Local probabilities for random walks conditioned to stay positive

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Abstract Let $S_0 = 0, \{S_n,\ n \geq 1\}$ be a random walk generated by a sequence of i.i.d. random variables $X_1, X_2, \ldots$ and let $\tau^- = \min\{n \geq 1 : S_n \leq 0\}$ and $\tau^+ = \min\{n \geq 1 : S_n > 0\}$. Assuming that the distribution of X_1 belongs to the domain of attraction of an α-stable law we study the asymptotic behavior, as $n \to \infty$, of the local probabilities $\mathbf{P}(\tau^{\pm} = n)$ and prove the Gnedenko and Stone type conditional local limit theorems for the probabilities $\mathbf{P}(S_n \in [x, x + \Delta)|\tau^- > n)$ with fixed Δ and $x = x(n) \in (0, \infty)$.

Keywords Limit theorems · Random walks · Stable laws

Mathematics Subject Classification (2000) 60G50 · 60G52 · 60E07

1 Introduction and main result

Let $S_0 := 0,\ S_n := X_1 + \cdots + X_n,\ n \geq 1$, be a random walk, where the X_i are independent copies of a random variable X and

$$\tau^- = \min\{n \geq 1 :\ S_n \leq 0\} \quad \text{and} \quad \tau^+ = \min\{n \geq 1 :\ S_n > 0\}$$

Supported by the Russian Foundation for Basic Research grant 08-01-00078 and by the GIF.

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be the first weak descending and first strict ascending ladder epochs of $\{S_n,\ n \geq 0\}$. The aim of this paper is to study, as $n \to \infty$, the asymptotic behavior of the local probabilities $\mathbf{P}\left(\tau^{\pm} = n\right)$ and conditional local probabilities $\mathbf{P}(S_n \in [x, x + \Delta)|\tau^- > n)$ for fixed $\Delta > 0$ and $x = x(n) \in (0, \infty)$.

To formulate our results we let

$$\mathcal{A} := \{0 < \alpha < 1;\ |\beta| < 1\} \cup \{1 < \alpha < 2;\ |\beta| \leq 1\} \cup \{\alpha = 1, \beta = 0\} \cup \{\alpha = 2, \beta = 0\}$$

be a subset in $\mathbb{R}^2$. For $(\alpha, \beta) \in \mathcal{A}$ and a random variable X write $X \in \mathcal{D}(\alpha, \beta)$ if the distribution of X belongs to the domain of attraction of a stable law with characteristic function

$$G_{\alpha,\beta}(t) := \exp\left\{-c|t|^{\alpha}\left(1 - i\beta\frac{t}{|t|}\tan\frac{\pi\alpha}{2}\right)\right\} = \int_{-\infty}^{+\infty} e^{itx} g_{\alpha,\beta}(u)du, \quad c > 0, \quad (1)$$

and, in addition, $\mathbf{E}X = 0$ if this moment exists.

Denote $\mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}$, $\mathbb{Z}_+ := \{1, 2, \ldots\}$ and let $\{c_n, n \geq 1\}$ be a sequence of positive integers specified by the relation

$$c_n := \inf\left\{u \geq 0 : \mu(u) \leq n^{-1}\right\}, \quad (2)$$

where

$$\mu(u) := \frac{1}{u^2}\int_{-u}^{u} x^2 \mathbf{P}(X \in dx).$$

It is known (see, for instance [15, Chap. XVII, Sect. 5]) that for every $X \in \mathcal{D}(\alpha, \beta)$ the function $\mu(u)$ is regularly varying with index $(-\alpha)$. This implies that $\{c_n, n \geq 1\}$ is a regularly varying sequence with index α^{-1}, i.e. there exists a function $l_1(n)$, slowly varying at infinity, such that

$$c_n = n^{1/\alpha} l_1(n). \quad (3)$$

In addition, the scaled sequence $\{S_n/c_n,\ n \geq 1\}$ converges in distribution, as $n \to \infty$, to the stable law given by (1).

The following conditional limit theorem will be crucial for the rest of this article.

Theorem 1 *If $X \in \mathcal{D}(\alpha, \beta)$, then there exists a non-negative random variable $M_{\alpha,\beta}$ with density $p_{\alpha,\beta}(u)$ such that, for all $u_2 > u_1 \geq 0$,*

$$\lim_{n \to \infty} \mathbf{P}\left(\frac{S_n}{c_n} \in [u_1, u_2) \,\Big|\, \tau^- > n\right) = \mathbf{P}(M_{\alpha,\beta} \in [u_1, u_2)) = \int_{u_1}^{u_2} p_{\alpha,\beta}(v)dv. \quad (4)$$

Remark 2 The validity of the first equality in (4) was established by Durrett [13]. We failed to find any reference for the absolutely continuity of $M_{\alpha,\beta}$. As was pointed out by the referee, the needed statement can be justified using the following arguments. First,

Eqs. (4.12) and (5.5) in Alili and Chaumont [1] imply that the distribution of $M_{\alpha,\beta}$ is a multiple of the entrance law of the measure of excursions away from 0 of the stable process reflected by its minimum. Second, Monrad and Silverstein [18] established the absolutely continuity of this entrance law. In fact, the absolutely continuity of $M_{\alpha,\beta}$ will be a by-product of our proofs and we include it in (4) to simplify the statements of the main theorems of the present paper.
It is necessary to mention that functional limit theorems for random walks conditioned to stay positive were established by Doney [11] and by Caravenna and Chaumont [7].

Our first result is an analog of the classical Stone local limit theorem.

Theorem 3 *Suppose $X \in \mathcal{D}(\alpha, \beta)$ and the distribution of X is non-lattice. Then, for every $\Delta > 0$,*

$$c_n\mathbf{P}(S_n \in [x, x+\Delta)|\tau^- > n) - \Delta p_{\alpha,\beta}(x/c_n) \to 0 \quad \textit{as } n \to \infty \tag{5}$$

uniformly in $x \in (0, \infty)$.

For the case when the distribution of X belongs to the domain of attraction of the normal law, that is, when $X \in \mathcal{D}(2, 0)$ relation (5) has been proved by Caravenna [6].
If the ratio x/c_n varies with n in such a way that $x/c_n \in (b_1, b_2)$ for some $0 < b_1 < b_2 < \infty$, we can rewrite (5) as

$$c_n\mathbf{P}(S_n \in [x, x+\Delta)|\tau^- > n) \sim \Delta p_{\alpha,\beta}(x/c_n) \quad \text{as } n \to \infty.$$

However, if $x/c_n \to 0$, then, in view of

$$\lim_{z \downarrow 0} p_{\alpha,\beta}(z) = 0$$

(see (80) below), relation (5) gives only

$$c_n\mathbf{P}(S_n \in [x, x+\Delta)|\tau^- > n) = o(1) \quad \text{as } n \to \infty. \tag{6}$$

Our next theorem refines (6) in the mentioned domain of small deviations, i.e. when $x/c_n \to 0$. To formulate the desired statement we need some additional notation.
Set $\chi^+ := S_{\tau^+}$ and introduce the renewal function

$$H(u) := \mathrm{I}\{u > 0\} + \sum_{k=1}^{\infty} \mathbf{P}(\chi_1^+ + \cdots + \chi_k^+ < u). \tag{7}$$

Clearly, H is a left-continuous function.

Theorem 4 *Suppose $X \in \mathcal{D}(\alpha, \beta)$ and the distribution of X is non-lattice. Then*

$$c_n\mathbf{P}(S_n \in [x, x+\Delta)|\tau^- > n) \sim g_{\alpha,\beta}(0)\frac{\int_x^{x+\Delta} H(u)du}{n\mathbf{P}\left(\tau^- > n\right)} \quad \textit{as } n \to \infty \tag{8}$$

uniformly in $x \in (0, \delta_n c_n]$, where $\delta_n \to 0$ as $n \to \infty$.

We continue by considering the lattice case and say that a random variable X is (h,a)-lattice if the distribution of X is lattice with span $h>0$ and shift $a\in[0,h)$, i.e. the h is the maximal number such that the support of the distribution of X is contained in the set $\{a+kh,\ k=0,\pm1,\pm2,\dots\}$.

Theorem 5 *Suppose $X\in\mathcal{D}(\alpha,\beta)$ and is (h,a)-lattice. Then*

$$c_n\mathbf{P}(S_n=an+x|\tau^->n)-hp_{\alpha,\beta}((an+x)/c_n)\to0\quad as\ n\to\infty\tag{9}$$

uniformly in $x\in(-an,\infty)\cap h\mathbb{Z}$.

For $X\in\mathcal{D}(2,0)$ and being $(h,0)$-lattice relation (9) has been obtained by Bryn-Jones and Doney [5].

Theorem 6 *Suppose $X\in\mathcal{D}(\alpha,\beta)$ and is (h,a)-lattice. Then*

$$c_n\mathbf{P}(S_n=an+x|\tau^->n)\sim hg_{\alpha,\beta}(0)\frac{H(an+x)}{n\mathbf{P}\left(\tau^->n\right)}\quad as\ n\to\infty\tag{10}$$

uniformly in $x\in(-an,-an+\delta_nc_n]\cap h\mathbb{Z}$, where $\delta_n\to0$ as $n\to\infty$.

Note that Alili and Doney [2] established (10) under the assumptions X is $(h,0)$-lattice and $\mathbf{E}X^2<\infty$. Bryn-Jones and Doney [5] generalized their results to the $(h,0)$-lattice $X\in\mathcal{D}(2,0)$.

The next theorem describes the asymptotic behavior of the density function $p_{\alpha,\beta}$ at zero. The explicit form of $p_{\alpha,\beta}$ is known only for $\alpha=2$, $\beta=0$: $p_{2,0}(x)=xe^{-x^2/2}\mathrm{I}$ $(x>0)$. For this reason we deduce an integral equation for $p_{\alpha,\beta}$ (see (79) below) and using Theorems 3–6 find the asymptotic behavior of $p_{\alpha,\beta}(z)$ at zero.

Theorem 7 *For every $(\alpha,\beta)\in\mathcal{A}$ there exists a constant $C>0$ such that*

$$p_{\alpha,\beta}(z)\sim Cz^{\alpha\rho}\quad as\ z\downarrow0,$$

where $\rho:=\int_{0+}^{\infty}g_{\alpha,\beta}(u)du$.

One of our main motivations to be interested in the local probabilities of conditioned random walks is the question of the asymptotic behavior of the local probabilities of the ladder epochs τ^- and τ^+. Before formulating the relevant results we recall some known facts concerning the properties of these random variables, given

$$\sum_{n=1}^{\infty}\frac{1}{n}\mathbf{P}(S_n>0)=\sum_{n=1}^{\infty}\frac{1}{n}\mathbf{P}(S_n\le0)=\infty.$$

The last means that $\{S_n,n\ge0\}$ is an oscillating random walk, and, in particular, the stopping moments τ^- and τ^+ are well-defined proper random variables. Moreover, it follows from the Wiener–Hopf factorization (see, for example, [4, Theorem 8.9.1, p. 376]) that for all $z\in[0,1)$,

$$1 - \mathbf{E}z^{\tau^-} = \exp\left\{-\sum_{n=1}^{\infty}\frac{z^n}{n}\mathbf{P}(S_n \le 0)\right\} \tag{11}$$

and

$$1 - \mathbf{E}z^{\tau^+} = \exp\left\{-\sum_{n=1}^{\infty}\frac{z^n}{n}\mathbf{P}(S_n > 0)\right\}. \tag{12}$$

Rogozin [20] investigated properties of τ^+ and demonstrated that the Spitzer condition

$$n^{-1}\sum_{k=1}^{n}\mathbf{P}(S_k > 0) \to \rho \in (0,1) \quad \text{as } n \to \infty \tag{13}$$

holds if and only if τ^+ belongs to the domain of attraction of a positive stable law with parameter ρ. In particular, if $X \in \mathcal{D}(\alpha,\beta)$ then (see, for instance [24]) condition (13) holds with

$$\rho = \int_{0+}^{\infty} g_{\alpha,\beta}(u)du = \begin{cases} \frac{1}{2}, & \alpha = 1, \\ \frac{1}{2} + \frac{1}{\pi\alpha}\arctan\left(\beta\tan\frac{\pi\alpha}{2}\right), & \text{otherwise.} \end{cases} \tag{14}$$

Since (11) and (12) imply

$$(1 - \mathbf{E}z^{\tau^+})(1 - \mathbf{E}z^{\tau^-}) = 1 - z \quad \text{for all } z \in (0,1),$$

one can deduce by Rogozin's result that (13) holds if and only if there exists a function $l(n)$ slowly varying at infinity such that, as $n \to \infty$,

$$\mathbf{P}\left(\tau^- > n\right) \sim \frac{l(n)}{n^{1-\rho}}, \quad \mathbf{P}\left(\tau^+ > n\right) \sim \frac{1}{\Gamma(\rho)\Gamma(1-\rho)n^{\rho}l(n)}. \tag{15}$$

We also would like to mention that, according to Doney [12], the Spitzer condition is equivalent to

$$\mathbf{P}(S_n > 0) \to \rho \in (0,1) \quad \text{as } n \to \infty. \tag{16}$$

Therefore, both relations in (15) are valid under condition (16).

