t-s}(y-x) - p^{\alpha}_{t-s}(y) \big|^{1+\beta}, \tag{3.41}$$

applying [4, Lemma 2.12] with

$$\theta = 1+\beta \quad \text{and} \quad \delta = \mathsf{1}_{\beta<(\alpha-1)/2} + \frac{\alpha-\beta-\varepsilon}{1+\beta}\, \mathsf{1}_{\beta\ge(\alpha-1)/2}, \tag{3.42}$$

we may fix $\varepsilon_1 \in (0, \alpha\gamma\beta)$ to get the bound

$$T_{\pm} \le c_{(3.43)} \big(x^{1+\beta} \mathsf{1}_{\beta<(\alpha-1)/2} + x^{\alpha-\beta-\varepsilon_1} \mathsf{1}_{\beta\ge(\alpha-1)/2} \big) \tag{3.43}$$

on $\{V \le c_\varepsilon\}$ for some constant $c_{(3.43)} = c_{(3.43)}(\varepsilon)$. Consequently,

$$\begin{aligned} &\mathbf{P}\big(L_{T_\pm} \ge kx^\eta, A^\varepsilon \big) \\ &\quad \le \mathbf{P}\Big(\sup_{0<v\le c_{(3.43)}(x^{1+\beta}\mathsf{1}_{\beta<(\alpha-1)/2} + x^{\alpha-\beta-\varepsilon_1}\mathsf{1}_{\beta\ge(\alpha-1)/2})} L_v \\ &\qquad \times \mathsf{1}\Big\{ \sup_{0<u<v} \Delta L_u \le c_{(3.39)} x^{\bar\eta_c - (2\alpha+1)\gamma} \Big\} \ge kx^\eta \Big). \end{aligned} \tag{3.44}$$

Now use Lemma 2.4 with $\kappa = 1+\beta$, $t = c_{(3.43)}(x^{1+\beta}\mathsf{1}_{\beta<(\alpha-1)/2} + x^{\alpha-\beta-\varepsilon_1} \times \mathsf{1}_{\beta\ge(\alpha-1)/2})$, kx^η instead of x, and $y = c_{(3.39)} x^{\bar\eta_c-(2\alpha+1)\gamma}$. This gives

$$\begin{aligned} &\mathbf{P}\big(L_{T_\pm} \ge kx^\eta, A^\varepsilon \big) \\ &\quad \le \left(\frac{C c_{(3.43)} (x^{1+\beta}\mathsf{1}_{\beta<(\alpha-1)/2} + x^{\alpha-\beta-\varepsilon_1}\mathsf{1}_{\beta\ge(\alpha-1)/2})}{kx^\eta (c_{(3.39)} x^{\bar\eta_c-(2\alpha+1)\gamma})^\beta} \right)^{\frac{x^{\eta-\bar\eta_c+(2\alpha+1)\gamma}}{c_{(3.39)}}}. \end{aligned} \tag{3.45}$$

Now we need additionally the following simple inequalities, which are easy to derive:

$$-\eta - \beta\big(\bar\eta_c - (2\alpha+1)\gamma\big) + 1 + \beta \ge (2\alpha+1)\gamma\beta \quad \text{on } \beta < \frac{\alpha-1}{2} \tag{3.46}$$

and

$$\begin{aligned} &-\eta - \beta\big(\bar\eta_c - (2\alpha+1)\gamma\big) + \alpha - \beta - \varepsilon_1 \ge (2\alpha+1)\gamma\beta - \varepsilon_1 \\ &\quad \ge \alpha\gamma\beta \quad \text{on } \beta \ge \frac{\alpha-1}{2}. \end{aligned} \tag{3.47}$$

In fact, $\bar\eta_c = 1$ under $\beta < (\alpha-1)/2$, whereas the other case in the definition of $\bar\eta_c$ applies under $\beta \ge (\alpha-1)/2$. Then, using the above inequalities and (3.45), we obtain

$$\mathbf{P}\big(L_{T_\pm} \ge kx^\eta, A^\varepsilon \big) \le \big(c_{(3.48)} k^{-1} x^{\alpha\gamma\beta} \big)^{(c^{-1}_{(3.39)} kx^{\eta-\bar\eta_c+(2\alpha+1)\gamma})} \tag{3.48}$$

for some constant $c_{(3.48)} = c_{(3.48)}(\varepsilon)$. Applying this bound with $\gamma = \frac{\bar\eta_c-\eta}{2(2\alpha+1)}$ to the summands at the right-hand side in inequality (3.12) and noting that $\alpha\gamma\beta$ is also a positive constant here, we have

$$\mathbf{P}\big(\big|Z_t^{2,\varepsilon}(x) - Z_t^{2,\varepsilon}(0)\big| \geq 2kx^{\eta}\big) \leq 2\big(c_{(3.48)}k^{-1}x\big)^{(c_{(3.49)}kx^{(\eta-\bar{\eta}_c)/2})} \tag{3.49}$$

for some constant $c_{(3.49)}$. This inequality yields

$$\lim_{k\to\infty}\sum_{n=1}^{\infty}\mathbf{P}\big(\big|Z_t^{2,\varepsilon}(n^{-q}) - Z_t^{2,\varepsilon}(0)\big| \geq kn^{-q\eta}\big) = 0 \tag{3.50}$$

for every positive q.

Recall that our purpose is to show that

$$\sup_{0<x<1}\frac{|Z_t^2(x) - Z_t^2(0)|}{x^{\eta}} < \infty \quad \text{almost surely} \tag{3.51}$$

or, in other words,

$$\lim_{k\uparrow\infty}\mathbf{P}\bigg(\sup_{0<x<1}\frac{|Z_t^2(x) - Z_t^2(0)|}{x^{\eta}} > k\bigg) = 0. \tag{3.52}$$

It is easy to see that

$$\bigg\{\sup_{0<x<1}\frac{|Z_t^2(x) - Z_t^2(0)|}{x^{\eta}} > k\bigg\} \subseteq \bigcup_{n=1}^{\infty}\bigg\{\sup_{x\in I_n}\big|Z_t^2(x) - Z_t^2(0)\big| > \frac{k}{2^q}n^{-q\eta}\bigg\}, \tag{3.53}$$

where $I_n := \{x\colon\ (n+1)^{-q} \leq x < n^{-q}\}$. Moreover, by the triangle inequality,

$$\big|Z_t^2(x) - Z_t^2(0)\big| \leq \big|Z_t^2(x) - Z_t^2(n^{-q})\big| + \big|Z_t^2(n^{-q}) - Z_t^2(0)\big|, \quad x \in I_n. \tag{3.54}$$

Furthermore, for all $R > 0$,

$$\begin{aligned}&\bigg\{\sup_{0<x<y<1}\frac{|Z_t^2(x) - Z_t^2(y)|}{|x-y|^{q\eta/(q+1)}} \leq R\bigg\}\\ &\quad\subseteq \big\{\big|Z_t^2(x) - Z_t^2(n^{-q})\big| \leq Rq^{q\eta/(q+1)}n^{-q\eta},\ x \in I_n\big\}.\end{aligned} \tag{3.55}$$

