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# TRANSITION PHENOMENA FOR LADDER EPOCHS OF RANDOM WALKS WITH SMALL NEGATIVE DRIFT

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## Abstract

For a family of random walks  $\{S^{(a)}\}$  satisfying  $\mathbb{E} S_1^{(a)} = -a < 0$ , we consider ladder epochs  $\tau^{(a)} = \min\{k \geq 1 : S_k^{(a)} < 0\}$ . We study the asymptotic behaviour, as  $a \rightarrow 0$ , of  $\mathbb{P}(\tau^{(a)} > n)$  in the case when  $n = n(a) \rightarrow \infty$ . As a consequence, we also obtain the growth rates of the moments of  $\tau^{(a)}$ .

*Keywords:* Random walk; ladder epoch; transition phenomena

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## 1. Introduction and statement of results

### 1.1. Background and purpose

Let  $X, X_1, X_2, \dots$  be independent, identically distributed random variables. Let  $S = \{S_n, n \geq 0\}$  denote the random walk with increments  $X_i$ , that is,

$$S_0 := 0, \quad S_n := \sum_{i=1}^n X_i.$$

Let us first recall what is known about the first descending ladder epoch  $\tau$  of  $S$ , i.e.

$$\tau := \min\{k \geq 1 : S_k < 0\}. \quad (1)$$

It is well known (see, for example, [17, Theorem 17.1]) that

$$\mathbb{P}(\tau < \infty) = 1 \iff \sum_{k=1}^{\infty} k^{-1} \mathbb{P}(S_k < 0) = \infty.$$

Under the latter condition, Rogozin [15] studied the asymptotic behaviour, as  $n \rightarrow \infty$ , of the tail probability  $\mathbb{P}(\tau > n)$ . In particular,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \mathbb{P}(S_k \geq 0) = \rho \in (0, 1] \iff \mathbb{P}(\tau > n) = n^{\rho-1} \ell(n), \quad (2)$$

where  $\ell$  is slowly varying at infinity. Also,  $\lim_{n \rightarrow \infty} (1/n) \sum_{k=1}^n \mathbb{P}(S_k \geq 0) = 0$  is equivalent to the relative stability of  $\tau$ . The latter means that the function  $x \mapsto \int_0^x \mathbb{P}(\tau > u) du$  is slowly

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varying at infinity. But this statement does not give any information about the asymptotic behaviour of  $P(\tau > n)$  in this case.

The situation when  $E\tau < \infty$ , which is a particular case of the relative stability, was considered by Embrechts and Hawkes [5]. There it was shown that

$$P(\tau > n) \sim n^{-1} P(S_n > 0) \exp \left\{ \sum_{j=1}^{\infty} j^{-1} P(S_j \geq 0) \right\},$$

under certain conditions on the sequence  $\{P(S_n > 0), n \geq 1\}$ .

If the expectation  $EX$  is finite then the condition  $\sum_{k=1}^{\infty} k^{-1} P(S_k \leq 0) = \infty$  is equivalent to the inequality  $EX \leq 0$ ; see again [17, Theorem 17.1]. If  $EX = 0$  and  $X$  belongs to the domain of attraction of a stable law of index  $\alpha > 1$ , then  $\lim_{n \rightarrow \infty} P(S_n \geq 0) \in (0, 1)$ . (For details, see the paragraph after (11).) This yields  $\lim_{n \rightarrow \infty} (1/n) \sum_{k=1}^n P(S_k \geq 0) = \rho \in (0, 1)$ . Then, using (2), we conclude that

$$P(\tau > n) = n^{\rho-1} \ell(n). \quad (3)$$

If  $EX < 0$  then  $E\tau$  is finite; see [17, Proposition 18.1]. In this case of negative drift, Doney [4] applied the results from [5] to two special classes of random walks. He showed that if  $EX \in (-\infty, 0)$  and  $P(X > x)$  is regularly varying at  $\infty$  with index  $\alpha < -1$ , then, as  $n \rightarrow \infty$ ,

$$P(\tau > n) \sim E\tau P(X > -nEX) \quad \text{as } n \rightarrow \infty. \quad (4)$$