The asymptotic representations (15) include a slowly varying function $l(x)$ which is of interest as well. Unfortunately, to get a more detailed information about the asymptotic properties of $l(x)$ it is necessary to impose additional hypotheses on the distribution of X. Thus, Rogozin [20] has shown that $l(x)$ is asymptotically a constant if and only if

$$\sum_{n=1}^{\infty}\frac{1}{n}\left(\mathbf{P}(S_n > 0) - \rho\right) < \infty. \tag{17}$$

It follows from the Spitzer–Rósen theorem (see [4, Theorem 8.9.23, p. 382]) that if $\mathbf{E}X^2 < \infty$, then (17) holds with $\rho = 1/2$, and, consequently,

$$\mathbf{P}(\tau^{\pm} > n) \sim \frac{C^{\pm}}{n^{1/2}} \quad \text{as } n \to \infty, \tag{18}$$

where $C^{\pm}$ are positive constants. Much less is known about the form of $l(x)$ if $\mathbf{E}X^2 = \infty$. For instance, if the distribution of X is symmetric, then, clearly,

$$\left|\mathbf{P}\left(S_n > 0\right) - \frac{1}{2}\right| = \frac{1}{2}\mathbf{P}\left(S_n = 0\right). \tag{19}$$

Furthermore, according to [19, Theorem III.9, p. 49], there exists $C > 0$ such that for all $n \geq 1$,

$$\mathbf{P}\left(S_n = 0\right) \leq \frac{C}{\sqrt{n}}.$$

By this estimate and (19) we conclude that (17) holds with $\rho = 1/2$ and, therefore, (18) is valid for all symmetric random walks.

One more situation was analyzed by Doney [9]. Assuming that $\mathbf{P}(X > x) = (x^{\alpha} l_0(x))^{-1}$, $x > 0$, with $1 < \alpha < 2$ and $l_0(x)$ slowly varying at infinity, he established some relationships between the asymptotic behavior of $l_0(x)$ and $l(x)$ at infinity for a number of cases.

Thus, up to now there is a group of results describing the behavior of the probabilities $\mathbf{P}(\tau^{\pm} > n)$ as $n \to \infty$ and the functions involved in their asymptotic representations. We complement the mentioned statements by the following two theorems describing the behavior of the local probabilities $\mathbf{P}(\tau^{\pm} = n)$ as $n \to \infty$.

Theorem 8 *If $X \in \mathcal{D}(\alpha, \beta)$ then there exists a sequence $\{Q_n^-, n \geq 1\}$ such that*

$$\mathbf{P}(\tau^- = n) = Q_n^- \frac{l(n)}{n^{2-\rho}}(1 + o(1)) \quad \text{as } n \to \infty. \tag{20}$$

The sequence $\{Q_n^-, n \geq 1\}$ is bounded from above, and there exists a positive constant Q_*^- such that $Q_n^- \mathrm{I}(Q_n^- > 0) \geq Q_*^-$ for all $n \geq 1$. Moreover, we may choose $Q_n^- \equiv 1 - \rho$ if and only if one of the following conditions holds:

(a) $\mathbf{E}(-S_{\tau^-}) = \infty$,
(b) $\mathbf{E}(-S_{\tau^-}) < \infty$ *and the distribution of X is $(h, 0)$-lattice,*
(c) $\mathbf{E}(-S_{\tau^-}) < \infty$ *and the distribution of X is non-lattice.*

Remark 9 The statement of the theorem includes the quantity $\mathbf{E}(-S_{\tau^-})$, which depends on τ^-, a random variable being the objective of the theorem. This is done only to simplify the form of the theorem. In fact, Chow [8] has shown that $\mathbf{E}(-S_{\tau^-})$ is finite if and only if

$$\int_0^{\infty} \frac{x^2}{\int_0^{\infty} y \min\{x, y\} \mathbf{P}(X^+ \in dy)} \mathbf{P}(X^- \in dx) < \infty,$$

where $X^+ := \max\{0, X\}$ and $X^- := -\min\{0, X\}$.

Remark 10 The simple random walk in which $P(X = \pm 1) = 1/2$ is the most natural example with $Q_n^- \neq 1-\rho$. Here $P(\tau^- = 2k+1) = 0$ and, consequently, $Q_{2k+1}^- = 0$ for all $k \geq 1$. On the other hand, $\lim_{k\to\infty} Q_{2k}^-$ exists and is strictly positive. This result is in complete agreement with Theorem 8: the step-distribution of the simple random walk is (2, 1)-lattice.

For the stopping time τ^+ we have a similar statement:

Theorem 11 *If $X \in \mathcal{D}(\alpha, \beta)$ then there exists a sequence $\{Q_n^+, n \geq 1\}$ such that*

$$\mathbf{P}(\tau^+ = n) = Q_n^+ \frac{l(n)}{n^{1+\rho}}(1 + o(1)) \quad \text{as } n \to \infty. \tag{21}$$

The sequence $\{Q_n^+, n \geq 1\}$ is bounded from above, and there exists a positive constant Q_*^+ such that $Q_n^+ \mathrm{I}(Q_n^+ > 0) \geq Q_*^+$ for all $n \geq 1$. Moreover, we may choose $Q_n^+ \equiv \rho/(\Gamma(\rho)\Gamma(1-\rho))$ if and only if one of the following conditions holds:

(a) $\mathbf{E}S_{\tau^+} = \infty$,
(b) $\mathbf{E}S_{\tau^+} < \infty$ *and the distribution of X is $(h, 0)$-lattice,*
(c) $\mathbf{E}S_{\tau^+} < \infty$ *and the distribution of X is non-lattice.*

In some special cases the asymptotic behavior of $\mathbf{P}\left(\tau^\pm = n\right)$ is already known from the literature. Eppel [14] proved that if $\mathbf{E}X = 0$, $\mathbf{E}X^2$ is finite, and the distribution of X is non-lattice, then

$$\mathbf{P}\left(\tau^\pm = n\right) \sim \frac{C^\pm}{n^{3/2}} \quad \text{as } n \to \infty. \tag{22}$$

Clearly, $X \in \mathcal{D}(2, 0)$ in this case. For aperiodic random walks on integers with $EX = 0$ and $EX^2 < \infty$ estimate (22) was obtained by Alili and Doney [2].

Asymptotic relation (22) is valid for all continuous symmetric (implying $\rho = 1/2$ in (16)) random walks (see [15, Chap. XII, Sect. 7]). Note that the restriction $X \in \mathcal{D}(\alpha, \beta)$ is superfluous in this situation.

Recently Borovkov [3] has shown that if (13) is valid and

$$n^{1-\rho}\left(\mathbf{P}(S_n > 0) - \rho\right) \to \text{const} \in (-\infty, \infty) \quad \text{as } n \to \infty, \tag{23}$$

then (20) holds with $l(n) \equiv \text{const} \in (0, \infty)$. Proving the mentioned result Borovkov does not assume that the distribution of X is taken from the domain of attraction of a stable law. However, he gives no explanations how one can check the validity of (23) in the general situation.

Further, Alili and Doney [2, Remark 1, p. 98] have demonstrated that if X is $(h, 0)$-lattice and $\mathbf{E}S_{\tau^+} < \infty$ then (21) holds with $Q_n^+ \sim \rho/(\Gamma(\rho)\Gamma(1-\rho))$.

Finally, Mogulskii and Rogozin [17] established (20) for X satisfying the conditions $\mathbf{E}X = 0$ and $\mathbf{E}|X|^3 < \infty$. Moreover, they proved that $Q_n^+ \sim$ const if and only if the distribution of X is either non-lattice or $(h, 0)$-lattice. Observe that $\mathbf{E}(-S_{\tau^-}) < \infty$ under their conditions.

2 Auxiliary results

2.1 Notation

In what follows we denote by $C, C_1, C_2, \ldots$ finite positive constants which may be *different* from formula to formula and by $l(x), l_0(x), l_1(x), l_2(x), \ldots$ functions slowly varying at infinity which are, as a rule, *fixed once and forever*.

It is known that if $X \in \mathcal{D}(\alpha, \beta)$ with $\alpha \in (0, 2)$, and $F(x) := \mathbf{P}(X < x)$, then

$$1 - F(x) + F(-x) \sim \frac{1}{x^{\alpha} l_0(x)} \quad \text{as } x \to \infty, \tag{24}$$

where $l_0(x)$ is a function slowly varying at infinity. Besides, for $\alpha \in (0, 2)$,

$$\frac{F(-x)}{1 - F(x) + F(-x)} \to q, \quad \frac{1 - F(x)}{1 - F(x) + F(-x)} \to p \quad \text{as } x \to \infty, \tag{25}$$

with $p + q = 1$ and $\beta = p - q$ in (1). It is easy to see that (24) implies

$$\mu(u) \sim \frac{\alpha}{2 - \alpha} \mathbf{P}(|X| > u) \quad \text{as } u \to \infty. \tag{26}$$

By this relation and the definition of c_n we deduce

$$\mathbf{P}(|X| > c_n) \sim \frac{2 - \alpha}{\alpha} \frac{1}{n} \quad \text{as } n \to \infty. \tag{27}$$

2.2 Some results from fluctuation theory

Now we formulate a number of statements concerning the distributions of the random variables τ^-, τ^+ and χ^+. Recall that a random variable ζ is called relatively stable if there exists a non-random sequence $d_n \to \infty$ as $n \to \infty$ such that

$$\frac{1}{d_n} \sum_{k=1}^{n} \zeta_k \xrightarrow{p} 1 \quad \text{as } n \to \infty,$$

where $\zeta_k \stackrel{d}{=} \zeta, \ k = 1, 2, \ldots,$ and are independent.

Lemma 12 (see [20, Theorem 9]) *Assume* $X \in \mathcal{D}(\alpha, \beta)$. *Then, as* $x \to \infty$,

$$\mathbf{P}\left(\chi^+ > x\right) \sim \frac{1}{x^{\alpha\rho} l_2(x)} \quad \text{if } \alpha\rho < 1, \tag{28}$$

and χ^+ *is relatively stable if* $\alpha\rho = 1$.

Lemma 13 *Suppose $X \in \mathcal{D}(\alpha, \beta)$. Then, as $x \to \infty$,*

$$H(x) \sim \frac{x^{\alpha\rho} l_2(x)}{\Gamma(1-\alpha\rho)\Gamma(1+\alpha\rho)} \tag{29}$$

if $\alpha\rho < 1$, and

$$H(x) \sim x l_3(x) \tag{30}$$

if $\alpha\rho = 1$, where

$$l_3(x) := \left(\int_0^x \mathbf{P}\left(\chi^+ > y\right) dy \right)^{-1}, \quad x > 0.$$

In addition, there exists a constant $C > 0$ such that, in both cases

$$H(c_n) \sim Cn\mathbf{P}(\tau^- > n) \quad \text{as } n \to \infty. \tag{31}$$

Proof If $\alpha\rho < 1$, then by [15, Chap. XIV, formula (3.4)]

$$H(x) \sim \frac{1}{\Gamma(1-\alpha\rho)\Gamma(1+\alpha\rho)} \frac{1}{\mathbf{P}(\chi^+ > x)} \quad \text{as } x \to \infty.$$

Hence, recalling (28), we obtain (29).

If $\alpha\rho = 1$, then (30) follows from Theorem 2 in [20].

Let us demonstrate the validity of (31). We know from [20] (see also [16]) that $\tau^+ \in \mathcal{D}(\rho, 1)$ under the conditions of the lemma and, in addition, $\chi^+ \in \mathcal{D}(\alpha\rho, 1)$ if $\alpha\rho < 1$. This means, in particular, that for sequences $\{a_n, n \ge 1\}$ and $\{b_n, n \ge 1\}$ specified by

$$\mathbf{P}(\tau^+ > a_n) \sim \frac{1}{n} \quad \text{and} \quad \mathbf{P}(\chi^+ > b_n) \sim \frac{1}{n} \quad \text{as } n \to \infty, \tag{32}$$

and vectors (τ_k^+, χ_k^+), $k = 1, 2, \ldots$, being independent copies of (τ^+, χ^+), we have

$$\frac{1}{a_n} \sum_{k=1}^n \tau_k^+ \xrightarrow{d} Y_\rho \quad \text{and} \quad \frac{1}{b_n} \sum_{k=1}^n \chi_k^+ \xrightarrow{d} Y_{\alpha\rho} \quad \text{as } n \to \infty. \tag{33}$$

Moreover, it was established by Doney (see Lemma in [11, p. 358]) that

$$b_n \sim C c_{[a_n]} \quad \text{as } n \to \infty, \tag{34}$$

where $[x]$ stands for the integer part of x. Therefore, $c_n \sim C b_{[a^{-1}(n)]}$, where, with a slight abuse of notation, $a^{-1}(n)$ is the inverse function to a_n. Hence, on account of (32),

$$\mathbf{P}(\chi^+ > c_n) \sim C_1\mathbf{P}(\chi^+ > b_{[a^{-1}(n)]}) \sim \frac{C_1}{a^{-1}(n)}$$
$$\sim C_2\mathbf{P}(\tau^+ > a_{[a^{-1}(n)]}) \sim C_3\mathbf{P}(\tau^+ > n). \quad (35)$$

If $\alpha\rho = 1$, then, instead of the second equivalence in (32), one should define b_n by

$$\frac{1}{b_n}\int_0^{b_n} \mathbf{P}(\chi^+ > y)dy \sim \frac{1}{n} \quad \text{as } n \to \infty$$

(see [20, p. 595]). In this case the second convergence in (33) transforms to

$$\frac{1}{b_n}\sum_{k=1}^{n} \chi_k^+ \xrightarrow{p} 1 \quad \text{as } n \to \infty,$$

while (35) should be changed to

$$\frac{1}{c_n}\int_0^{c_n} \mathbf{P}(\chi^+ > y)dy \sim \frac{C_1}{b_{[a^{-1}(n)]}}\int_0^{b_{[a^{-1}(n)]}} \mathbf{P}(\chi^+ > y)dy \sim \frac{C_1}{a^{-1}(n)}$$
$$\sim C_1\mathbf{P}(\tau^+ > a_{[a^{-1}(n)]}) \sim C_2\mathbf{P}(\tau^+ > n). \quad (36)$$

Combining (35) and (36) with (29) and (30) gives

$$H(c_n) \sim C\mathbf{P}(\tau^+ > n) \quad \text{as } n \to \infty$$

for all $X \in \mathcal{D}(\alpha, \beta)$. Using (15) finishes the proof of the lemma. □