Consequently, for all $n \geq 1$,

$$\begin{aligned}&\bigg\{\sup_{x\in I_n}\big|Z_t^2(x) - Z_t^2(0)\big| > \frac{k}{2^q}n^{-q\eta}\bigg\}\\ &\subseteq \bigg\{\sup_{0<x<y<1}\frac{|Z_t^2(x) - Z_t^2(y)|}{|x-y|^{q\eta/(q+1)}} > c(q)k\bigg\} \cup \bigg\{\big|Z_t^2(n^{-q}) - Z_t^2(0)\big| > \frac{k}{2^{q+1}}n^{-q\eta}\bigg\},\end{aligned}$$

where $c(q)$ is some positive constant. If we choose q so small that $\eta q/(q+1) < \eta_c$, then

$$\lim_{k\to\infty}\mathbf{P}\bigg(\sup_{0<x<y<1}\frac{|Z_t^2(x) - Z_t^2(y)|}{|x-y|^{q\eta/(q+1)}} > c(q)k\bigg) = 0, \tag{3.56}$$

since, by Theorem 1.2(a) of [4], Z_t^2 is locally Hölder continuous of every index smaller than η_c. Therefore, it suffices to show that

$$\lim_{k\to\infty} \mathbf{P}\Bigg(\bigcup_{n=1}^{\infty}\Big\{\big|Z_t^2(n^{-q}) - Z_t^2(0)\big| > \frac{k}{2^{q+1}}n^{-q\eta}\Big\}\Bigg) = 0. \tag{3.57}$$

But

$$\begin{aligned}\mathbf{P}&\Bigg(\bigcup_{n=1}^{\infty}\Big\{\big|Z_t^2(n^{-q}) - Z_t^2(0)\big| > \frac{k}{2^{q+1}}n^{-q\eta}\Big\}\Bigg)\\ &\le \mathbf{P}\Bigg(\bigcup_{n=1}^{\infty}\Big\{\big|Z_t^{2,\varepsilon}(n^{-q}) - Z_t^{2,\varepsilon}(0)\big| > \frac{k}{2^{q+1}}n^{-q\eta}\Big\}\Bigg) + \mathbf{P}\big(A^{\varepsilon,\mathrm{c}}\big),\end{aligned} \tag{3.58}$$

where $A^{\varepsilon,\mathrm{c}}$ denotes the complement of A^{ε}. It follows from (3.50) that

$$\lim_{k\to\infty} \mathbf{P}\Bigg(\bigcup_{n=1}^{\infty}\Big\{\big|Z_t^{2,\varepsilon}(n^{-q}) - Z_t^{2,\varepsilon}(0)\big| > \frac{k}{2^{q+1}}n^{-q\eta}\Big\}\Bigg) = 0. \tag{3.59}$$

Moreover, $\mathbf{P}(A^{\varepsilon,\mathrm{c}}) \le 3\varepsilon$, see (3.11). As a result, we have

$$\limsup_{k\uparrow\infty} \mathbf{P}\Bigg(\bigcup_{n=1}^{\infty}\Big\{\big|Z_t^2(n^{-q}) - Z_t^2(0)\big| > \frac{k}{2^{q+1}}n^{-q\eta}\Big\}\Bigg) \le 3\varepsilon. \tag{3.60}$$

Since ε may be arbitrarily small, this implies (3.52). This yields the desired Hölder continuity of Z_t^2 at 0 for all $\eta < \bar{\eta}_c$. Since Z_t^1 and Z_t^3 are a.s. Lipschitz continuous at 0 (cf. [4, Remark 2.13]), recalling (2.5), the proof of Theorem 1.1(a) is complete. □

4 Optimality of $\bar{\eta}_c$: Proof of Theorem 1.1(b)

We continue to consider $d = 1$, to fix $t, z, \mu, \alpha, \beta, \eta$ as in the theorem, and to assume $0 < t < 1$ and $z = 0$.

In analogy to the proof of optimality of η_c in [4, Sect. 5], our strategy is to find a sequence of "big" jumps that occur close to time t. But in contrast to the case of the local Hölder continuity, we need to find these "big" jumps in the vicinity of 0, where these jumps should destroy the Hölder continuity of any index greater than or equal to $\bar{\eta}_c$. This needs to overcome some new technical difficulties.

Recall that we need to prove the optimality in the case $\beta > (\alpha-1)/2$ only. This implies that $\bar{\eta}_c = \frac{\alpha+1}{\beta+1} - 1 < 1$.

First let us give two technical lemmas that we need for the proof.

Lemma 4.1 (Some Left-Hand Continuity) *For all* $c, \theta > 0$,

$$\mathbf{P}\Big(X_t(0) > \theta,\ \liminf_{s\uparrow t} S^{\alpha}_{t-s}X_s\big(c(t-s)^{1/\alpha}\big) \le \theta\Big) = 0. \tag{4.1}$$

Proof For brevity, set

$$A := \Big\{\liminf_{s\uparrow t} S^{\alpha}_{t-s} X_s\big(c(t-s)^{1/\alpha}\big) \le \theta\Big\}, \tag{4.2}$$

and for $n > 1/t$, define the stopping times

$$\tau_n := \begin{cases} \inf\{s \in (t-1/n, t)\colon\ S^{\alpha}_{t-s} X_s(c(t-s)^{1/\alpha}) \le \theta + 1/n\}, & \omega \in A, \\ t, & \omega \in A^{\mathrm{c}}. \end{cases} \tag{4.3}$$

Define also

$$x_n := c(t-\tau_n)^{1/\alpha}. \tag{4.4}$$

Then, using the strong Markov property, we get

$$\mathbf{E}\big[X_t(x_n) \,\big|\, \mathcal{F}_{\tau_n}\big] = S^{\alpha}_{t-\tau_n} X_{\tau_n}(x_n) = X_t(0)1_{A^{\mathrm{c}}} + S^{\alpha}_{t-\tau_n} X_{\tau_n}(x_n)1_A. \tag{4.5}$$

We next note that $x_n \to 0$ almost surely as $n \uparrow \infty$. This implies, in view of the continuity of X_t at zero, that $X_t(x_n) \to X_t(0)$ almost surely. Recalling that $\mathbf{E}\sup_{|x|\le 1} X_t(x) < \infty$ in view of Corollary 2.8 of [4], we conclude that

$$X_t(x_n) \underset{n\uparrow\infty}{\longrightarrow} X_t(0) \quad \text{in } \mathcal{L}_1. \tag{4.6}$$

This, in turn, implies that

$$\mathbf{E}\big[X_t(x_n) \,\big|\, \mathcal{F}_{\tau_n}\big] - \mathbf{E}\big[X_t(0) \,\big|\, \mathcal{F}_{\tau_n}\big] \underset{n\uparrow\infty}{\longrightarrow} 0 \quad \text{in } \mathcal{L}_1. \tag{4.7}$$

Furthermore, it follows from the well-known Levy theorem on convergence of conditional expectations that

$$\mathbf{E}\big[X_t(0) \,\big|\, \mathcal{F}_{\tau_n}\big] \underset{n\uparrow\infty}{\longrightarrow} \mathbf{E}\big[X_t(0) \,\big|\, \mathcal{F}_{\infty}\big] \quad \text{in } \mathcal{L}_1, \tag{4.8}$$

where $\mathcal{F}_\infty := \sigma(\bigcup_{n>1/t} \mathcal{F}_{\tau_n})$.