Besides this regularly varying tail case, Doney found the asymptotics of  $P(\tau > n)$  for random walks having negative drift and satisfying the following condition. If the equation  $(d/dh) Ee^{hX} = 0$  has a positive solution, say  $h_0$ , then

$$P(\tau > n) \sim C \left( \frac{E\mu^\tau - 1}{\mu - 1} \right) \mu^{-n} n^{-3/2} \quad \text{as } n \rightarrow \infty, \quad (5)$$

where  $\mu = 1/Ee^{h_0X}$  and  $C$  is a constant depending on  $Ee^{hX}$ . The latter relation was generalised by Bertoin and Doney [2] to the case where  $(d/dh) Ee^{hX} < 0$  for all  $h > 0$  such that  $Ee^{hX} < \infty$ .

It should be noted that [2] and [4] are devoted to the study of the asymptotic behaviour of  $P(\tau_x > n)$  for any fixed  $x \geq 0$ , where  $\tau_x := \min\{k \geq 1: S_k < -x\}$ . The main result can be stated as follows. If  $X$  satisfies the conditions stated before (4) or (5), then there exists a function  $U$  such that

$$\lim_{n \rightarrow \infty} \frac{P(\tau_x > n)}{P(\tau > n)} \rightarrow U(x).$$

By studying the asymptotic behaviour, as  $n \rightarrow \infty$ , of  $P(\tau > n)$ , we hope to get a good approximation for large but finite values of  $n$ . The quality of such an approximation depends on different parameters of the random walk. It follows from the papers mentioned above that the asymptotic behaviour of  $P(\tau > n)$  depends crucially on whether  $EX = 0$  or  $EX < 0$ . Therefore, it would be very useful to clarify the influence of  $EX$  on  $P(\tau > n)$  in the case when that expectation is quite small. We illustrate the problem with the following concrete example. Let  $S$  be a random walk with  $EX = 10^{-3}$ ; we want to calculate the quantity  $P(\tau > 10^5)$ . Here we have two possibilities. On the one hand, we can say that the expectation is so small that we may apply asymptotic relations for zero-mean random walks. On the other hand, we can say that the expectation is negative and we should use (4) or (5), depending on the tail behaviour of  $X$ . But how do we decide which approximation is better for these values of  $EX$  and  $n$ ? This

question leads to the following mathematical problem. What can be said about the asymptotic behaviour of  $P(\tau > n)$  in the case when  $E X \rightarrow 0$  and  $n \rightarrow \infty$  simultaneously?

In the present paper we consider this problem in the case when the random walk's increment belongs to the domain of attraction of a stable law. We shall show that there exists a function  $f$  such that

- (a) if  $n \ll f(E X)$  then we have to use (3),
- (b) if  $n \gg f(E X)$  then we have to use formulae for random walks with negative drift,
- (c) if  $n \sim v f(E X)$ ,  $v \in (0, \infty)$ , then we have to use (3), but with a correction factor depending on  $v$ .

The last point seems to be the most interesting one. It describes *transition phenomena* for the ladder epoch  $\tau$ , which appear in the case of small drift.

Our main result, Theorem 1, is devoted to the study of this transition. There it will be clarified what the function  $f$  and the correction factor look like. As a consequence, we shall obtain the claim in (a). Furthermore, Theorem 1 allows us to determine the asymptotic behaviour, as  $E X \rightarrow 0$ , of some moments  $E \tau^r$ ; see Theorem 2. The expectation  $E \tau$  is of particular interest, since it appears in asymptotic relations connected to the claim in (b); see Theorems 3, 4, and 5.