Lemma 14 *If* $\mathbf{E}(-S_{\tau^-}) < \infty$*, then there exists a positive constant* C_0 *such that*

$$c_n \sim C_0\frac{n^{1-\rho}}{l(n)}. \quad (37)$$

Proof Let $T^- := \min\{k \geq 1 : -S_k > 0\}$ and $\chi^- = -S_{T^-}$ be the first strict ladder height for the random walk $\{-S_n, n \geq 0\}$. Applying (36) to $\{-S_n, n \geq 0\}$, we have

$$\frac{1}{c_n}\int_0^{c_n} \mathbf{P}(\chi^- > y)dy \sim C\mathbf{P}(T^- > n). \quad (38)$$

Obviously, $\mathbf{E}(-S_{\tau^-}) < \infty$ yields $\mathbf{E}\chi^- < \infty$. Therefore $\int_0^{c_n} \mathbf{P}(\chi^- > y)dy \to \mathbf{E}\chi^-$ as $n \to \infty$. Combining this with (38), and recalling that $\mathbf{P}(T^- > n) \sim C\mathbf{P}(\tau^- > n)$

in view of the equality

$$\sum_{n=1}^{\infty} \mathbf{P}(T^- > n)z^n = \sum_{n=1}^{\infty} \mathbf{P}(\tau^- > n)z^n \exp\left\{\sum_{k=1}^{\infty} \frac{z^k}{k}\mathbf{P}(S_k = 0)\right\},$$

asymptotic representation (15), and the estimate

$$\sum_{k=1}^{\infty} \frac{1}{k}\mathbf{P}(S_k = 0) < \infty,$$

we obtain

$$\lim_{n\to\infty} c_n \mathbf{P}(\tau^- > n) =: C_0 \in (0, \infty).$$

On account of (15) this proves (37). □

2.3 Upper estimates for local probabilities

For $x \geq 0$ and $n = 0, 1, 2, \ldots,$ let

$$B_n(x) := \mathbf{P}\left(S_n \in (0, x); \tau^- > n\right),$$
$$b_n(x) := B_n(x+1) - B_n(x) = \mathbf{P}\left(S_n \in [x, x+1); \tau^- > n\right).$$

Note that by the duality principle for random walks

$$\begin{aligned}1 + \sum_{j=1}^{\infty} B_j(x) &= 1 + \sum_{j=1}^{\infty} \mathbf{P}\left(S_j \in (0, x); \tau^- > j\right)\\ &= 1 + \sum_{j=1}^{\infty} \mathbf{P}\left(S_j \in (0, x); S_j > S_0, S_j > S_1, \ldots, S_j > S_{j-1}\right)\\ &= H(x), \quad x > 0. \end{aligned} \tag{39}$$

Lemma 15 *The sequence of functions $\{B_n(x),\ n \geq 1\}$ satisfies the recurrence equations*

$$nB_n(x) = \mathbf{P}(S_n \in (0, x)) + \sum_{k=1}^{n-1} \int_0^x \mathbf{P}(S_k \in (0, x-y))dB_{n-k}(y) \tag{40}$$

and

$$nB_n(x) = \mathbf{P}\left(S_n \in (0, x)\right) + \sum_{k=1}^{n-1} \int_0^x B_{n-k}(x-y)\mathbf{P}\left(S_k \in dy\right). \tag{41}$$

Remark 16 The proof of (41) is contained in Eppel [14] (see formula (5) there). Representation (40) is not given by Eppel. However, it can be easily obtained by the same method. Here we demonstrate the mentioned relations only for the completeness of the presentation.

Proof Let

$$\mathcal{B}_n(t) := \mathbf{E}\left[e^{itS_n}; \tau^- > n\right] = \int_0^\infty e^{itx}\mathbf{P}\left(S_n \in dx; \tau^- > n\right), \quad t \in (-\infty, \infty),$$

be the Fourier transform of the measure B_n. It is known (see, for instance [22, Chap. 4, Sect. 17]) that

$$1 + \sum_{n=1}^{\infty} z^n \mathcal{B}_n(t) = \exp\left\{\sum_{k=1}^{\infty} \frac{z^k}{k}\mathcal{S}_k(t)\right\}, \ |z| < 1,$$

where $\mathcal{S}_k(t) := \mathbf{E}\left[e^{itS_k}; S_k > 0\right]$. Differentiation with respect to z gives

$$\sum_{n=1}^{\infty} z^{n-1} n\mathcal{B}_n(t) = \left(1 + \sum_{n=1}^{\infty} z^n \mathcal{B}_n(t)\right)\sum_{k=1}^{\infty} z^{k-1}\mathcal{S}_k(t).$$

Comparing the coefficients of z^{n-1} in the both sides of this equality, we get

$$n\mathcal{B}_n(t) = \mathcal{S}_n(t) + \sum_{k=1}^{n-1} \mathcal{B}_{n-k}(t)\mathcal{S}_k(t). \tag{42}$$

Going back to the distributions, we obtain the desired representations. □

From now on we assume *without loss of generality* that $h = 1$ in the lattice case and, to study the asymptotic behavior of the probabilities of small deviations when X is $(1, a)$-lattice, introduce a shifted sequence $\bar{S}_n := S_n - an$ and probabilities $\bar{b}_n(x) := \mathbf{P}(\bar{S}_n = x) = b_n(an + x)$. Further, for fixed $x \in \mathbb{Z}$ and $1 \le k \le n - 1$ set

$$\mathcal{I}_x(k, n) := (-a(n - k), ak + x) \cap \mathbb{Z}.$$

Lemma 17 *The sequence of functions* $\{\bar{b}_n(x),\ n \ge 1\}$ *satisfies the recurrence equation*

$$n\bar{b}_n(x) = \mathbf{P}(\bar{S}_n = x) + \sum_{k=1}^{n-1} \sum_{y \in \mathcal{I}_x(k,n)} \bar{b}_k(x - y)\mathbf{P}(\bar{S}_{n-k} = y). \tag{43}$$

Remark 18 Alili and Doney [2] obtained this representation in the case when X is $(h, 0)$-lattice.

Proof It follows from (42) that

$$n\bar{b}_n(x) = \mathbf{P}(\bar{S}_n = x) + \sum_{k=1}^{n-1} \sum \bar{b}_k(x-y)\mathbf{P}(\bar{S}_{n-k} = y),$$

where the second sum is taken over all $y \in \mathbb{Z}$ satisfying the conditions $ak + x - y > 0$, $a(n-k) + y > 0$. This proves the lemma. □

Lemma 19 *Assume $X \in \mathcal{D}(\alpha, \beta)$. Then there exists $C > 0$ such that, for all $y > 0$ and all $n \ge 1$,*

$$b_n(y) \le \frac{C}{c_n} \frac{l(n)}{n^{1-\rho}} \tag{44}$$

and

$$B_n(y) \le \frac{C\,(y+1)}{c_n} \frac{l(n)}{n^{1-\rho}}. \tag{45}$$

Proof For $n = 1$ the statement of the lemma is obvious. Let $\left\{S_n^*, n \ge 0\right\}$ be a random walk distributed as $\{S_n, n \ge 0\}$ and independent of it. One can easily check that for each $n \ge 2$,

$$\begin{aligned}
b_n(y) &= \mathbf{P}\left(y \le S_n < y+1; \tau^- > n\right) \\
&= \int_0^\infty \mathbf{P}\left(y - S_{[n/2]} \le S_n - S_{[n/2]} < y + 1 - S_{[n/2]}; S_{[n/2]} \in dz; \tau^- > n\right) \\
&\le \int_0^\infty \mathbf{P}\left(y - z \le S^*_{n-[n/2]} < y + 1 - z; S_{[n/2]} \in dz; \tau^- > [n/2]\right) \\
&\le \mathbf{P}\left(\tau^- > [n/2]\right) \sup_z \mathbf{P}\left(z \le S^*_{n-[n/2]} < z + 1\right).
\end{aligned} \tag{46}$$

Since the density of any α-stable law is bounded, it follows from the Gnedenko and Stone local limit theorems that there exists a constant $C > 0$ such that for all $n \ge 1$ and all $z \ge 0$,

$$\mathbf{P}\left(S_n \in [z, z+\Delta)\right) \le \frac{C\Delta}{c_n}. \tag{47}$$

Hence it follows, in particular, that, for any $z > 0$,

$$\mathbf{P}\left(S_n \in [0, z)\right) \le \frac{C(z+1)}{c_n}. \tag{48}$$

Substituting (47) into (46), and recalling (3) and properties of regularly varying functions, we get (44). Estimate (45) follows from (44) by summation. □

Lemma 20 *If $X \in \mathcal{D}(\alpha, \beta)$ then there exists a constant $C \in (0, \infty)$ such that*

$$b_n(x) \le C \frac{H(x)}{nc_n} \tag{49}$$

and

$$B_n(x) \le C \frac{xH(x)}{nc_n} \tag{50}$$

for all $n \ge 1$ *and all* $x \in (0, c_n]$.

Remark 21 Comparing (49) and (10) (to be proved later), we see that, in the domain of small deviations, the estimates given by the lemma are optimal up to a constant factor.

Proof By (41) we get

$$\begin{aligned} nb_n(x) = \mathbf{P}(S_n \in [x, x+1)) &+ \sum_{k=1}^{n-1} \int_0^x b_{n-k}(x-y)\mathbf{P}(S_k \in dy) \\ &+ \sum_{k=1}^{n-1} \int_x^{x+1} B_{n-k}(x+1-y)\mathbf{P}(S_k \in dy). \end{aligned} \tag{51}$$

Using (44), (48) and properties of slowly varying functions, we deduce

$$\begin{aligned} \sum_{k=1}^{[n/2]} \int_0^x b_{n-k}(x-y)\mathbf{P}(S_k \in dy) &\le C \sum_{k=1}^{[n/2]} \frac{l(n-k)}{c_{n-k}\,(n-k)^{1-\rho}} \mathbf{P}(S_k \in [0, x)) \\ &\le \frac{C_1}{c_n} \frac{l(n)}{n^{1-\rho}} \sum_{k=1}^{[n/2]} \mathbf{P}(S_k \in [0, x)). \end{aligned} \tag{52}$$

On the other hand, in view of (47) and monotonicity of $B_k(x)$ in x we conclude (assuming that x is integer without loss of generality and letting $B_k(-1) = 0$ and $H(-1) = 0$) that

$$\begin{aligned} &\sum_{k=[n/2]+1}^{n} \int_0^x b_{n-k}(x-y)\mathbf{P}(S_k \in dy) \\ &\le \sum_{k=[n/2]+1}^{n} \sum_{j=0}^{x} \left(B_{n-k}(x-j+1) - B_{n-k}(x-j-1)\right) \mathbf{P}(S_k \in [j, j+1)) \\ &\le \sum_{k=[n/2]+1}^{n} \sum_{j=0}^{x} \left(B_{n-k}(x-j+1) - B_{n-k}(x-j-1)\right) \frac{C}{c_k} \\ &\le \frac{C}{c_n} \sum_{j=0}^{x} \sum_{k=0}^{\infty} \left(B_k(x-j+1) - B_k(x-j-1)\right) \\ &= \frac{C}{c_n} \sum_{j=0}^{x} \left(H(x-j+1) - H(x-j-1)\right) \end{aligned}$$

$$\le \frac{C}{c_n}(H(x)+H(x+1)) \le \frac{2C}{c_n}H(x+1),$$

where for the intermediate equality we have used (39). This gives

$$\sum_{k=[n/2]+1}^{n} \int_0^x b_{n-k}(x-y)\mathbf{P}(S_k \in dy) \le \frac{C}{c_n}H(x+1). \tag{53}$$

Since $x \mapsto B_n(x)$ increases for every n,

$$\sum_{k=1}^{n-1} \int_x^{x+1} B_{n-k}(x+1-y)\mathbf{P}(S_k \in dy) \le \sum_{k=1}^{n-1} B_{n-k}(1)\mathbf{P}(S_k \in [x, x+1)). \tag{54}$$

Further, in view of (45) and (47) we have

$$\sum_{k=1}^{[n/2]} B_{n-k}(1)\mathbf{P}(S_k \in [x, x+1)) \le \frac{C_1}{c_n}\frac{l(n)}{n^{1-\rho}} \sum_{k=1}^{[n/2]} \mathbf{P}(S_k \in [x, x+1)). \tag{55}$$

Applying (47) once again yields

$$\sum_{k=[n/2]+1}^{n-1} B_{n-k}(1)\mathbf{P}(S_k \in [x, x+1)) \le \frac{C}{c_n} \sum_{k=[n/2]+1}^{n-1} B_{n-k}(1) \le \frac{C}{c_n}H(1). \tag{56}$$

Combining (51)–(56) and using the monotonicity of $H(x)$, we obtain the estimate

$$nb_n(x) \le \frac{C}{c_n}\left(H(x+1) + \frac{l(n)}{n^{1-\rho}} \sum_{k=1}^{[n/2]} \mathbf{P}(S_k \in [0, x+1))\right).$$

Therefore, to complete the proof of (49) it remains to show that

$$\frac{l(n)}{n^{1-\rho}} \sum_{k=1}^{[n/2]} \mathbf{P}(S_k \in [0, x+1)) \le CH(x+1). \tag{57}$$

This will be done separately for the cases $\alpha \in (1, 2]$, $\alpha \in (0, 1)$, and $\alpha = 1$. Consider first the case $\alpha \in (1, 2]$. It follows from (48) that

$$\sum_{k=1}^{[n/2]} \mathbf{P}(0 \le S_k < x+1) \le C(x+1)\sum_{k=1}^{n} \frac{1}{c_k} \le C(x+1)\frac{n}{c_n}, \tag{58}$$

where at the last step we have used the relation

$$\sum_{k=1}^{n} \frac{1}{c_k} \sim \frac{\alpha}{\alpha-1} \frac{n}{c_n} \quad \text{as } n \to \infty. \tag{59}$$

By Lemma 13 and properties of regularly varying functions we conclude that there exists a non-decreasing function $\phi(u)$ such that $u/H(u) \sim \phi(u)$ as $u \to \infty$. Therefore, for any $\varepsilon \in (0, 1/2)$ there exists a $u_0 = u_0(\varepsilon)$ such that, for all $u \geq u_0$,

$$(1-\varepsilon)\phi(u) \leq \frac{u}{H(u)} \leq (1+\varepsilon)\phi(u).$$

From this estimate it is not difficult to conclude that there exists a constant C such that, for all $n \geq 1$ and all $x \in (0, c_n]$,

$$\frac{x}{H(x)} \leq C \frac{c_n}{H(c_n)}.$$

Hence we see that the right-hand side of (58) is bounded from above by

$$C \frac{nH(x+1)}{H(c_n)}.$$

Recalling that $H(x)$ is regularly varying as $x \to \infty$, and applying (31) and (15), we finally arrive at the inequality

$$\sum_{k=1}^{[n/2]} \mathbf{P}(0 \leq S_k < x+1) \leq CH(x+1) \frac{n^{1-\rho}}{l(n)}.$$

This justifies (57) for $\alpha \in (1, 2]$.