Noting that $\tau_n \uparrow t$, we conclude that

$$\mathcal{F}_{t-} \subseteq \mathcal{F}_\infty \subseteq \mathcal{F}_t. \tag{4.9}$$

Since $X.(0)$ is continuous at fixed t a.s., we have $X_t(0) = \mathbf{E}[X_t(0)|\mathcal{F}_{t-}]$ almost surely. Consequently, $\mathbf{E}[X_t(0)|\mathcal{F}_\infty] = X_t(0)$ almost surely, and we get, as a result,

$$\mathbf{E}\big[X_t(0) \,\big|\, \mathcal{F}_{\tau_n}\big] \underset{n\uparrow\infty}{\longrightarrow} X_t(0) \quad \text{in } \mathcal{L}_1. \tag{4.10}$$

Combining (4.7) and (4.10), we have

$$\mathbf{E}\big[X_t(x_n) \,\big|\, \mathcal{F}_{\tau_n}\big] \underset{n\uparrow\infty}{\longrightarrow} X_t(0) \quad \text{in } \mathcal{L}_1. \tag{4.11}$$

From this convergence and from (4.5) we finally get

$$\mathbf{E}\big(1_A\big|X_t(0) - S^\alpha_{t-\tau_n}X_{\tau_n}(x_n)\big|\big) \underset{n\uparrow\infty}{\longrightarrow} 0. \tag{4.12}$$

Since $S^\alpha_{t-\tau_n}X_{\tau_n}(x_n) \le \theta + 1/n$ on A for all $n > 1/t$, the latter convergence implies that $X_t(0) \le \theta$ almost surely on the event A. Thus, the proof is finished. □

Lemma 4.2 (Some Local Boundedness) *Fix any nonempty bounded* $B \subset \mathsf{R}$. *Then*

$$W_B := \sup_{(c,s,x):\ c\ge 1,\ 0\vee(t-c^{-\alpha})\le s<t,\ x\in B} \frac{X_s(B_{c(t-s)^{1/\alpha}}(x))}{c(t-s)^{1/\alpha}} < \infty \quad a.s. \tag{4.13}$$

Proof Every ball of radius $c(t-s)^{1/\alpha}$ can be covered with at most $[c]+1$ balls of radius $(t-s)^{1/\alpha}$. Therefore,

$$\sup_{(c,s,x):\ c\ge 1,\ 0\vee(t-c^{-1/\alpha})\le s<t,\ x\in B} \frac{X_s(B_{c(t-s)^{1/\alpha}}(x))}{c(t-s)^{1/\alpha}} \le 2 \sup_{(s,x):\ 0<s\le t,\ x\in B_1} \frac{X_s(B_{(t-s)^{1/\alpha}}(x))}{(t-s)^{1/\alpha}}, \tag{4.14}$$

where $B_1 := \{x\colon \operatorname{dist}(x,\overline{B}) \le 1\}$ with $\overline{B}$ denoting the closure of B. (The restriction $s \ge t - c^{-1/\alpha}$ is imposed to have all centers x of the balls $B_{(t-s)^{1/\alpha}}(x)$ in B_1.) We further note that

$$S^\alpha_{t-s}X_s(x) = \int_{\mathsf{R}} \mathrm{d}y\, p^\alpha_{t-s}(x-y)X_s(y) \ge \int_{B_{(t-s)^{1/\alpha}}(x)} \mathrm{d}y\, p^\alpha_{t-s}(x-y)X_s(y). \tag{4.15}$$

Using the monotonicity and the scaling property of p^α, we get the bound

$$S^\alpha_{t-s}X_s(x) \ge (t-s)^{-1/\alpha} p^\alpha_1(1) X_s\big(B_{(t-s)^{1/\alpha}}(x)\big). \tag{4.16}$$

Consequently,

$$\sup_{(s,x):\ 0<s\le t,\ x\in B_1} \frac{X_s(B_{(t-s)^{1/\alpha}}(x))}{(t-s)^{1/\alpha}} \le \frac{1}{p^\alpha_1(1)} \sup_{(s,x):\ 0<s\le t,\ x\in B_1} S^\alpha_{t-s}X_s(x). \tag{4.17}$$

It was proved in Lemma 2.11 of [4] that the random variable at the right-hand side is finite. Thus, the lemma is proved. □

Introduce the event

$$D_\theta := \Big\{X_t(0) > \theta,\ \sup_{0<s\le t} X_s(\mathsf{R}) \le \theta^{-1},\ W_{B_3(0)} \le \theta^{-1}\Big\}. \tag{4.18}$$

For the rest of the paper, take an arbitrary $\varepsilon \in (0, t \wedge 1/8)$. For constants c, $Q > 0$, define the stopping time

$$\tau_{\varepsilon,c,Q} := \inf\Big\{ s \in (t-\varepsilon, t)\colon\ \Delta X_s(y) > Q\big(y(t-s)\big)^{1/(1+\beta)} \log^{1/(1+\beta)}\big((t-s)^{-1}\big) \\ \text{for some } \frac{c}{2}(t-s)^{1/\alpha} \le y \le \frac{3c}{2}(t-s)^{1/\alpha} \Big\}. \tag{4.19}$$

In the next lemma we are going to show the finiteness of $\tau_{\varepsilon,c,Q}$, which means that there is a "big" jump close to time t and to the spatial point $z = 0$.

Lemma 4.3 (Finiteness of $\tau_{\varepsilon,c_{(4.20)},Q}$) *For each* $\theta > 0$, *there exists a constant* $c_{(4.20)} = c_{(4.20)}(\theta) \ge 1$ *such that*

$$\mathbf{P}(\tau_{\varepsilon,c_{(4.20)},Q} = \infty | D_\theta) = 0, \quad \varepsilon \in (0, t \wedge 1/8),\ Q > 0. \tag{4.20}$$

Proof Analogously to the proof of Lemma 4.3 in [4], to demonstrate that the number of jumps is greater than zero almost surely on some event, it is enough to show the divergence of a certain integral on that event or even on a bigger one. Specifically here, it suffices to verify that

$$I_{\varepsilon,c} := \int_{t-\varepsilon}^{t} \frac{\mathrm{d}s}{(t-s)\log((t-s)^{-1})} \int_{\frac{c}{2}(t-s)^{1/\alpha}}^{\frac{3c}{2}(t-s)^{1/\alpha}} \mathrm{d}y\, y^{-1} X_s(y) = \infty \tag{4.21}$$

almost surely on the event $\{X_t(0) > \theta,\ \sup_{0<s\le t} X_s(\mathsf{R}) \le \theta^{-1}\}$.