## 1.2. Transition phenomena

We start with a more precise description of our model of random walks with asymptotically small drift. We shall consider a family of random walks  $\{S^{(a)}, a \in [0, a_0]\}$  with drift  $-a$ , that is,  $E S_1^{(a)} = -a$ , and investigate the asymptotic behaviour, as  $a \rightarrow 0$ , of the probability  $P(\tau^{(a)} > n)$  for  $n = n(a)$ , where  $\tau^{(a)}$  is the first descending ladder epoch of  $S^{(a)}$ , as in (1).

Let  $X^{(a)}$  denote a random variable that is distributed as the increments of the random walk  $S^{(a)}$ . It is easy to see that if  $X^{(a)}$  converges in distribution, as  $a \rightarrow 0$ , to  $X^{(0)}$  then, for every fixed  $n$ ,

$$P(\tau^{(a)} > n) \sim P(\tau^{(0)} > n) \quad \text{as } a \rightarrow 0. \quad (6)$$

A more interesting problem consists in investigating the asymptotic behaviour of the tail probability  $P(\tau^{(a)} > n)$  when  $n = n(a) \rightarrow \infty$  as  $a \rightarrow 0$ . The solution to this problem depends on the structure of the family  $\{S^{(a)}, a \in [0, a_0]\}$ .

In this paper we shall assume that there exists a random variable  $X$  with zero mean such that the random variables  $X^{(a)}$  and  $X - a$  have the same distribution for all  $a \in [0, a_0]$ . Then the random variables  $S_n^{(a)}$  and  $S_n^{(0)} - na$  are equal in distribution for all  $a \in [0, a_0]$  and  $n \geq 1$ . Furthermore, we restrict ourselves from now on to so-called *asymptotically stable random walks*. Namely, we shall always assume that the distribution of  $X$  belongs to the domain of attraction of a stable law with characteristic function

$$G_{\alpha, \beta}(t) := \exp \left\{ -|t|^\alpha \left( 1 - i\beta \frac{t}{|t|} \tan \frac{\pi\alpha}{2} \right) \right\} \quad (7)$$

with  $\alpha \in (1, 2]$  and  $|\beta| \leq 1$ . In this case we write  $X \in \mathcal{D}(\alpha, \beta)$ .

Let  $\{c_n, n \geq 1\}$  denote the sequence of positive integers specified by the relation

$$c_n := \inf \{u \geq 0 : u^{-2} V(u) \leq n^{-1}\}, \quad (8)$$

where

$$V(u) := \int_{-u}^u x^2 P(X \in dx), \quad u > 0.$$

It is known (see, for instance, [6, Chapter XVII, Section 5]) that the function  $V$  is regularly varying at infinity with index  $2 - \alpha$  for every  $X \in \mathcal{D}(\alpha, \beta)$ . This implies that  $\{c_n, n \geq 1\}$  is regularly varying with index  $\alpha^{-1}$ , i.e. there exists a function  $l_1$ , slowly varying at infinity, such that

$$c_n = n^{1/\alpha} l_1(n). \quad (9)$$

In addition, the scaled sequence  $\{S_n^{(0)}/c_n, n \geq 1\}$  converges in distribution, as  $n \rightarrow \infty$ , to the stable law corresponding to  $G_{\alpha, \beta}$  in (7).

Let  $\{Y_{\alpha, \beta}(t), t \geq 0\}$  denote a stable Lévy process such that  $Y_{\alpha, \beta}(1)$  is distributed according to (7).