Now we turn to the case $\alpha \in (0, 1)$. Letting $N_x := \max\{k \geq 1 : c_k \leq x+1\}$ and applying (47), we get

$$\begin{aligned} \sum_{k=1}^{[n/2]} \mathbf{P}(0 \leq S_k < x+1) &\leq N_x + C(x+1) \sum_{k=N_x+1}^{n} \frac{1}{c_k} \\ &\leq N_x + C(x+1) \frac{N_x}{c_{N_x+1}}. \end{aligned} \tag{60}$$

Here we have used the asymptotic representation

$$\sum_{k=n+1}^{\infty} \frac{1}{c_k} \sim \frac{\alpha}{1-\alpha} \frac{n}{c_{n+1}} \quad \text{as } n \to \infty.$$

If $\alpha = 1$, then, in view of (3),

$$\sum_{k=N_x+1}^{n} \frac{1}{c_k} = \frac{N_x + 1}{c_{N_x+1}} \sum_{k=N_x+1}^{n} \frac{l_1(N_x + 1)}{l_1(k)} \frac{1}{k}.$$

From the Karamata representation for slowly varying functions (see [21, Theorem 1.2]) we conclude that for every slowly varying function $l^*(x)$ and every $\gamma > 0$ there exists a constant $C = C(\gamma)$ such that

$$\frac{l^*(x)}{l^*(y)} \le C \max\left\{\left(\frac{x}{y}\right)^{\gamma}, \left(\frac{x}{y}\right)^{-\gamma}\right\} \quad \text{for all } x, y > 0. \tag{61}$$

Applying this inequality to $l_1(x)$, we obtain

$$\sum_{k=N_x+1}^{n} \frac{1}{c_k} \le C \frac{N_x + 1}{c_{N_x+1}} \left(\frac{n}{N_x + 1}\right)^{\gamma} \log\left(\frac{n}{N_x + 1}\right).$$

Combining this bound with (60), and using the inequality $c_{N_x+1} \ge x + 1$, we conclude that

$$\sum_{k=1}^{[n/2]} \mathbf{P}(0 \le S_k < x + 1) \le C_1 N_x \left(\frac{n}{N_x}\right)^{2\gamma} \tag{62}$$

for all $\alpha \in (0, 1]$. Consequently,

$$\frac{l(n)}{n^{1-\rho}} \sum_{k=1}^{[n/2]} \mathbf{P}(0 \le S_k < x + 1) \le C_1 H(x + 1) \left(\frac{n}{N_x}\right)^{2\gamma} \frac{l(n) N_x}{n^{1-\rho} H(x + 1)}.$$

The definition of N_x, (31), and (15) imply

$$H(x + 1) \ge H(c_{N_x}) \ge C l(N_x) N_x^{\rho}.$$

Therefore,

$$\frac{l(n)}{n^{1-\rho}} \sum_{k=1}^{[n/2]} \mathbf{P}(0 \le S_k < x + 1) \le C_1 H(x + 1) \left(\frac{N_x}{n}\right)^{1-\rho-2\gamma} \frac{l(n)}{l(N_x)}.$$

Applying (61) to $l(x)$ and choosing $\gamma = (1 - \rho)/4$, we finally arrive at the inequality

$$\frac{l(n)}{n^{1-\rho}} \sum_{k=1}^{[n/2]} \mathbf{P}(0 \le S_k < x + 1) \le C H(x + 1) \left(\frac{N_x}{n}\right)^{(1-\rho)/4} \le C H(x + 1) \tag{63}$$

establishing (57) for $\alpha \in (0, 1]$. Thus, (57) is justified for all $X \in \mathcal{D}(\alpha, \beta)$, and, consequently, (49) is proved.

The second statement of the lemma follows by summation. □

Later on we need the following refined version of Lemma 20:

Corollary 22 *Suppose $X \in \mathcal{D}(\alpha, \beta)$. Then there exists a constant $C \in (0, \infty)$ such that, for all $n \geq 1$,*

$$b_n(x) \leq C \frac{H(\min(c_n, x))}{nc_n} \tag{64}$$

and

$$B_n(x) \leq C \frac{\min(c_n, x) H(\min(c_n, x))}{nc_n}. \tag{65}$$

Proof The desired estimates follow from (44), (45) and Lemma 20. □

Lemma 23 *There exists a constant $C \in (0, \infty)$ such that, for all $z \in [0, \infty)$,*

$$\limsup_{\varepsilon \downarrow 0} \varepsilon^{-1} \mathbf{P}\left(M_{\alpha,\beta} \in [z, z+\varepsilon)\right) \leq C \min\{1, z^{\alpha\rho}\}.$$

In particular,

$$\lim_{z \downarrow 0} \limsup_{\varepsilon \downarrow 0} \varepsilon^{-1} \mathbf{P}\left(M_{\alpha,\beta} \in [z, z+\varepsilon)\right) = 0.$$

Proof For all $z \geq 0$ and all $\varepsilon > 0$ we have

$$\mathbf{P}\left(M_{\alpha,\beta} \in [z, z+\varepsilon)\right) \leq \lim \sup_{n \to \infty} \mathbf{P}\left(S_n \in [c_n z, c_n(z+\varepsilon))|\tau^- > n\right).$$

Applying (64) gives

$$\mathbf{P}\left(S_n \in [c_n z, c_n(z+\varepsilon))|\tau^- > n\right) \leq C \frac{H(\min(c_n, (z+\varepsilon)c_n))}{nc_n \mathbf{P}(\tau^- > n)} \varepsilon c_n.$$

Recalling that $H(x)$ is regularly varying with index $\alpha\rho$ by Lemma 13 and taking into account (31), we get

$$\begin{aligned}\mathbf{P}\left(S_n \in [c_n z, c_n(z+\varepsilon))|\tau^- > n\right) &\leq C\varepsilon \min\{1, (z+\varepsilon)^{\alpha\rho}\} \frac{H(c_n)}{n\mathbf{P}(\tau^- > n)} \\ &\leq C\varepsilon \min\{1, (z+\varepsilon)^{\alpha\rho}\}.\end{aligned}$$

Consequently,

$$\mathbf{P}(M_{\alpha,\beta} \in [z, z+\varepsilon)) \leq C\varepsilon \min\{1, (z+\varepsilon)^{\alpha\rho}\}. \tag{66}$$

This inequality shows that there exists a constant $C \in (0, \infty)$ such that

$$\limsup_{\varepsilon \downarrow 0} \varepsilon^{-1} \mathbf{P}\left(M_{\alpha,\beta} \in [z, z+\varepsilon)\right) \leq C \min\{1, z^{\alpha\rho}\} \text{ for all } z \geq 0$$

as desired. □

3 Probabilities of normal deviations: proofs of Theorems 3 and 5

The first part of the proof to follow is one and the same for non-lattice (Theorem 3) and lattice (Theorem 5) cases.

It follows from (40) that

$$\begin{aligned} nb_n(x) &= \mathbf{P}(S_n \in [x, x+1)) + \sum_{k=1}^{n-1} \int_0^x \mathbf{P}(S_k \in [x-y, x-y+1)) dB_{n-k}(y) \\ &\quad + \sum_{k=1}^{n-1} \int_x^{x+1} \mathbf{P}(S_k \in (0, x-y+1)) dB_{n-k}(y) \\ &=: R_\varepsilon^{(1)}(x) + R_\varepsilon^{(2)}(x) + R_\varepsilon^{(3)}(x) + R^{(0)}(x), \end{aligned} \tag{67}$$

where, for any fix $\varepsilon \in (0, 1/2)$ and with a slight abuse of notation

$$R_\varepsilon^{(1)}(x) := \sum_{k=1}^{[\varepsilon n]} \int_0^x \mathbf{P}(S_k \in [x-y, x-y+1)) dB_{n-k}(y),$$

$$R_\varepsilon^{(2)}(x) := \sum_{k=[\varepsilon n]+1}^{[(1-\varepsilon)n]} \int_0^x \mathbf{P}(S_k \in [x-y, x-y+1)) dB_{n-k}(y),$$

$$R_\varepsilon^{(3)}(x) := \mathbf{P}(S_n \in [x, x+1)) + \sum_{k=[(1-\varepsilon)n]+1}^{n-1} \int_0^x \mathbf{P}(S_k \in [x-y, x-y+1)) dB_{n-k}(y),$$

and

$$R^{(0)}(x) := \sum_{k=1}^{n-1} \int_x^{x+1} \mathbf{P}(S_k \in (0, x-y+1)) dB_{n-k}(y).$$

First observe that

$$R^{(0)}(x) \le \sum_{k=1}^{n-1} \mathbf{P}(S_k \in (0, 1)) b_{n-k}(x).$$

Applying Corollary 22 we may simplify the estimate above to

$$\begin{aligned} R^{(0)}(x) &\le C \sum_{k=1}^{n-1} \mathbf{P}(S_k \in (0, 1)) \frac{H_{n-k}(c_{n-k})}{(n-k)c_{n-k}} \\ &\le C \frac{H(c_n)}{nc_n} \sum_{k=1}^{[n/2]} \mathbf{P}(S_k \in (0, 1)) + \frac{C}{c_n} \sum_{k=1}^{[n/2]} \frac{H(c_k)}{kc_k}, \end{aligned} \tag{68}$$

where at the last step we have used the properties of c_n and inequality (47).

Since $H(x) \le Cx$, we have

$$\sum_{k=1}^{[n/2]} \frac{H(c_k)}{kc_k} \le C \sum_{k=1}^{[n/2]} \frac{1}{k} \le C \log n.$$

Further, by (58) and (62) with $x = 0$ we know that

$$\sum_{k=1}^{[n/2]} \mathbf{P}(S_k \in (0,1)) \le C \left(\frac{n}{c_n} \mathrm{I}(\alpha \in (1,2]) + n^{\gamma} \mathrm{I}(\alpha \in (0,1]) \right) \le C \left(\frac{n}{c_n} + n^{\gamma} \right).$$

Substituting these estimates into (68) leads to the inequality

$$R^{(0)}(x) \le \frac{C}{c_n} \left(\frac{H(c_n)}{c_n} + \frac{H(c_n)}{n^{1-\gamma}} + \log n \right).$$

By this relation, recalling that $\mathbf{P}(\tau^- > n)$ is regularly varying with index $\rho - 1 > -1$ (see (15)) and using (31), we obtain

$$\limsup_{n\to\infty} \frac{c_n}{n\mathbf{P}(\tau^- > n)} R^{(0)}(x) = 0. \tag{69}$$

Now we evaluate the remaining terms in (67).
In view of (47)

$$R_{\varepsilon}^{(3)}(x) \le \frac{C}{c_n} \left(1 + \sum_{k=1}^{[\varepsilon n]} B_k(x) \right) \le \frac{C}{c_n} \sum_{k=0}^{[\varepsilon n]} \mathbf{P}(\tau^- > k)$$

for all $x > 0$. Further, by (15)

$$\sum_{k=0}^{[\varepsilon n]} \mathbf{P}(\tau^- > k) \sim \rho^{-1} \varepsilon^{\rho} n \mathbf{P}(\tau^- > n) \quad \text{as } n \to \infty.$$

As a result we obtain

$$\limsup_{n\to\infty} \frac{c_n}{n\mathbf{P}(\tau^- > n)} \sup_{x>0} R_{\varepsilon}^{(3)}(x) \le C\varepsilon^{\rho}. \tag{70}$$

Using the inequalities

$$\int_j^{j+1} \mathbf{P}(S_k \in [x-y, x-y+1)) dB_{n-k}(y) \le \mathbf{P}(S_k \in [x-j-1, x-j+1)) b_{n-k}(j) \tag{71}$$

and

$$\int_{[x]}^{x} \mathbf{P}(S_k \in [x-y, x-y+1))dB_{n-k}(y) \le \mathbf{P}(S_k \in [0,2))b_{n-k}([x]), \qquad (72)$$

and applying Corollary 22, we get

$$R_\varepsilon^{(1)}(x) \le C\frac{H(c_n)}{nc_n}\sum_{k=1}^{[\varepsilon n]} \mathbf{P}(0 < S_k < x) \le \varepsilon C \frac{H(c_n)}{c_n}.$$

From this estimate and (31) we deduce

$$\limsup_{n\to\infty} \frac{c_n}{n\mathbf{P}(\tau^- > n)} R_\varepsilon^{(1)}(x) \le C\varepsilon. \qquad (73)$$

Evaluating $R_\varepsilon^{(2)}(x)$ we have to distinguish the non-lattice (Theorem 3) and lattice (Theorem 5) cases. Detailed estimates are given for the non-lattice case only. To deduce the respective estimates for the lattice case one should use the Gnedenko local limit theorem instead of the Stone local limit theorem.