The mapping $\varepsilon \mapsto I_{\varepsilon,c}$ is nonincreasing. Therefore, we shall additionally assume, without loss of generality, that $\varepsilon \le c^{-1/\alpha}$, and this in turn implies that $c(t-s)^{1/\alpha} \le 1$ for all $s \in (t-\varepsilon, t)$. So, in what follows, in the proof of the lemma we will assume without loss of generality that given c, we choose ε so that

$$c(t-s)^{1/\alpha} \le 1, \quad s \in (t-\varepsilon, t). \tag{4.22}$$

Since $y \le \frac{3c}{2}(t-s)^{1/\alpha}$ and $p_s^\alpha(x) \le p_s^\alpha(0)$ for all $x \in \mathsf{R}$, we have

$$I_{\varepsilon,c} \ge \frac{2}{3c} \int_{t-\varepsilon}^{t} \frac{\mathrm{d}s}{(t-s)^{1+1/\alpha}\log((t-s)^{-1})} \\ \times \int_{\frac{c}{2}(t-s)^{1/\alpha}}^{\frac{3c}{2}(t-s)^{1/\alpha}} \mathrm{d}y \frac{p_{t-s}^\alpha(c(t-s)^{1/\alpha} - y)}{p_{t-s}^\alpha(0)} X_s(y). \tag{4.23}$$

Then, using the scaling property of p^α, we obtain

$$I_{\varepsilon,c} \ge \frac{2}{3cp_1^\alpha(0)} \int_{t-\varepsilon}^{t} \frac{\mathrm{d}s}{(t-s)\log((t-s)^{-1})} \Big(S_{t-s}^\alpha X_s\big(c(t-s)^{1/\alpha}\big) \\ - \int_{|y-c(t-s)^{1/\alpha}| > \frac{c}{2}(t-s)^{1/\alpha}} \mathrm{d}y\, p_{t-s}^\alpha\big(c(t-s)^{1/\alpha} - y\big) X_s(y) \Big). \tag{4.24}$$

Since we are in dimension one, if

$$y \in \widetilde{D}_{s,j} := \Big\{z\colon c\Big(\frac{1}{2}+j\Big)(t-s)^{1/\alpha} < \big|z - c(t-s)^{1/\alpha}\big| \\ < c\Big(2+\frac{1}{2}+j\Big)(t-s)^{1/\alpha}\Big\}, \tag{4.25}$$

then

$$\begin{aligned} p^{\alpha}_{t-s}\big(c(t-s)^{1/\alpha}-y\big) &\le p^{\alpha}_{t-s}\big(c(j+1/2)(t-s)^{1/\alpha}\big) \\ &= (t-s)^{-1/\alpha} p^{\alpha}_{1}\big(c(j+1/2)\big) \\ &\le c_{(3.23)} c^{-\alpha-1}(t-s)^{-1/\alpha}(1/2+j)^{-\alpha-1}. \end{aligned} \tag{4.26}$$

From this bound we conclude that

$$\int_{|y-c(t-s)^{1/\alpha}|>\frac{c}{2}(t-s)^{1/\alpha}} \mathrm{d}y\, p^{\alpha}_{t-s}\big(c(t-s)^{1/\alpha}-y\big) 1_{B_2(0)}(y) X_s(y) \\ \le c_{(3.23)} c^{-\alpha-1}(t-s)^{-1/\alpha}\sum_{j=0}^{\infty}(1/2+j)^{-\alpha-1}\int_{\widetilde{D}_{s,j}} \mathrm{d}y\, 1_{B_2(0)}(y)X_s(y). \tag{4.27}$$

Now recall again that the spatial dimension equals to one, and hence for any $j \ge 0$, the set $\widetilde{D}_{s,j}$ in (4.25) is the union of two balls of radius $c(t-s)^{1/\alpha}$. If furthermore $\widetilde{D}_{s,j} \cap B_2(0) \ne \emptyset$, then, in view of the assumption $c(t-s)^{1/\alpha} \le 1$, the centers of those balls lie in $B_3(0)$. Therefore, we can apply Lemma 4.2 to bound the integral $\int_{\widetilde{D}_{s,j}} \mathrm{d}y\, 1_{B_2(0)}(y)X_s(y)$ by $2c(t-s)^{1/\alpha} W_{B_3(0)}$ and obtain

$$\int_{|y-c(t-s)^{1/\alpha}|>\frac{c}{2}(t-s)^{1/\alpha}} \mathrm{d}y\, p^{\alpha}_{t-s}\big(c(t-s)^{1/\alpha}-y\big) 1_{B_2(0)}(y) X_s(y) \\ \le 2W_{B_3(0)} c_{(3.23)} c^{-\alpha}\sum_{j=0}^{\infty}(1/2+j)^{-\alpha-1} \le C\, W_{B_3(0)} c^{-\alpha}. \tag{4.28}$$

Furthermore, if $|y| \ge 2$ and $(t-s) \le c^{-\alpha}$, then

$$\begin{aligned} p^{\alpha}_{t-s}\big(c(t-s)^{1/\alpha}-y\big) \le p^{\alpha}_{t-s}(1) &= (t-s)^{-1/\alpha} p^{\alpha}_{1}\big((t-s)^{-1/\alpha}\big) \\ &\le c_{(3.23)}(t-s). \end{aligned} \tag{4.29}$$

This implies that

$$\begin{aligned} \int_{\mathsf{R}\setminus B_2(0)} \mathrm{d}y\, p^{\alpha}_{t-s}\big(c(t-s)^{1/\alpha}-y\big) X_s(y) &\le c_{(3.23)}(t-s) X_s(\mathsf{R}) \\ &\le c_{(3.23)} c^{-\alpha} X_s(\mathsf{R}). \end{aligned} \tag{4.30}$$

Combining this bound with (4.28), we obtain

$$\int_{|y-c(t-s)^{1/\alpha}|>\frac{c}{2}(t-s)^{1/\alpha}} \mathrm{d}y\, p^{\alpha}_{t-s}\big(c(t-s)^{1/\alpha}-y\big)X_s(y)$$
$$\leq Cc^{-\alpha}\Big(W_{B_3(0)} + \sup_{0<s\leq t} X_s(\mathsf{R})\Big). \tag{4.31}$$

Thus, we can choose c so large that the right-hand side in the previous inequality does not exceed $\theta/2$. Since, in view of Lemma 4.1,

$$\liminf_{s\uparrow t} S^{\alpha}_{t-s}X_s\big(c(t-s)^{1/\alpha}\big) > \theta, \tag{4.32}$$

we finally get

$$\liminf_{s\uparrow t}\Big(S^{\alpha}_{t-s}X_s\big(c(t-s)^{1/\alpha}\big)$$
$$-\int_{|y-c(t-s)^{1/\alpha}|>\frac{c}{2}(t-s)^{1/\alpha}} \mathrm{d}y\, p^{\alpha}_{t-s}\big(c(t-s)^{1/\alpha}-y\big)X_s(y)\Big) \geq \theta/2. \tag{4.33}$$

From this bound and from (4.24) the desired property of $I_{\varepsilon,c}$ follows. □

Fix any $\theta > 0$ and, to simplify the notation, write $c := c_{(4.20)}$. For all n sufficiently large, say $n \geq N_0$, define

$$A_n := \Big\{\Delta X_s\Big(\Big(\frac{c}{2}2^{-n}, \frac{3c}{2}2^{-n}\Big)\Big) \geq 2^{-(\bar{\eta}_c+1)n} n^{1/(1+\beta)}$$
$$\text{for some } s \in \big(t-2^{-\alpha n}, t-2^{-\alpha(n+1)}\big)\Big\}. \tag{4.34}$$

Based on Lemma 4.3, we will show in the following lemma that, conditionally on D_θ, infinitely many of the A_n's occur. This then gives us a bit more precise information on the "big" jumps we are looking for.