It is known (see Proposition 17.5 of [17]) that the generating function of the sequence  $\{P(\tau^{(a)} > n), n \geq 0\}$  satisfies the identity

$$\sum_{n=0}^{\infty} P(\tau^{(a)} > n) z^n = \exp \left\{ \sum_{n=1}^{\infty} \frac{z^n}{n} P(S_n^{(a)} \geq 0) \right\}, \quad z \in (0, 1). \quad (10)$$

Thus, for every  $n \geq 1$ , the probability  $P(\tau^{(a)} > n)$  is determined by  $\{P(S_k^{(a)} \geq 0), 1 \leq k \leq n\}$ . From the definition of the family  $S^{(a)}$  and from the asymptotic stability of  $\{S_n^{(0)}, n \geq 0\}$ , we conclude that

$$P(S_n^{(a)} \geq 0) \sim P(S_n^{(0)} \geq 0) \sim P(Y_{\alpha, \beta}(1) \geq 0) =: \rho \quad (11)$$

for  $n = n(a) \rightarrow \infty$  satisfying  $na/c_n \rightarrow 0$ . It is known (see [18]) that

$$\rho = \frac{1}{2} + \frac{1}{\pi\alpha} \arctan \left( \beta \tan \frac{\pi\alpha}{2} \right)$$

for all  $\alpha \in (1, 2]$  and  $|\beta| \leq 1$ . We can easily verify that  $\rho \in (0, 1)$  for all  $\alpha \in (1, 2]$  and  $|\beta| \leq 1$ . Hence, we can expect that

$$P(\tau^{(a)} > n) \sim P(\tau^{(0)} > n) = n^{\rho-1} \ell(n), \quad (12)$$

where in the second step we have used (2). Furthermore, if  $na/c_n \rightarrow u \in (0, \infty)$  then

$$P(S_n^{(a)} \geq 0) \sim P(Y_{\alpha, \beta}(1) \geq u) > 0.$$

In this case we expect, although this conjecture is not as obvious as (12), that

$$P(\tau^{(a)} > n) \sim P(\tau^{(0)} > n) G(u) \quad (13)$$

for some function  $G$ .

The following theorem confirms conjectures (12) and (13).

**Theorem 1.** *Suppose that  $X \in \mathcal{D}(\alpha, \beta)$ . If  $n = n(a)$  is such that*

$$\lim_{a \rightarrow 0} \frac{an}{c_n} = u \in [0, \infty) \quad (14)$$

*then*

$$\lim_{a \rightarrow 0} \frac{P(\tau^{(a)} > n)}{P(\tau^{(0)} > n)} = 1 - F_{\alpha, \beta}(u), \quad (15)$$

where the distribution function  $F_{\alpha,\beta}$  can be described by the equality

$$\begin{aligned} & \int_0^\infty e^{-\lambda x} x^{\rho-1} (1 - F_{\alpha,\beta}(x^{1-1/\alpha})) dx \\ &= C \exp \left\{ - \int_0^\infty \frac{1 - e^{-\lambda t}}{t} P(Y_{\alpha,\beta}(t) - t > 0) dt \right\}, \quad \lambda \geq 0 \end{aligned} \quad (16)$$

with  $\rho$  defined as in (11) and  $C$  specified by the condition  $F_{\alpha,\beta}(0) = 0$ .

The existence of the limit in (15) is an easy consequence of the invariance principle for random walks conditioned to stay positive, which was proved in [3]. The most difficult part of the proof is the derivation of characterisation (16) of the limiting distribution  $F_{\alpha,\beta}$ ; see Section 3.

It follows from (9) that (14) is equivalent to

$$n \sim u^{\alpha/(\alpha-1)} \left( \frac{1}{a} \right)^{\alpha/(\alpha-1)} l^* \left( \frac{1}{a} \right) \quad \text{as } a \rightarrow 0,$$

where  $l^*$  is slowly varying at infinity, which is determined by  $l_1$ . Therefore, the statement of Theorem 1 can be reformulated as follows. If  $n = n(a)$  satisfies

$$n \sim v \left( \frac{1}{a} \right)^{\alpha/(\alpha-1)} l^* \left( \frac{1}{a} \right) \quad \text{as } a \rightarrow 0 \quad (17)$$

for some  $v \geq 0$  then

$$\lim_{a \rightarrow 0} \frac{P(\tau^{(a)} > n)}{P(\tau^{(0)} > n)} = 1 - F_{\alpha,\beta}(v^{1-1/\alpha}). \quad (18)$$