Thus, in the non-lattice case we combine the Stone local limit theorem with the first equality in (4) and obtain, uniformly in $x > 0$, as $n \to \infty$,

$$\begin{aligned} R_\varepsilon^{(2)}(x) &= \sum_{k=[\varepsilon n]+1}^{[(1-\varepsilon)n]} \frac{1}{c_{n-k}} \int_0^x g_{\alpha,\beta}\left(\frac{x-y}{c_{n-k}}\right) dB_k(y) + o\left(\frac{1}{c_{n\varepsilon}}\sum_{k=1}^{n} B_k(x)\right) \\ &= \sum_{k=[\varepsilon n]+1}^{[(1-\varepsilon)n]} \frac{\mathbf{P}(\tau^- > k)}{c_{n-k}} \int_0^{x/c_k} g_{\alpha,\beta}\left(\frac{x-c_k u}{c_{n-k}}\right) \mathbf{P}(M_{\alpha,\beta} \in du) \\ &\quad + o\left(\frac{1}{c_{n\varepsilon}}\sum_{k=1}^{n} B_k(x) + \sum_{k=1}^{n-1}\frac{\mathbf{P}(\tau^- > k)}{c_{n\varepsilon}}\right). \end{aligned}$$

According to (15)

$$\sum_{k=1}^{n} B_k(x) \le \sum_{k=1}^{n} \mathbf{P}(\tau^- > k) \le Cn\mathbf{P}(\tau^- > n).$$

Hence it follows that

$$R_{\varepsilon}^{(2)}(x) = \sum_{k=[\varepsilon n]+1}^{[(1-\varepsilon)n]} \frac{\mathbf{P}(\tau^- > k)}{c_{n-k}} \int_0^{x/c_k} g_{\alpha,\beta}\left(\frac{x - c_k u}{c_{n-k}}\right) \mathbf{P}(M_{\alpha,\beta} \in du)$$
$$+ o\left(\frac{n\mathbf{P}(\tau^- > n)}{c_{n\varepsilon}}\right).$$

Since c_k and $\mathbf{P}(\tau^- > k)$ are regularly varying and $g_{\alpha,\beta}(x)$ is uniformly continuous in $(-\infty, \infty)$, we let, for brevity, $v = x/c_n$ and continue the previous estimates for $R_{\varepsilon}^{(2)}(x)$ with

$$= \frac{\mathbf{P}(\tau^- > n)}{c_n} \sum_{k=[\varepsilon n]+1}^{[(1-\varepsilon)n]} \frac{(k/n)^{\rho-1}}{(1-k/n)^{1/\alpha}} \int_0^{v/(k/n)^{1/\alpha}} g_{\alpha,\beta}\left(\frac{v - (k/n)^{1/\alpha} u}{(1-k/n)^{1/\alpha}}\right) \mathbf{P}(M_{\alpha,\beta} \in du)$$
$$+ o\left(\frac{n\mathbf{P}(\tau^- > n)}{c_{n\varepsilon}}\right)$$
$$= \frac{n\mathbf{P}(\tau^- > n)}{c_n} f(\varepsilon, 1-\varepsilon; v) + o\left(\frac{n\mathbf{P}(\tau^- > n)}{c_{n\varepsilon}}\right),$$

where, for $0 \le w_1 \le w_2 \le 1$,

$$f(w_1, w_2; v) := \int_{w_1}^{w_2} \frac{t^{\rho-1} dt}{(1-t)^{1/\alpha}} \int_0^{v/t^{1/\alpha}} g_{\alpha,\beta}\left(\frac{v - t^{1/\alpha} u}{(1-t)^{1/\alpha}}\right) \mathbf{P}(M_{\alpha,\beta} \in du). \tag{74}$$

Observe that by boundness of $g_{\alpha,\beta}(y)$

$$f(0, \varepsilon; v) \le C \int_0^{\varepsilon} t^{\rho-1} dt \le C\varepsilon^{\rho}.$$

Further, it follows from (66) that $\int \phi(u)\mathbf{P}(M_{\alpha,\beta} \in du) \le C \int \phi(u) du$ for every non-negative integrable function ϕ. Therefore,

$$f(1-\varepsilon, 1; v)$$
$$\le C \int_{1-\varepsilon}^{1} \frac{t^{\rho-1} dt}{(1-t)^{1/\alpha}} \int_0^{v/t^{1/\alpha}} g_{\alpha,\beta}\left(\frac{v - t^{1/\alpha} u}{(1-t)^{1/\alpha}}\right) du = \left(z = \frac{v - t^{1/\alpha} u}{(1-t)^{1/\alpha}}\right)$$
$$= C \int_{1-\varepsilon}^{1} t^{\rho-1-1/\alpha} dt \int_0^{v/(1-t)^{1/\alpha}} g_{\alpha,\beta}(z)\, dz \le C\varepsilon.$$

As a result we have

$$\limsup_{n\to\infty} \sup_{x>0} \left| \frac{c_n}{n\mathbf{P}(\tau^- > n)} R_\varepsilon^{(2)}(x) - f(0,1;x/c_n) \right| \le C\varepsilon^\rho. \tag{75}$$

Combining (69)–(75) with representation (67) leads to

$$\limsup_{n\to\infty} \sup_{x>0} \left| \frac{c_n}{\mathbf{P}(\tau^- > n)} b_n(x) - f(0,1;x/c_n) \right| \le C\varepsilon^\rho. \tag{76}$$

Since $\varepsilon > 0$ is arbitrary, it follows that, as $n \to \infty$

$$\frac{c_n}{\mathbf{P}(\tau^- > n)} b_n(x) - f(0,1;x/c_n) \to 0 \tag{77}$$

uniformly in $x > 0$. Recalling (4), we deduce by integration of (77) and evident transformations that

$$\int_{u_1}^{u_2} f(0,1;z)dz = \mathbf{P}(M_{\alpha,\beta} \in [u_1,u_2]) \tag{78}$$

for all $0 < u_1 < u_2 < \infty$. This means, in particular, that the distribution of $M_{\alpha,\beta}$ is absolutely continuous. Furthermore, it is not difficult to see that $z \mapsto f(0,1;z)$ is a continuous mapping. Hence, in view of (78), we may consider $f(0,1;z)$ as a continuous version of the density of the distribution of $M_{\alpha,\beta}$ and let $p_{\alpha,\beta}(z) := f(0,1;z)$. This and (77) imply the statement of Theorem 3 for $\Delta = 1$. To establish the desired result for arbitrary $\Delta > 0$ it suffices to consider the random walk S_n/Δ and to observe that

$$c_n^\Delta := \inf\left\{ u \ge 0 : \frac{1}{u^2} \int_{-u}^{u} x^2 P\left(\frac{X}{\Delta} \in dx\right) \right\} = c_n/\Delta.$$

Note that (74) gives an interesting representation for $p_{\alpha,\beta}(v)$:

$$p_{\alpha,\beta}(z) = \int_0^1 \frac{t^{\rho-1}dt}{(1-t)^{1/\alpha}} \int_0^{z/t^{1/\alpha}} g_{\alpha,\beta}\left(\frac{z - t^{1/\alpha}u}{(1-t)^{1/\alpha}}\right) p_{\alpha,\beta}(u)du. \tag{79}$$

For the case $\alpha = 2$ this equation coincides (up to a change of variables) with Eq. (B.1) in [6]. In the general case it may be considered as a version of Eq. (4.9) in [1]. Observe that in view of Lemma 23

$$p_{\alpha,\beta}(z) \le C \min\{1, z^{\alpha\rho}\}$$

and

$$\lim_{z\downarrow 0} p_{\alpha,\beta}(z) = 0, \tag{80}$$

which is not surprising. In Sect. 4.3 we refine these statements.

4 Probabilities of small deviations

4.1 Lattice case: proof of Theorem 6

Recall that the span $h = 1$ according to our agreement. Fix any $\varepsilon \in (0, 1/2)$ and, using Lemma 17, write

$$n\bar{b}_n(x) = R_{\varepsilon n}(x) + \bar{R}_{\varepsilon n}(x), \tag{81}$$

where

$$R_{\varepsilon n}(x) := \mathbf{P}(\bar{S}_n = x) + \sum_{k=1}^{[\varepsilon n]} \sum_{y \in \mathcal{I}_x(k,n)} \bar{b}_k(x-y)\mathbf{P}(\bar{S}_{n-k} = y)$$

and

$$\bar{R}_{\varepsilon n}(x) := \sum_{k=[\varepsilon n]+1}^{n-1} \sum_{y \in \mathcal{I}_x(k,n)} \bar{b}_k(x-y)\mathbf{P}(\bar{S}_{n-k} = y).$$

In view of Lemma 20,

$$\begin{aligned}
\bar{R}_{\varepsilon n}(x) &\le C \sum_{k=[\varepsilon n]+1}^{n-1} \sum_{y \in \mathcal{I}_x(k,n)} \frac{H(ak + x - y)}{kc_k}\mathbf{P}(\bar{S}_{n-k} = y)\\
&\le C(\varepsilon)\frac{H(an+x)}{nc_n} \sum_{k=1}^{n-[\varepsilon n]} \mathbf{P}(0 \le S_k < an + x).
\end{aligned}$$

Introduce the set

$$\mathcal{G}_n := (-an, -an + \delta_n c_n] \cap \mathbb{Z}.$$

Taking into account estimate (58) (with $[n/2]$ replaced by $n - [n\varepsilon]$), we see that for $\alpha \in (1, 2]$

$$\begin{aligned}
\limsup_{n\to\infty} \sup_{x\in\mathcal{G}_n} \frac{c_n \bar{R}_{\varepsilon n}(x)}{H(an+x)} &\le C(\varepsilon) \limsup_{n\to\infty} \sup_{x\in\mathcal{G}_n} \frac{an+x}{c_n}\\
&= C(\varepsilon) \limsup_{n\to\infty} \delta_n = 0.
\end{aligned} \tag{82}$$

Similarly, writing $c^{-1}(n)$ for the inverse function of c_n we conclude by (62) (with $[n/2]$ replaced by $n - [n\varepsilon]$) that for $\alpha \in (0, 1)$ and every $\gamma < 1/2$.

$$\begin{aligned}
\limsup_{n\to\infty} \sup_{x\in\mathcal{G}_n} \frac{c_n \bar{R}_{\varepsilon n}(x)}{H(an+x)} &\le C(\varepsilon) \limsup_{n\to\infty} \left(\frac{N_{\delta_n c_n}}{n}\right)^{1-2\gamma}\\
&= C(\varepsilon) \limsup_{n\to\infty} \left(\frac{c^{-1}(\delta_n c_n)}{c^{-1}(c_n)}\right)^{1-2\gamma} = 0.
\end{aligned} \tag{83}$$

According to the Gnedenko local limit theorem

$$\sup_{k\in[1,n(1-\varepsilon)]}\ \sup_{y\in\mathcal{I}_x(k,n)} \left|c_{n-k}\mathbf{P}(\bar{S}_{n-k}=y)-g_{\alpha,\beta}(0)\right| \to 0 \quad \text{as } n\to\infty.$$

Therefore,

$$\begin{aligned}\sum_{y\in\mathcal{I}_x(k,n)} &\bar{b}_k(x-y)\mathbf{P}(\bar{S}_{n-k}=y)\\ &=\frac{g_{\alpha,\beta}(0)+\Delta_1(x,n-k)}{c_{n-k}}\sum_{y\in\mathcal{I}_x(k,n)}\bar{b}_k(x-y),\end{aligned}$$

where $\Delta_1(x,n-k)\to 0$ as $n\to\infty$ uniformly in $x\in\mathcal{G}_n$ and $k\in[1,n(1-\varepsilon)]$. Hence, by the identity

$$\sum_{y\in\mathcal{I}_x(k,n)}\bar{b}_k(x-y)=B_k(a(n-k)+x),$$

we see that

$$R_{\varepsilon n}(x)=(g_{\alpha,\beta}(0)+\Delta_2(x,n))\left(\frac{1}{c_n}+\sum_{k=1}^{[\varepsilon n]}\frac{1}{c_{n-k}}B_k(a(n-k)+x)\right), \tag{84}$$

where $\Delta_2(x,n)\to 0$ as $n\to\infty$ uniformly in $x\in\mathcal{G}_n$. Since the sequence $\{c_n, n\ge 1\}$ is non-decreasing and varies regularly with index $1/\alpha$ as $n\to\infty$, we have

$$\begin{aligned}\sum_{k=1}^{[\varepsilon n]}B_k(a(n-k)+x)&\le c_n\sum_{k=n-[\varepsilon n]}^{n-1}\frac{1}{c_k}B_{n-k}(ak+x)\\ &\le\left((1-\varepsilon)^{-1/\alpha}+\Delta_3(x,n)\right)\sum_{k=1}^{[\varepsilon n]}B_k(a(n-k)+x), \end{aligned}\tag{85}$$

where $\Delta_3(x,n)\to 0$ as $n\to\infty$ uniformly in $x\in\mathcal{G}_n$. On the other hand, for all $x>-an$,

$$H(an+x)-\sum_{k=[\varepsilon n]+1}^{\infty}B_k(a\,(n-k)+x)\le 1+\sum_{k=1}^{[\varepsilon n]}B_k(a(n-k)+x)\le H(an+x). \tag{86}$$

Applying (50) gives for some constant $C_1 = C_1(\varepsilon)$

$$\sum_{k=[\varepsilon n]+1}^{\infty} B_k(a(n-k)+x) \le (an+x)H(an+x) \sum_{k=[\varepsilon n]+1}^{\infty} \frac{C}{kc_k}$$
$$\le C_1 \frac{(an+x)H(an+x)}{c_n} \le C_1\delta_n H(an+x) \qquad (87)$$

for all $x \in \mathcal{G}_n$. From (86) and (87) we conclude that

$$\frac{1}{H(an+x)}\left(1+\sum_{k=1}^{[\varepsilon n]} B_k(a(n-k)+x)\right) - 1 \to 0 \qquad (88)$$

uniformly in $x \in \mathcal{G}_n$. Combining (84), (85), and (88) leads to

$$\limsup_{n\to\infty} \sup_{x\in\mathcal{G}_n} \left| \frac{c_n R_{\varepsilon n}(x)}{H(an+x)} - g_{\alpha,\beta}(0)\right| \le r(\varepsilon),$$

where $r(\varepsilon) \to 0$ as $\varepsilon \to 0$. This estimate, (82) and (83) show that

$$\limsup_{n\to\infty} \sup_{x\in\mathcal{G}_n} \left| \frac{c_n n}{H(an+x)} \bar{b}_n(x) - g_{\alpha,\beta}(0)\right| \le r(\varepsilon).$$

Letting $\varepsilon \to 0$ and recalling that

$$\bar{b}_n(x) = \mathbf{P}(S_n = an+x | \tau^- > n)\mathbf{P}(\tau^- > n)$$

we finish the proof of Theorem 6.