Lemma 4.4 (Existence of Big Jumps) *We have*

$$\mathbf{P}(A_n \textit{ infinitely often} \mid D_\theta) = 1. \tag{4.35}$$

Proof If $y \in (\frac{c}{2}(t-s)^{1/\alpha}, \frac{3c}{2}(t-s)^{1/\alpha})$ and $s \in (t-2^{-\alpha n}, t-2^{-\alpha(n+1)})$, then

$$\big((t-s)y\log\big((t-s)^{-1}\big)\big)^{1/(1+\beta)} \geq \Big(2^{-\alpha(n+1)}\frac{c}{2}2^{-n-1}\alpha n\log 2\Big)^{1/(1+\beta)}$$
$$= c^{-1}_{(4.36)} 2^{-(\bar{\eta}_c+1)n} n^{1/(1+\beta)}. \tag{4.36}$$

This implies that

$$A_n \supseteq \Big\{\Delta X_s\Big(\Big(\frac{c}{2}(t-s)^{1/\alpha}, \frac{3c}{2}(t-s)^{1/\alpha}\Big)\Big)$$

$$\geq c_{(4.36)}\big((t-s)y\log\big((t-s)^{-1}\big)\big)^{1/(1+\beta)}$$
$$\text{for some } s\in\big(t-2^{-\alpha n}, t-2^{-\alpha(n+1)}\big)\Big\}. \quad (4.37)$$

In what follows, with some abuse of notation, we denote $\tau_{\varepsilon,c} := \tau_{\varepsilon,c,c_{(4.36)}}$. Consequently, from (4.37) we get

$$\bigcup_{n=N}^{\infty} A_n \supseteq \{\tau_{2^{-\alpha N},c} < \infty\} \quad \text{for all } N > N_0 \vee \alpha^{-1}\log_2(t\wedge 1/8). \quad (4.38)$$

Applying Lemma 4.3 and using the monotonicity of the union in N, we get

$$\mathbf{P}\Bigg(\bigcup_{n=N}^{\infty} A_n \,\Bigg|\, D_\theta\Bigg) = 1 \quad \text{for all } N \geq N_0. \quad (4.39)$$

This completes the proof. $\square$

Now it is time to explain our

Detailed strategy of proof of Theorem 1.1(b) Define

$$A^\varepsilon := \Big\{\Delta X_s(y) \leq c_{(2.7)}\big((t-s)|y|\big)^{1/(1+\beta)}(f_{s,x})^\ell \text{ for all } s<t \text{ and } y\in B_{1/\mathrm{e}}(0)\Big\}$$
$$\cap\Big\{\Delta X_s(y) \leq c^*(t-s)^{1/(1+\beta)-\gamma} \text{ for all } s<t \text{ and } y\in\mathsf{R}\Big\}\cap\{V\leq c_\varepsilon\}, \quad (4.40)$$

where $f_{s,x}$, ℓ and c^* are defined in (2.6), (2.8), and (3.9), respectively. Note that $D_\theta \uparrow \{X_t(0)>0\}$ as $\theta\downarrow 0$, and by (3.6), (3.9), and Lemma 2.1 we have that $A^\varepsilon \uparrow \Omega$ as $\varepsilon\downarrow 0$. Hence, for the proof of Theorem 1.1(b), *it is sufficient* to show that

$$\mathbf{P}\Bigg(\sup_{x\in B_\epsilon(0),\ x\neq 0} \frac{|\tilde{X}_t(x)-\tilde{X}_t(0)|}{|x|^{\bar{\eta}_c}} = \infty \,\Bigg|\, D_\theta\cap A^\varepsilon\Bigg) = 1. \quad (4.41)$$

Moreover, since Z_t^1 and Z_t^3 are a.s. Lipschitz continuous at 0, the latter will follow from the equality

$$\mathbf{P}\big(Z_t^2(c2^{-n-2}) - Z_t^2(0) \geq 2^{-\bar{\eta}_c n} n^{1/(1+\beta)-\varepsilon} \text{ infinitely often} | D_\theta\cap A^\varepsilon\big) = 1. \quad (4.42)$$

To verify (4.42), we will again exploit our method of representing Z_t^2 using a time-changed stable process. To be more precise, applying (3.2) with $x=c2^{-n-2}$ (for n sufficiently large) and using n-dependent notation as $L_n^\pm$, $T_{n,\pm}$ (and later $\varphi_{n,\pm}$), we have

$$Z_t^2\big(c2^{-n-2}\big) - Z_t^2(0) = L_n^+(T_{n,+}) - L_n^-(T_{n,-}). \quad (4.43)$$

Let us define the events

$$B_n^+ := \{L_n^+(T_{n,+}) \geq 2^{1-\bar{\eta}_c n} n^{1/(1+\beta)-\varepsilon}\},$$
$$B_n^- := \{L_n^-(T_{n,-}) \leq 2^{-\bar{\eta}_c n} n^{1/(1+\beta)-\varepsilon}\},$$

and

$$B_n := B_n^+ \cap B_n^-. \tag{4.44}$$

Then, obviously,

$$\{Z_t^2(c2^{-n-2}) - Z_t^2(0) \geq 2^{-\bar{\eta}_c n} n^{1/(1+\beta)-\varepsilon}\} \supseteq B_n \supseteq B_n \cap A_n. \tag{4.45}$$

Thus, (4.42) will follow once we verify that

$$\lim_{N\uparrow\infty} \mathbf{P}\Bigg(\bigcup_{n=N}^{\infty} (B_n \cap A_n) \,\Bigg|\, D_\theta \cap A^\varepsilon\Bigg) = 1. \tag{4.46}$$

Taking into account Lemma 4.4, we conclude that to get (4.46) we have to show

$$\lim_{N\uparrow\infty} \mathbf{P}\Bigg(\bigcup_{n=N}^{\infty} \big(B_n^{\mathrm{c}} \cap A_n\big) \,\Bigg|\, D_\theta \cap A^\varepsilon\Bigg) = 0. \tag{4.47}$$

Hence, the proof of Theorem 1.1(b) will be complete once we demonstrate statement (4.47). □

Now we will present two lemmas, from which (4.47) will follow immediately. To this end, split

$$B_n^{\mathrm{c}} \cap A_n = \big(B_n^{+,\mathrm{c}} \cap A_n\big) \cup \big(B_n^{-,\mathrm{c}} \cap A_n\big). \tag{4.48}$$

Lemma 4.5 (First Term in (4.48)) *We have*

$$\lim_{N\uparrow\infty} \sum_{n=N}^{\infty} \mathbf{P}\big(B_n^{+,\mathrm{c}} \cap A_n \cap A^\varepsilon\big) = 0. \tag{4.49}$$

The proof of this lemma is a word-for-word repetition of the proof of Lemma 5.3 in [4] (it is even simpler as we do not need additional indexing in k here), and we omit it. The idea behind the proof is simple: Whenever X has a "big" jump guaranteed by A_n, this jump corresponds to the jump of L_n^+, and then it is very difficult for a spectrally positive process L_n^+ to come down, which is required by $B_n^{+,\mathrm{c}}$.