In particular, if (17) holds with  $v = 0$  then  $P(\tau^{(a)} > n) \sim P(\tau^{(0)} > n)$ . Roughly speaking, (3) gives a rather good approximation in the case when  $n$  is much smaller than  $(1/a)^{\alpha/(\alpha-1)} l^*(1/a)$ . But if  $(1/a)^{\alpha/(\alpha-1)} l^*(1/a)$  and  $n$  are comparable, then we have to use a correction factor, given by the right-hand side of (18). To calculate this correction for concrete values of  $v$ , we need to know the form of the distribution function  $F_{\alpha,\beta}$ . We are able to give an explicit expression for  $F_{\alpha,\beta}$  only in some special cases. We shall see in the proof of Theorem 1 that

$$1 - F_{\alpha,\beta}(u) = P\left(\inf_{t \leq 1} (M_{\alpha,\beta}(t) - ut) \geq 0\right),$$

where  $\{M_{\alpha,\beta}(t), t \in [0, 1]\}$  is the meander of  $Y_{\alpha,\beta}$ . Using the construction of the meander via the limit of conditioned distributions of the original process  $Y_{\alpha,\beta}$ , we shall show that

$$1 - F_{2,0}(u) = u \int_u^\infty v^{-2} e^{-v^2/2} dv$$

and

$$1 - F_{\alpha,1}(u) = \frac{u^{1/(\alpha-1)}}{(\alpha-1)g_{\alpha,1}(0)} \int_u^\infty v^{-\alpha/(\alpha-1)} g_{\alpha,1}(v) dv, \quad \alpha \in (1, 2),$$

where  $g_{\alpha,\beta}$  denotes the density function of the random variable  $Y_{\alpha,\beta}(1)$ . For all other values of  $\alpha$  and  $\beta$ , the explicit form of  $F_{\alpha,\beta}$  remains unknown.

**Remark 1.** The expression on the right-hand side of (16) is known (see [1, p. 168]) to be the Laplace transform of the random variable

$$T_{\max} := \sup \left\{ t > 0 : Y_{\alpha, \beta}(t) - t = \max_{u \geq 0} (Y_{\alpha, \beta}(u) - u) \right\}.$$

Let  $f_{\max}$  denote the density function of this random variable. Then from (16) we can obtain the equality

$$1 - F_{\alpha, \beta}(x) = C x^{\alpha(1-\rho)/(\alpha-1)} f_{\max}(x^{\alpha/(\alpha-1)}), \quad x > 0.$$

Having this relation we can obtain the explicit form of  $f_{\max}$  in the case of Brownian motion ( $\alpha = 2$  and  $\beta = 0$ ) and in the case of spectrally positive Lévy processes ( $\alpha \in (1, 2)$  and  $\beta = 1$ ).

We now turn our attention to the moments of  $\tau^{(a)}$ .

It was shown by Gut [7] that the condition  $E(\max\{0, X\})^r < \infty$  for some  $r > 0$  is necessary and sufficient for the finiteness of  $E(\tau^{(a)})^r$ . Therefore, the condition  $X \in \mathcal{D}(\alpha, \beta)$  yields the finiteness of  $E(\tau^{(a)})^r$  for all  $r < \alpha$ .

From the bound

$$P(\tau^{(a)} > n) \leq P(\tau^{(0)} > n) \quad \text{for all } n \geq 0$$

and (6), using dominated convergence, we infer that

$$\lim_{a \rightarrow 0} E(\tau^{(a)})^r = E(\tau^{(0)})^r < \infty$$

for all  $r \in (0, 1 - \rho)$ . Furthermore, it easily follows from Theorem 1 and (12) that

$$\lim_{a \rightarrow 0} E(\tau^{(a)})^r = \infty \quad \text{for all } r > 1 - \rho.$$

Theorem 1 allows us to determine the rate of growth as  $a \rightarrow 0$  of  $E(\tau^{(a)})^r$  for  $r \in (1 - \rho, \alpha)$ .