4.2 Non-lattice case: proof of Theorem 4

As in the proof of Theorem 3 we restrict our attention to the case $\Delta = 1$. Some of our subsequent arguments are similar to those used in the proof of Theorem 6, and we skip the respective details.

Using (71), (72) and Lemma 20 gives (in the notation introduced after formula (67))

$$R_\varepsilon^{(1)}(x) + R_\varepsilon^{(2)}(x) = \sum_{k=1}^{[(1-\varepsilon)n]} \int_0^x \mathbf{P}(S_k \in [x-y, x-y+1))dB_{n-k}(y)$$
$$\le C(\varepsilon)\frac{H(x)}{nc_n} \sum_{k=1}^{[(1-\varepsilon)n]} \mathbf{P}(0 \le S_k < x+1).$$

By the arguments mimicking those used in the lattice case one can easily show that

$$\lim_{n\to\infty}\sup_{0<x\le\delta_n c_n}\frac{c_n}{H(x)}\left(R_\varepsilon^{(1)}(x)+R_\varepsilon^{(2)}(x)\right)=0. \tag{89}$$

Further, by the Stone local limit theorem

$$\int_0^x \mathbf{P}(S_k\in[x-y,x-y+1))dB_{n-k}(y)=\frac{g_{\alpha,\beta}(0)+\Delta_1(k,x)}{c_k}B_{n-k}(x),$$

where $\Delta_1(k,x)\to 0$ uniformly in $x\in(0,\delta_n c_n)$ and $k\in[(1-\varepsilon)n,n]$. Therefore,

$$\begin{aligned}R_\varepsilon^{(3)}(x)&=\mathbf{P}(S_n\in[x,x+1))+\sum_{k=[(1-\varepsilon)n]+1}^{n-1}\int_0^x \mathbf{P}(S_k\in[x-y,x-y+1))dB_{n-k}(y)\\&=(g_{\alpha,\beta}(0)+\Delta_2(n,x))\left(\frac{1}{c_n}+\sum_{k=1}^{[\varepsilon n]}\frac{1}{c_{n-k}}B_k(x)\right),\end{aligned}$$

where $\Delta_2(n,x)\to 0$ uniformly in $x\in(0,\delta_n c_n)$. Therefore, as in the lattice case,

$$\limsup_{n\to\infty}\sup_{0<x\le\delta_n c_n}\left|\frac{c_n}{H(x)}R_\varepsilon^{(3)}(x)-g_{\alpha,\beta}(0)\right|\le r(\varepsilon), \tag{90}$$

where $r(\varepsilon)\to 0$ as $\varepsilon\to 0$. Combining (89) and (90), we get

$$\limsup_{n\to\infty}\sup_{0<x\le\delta_n c_n}\left|\frac{c_n}{H(x)}\left(R_\varepsilon^{(1)}(x)+R_\varepsilon^{(2)}(x)+R_\varepsilon^{(3)}(x)\right)-g_{\alpha,\beta}(0)\right|=0. \tag{91}$$

Now using definition (67) we write $R^{(0)}(x)=R_\varepsilon^{(4)}(x)+R_\varepsilon^{(5)}(x)$, where

$$R_\varepsilon^{(4)}(x):=\sum_{k=1}^{[(1-\varepsilon)n]}\int_x^{x+1}\mathbf{P}(S_k\in(0,x-y+1))dB_{n-k}(y)$$

and

$$R_\varepsilon^{(5)}(x):=\sum_{k=[(1-\varepsilon)n]+1}^{n-1}\int_x^{x+1}\mathbf{P}(S_k\in(0,x-y+1))dB_{n-k}(y).$$

Evidently,

$$R_\varepsilon^{(4)}(x)\le\sum_{k=1}^{[(1-\varepsilon)n]}\mathbf{P}(S_k\in(0,1))b_{n-k}(x).$$

Applying (47) and (49), we see that

$$R_\varepsilon^{(4)}(x) \le H(x) \sum_{k=1}^{[(1-\varepsilon)n]} \frac{1}{c_k} \frac{1}{(n-k)c_{n-k}} \le \frac{C(\varepsilon)}{nc_n} \sum_{k=1}^{n} \frac{1}{c_k}.$$

Observing that $\sum_{k=1}^n c_k^{-1} \le C(1+n/c_n)$, we conclude that

$$\lim_{n\to\infty} \sup_{0<x\le\delta_n c_n} \frac{c_n}{H(x)} R_\varepsilon^{(4)}(x) = 0. \tag{92}$$

Further, by the Stone local limit theorem,

$$\int_x^{x+1} \mathbf{P}(S_k \in [0, x-y+1)) dB_{n-k}(y)$$
$$= \frac{(g_{\alpha,\beta}(0) + \Delta_3(k,x))}{c_k} \int_x^{x+1} (x-y+1) dB_{n-k}(y),$$

where $\Delta_3(k,x) \to 0$ uniformly in $x \in (0, \delta_n c_n]$ and $k \in [(1-\varepsilon)n, n]$. Integration by parts gives

$$\int_x^{x+1} (x-y+1) dB_{n-k}(y) = -B_{n-k}(x) + \int_x^{x+1} B_{n-k}(y) dy.$$

Consequently,

$$R_\varepsilon^{(5)}(x) = (g_{\alpha,\beta}(0) + \Delta_4(n,x)) \sum_{k=1}^{[\varepsilon n]} \frac{1}{c_{n-k}} \left(\int_x^{x+1} B_k(y) dy - B_k(x) \right), \tag{93}$$

where $\Delta_4(n,x) \to 0$ uniformly in $x \in (0, \delta_n c_n]$.

Setting

$$I(x) := \int_x^{x+1} H(y) dy - H(x)$$

we see, similarly to the proof in the lattice case, that

$$\limsup_{n\to\infty} \sup_{0<x\le\delta_n c_n} \left| \frac{c_n}{I(x)} \sum_{k=1}^{[\varepsilon n]} \frac{1}{c_{n-k}} \left(\int_x^{x+1} B_k(y) dy - B_k(x) \right) - g_{\alpha,\beta}(0) \right| \le r(\varepsilon), \tag{94}$$

where $r(\varepsilon) \to 0$ as $\varepsilon \to 0$. By (92)–(94) we deduce

$$\lim_{n\to\infty} \sup_{0<x\le\delta_n c_n} \left| \frac{c_n}{I(x)} R^{(0)}(x) - g_{\alpha,\beta}(0) \right| = 0. \tag{95}$$

Substituting (91) and (95) into (67) finishes the proof.

4.3 Proof of Theorem 7

It is sufficient to show that there exists a constant $C > 0$ such that

$$p_{\alpha,\beta}(\varepsilon_m) \sim C\varepsilon_m^{\alpha\rho} \quad \text{as } m \to \infty \tag{96}$$

for every sequence $\varepsilon_m \to 0$. Since $H(x)$ is regularly varying with index $\alpha\rho$, there exists a sequence $n_1(m) \to \infty$ as $m \to \infty$ such that

$$\sup_{n\ge n_1(m)} \left| \frac{\varepsilon_m^{\alpha\rho} H(c_n)}{H(\varepsilon_m c_n)} - 1 \right| \to 0 \quad \text{as } m \to \infty.$$

From this fact and Theorem 4 we deduce:

$$\begin{aligned} c_n \mathbf{P}(S_n \in [\varepsilon_m c_n, \varepsilon_m c_n + 1) | \tau^- > n) &= g_{\alpha,\beta}(0) \frac{H(\varepsilon_m c_n)}{n\mathbf{P}\left(\tau^- > n\right)} (1 + \varphi_{n,m}^{(1)}) \\ &= g_{\alpha,\beta}(0) \frac{\varepsilon_m^{\alpha\rho} H(c_n)}{n\mathbf{P}\left(\tau^- > n\right)} (1 + \varphi_{n,m}^{(2)}), \end{aligned} \tag{97}$$

where, for $i = 1, 2$,

$$\sup_{n\ge n_1(m)} |\varphi_{n,m}^{(i)}| \to 0 \quad \text{as } m \to \infty.$$

Further, according to Theorem 3,

$$c_n \mathbf{P}(S_n \in [\varepsilon_m c_n, \varepsilon_m c_n + 1) | \tau^- > n) = p_{\alpha,\beta}(\varepsilon_m) + \varphi_{n,m}, \tag{98}$$

where $\varphi_{n,m} = \varphi_{n,m}(\varepsilon_m) \to 0$ as $n \to \infty$ uniformly for all possible choices of ε_m, that is,

$$\sup_{\{\varepsilon_m\}} |\varphi_{n,m}| \le \Phi_n \quad \text{and} \quad \lim_{n\to\infty} \Phi_n = 0. \tag{99}$$

Comparing (97) and (98) gives

$$p_{\alpha,\beta}(\varepsilon_m) = g_{\alpha,\beta}(0) \frac{\varepsilon_m^{\alpha\rho} H(c_{n(m)})}{n(m)\mathbf{P}\left(\tau^- > n(m)\right)} (1 + \varphi_{n(m),m}^{(2)}) - \varphi_{n(m),m}, \tag{100}$$

where $n(m)$ is any sequence satisfying $n(m) \geq n_1(m)$ for all $m \geq 1$. Let $n_2(m)$ be defined by the relation

$$n_2(m) := \min\{n \geq 1 : \sup_{k \geq n} \Phi_k < \varepsilon_m^{\alpha\rho+1}\}.$$

Now, if $n(m) \geq \max\{n_1(m), n_2(m)\}$, then from the definition of $n_2(m)$ and (100) we have

$$p_{\alpha,\beta}(\varepsilon_m) = g_{\alpha,\beta}(0)\varepsilon_m^{\alpha\rho} \frac{H(c_{n(m)})}{n(m)\mathbf{P}(\tau^- > n(m))}(1 + \varphi^{(2)}_{n(m),m}) + O(\varepsilon_m^{\alpha\rho+1}).$$

Taking into account (31), we obtain (96). The theorem is proved.

5 Proof of Theorem 8

We start with the following technical lemma which may be known from the literature.

Lemma 24 *Let $w(n)$ be a monotone increasing function. If, for some $\gamma > 0$, there exist slowly varying functions $l^*(n)$ and $l^{**}(n)$ such that, as $n \to \infty$,*

$$\sum_{k=n}^{\infty} \frac{w(k)}{k^{\gamma+1} l^*(k)} \sim \frac{1}{n^\gamma l^{**}(n)},$$

then, as $n \to \infty$,

$$w(n) \sim \gamma \frac{l^*(n)}{l^{**}(n)}.$$

Proof Let, for this lemma only, $r_i(n), n = 1, 2, \ldots; i = 1, 2, 3, 4$ be sequences of real numbers vanishing as $n \to \infty$. For $\delta \in (0, 1)$ we have by monotonicity of $w(n)$ and properties of slowly varying functions

$$\begin{aligned}
w([\delta n]) \sum_{k=[\delta n]}^{n} \frac{1}{k^{\gamma+1} l^*(k)} &= w([\delta n]) \frac{1 + r_2(n)}{\gamma n^\gamma l^*(n)} \left(\delta^{-\gamma} - 1\right) \\
&\leq \sum_{k=[\delta n]}^{n} \frac{w(k)}{k^{\gamma+1} l^*(k)} = \frac{1 + r_1(n)}{n^\gamma l^{**}(n)} \left(\delta^{-\gamma} - 1\right) \\
&\leq w(n) \sum_{k=[\delta n]}^{n} \frac{1}{k^{\gamma+1} l^*(k)} \\
&= w(n) \frac{1 + r_2(n)}{\gamma n^\gamma l^*(n)} \left(\delta^{-\gamma} - 1\right).
\end{aligned}$$

Hence it follows that

$$w([\delta n]) \leq \frac{1 + r_1(n)}{1 + r_2(n)} \frac{\gamma l^*(n)}{l^{**}(n)} \leq w(n)$$

and, therefore,

$$\frac{1+r_1(n)}{1+r_2(n)}\frac{\gamma l^*(n)}{l^{**}(n)} \le w(n) \le \frac{1+r_3([n\delta^{-1}])}{1+r_4([n\delta^{-1}])}\frac{\gamma l^*([n\delta^{-1}])}{l^{**}([n\delta^{-1}])}.$$

Since l^* and l^{**} are slowly varying functions, we get

$$\lim_{n\to\infty}\frac{w(n)l^{**}(n)}{\gamma l^*(n)} = 1,$$

as desired. □

Remark 25 By the same arguments one can show that if $w(x)$ is a monotone increasing function and, for some $\gamma > 0$, there exist slowly varying functions $l^*(x)$ and $l^{**}(x)$ such that, as $x \to \infty$,

$$\int_x^\infty \frac{w(y)dy}{y^{\gamma+1}l^*(y)} \sim \frac{1}{x^\gamma l^{**}(x)},$$

then, as $x \to \infty$,

$$w(x) \sim \gamma\frac{l^*(x)}{l^{**}(x)}.$$

Note also that this statement for the case $l^*(x) \equiv \text{const}$ can be found in [15, Chap. VIII, Sect. 9].