Lemma 4.6 (Second Term in (4.48)) *We have*

$$\lim_{N\uparrow\infty} \sum_{n=N}^{\infty} \mathbf{P}\big(B_n^{-,\mathrm{c}} \cap A_n \cap A^\varepsilon \cap D_\theta\big) = 0. \tag{4.50}$$

The remaining part of the paper will be devoted to the proof of Lemma 4.6, and we prepare now for it.

One can easily see that $B_n^{-,\mathrm{c}}$ is a subset of a union of two events (with the obvious correspondence):

$$\begin{aligned} B_n^{-,\mathrm{c}} \subseteq U_n^1 \cup U_n^2 &:= \left\{\Delta L_n^- > 2^{-\bar{\eta}_{\mathrm{c}} n} n^{1/(1+\beta)-2\varepsilon}\right\} \\ &\cup \left\{\Delta L_n^- \leq 2^{-\bar{\eta}_{\mathrm{c}} n} n^{1/(1+\beta)-2\varepsilon},\ L_n^-(T_{n,-}) > 2^{-\bar{\eta}_{\mathrm{c}} n} n^{1/(1+\beta)-\varepsilon}\right\}, \end{aligned} \tag{4.51}$$

where

$$\Delta L_n^- := \sup_{0<s\leq T_{n,-}} \Delta L_n^-(s). \tag{4.52}$$

The occurrence of the event U_n^1 means that L_n^- has big jumps. If U_n^2 occurs, it means that L_n^- gets large without big jumps. It is well known that stable processes without big jumps cannot achieve large values. Thus, the statement of the next lemma is not surprising.

Lemma 4.7 (No Big Values of L_n^- in Case of Absence of "Big" Jumps) *We have*

$$\lim_{N\uparrow\infty} \sum_{n=N}^{\infty} \mathbf{P}\big(U_n^2 \cap A^\varepsilon\big) = 0. \tag{4.53}$$

We omit the proof of this lemma as well, since its crucial part related to bounding of $\mathbf{P}(U_n^2 \cap A^\varepsilon)$ is a repetition of the proof of Lemma 5.6 in [4] (again with obvious simplifications).

Lemma 4.8 (Big Jumps of L_n^- Caused by Several Big Jumps of M) *There exist constants ρ and ξ such that, for all sufficiently large values of n,*

$$A^\varepsilon \cap A_n \cap U_n^1 \subseteq A^\varepsilon \cap E_n(\rho,\xi), \tag{4.54}$$

where

$$\begin{aligned} E_n(\rho,\xi) := \Big\{ &\textit{There exist at least two jumps of } M \textit{ of the form } r\delta_{(s,y)} \textit{ such that} \\ &r \geq \big((t-s)\max\{(t-s)^{1/\alpha}, |y|\}\big)^{1/(1+\beta)} \log^{1/(1+\beta)-2\varepsilon}\big((t-s)^{-1}\big), \\ &|y| \leq (t-s)^{1/\alpha} \log^{\xi}\big((t-s)^{-1}\big),\ s \in \big[t-2^{-\alpha n} n^{\rho}, t-2^{-\alpha n} n^{-\rho}\big]\Big\}. \end{aligned} \tag{4.55}$$

Proof By the definition of A_n, there exists a jump of M of the form $r\delta_{(s,y)}$ with r, s as in $E_n(\rho,\xi)$, and $y > c2^{-n-1}$. Furthermore, noting that $\varphi_{n,-}(y) = 0$ for $y \geq c2^{-n-3}$, we see that the jumps $r\delta_{(s,y)}$ of M contribute to $L_n^-(T_{n,-})$ if and only if $y < c2^{-n-3}$. Thus, to prove the lemma, it is sufficient to show that U_n^1 yields the existence of at least one further jump of M on the half-line $\{y < c2^{-n-3}\}$ with properties mentioned in the statement. Denote

$$\begin{aligned} D := \Big\{&(r,s,y)\colon\ r \geq \big((t-s)\max\{(t-s)^{1/\alpha}, |y|\}\big)^{1/(1+\beta)} \log^{1/(1+\beta)-2\varepsilon}\big((t-s)^{-1}\big),\\ &y \in \big(-(t-s)^{1/\alpha}\log^{\xi}\big((t-s)^{-1}\big), c2^{-n-3}\big),\\ &s \in \big[t-2^{-\alpha n}n^{\rho}, t-2^{-\alpha n}n^{-\rho}\big]\Big\}. \end{aligned} \tag{4.56}$$

Then we need to show that U_n^1 implies the existence of a jump $r\delta_{(s,y)}$ of M with $(r,s,y) \in D$.

Note that

$$\begin{aligned} D \;=\; & D_1 \cap D_2 \cap D_3\\ := & \Big\{(r,s,y)\colon\ r \geq 0,\ s \in [0,t],\ y \in \big(-(t-s)^{1/\alpha}\log^{\xi}\big((t-s)^{-1}\big), c2^{-n-3}\big)\Big\}\\ & \cap \Big\{(r,s,y)\colon\ r \geq 0,\ y \in \big(-\infty, c2^{-n-3}\big),\ s \in \big[t-2^{-\alpha n}n^{\rho},\ t-2^{-\alpha n}n^{-\rho}\big]\Big\}\\ & \cap \Big\{(r,s,y)\colon\ y \in \big(-\infty, c2^{-n-3}\big),\ s \in [0,t],\\ & \quad r \geq \big((t-s)\max\{(t-s)^{1/\alpha}, |y|\}\big)^{1/(1+\beta)} \log^{1/(1+\beta)-2\varepsilon}\big((t-s)^{-1}\big)\Big\}. \end{aligned}$$

Therefore,

$$D^{\mathrm{c}} \cap \big\{y < c2^{-n-3}\big\} = \big(D_1^{\mathrm{c}} \cap \big\{y < c2^{-n-3}\big\}\big) \cup \big(D_1 \cap D_2^{\mathrm{c}}\big) \cup \big(D_1 \cap D_2 \cap D_3^{\mathrm{c}}\big), \tag{4.57}$$

where the complements are defined with respect to the set

$$\big\{(r,s,y)\colon\ r \geq 0,\ s \in [0,t], y \in \mathsf{R}\big\}. \tag{4.58}$$

We first show that any jumps of M in $D_1^{\mathrm{c}} \cap \{y < c2^{-n-3}\}$ cannot be the course of a jump of L_n^- such that U_n^1 holds. Indeed, using Lemma 2.3 with $\delta = \bar{\eta}_{\mathrm{c}}$, we get for $y < 0$ the inequality

$$\begin{aligned} \varphi_{n,-}(y) &= p_{t-s}^{\alpha}(y) - p_{t-s}^{\alpha}\big(y - c2^{-n-1}\big) \leq 2^{1-\bar{\eta}_{\mathrm{c}} n}(t-s)^{-\bar{\eta}_{\mathrm{c}}/\alpha} p_{t-s}^{\alpha}(y)\\ &\leq C2^{-\bar{\eta}_{\mathrm{c}} n}(t-s)^{-(1+\bar{\eta}_{\mathrm{c}})/\alpha}\left(\frac{y}{(t-s)^{1/\alpha}}\right)^{-\alpha-1}\\ &= C2^{-\bar{\eta}_{\mathrm{c}} n}(t-s)^{1-\bar{\eta}_{\mathrm{c}}/\alpha}|y|^{-\alpha-1}, \end{aligned} \tag{4.59}$$

where in the second step we used the scaling property and (3.23).

Further, by (4.40), on the set A^{ε} we have

$$\Delta X_s(y) \leq C\big(|y|(t-s)\big)^{1/(1+\beta)}(f_{s,y})^{\ell}, \quad |y| \leq 1/\mathrm{e}, \tag{4.60}$$

and

$$\Delta X_s(y) \le C(t-s)^{1/(1+\beta)-\gamma}, \quad |y| > 1/\mathrm{e}, \tag{4.61}$$

and recall that $f_{s,x} = \log((t-s)^{-1}) 1_{\{x\neq 0\}} \log(|x|^{-1})$. Combining (4.59) and (4.60), we conclude that the corresponding jump of L_n^-, henceforth denoted by $\Delta L_n^-[r\delta_{(s,y)}]$, is bounded by

$$C2^{-\bar{\eta}_c n}(t-s)^{1-\bar{\eta}_c/\alpha+\frac{1}{1+\beta}} \log^{\frac{1}{1+\beta}+q}\big((t-s)^{-1}\big)|y|^{-\alpha-1+\frac{1}{1+\beta}} \log^{\frac{1}{1+\beta}+q}\big(|y|^{-1}\big). \tag{4.62}$$

Since $|y|^{-\alpha-1+\frac{1}{1+\beta}} \log^{\frac{1+\gamma}{1+\beta}}(|y|^{-1})$ is decreasing, we get, maximizing over y, for $y < -(t-s)^{1/\alpha} \log^{\xi}((t-s)^{-1})$ the bound

$$\Delta L_n^-[r\delta_{(s,y)}] \le C2^{-\bar{\eta}_c n} \log^{\frac{2}{1+\beta}+2q-\xi(\alpha+1-\frac{1}{1+\beta})}\big(|y|^{-1}\big). \tag{4.63}$$

Choosing $\xi \ge \frac{2+2q(1+\beta)}{(1+\beta)(1+\alpha)-1}$, we see that

$$\Delta L_n^-[r\delta_{(s,y)}] \le C2^{-\bar{\eta}_c n}, \quad |y| < 1/\mathrm{e}. \tag{4.64}$$

Moreover, if $y < -1/\mathrm{e}$, then it follows from (4.59) and (4.61) that the jump $\Delta L_n^-[r\delta_{(s,y)}]$ is bounded by

$$C2^{-\bar{\eta}_c n}(t-s)^{1-\bar{\eta}_c/\alpha+\frac{1}{1+\beta}-\gamma}|y|^{-\alpha-1} \le C2^{-\bar{\eta}_c n}. \tag{4.65}$$

Combining (4.64) and (4.65), we see that all the jumps of M in $D_1^c \cap \{y < c2^{-n-3}\}$ do not produce jumps of L_n^- such that U_n^1 holds.

We next assume that M has a jump $r\delta_{(s,y)}$ in $D_1 \cap D_2^c$. If, additionally, $s \le t - 2^{-\alpha n} n^{\rho}$, then, using Lemma 2.3 with $\delta = 1$, we get

$$\varphi_{n,-}(y) = p_{t-s}^{\alpha}(y) - p_{t-s}^{\alpha}\big(y - c2^{-n-1}\big) \le 2^{1-n}(t-s)^{-2/\alpha}. \tag{4.66}$$

From this bound and (4.60) we obtain

$$\begin{aligned}\Delta L_n^-[r\delta_{(s,y)}] &\le C2^{-n}(t-s)^{-2/\alpha+\frac{1}{1+\beta}} \log^{\frac{1}{1+\beta}+q}\big((t-s)^{-1}\big)|y|^{\frac{1}{1+\beta}} \log^{\frac{1}{1+\beta}+q}\big(|y|^{-1}\big) \\ &\le C2^{-n}(t-s)^{(\frac{1+\alpha}{1+\beta}-2)/\alpha} \log^{\frac{2+\xi}{1+\beta}+2q}\big((t-s)^{-1}\big).\end{aligned} \tag{4.67}$$

Using the assumption $t - s \ge 2^{-\alpha n} n^{\rho}$, we arrive at the inequality

$$\Delta L_n^-[r\delta_{(s,y)}] \le C2^{-\bar{\eta}_c n} n^{-\rho(1-\bar{\eta}_c)/\alpha+\frac{2+\xi}{1+\beta}+2q}. \tag{4.68}$$

From this we see that if we choose $\rho \ge \frac{\alpha(\xi+2+2q(1+\beta))}{(1+\beta)(1-\bar{\eta}_c)}$, then the jumps of L_n^- are bounded by $C2^{-\bar{\eta}_c n}$, and hence U_n^1 does not occur.

If M has a jump in $D_1 \cap D_2^c$ at time $s \geq t - 2^{-\alpha n} n^{-\rho}$, then, using (4.60) and the bound

$$\varphi_{n,-}(y) = p^{\alpha}_{t-s}(y) - p^{\alpha}_{t-s}\big(y - c2^{-n-1}\big) \leq p^{\alpha}_{t-s}(0) \leq C(t-s)^{-1/\alpha}, \quad (4.69)$$

we get for $y \in (-(t-s)^{1/\alpha} \log^{\xi}((t-s)^{-1}),\ c2^{-n-3})$ and $t - s \leq 2^{-\alpha n} n^{-\rho}$ the inequality

$$\begin{aligned}\Delta L_n^-[r\delta_{(s,y)}] &\leq C(t-s)^{(\frac{1+\alpha}{1+\beta}-1)/\alpha} \log^{\frac{2+\xi}{1+\beta}+2q}\big((t-s)^{-1}\big) \\ &\leq C2^{-\bar{\eta}_c n} n^{-\rho(\bar{\eta}_c/\alpha)+\frac{2+\xi}{1+\beta}+2q}. \end{aligned} \quad (4.70)$$

Choosing $\rho \geq \frac{\alpha(\xi+2+2q(1+\beta))}{(1+\beta)\bar{\eta}_c}$, we conclude that $\Delta L_n^-[r\delta_{(s,y)}] \leq C2^{-\bar{\eta}_c n}$, and again U_n^1 does not occur.