5.1 Proof of Theorem 8 for $\{0 < \alpha < 2,\ \beta < 1\}$

For a fixed $\varepsilon \in (0, 1)$ write

$$\mathbf{P}\left(\tau^- = n\right) = \mathbf{P}\left(S_n \le 0; \tau^- > n-1\right) =: J_1(\varepsilon c_n) + J_2(\varepsilon c_n)$$

where

$$J_1(\varepsilon c_n) := \int_\varepsilon^\infty \mathbf{P}\left(X \le -yc_n\right)\mathbf{P}\left(S_{n-1} \in c_n dy; \tau^- > n-1\right).$$

and

$$J_2(\varepsilon c_n) := \int_0^{\varepsilon c_n} \mathbf{P}\left(X \le -y\right)\mathbf{P}\left(S_{n-1} \in dy; \tau^- > n-1\right).$$

First we study properties of $J_1(\varepsilon c_n)$.

We know from (24) and (25) that if $X \in \mathcal{D}(\alpha, \beta)$ with $0 < \alpha < 2$ and $\beta < 1$, then, for a $q \in (0, 1]$,

$$\mathbf{P}\left(X \le -y\right) \sim \frac{q}{y^\alpha l_0(y)} \quad \text{as } y \to \infty, \tag{101}$$

and, according to (27),

$$\mathbf{P}(X \le -c_n) \sim \frac{q(2-\alpha)}{\alpha n} \quad \text{as } n \to \infty.$$

Moreover, for any $\varepsilon > 0$,

$$\frac{\mathbf{P}(X \le -yc_n)}{\mathbf{P}(X \le -c_n)} \to y^{-\alpha} \quad \text{as } n \to \infty, \tag{102}$$

uniformly in $y \in (\varepsilon, \infty)$.

It easily follows from (102) and (4) that, as $n \to \infty$,

$$\begin{aligned} J_1(\varepsilon c_n) &= \mathbf{P}(X \le -c_n)\,\mathbf{P}\left(\tau^- > n-1\right) \int_\varepsilon^\infty \frac{\mathbf{P}(X \le -yc_n)}{\mathbf{P}(X \le -c_n)} \\ &\quad \times \mathbf{P}\left(\frac{S_{n-1}}{c_n} \in dy \,|\, \tau^- > n-1\right) \\ &\sim \frac{q(2-\alpha)l(n)}{\alpha n^{2-\rho}} \int_\varepsilon^\infty \frac{\mathbf{P}(X \le -yc_n)}{\mathbf{P}(X \le -c_n)} \mathbf{P}\left(\frac{S_{n-1}}{c_n} \in dy \,|\, \tau^- > n-1\right) \\ &\sim \frac{q(2-\alpha)l(n)}{\alpha n^{2-\rho}} \int_\varepsilon^\infty \frac{\mathbf{P}\left(M_{\alpha,\beta} \in dy\right)}{y^\alpha}. \end{aligned} \tag{103}$$

From Theorem 7 it follows that $p_{\alpha,\beta}(y) \le Cy^{\alpha\rho}$ for some positive constant C and all $y \in (0, 1]$. Consequently,

$$\int_0^\infty \frac{\mathbf{P}\left(M_{\alpha,\beta} \in dy\right)}{y^\alpha} \le C \int_0^1 y^{-\alpha+\alpha\rho} dy + \mathbf{P}\left(M_{\alpha,\beta} > 1\right).$$

Noting that the condition $\beta < 1$ implies the bound $-\alpha + \alpha\rho > -1$, we conclude that

$$\int_0^\infty \frac{\mathbf{P}\left(M_{\alpha,\beta} \in dy\right)}{y^\alpha} < \infty.$$

Therefore,

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{\alpha n^{2-\rho}}{q(2-\alpha)l(n)} J_1(\varepsilon c_n) = \int_0^\infty \frac{\mathbf{P}\left(M_{\alpha,\beta} \in dy\right)}{y^\alpha}. \tag{104}$$

Now to complete the proof of Theorem 8 in the case $\{0 < \alpha < 2, \beta < 1\}$ it remains to demonstrate that

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{n^{2-\rho}}{l(n)} J_2(\varepsilon c_n) = 0. \tag{105}$$

To this aim we observe that

$$J_2(\varepsilon c_n) \le \sum_{j=0}^{[\varepsilon c_n]+1} \mathbf{P}(X \le -j)\, b_{n-1}(j) =: R(\varepsilon c_n)$$

and evaluate $R(\varepsilon c_n)$ separately for the following two cases:

(i) $\beta \in (-1, 1)$;
(ii) $\beta = -1$.

(i). In view of (49), equivalences (29) and (24), we have

$$\begin{aligned} R(\varepsilon c_n) &\le C \sum_{j=1}^{[\varepsilon c_n]+1} \frac{1}{j^{\alpha} l_0(j)} \frac{j^{\alpha\rho} l_2(j)}{n c_n} \le C_2 \frac{1}{n c_n} (\varepsilon c_n)^{1-\alpha(1-\rho)} \frac{l_2(\varepsilon c_n)}{l_0(\varepsilon c_n)} \\ &\le C_3 \frac{1}{n c_n} \varepsilon^{1-\alpha(1-\rho)-\gamma} c_n^{1-\alpha(1-\rho)} \frac{l_2(c_n)}{l_0(c_n)} \le C_4 \varepsilon^{1-\alpha(1-\rho)-\gamma} \frac{H(c_n)\mathbf{P}(|X| > c_n)}{n} \end{aligned}$$

for any fixed $\gamma \in (0, 1 - \alpha(1 - \rho))$ and all sufficiently large n. At the third step we have applied (61) to the function $l_2(x)/l_0(x)$. Using (27) and (31), we get

$$R(\varepsilon c_n) \le C \varepsilon^{1-\alpha(1-\rho)-\gamma} \frac{\mathbf{P}(\tau^- > n)}{n}.$$

Hence on account of (15) we conclude that

$$R(\varepsilon c_n) \le C \frac{l(n)}{n^{2-\rho}} \varepsilon^{1-\alpha(1-\rho)-\gamma}. \tag{106}$$

(ii). It follows from (14) that if $\beta = -1$, then $\alpha\rho = 1$. By Lemma 13, $H(x) \le C x l_3(x)$. Combining this estimate with (49) yields

$$b_n(j) \le C \frac{j l_3(j)}{n c_n}.$$

Recalling (101) and using (61), we obtain for any fixed $\gamma \in (0, 2-\alpha)$ and all $n \geq n(\gamma)$,

$$\begin{aligned} R(\varepsilon c_n) &\leq C \sum_{j=0}^{[\varepsilon c_n]+1} \mathbf{P}\left(X \leq -j\right) \frac{j l_3(j)}{n c_n} \\ &\leq C_1 \left(\varepsilon c_n\right)^{2-\alpha} \frac{1}{n c_n} \frac{l_3(\varepsilon c_n)}{l_0(\varepsilon c_n)} \\ &\leq C_2 \varepsilon^{2-\alpha-\gamma} \frac{1}{n} \frac{c_n l_3(c_n)}{c_n^{\alpha} l_0(c_n)} \leq C_3 \varepsilon^{2-\alpha-\gamma} \frac{l(n)}{n^{2-\rho}}, \end{aligned} \tag{107}$$

where at the last step we have applied the inequalities $H(c_n) \leq C c_n l_3(c_n) \leq C n^{\rho} l(n)$, following from (30), (31), and (15), and the relation

$$\frac{1}{n} \sim \frac{\alpha}{2-\alpha} \frac{1}{c_n^{\alpha} l_0(c_n)},$$

being a corollary of (26).

Estimates (106) and (107) imply (105). Combining (104) with (105) leads to

$$\mathbf{P}\left(\tau^{-} = n\right) \sim \frac{q(2-\alpha) l(n)}{\alpha n^{2-\rho}} \int_{0}^{\infty} \frac{\mathbf{P}\left(M_{\alpha,\beta} \in dy\right)}{y^{\alpha}} = \frac{q(2-\alpha) l(n)}{\alpha n^{2-\rho}} \mathbf{E}\left(M_{\alpha,\beta}\right)^{-\alpha}. \tag{108}$$

Summation over n gives

$$\mathbf{P}\left(\tau^{-} > n\right) = \sum_{k=n+1}^{\infty} \mathbf{P}\left(\tau^{-} = k\right) \sim \frac{q(2-\alpha)}{\alpha\,(1-\rho)} \frac{l(n)}{n^{1-\rho}} \mathbf{E}\left(M_{\alpha,\beta}\right)^{-\alpha}.$$

Comparing this with (15), we get an interesting identity

$$\mathbf{E}\left(M_{\alpha,\beta}\right)^{-\alpha} = \alpha(1-\rho)/q(2-\alpha), \tag{109}$$

which, in view of (108), completes the proof of Theorem 8 for $0 < \alpha < 2,\ \beta < 1$.

5.2 Proof of Theorem 8 for $\{1 < \alpha < 2, \beta = 1\} \cup \{\alpha = 2, \beta = 0\}$

We consider only the lattice random walks with $a \in (0, 1)$ and $h = 1$. The non-lattice case requires only minor changes. The main reason for the choice of the lattice situation is the fact that only in this case we can get oscillating sequences Q_n^{-}.

By the total probability formula,

$$\mathbf{P}(\tau^{-} = n+1) = \sum_{k > -an} \mathbf{P}(S_n = an + k; \tau^{-} > n) \mathbf{P}(X \leq -an - k). \tag{110}$$

One can easily verify that under the conditions imposed on the distribution of X there exists a sequence $\delta_n \to 0$ such that $\delta_n c_n \to \infty$ and

$$\mathbf{P}(X \le -\delta_n c_n) = o(n^{-1}) \quad \text{as } n \to \infty. \tag{111}$$

Using, as earlier, the notation $\mathcal{G}_n = (-an, -an + \delta_n c_n) \cap \mathbb{Z}$, and combining (110) with (111), we obtain

$$\mathbf{P}(\tau^- = n+1) = \sum_{k \in \mathcal{G}_n} \mathbf{P}(S_n = an + k; \tau^- > n)\mathbf{P}(X \le -an - k) + o\left(\frac{l(n)}{n^{2-\rho}}\right).$$

Let $\{an\}$ be the fractional part of an. By Theorem 6

$$\begin{aligned}\mathbf{P}(\tau^- = n+1) &= \frac{g_{\alpha,\beta}(0) + o(1)}{nc_n} \sum_{k \in \mathcal{G}_n} H(an + k)\mathbf{P}(X \le -an - k) + o\left(\frac{l(n)}{n^{2-\rho}}\right) \\ &= \frac{g_{\alpha,\beta}(0) + o(1)}{nc_n} \sum_{j=0}^{\delta_n c_n} H(\{an\} + j)\mathbf{P}(X \le -\{an\} - j) + o\left(\frac{l(n)}{n^{2-\rho}}\right).\end{aligned} \tag{112}$$

For $z \ge 0$ set

$$\omega(z; n) := \sum_{j=0}^{\delta_n c_n} H(z + j)\mathbf{P}(X \le -z - j), \qquad \omega(n) := \omega(0; n),$$

and using the equality

$$\mathbf{E}(-S_{\tau^-}) = \int_0^\infty H(x)\mathbf{P}(X \le -x)dx \tag{113}$$

(see Doney [10]) consider the "if" part of Theorem 8 under the hypotheses of points (a), (b), and (c) separately.

(a) Condition $E(-S_{\tau^-}) = \infty$ implies

$$\omega(n) \to \infty \quad \text{as } n \to \infty. \tag{114}$$

Since $H(u)$ is a renewal function, there exists a constant C such that

$$H(u + v) - H(u) \le C(v + 1) \quad \text{for all } u, v \ge 0. \tag{115}$$

By (115) and monotonicity of $H(u)$ and $\mathbf{P}(X \le -u)$ we conclude that

$$\omega(\{an\}; n) \le \sum_{j=0}^{\delta_n c_n} H(j+1)\mathbf{P}(X \le -j) \le \omega(n) + C \sum_{j=0}^{\delta_n c_n} \mathbf{P}(X \le -j)$$

and

$$\omega(\{an\}; n) \ge \sum_{j=0}^{\delta_n c_n} H(j)\mathbf{P}(X \le -j-1) \ge \omega(1; n) - C \sum_{j=0}^{\delta_n c_n} \mathbf{P}(X \le -j)$$
$$\ge \omega(n) - C \sum_{j=0}^{\delta_n c_n} \mathbf{P}(X \le -j).$$

From (114) and the fact that $H(x) \to \infty$ as $x \to \infty$ we deduce that

$$\sum_{j=0}^{\varepsilon_n c_n} \mathbf{P}(X \le -j) = o\left(\omega(n)\right) \quad \text{as } n \to \infty.$$

This yields $\omega(\{an\}; n) \sim \omega(n)$ as $n \to \infty$ which, combined with (112), gives

$$\mathbf{P}(\tau^- = n+1) = \frac{g_{\alpha,\beta}(0) + o(1)}{nc_n}\omega(n) + o\left(\frac{l(n)}{n^{2-\rho}}\right), \quad n \to \infty. \tag{116}$$

Summing over $n \ge k$, we get, as $k \to \infty$,

$$\frac{l(k)}{k^{1-\rho}} \sim \mathbf{P}(\tau^- > k) = (g_{\alpha,\beta}(0) + o(1)) \sum_{n=k}^{\infty} \frac{\omega(n)}{nc_n} + o\left(\frac{l(k)}{k^{1-\rho}}\right).$$

We know from (14) that $\rho = 1 - 1/\alpha$ if $\{1 < \alpha < 2, \beta = 1\}$ or $\{\alpha = 2, \beta = 0\}$. Since $\omega(n)$ is non-decreasing and, by (3), c_n is regularly varying with index $1/\alpha$, Lemma 24 implies

$$\frac{\omega(n)}{nc_n} \sim \frac{1-\rho}{g_{\alpha,\beta}(0)} \frac{l(n)}{n^{2-\rho}} \quad \text{as } n \to \infty.$$

Consequently,

$$\mathbf{P}(\tau^- = n) = (1-\rho)\frac{l(n)}{n^{2-\rho}}(1 + o(1)), \quad n \to \infty.$$

This finishes the proof of (20) given $\mathbf{E}(-S_{\tau^-}) = \infty$.