Finally, it remains to consider the jumps of M in $D_1 \cap D_2 \cap D_3^c$. If the value of the jump does not exceed $(t-s)^{\frac{\alpha+1}{\alpha(1+\beta)}} \log^{\frac{1}{1+\beta}-2\varepsilon}((t-s)^{-1})$, then it follows from Lemma 2.3 with $\delta = \bar{\eta}_c$ that

$$\Delta L_n^-[r\delta_{(s,y)}] \leq C2^{-\bar{\eta}_c n} \log^{\frac{1}{1+\beta}-2\varepsilon}\big((t-s)^{-1}\big). \quad (4.71)$$

Then, on D_2, that is, for $t - s > 2^{-\alpha n} n^{-\rho}$,

$$\Delta L_n^-[r\delta_{(s,y)}] \leq C2^{-\bar{\eta}_c n} n^{\frac{1}{1+\beta}-2\varepsilon}. \quad (4.72)$$

Furthermore, if $y < -(t-s)^{1/\alpha}$ and the value of the jump is less than $(|y|(t-s))^{\frac{1}{1+\beta}} \log^{\frac{1}{1+\beta}-2\varepsilon}((t-s)^{-1})$, then, using (4.59), we get

$$\begin{aligned}\Delta L_n^-[r\delta_{(s,y)}] &\leq C2^{-\bar{\eta}_c n}(t-s)^{1-\bar{\eta}_c/\alpha} \log^{\frac{1}{1+\beta}-2\varepsilon}\big((t-s)^{-1}\big)|y|^{-\alpha-1+\frac{1}{1+\beta}} \\ &\leq C2^{-\bar{\eta}_c n} \log^{\frac{1}{1+\beta}-2\varepsilon}\big((t-s)^{-1}\big). \end{aligned} \quad (4.73)$$

Then, on D_2, that is, for $t - s > 2^{-\alpha n} n^{-\rho}$,

$$\Delta L_n^-[r\delta_{(s,y)}] \leq C2^{-\bar{\eta}_c n} n^{\frac{1}{1+\beta}-2\varepsilon}. \quad (4.74)$$

By (4.72) and (4.74), we see that the jumps of M in $D_1 \cap D_2 \cap D_3^c$ do not produce jumps such that U_n^1 holds. Combining all the above, we conclude that to have $\Delta L_n^-[r\delta_{(s,y)}] > C2^{-\bar{\eta}_c n} n^{\frac{1}{1+\beta}-2\varepsilon}$, it is necessary to have a jump in $D_1 \cap D_2 \cap D_3$. Thus, the proof is finished. □

Proof of Lemma 4.6 In view of the Lemmas 4.7 and 4.8, it suffices to show that

$$\lim_{N\uparrow\infty} \sum_{n=N}^{\infty} \mathbf{P}\big(E_n(\rho,\xi) \cap A^{\varepsilon} \cap D_{\theta}\big) = 0. \quad (4.75)$$

The intensity of the jumps in D [the set defined in (4.56) and satisfying conditions in $E_n(\rho,\xi)$] is given by

$$\int_{t-2^{-\alpha n}n^{\rho}}^{t-2^{-\alpha n}n^{-\rho}} \mathrm{d}s \int_{|y|\le (t-s)^{1/\alpha}\log^{\xi}((t-s)^{-1})} X_s(\mathrm{d}y)\frac{\log^{2\varepsilon(1+\beta)-1}((t-s)^{-1})}{(t-s)\max\{(t-s)^{1/\alpha},|y|\}}. \quad (4.76)$$

Since in (4.75) we are interested in a limit as $N\uparrow\infty$, we may assume that n is such that $(t-s)^{1/\alpha}\log^{\xi}((t-s)^{-1})\le 1$ for $s\ge t-2^{-\alpha n}n^{\rho}$. We next note that

$$\int_{|y|\le(t-s)^{1/\alpha}} \frac{X_s(\mathrm{d}y)}{\max\{(t-s)^{1/\alpha},|y|\}} = (t-s)^{-1/\alpha}X_s\big((-(t-s)^{1/\alpha},(t-s)^{1/\alpha})\big)\le\theta^{-1} \quad (4.77)$$

on D_θ. Further, for every $j\ge 1$ satisfying $j\le\log^{\xi}((t-s)^{-1})$,

$$\int_{j(t-s)^{1/\alpha}\le|y|\le(j+1)(t-s)^{1/\alpha}} \frac{X_s(\mathrm{d}y)}{\max\{(t-s)^{1/\alpha},|y|\}} \le j^{-1}(t-s)^{-1/\alpha}X_s\big(\{y\colon\ j(t-s)^{1/\alpha}\le|y|\le(j+1)(t-s)^{1/\alpha}\}\big). \quad (4.78)$$

Since the set $\{y\colon\ j(t-s)^{1/\alpha}\le|y|\le(j+1)(t-s)^{1/\alpha}\}$ is the union of two balls with radius $\frac{1}{2}(t-s)^{-1/\alpha}$ and centers in $B_2(0)$, we can apply Lemma 4.2 with $c=1$ to get

$$\int_{j(t-s)^{1/\alpha}\le|y|\le(j+1)(t-s)^{1/\alpha}} \frac{X_s(\mathrm{d}y)}{\max\{(t-s)^{1/\alpha},|y|\}}\le 2\theta^{-1}j^{-1} \quad (4.79)$$

on D_θ. As a result, on the event D_θ we get the inequality

$$\int_{|y|\le(t-s)^{1/\alpha}\log^{\xi}((t-s)^{-1})} X_s(\mathrm{d}y)\frac{1}{\max\{(t-s)^{1/\alpha},|y|\}} \le C\theta^{-1}\log\big(\big|\log\big((t-s)^{-1}\big)\big|\big). \quad (4.80)$$

Substituting this into (4.76), we conclude that the intensity of the jumps is bounded by

$$C\theta^{-1}\int_{t-2^{-\alpha n}n^{\rho}}^{t-2^{-\alpha n}n^{-\rho}} \mathrm{d}s\,\frac{\log^{2\varepsilon(1+\beta)-1}((t-s)^{-1}\log\log((t-s)^{-1}))}{(t-s)}. \quad (4.81)$$

Simple calculations show that the latter expression is less than

$$C\theta^{-1}n^{2\varepsilon(1+\beta)-1}\log^{1+2\varepsilon(1+\beta)}n. \quad (4.82)$$

Consequently, since $E_n(\rho,\xi)$ holds when there are two jumps in D, we have

$$\mathbf{P}\big(E_n(\rho,\xi)\cap A^{\varepsilon}\cap D_\theta\big)\le C\theta^{-2}n^{4\varepsilon(1+\beta)-2}\log^{2+4\varepsilon(1+\beta)}n. \quad (4.83)$$

Because $\varepsilon<1/8\le 1/4(1+\beta)$, the sequence $\mathbf{P}(E_n(\rho,\xi)\cap A^{\varepsilon}\cap D_\theta)$ is summable, and the proof of the lemma is complete. □

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