(b) The assumption $\mathbf{E}(-S_{\tau^-}) < \infty$ and relations (29), (30), and (113) imply

$$\sum_{j > \delta_n c_n} H(\{an\} + j)\mathbf{P}(X \le -\{an\} - j) \to 0 \quad \text{as } n \to \infty$$

and, consequently,

$$\omega(\{an\}\,;n) = \Omega(\{an\}) + o(1) \quad \text{as } n \to \infty,$$

where

$$\Omega(\{an\}) := \sum_{j=0}^{\infty} H(\{an\}+j)\mathbf{P}(X \le -\{an\}-j).$$

Combining this representation with (112), observing that $\Omega(\{an\}) < C < \infty$ if $\mathbf{E}(-S_{\tau^-}) < \infty$, and recalling Lemma 14 we see that

$$\mathbf{P}(\tau^- = n+1) = \frac{g_{\alpha,\beta}(0)}{nc_n}\Omega(\{an\}) + o\left(\frac{l(n)}{n^{2-\rho}}\right), \quad n \to \infty. \tag{117}$$

Denote

$$\tilde{\Omega}(\{an\}) := \mathbf{P}(X \le -\{an\})\mathrm{I}(\{an\} > 0) + \mathbf{P}(X \le -1)\mathrm{I}(\{an\} = 0).$$

Since X is $(1,a)$-lattice, the quantity $\tilde{\Omega}(\{an\})$ is either 0 or not less than some positive number $\tilde{\Omega}_*$. Furthermore, one can easily verify that $\Omega(\{an\}) \ge \tilde{\Omega}(\{an\})$ and $\Omega(\{an\}) = 0$ if and only if $\tilde{\Omega}(\{an\}) = 0$. Consequently, $\Omega(\{an\})$ is either zero or not less than $\tilde{\Omega}_*$. Finally, in view of (110) $\tilde{\Omega}(\{an\}) = 0$ implies $\mathbf{P}(\tau^- = n+1) = 0$. Therefore, we can rewrite (117) in the form

$$\mathbf{P}(\tau^- = n+1) = \frac{g_{\alpha,\beta}(0)}{nc_n}\Omega(\{an\})(1+o(1)). \tag{118}$$

Now (118) and (37) give (20) with

$$Q_n^- := C_0 g_{\alpha,\beta}(0)\Omega(\{a(n-1)\}). \tag{119}$$

If $a = 0$, then, evidently,

$$Q_n^- \equiv C_0 g_{\alpha,\beta}(0)\Omega(0) = C_0 g_{\alpha,\beta}(0)\mathbf{E}(-S_{\tau^-}) := Q,$$

and, consequently,

$$\mathbf{P}(\tau^- = n) = Q\frac{l(n)}{n^{2-\rho}}(1+o(1)).$$

Comparing this asymptotic equality with the known tail behavior of the distribution of τ^-, we infer that Q should be equal to $1-\rho$.

This finishes the proof of (20) under the conditions of point **(b)**.

To demonstrate the validity of (20) under the conditions of point **(c)** one should made only evident minor changes of the just used arguments and we omit the respective details.

To justify the "only if" part of Theorem 8 we need to show that the sequence $\left\{Q_n^-, n \ge 1\right\}$ defined in (119) does not converge if $\mathbf{E}(-S_{\tau^-}) < \infty$ and X is $(1,a)$-lattice with some $a \in (0,1)$.

Assume first that a is rational, i.e. $a = i/j$ for some $1 \leq i < j < \infty$ with g.c.d.$(i, j) = 1$. Let $b = b(a)$ be the smallest natural number satisfying $\{ab\} = 1-a$. Then $\{a\,(kj + b)\} = 1 - a$ for all $k \geq 1$. Consequently,

$$\Omega(\{a\,(kj+b)\}) = \sum_{m=0}^{\infty} H((1-a)+m)\mathbf{P}(X \leq -(1-a)-m)$$

and

$$\Omega(\{akj\}) = \sum_{m=0}^{\infty} H(m)\mathbf{P}(X \leq -m).$$

Observing that $\mathbf{P}(X \leq -m) = \mathbf{P}(X \leq -(1-a)-m)$, we obtain

$$\begin{aligned}\Omega(\{a\,(kj+b)\}) - \Omega(\{akj\}) &= \sum_{m=0}^{\infty} (H((1-a)+m) - H(m))\,\mathbf{P}(X \leq -(1-a)-m)\\ &\geq (H(1-a) - H(0))\,\mathbf{P}(X \leq -(1-a))\\ &= H(1-a)\mathbf{P}(X < 0)\\ &> \mathbf{P}(X < 0).\end{aligned}$$

From this inequality it follows that the sequence $\{\Omega(\{an\}), n \geq 1\}$, does not converge.

Assume now that a is irrational. Define $\mathcal{N}_1 := \{n : \{an\} < (1-a)/3\}$ and $\mathcal{N}_2 := \{n : \{an\} \in (2(1-a)/3, (1-a))\}$. The cardinality of each of the sets is infinite. In addition, one can easily verify that

$$\begin{aligned}\Omega(\{an_2\}) - \Omega(\{an_2\}) &\geq (H(2(1-a)/3) - H((1-a)/3))\,\mathbf{P}(X < 0)\\ &\geq \mathbf{P}(X < 0)\mathbf{P}\left(\chi^+ \in ((1-a)/3, 2(1-a)/3)\right) > 0\end{aligned}$$

for all $n_1 \in \mathcal{N}_1$ and $n_2 \in \mathcal{N}_2$. Therefore, in the case of irrational shift the sequence $\Omega(\{an\})$, $n \geq 1$, is oscillating as well.

Theorem 8 is proved.

Remark 26 Analyzing the proof of Theorem 8 one can see that the sequence $\left\{Q_n^-, n \geq 1\right\}$ in (20) may be written in the form

$$Q_n^- := D(\{a(n-1)\}),$$

where $D(x)$, $0 \leq x < 1$, is a non-negative function and where we agree to take $a = 0$ for non-lattice distributions.

6 Discussion and concluding remarks

We see by (11) that the distribution of τ^- is completely specified by the sequence $\{\mathbf{P}\,(S_n > 0)\,, n \geq 1\}$. As we have mentioned in the introduction, the validity of condition (16) is sufficient to reveal the asymptotic behavior of $\mathbf{P}(\tau^- > n)$ as $n \to \infty$.

Thus, in view of (15), informal arguments based on the plausible smoothness of $l(n)$ immediately give the desired answer

$$\begin{aligned}\mathbf{P}(\tau^- = n) &= \mathbf{P}(\tau^- > n-1) - \mathbf{P}(\tau^- > n)\\ &= \frac{l(n-1)}{(n-1)^{1-\rho}} - \frac{l(n)}{n^{1-\rho}} \approx l(n)\left(\frac{1}{(n-1)^{1-\rho}} - \frac{1}{n^{1-\rho}}\right)\\ &\approx \frac{(1-\rho)l(n)}{n^{2-\rho}} \sim \frac{1-\rho}{n}\mathbf{P}(\tau^- > n)\end{aligned}$$

under the Doney condition only. In the present paper we failed to achieve such a generality. However, it is worth mentioning that the Doney condition, being formally weaker than the conditions of Theorem 8, requires in the general case the knowledge of the behavior of the whole sequence $\{\mathbf{P}(S_n > 0), n \geq 1\}$, while the assumptions of Theorem 8 concern a single summand only. Of course, imposing a stronger condition makes our life easier and allows us to give, in a sense, a constructive proof showing what happens in reality at the distant moment τ^- of the first jump of the random walk in question below zero. Indeed, our arguments for the case $\{0 < \alpha < 2, \ \beta < 1\}$ demonstrate (compare (101), (102), and (103)) that for any $x_2 > x_1 > 0$,

$$\begin{aligned}&\lim_{n\to\infty} \mathbf{P}(S_{n-1} \in (c_n x_1, c_n x_2] | \tau^- = n)\\ &= \lim_{n\to\infty} \frac{\mathbf{P}(\tau^- > n-1)}{\mathbf{P}(\tau^- = n)} \int_{x_1}^{x_2} \mathbf{P}(X < -yc_n)\mathbf{P}(S_{n-1} \in c_n dy | \tau^- > n-1)\\ &= \lim_{n\to\infty} \frac{\mathbf{P}(\tau^- > n-1)q(2-\alpha)}{\mathbf{P}(\tau^- = n)\alpha n} \int_{x_1}^{x_2} \frac{\mathbf{P}(X < -yc_n)}{\mathbf{P}(X < -c_n)}\mathbf{P}(S_{n-1} \in c_n dy | \tau^- > n-1)\\ &= \frac{q(2-\alpha)}{\alpha(1-\rho)} \int_{x_1}^{x_2} \frac{\mathbf{P}(M_{\alpha,\beta} \in dy)}{y^\alpha}.\end{aligned}$$

In view of (109) this means that the contribution of the trajectories of the random walk satisfying $S_{n-1}c_n^{-1} \to 0$ or $S_{n-1}c_n^{-1} \to \infty$ as $n \to \infty$ to the event $\{\tau^- = n\}$ is negligibly small in probability. A "typical" trajectory looks in this case as follows: it is located over the level zero up to moment $n-1$ with $S_{n-1} \in (\varepsilon c_n, \varepsilon^{-1} c_n)$ for sufficiently small $\varepsilon > 0$ and at moment $\tau^- = n$ the trajectory makes a big negative jump $X_n < -S_{n-1}$ of order $O(c_n)$.

On the other hand, if $\{1 < \alpha < 2, \ \beta = 1\}$ and $\mathbf{E}(-S_{\tau^-}) < \infty$, then, in the $(1, a)$-lattice case, for all $i \geq 0$,

$$\begin{aligned}&\mathbf{P}(S_{n-1} = \{a(n-1)\} + i | \tau^- = n)\\ &\quad = \frac{H(\{a(n-1)\} + i)\mathbf{P}(X \leq -\{a(n-1)\} - i)}{\Omega(\{a(n-1)\})}(1 + o(1))\end{aligned}$$

provided that $\Omega(\{a(n-1)\}) > 0$. Since

$$\sum_{i=0}^{\infty} H(\{a(n-1)\}+i)\mathbf{P}(X \le -\{a(n-1)\}-i) = \Omega(\{a(n-1)\}),$$

the main contribution to $\mathbf{P}\left(\tau^- = n\right)$ is given in this case by the trajectories located over the level zero up to moment $n-1$ with $S_{n-1} \in [0, N]$ for sufficiently big N and with not "too big" jump $X_n < -S_{n-1}$ of order $O(1)$.

Unfortunately, our approach to investigate the behavior of $\mathbf{P}(\tau^- = n)$ in the case when $\mathbf{E}(-S_{\tau^-}) = \infty$ and $\{1 < \alpha < 2,\ \beta = 1\} \cup \{\alpha = 2, \beta = 0\}$ is pure analytical and does not allow us to extract typical trajectories without further restrictions on the distribution of X. However, we can still deduce from our proof some properties of the random walk conditioned on $\{\tau^- = n\}$. Observe that, for any fixed $\varepsilon > 0$, the trajectories with $S_{n-1} > \varepsilon c_n$ give no essential contribution to $\mathbf{P}(\tau^- = n)$. More precisely, there exists a sequence $\delta_n \to 0$ such that

$$\mathbf{P}(S_{n-1} > \delta_n c_n | \tau^- = n) = o(1).$$

Furthermore, one can easily verify that if $\sum_{j=1}^{\infty} H(j)\mathbf{P}(X \le -j) = \infty$, then for every $N \ge 1$,

$$\sum_{j=1}^{N} \mathbf{P}(S_{n-1} = j; \tau^- > n-1)\mathbf{P}(X \le -j) = o\left(\frac{l(n)}{n^{3/2}}\right) \quad \text{as } n \to \infty,$$

i.e. the contribution of the trajectories with $S_{n-1} = O(1)$ to $\mathbf{P}(\tau^- = n)$ is negligible small. As a result we see that $S_{n-1} \to \infty$ but $S_{n-1} = o(c_n)$ for all "typical" trajectories meeting the condition $\{\tau^- = n\}$. Thus, in the case $\{1 < \alpha < 2, \beta = 1\} \cap \{\mathbf{E}(-S_{\tau^-}) = \infty\}$ we have a kind of "continuous transition" between the different strategies for $\{\beta < 1\}$ and $\{1 < \alpha < 2, \beta = 1\} \cap \{\mathbf{E}(-S_{\tau^-}) < \infty\}$.

Acknowledgments The first version of the paper was based on the preprint [23]. We are thankful to an anonymous referee who attracted our attention to the fact that by our methods one can prove not only local Theorems 8 and 11 but the Gnedenko and Stone type conditional local Theorems 3–6 as well. V.W. is thankful to Anatoly Mogulskii for simulating discussions on ladder epochs. This project was started during the visits of the first author to the Weierstrass Institute in Berlin and the second author to the Steklov Mathematical Institute in Moscow. The hospitality of the both institutes is greatly acknowledged.

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