**Theorem 2.** Suppose that  $X \in \mathcal{D}(\alpha, \beta)$ . Then, for every  $r \in (1 - \rho, \alpha)$ , there exists a function  $L_r$  slowly varying at infinity such that

$$E(\tau^{(a)})^r = L_r\left(\frac{1}{a}\right) a^{-\alpha(r+\rho-1)/(\alpha-1)}, \quad (19)$$

with  $\rho$  defined as in (11).

This is already known in some particular cases, which we now mention.

First of all we note that if the second moment of  $X$  is finite then, applying dominated convergence, we can show that  $E S_{\tau^{(a)}}^{(0)} \rightarrow E S_{\tau^{(0)}}^{(0)}$  as  $a \rightarrow 0$ . Thus, using the Wald identity and the well-known equality (see [17, Proposition 18.5])

$$-E S_{\tau^{(0)}}^{(0)} = \frac{(E X^2)^{1/2}}{\sqrt{2}} \exp \left\{ \sum_{k=1}^{\infty} k^{-1} \left( P(S_k^{(0)} \geq 0) - \frac{1}{2} \right) \right\},$$

we obtain, as  $a \rightarrow 0$ ,

$$E \tau^{(a)} \sim \frac{-E S_{\tau^{(0)}}^{(0)}}{a} = \frac{(E X^2)^{1/2}}{a\sqrt{2}} \exp \left\{ \sum_{k=1}^{\infty} k^{-1} \left( P(S_k^{(0)} \geq 0) - \frac{1}{2} \right) \right\}.$$

Furthermore, the asymptotic behaviour of  $E \tau^{(a)}$  in the case of a non-Gaussian stable limit, that is,  $\alpha < 2$ , was recently studied by Lotov [9]. He proved that

$$E \tau^{(a)} = a^{-\alpha\rho/(\alpha-1)+o(1)} \quad \text{as } a \downarrow 0$$

in this case. Moreover, he showed that (19) with  $r = 1$  holds under the additional condition that

$$\sum_{k=1}^{\infty} \frac{1}{k} \sup_{x \in \mathbb{R}} |P(S_k^{(0)} > c_k x) - P(Y_{\alpha,\beta} > x)| < \infty.$$

Having expressions for the expectation  $E \tau^{(a)}$  we can describe the asymptotic behaviour of some further characteristics of the random walk  $\{S_n^{(a)}, n \geq 0\}$ . First, from the Wald identity and Theorem 2, we obtain the equality

$$E S_{\tau^{(a)}}^{(a)} = -a E \tau^{(a)} = -L_1 \left( \frac{1}{a} \right) a^{1-\alpha\rho/(\alpha-1)}.$$

Second, it is well known that the stopping time  $\tau_+^{(a)} := \min\{k \geq 1 : S_k^{(a)} \geq 0\}$  is infinite with positive probability and that  $P(\tau_+^{(a)} = \infty) = 1/E \tau^{(a)}$ . Then, using Theorem 2 once again, we obtain

$$P(\tau_+^{(a)} = \infty) = \frac{a^{\alpha\rho/(\alpha-1)}}{L_1(1/a)}.$$

To conclude this subsection, we note that our assumption that the distributions of  $X^{(a)}$  and  $X - a$  are equal can be weakened. First of all we note that if  $X^{(a)}$  satisfies the conditions

$$E X^{(a)} = -a \quad \text{and} \quad \lim_{a \rightarrow 0} E(X^{(a)})^2 = \sigma^2 \in (0, \infty),$$

then the results of the present subsection still hold. Moreover, in the case of an infinite second moment, the results of the present subsection remain valid if  $X^{(a)} = X - a + Y^{(a)}$  in distribution, where  $X \in \mathcal{D}(\alpha, \beta)$  for some  $\alpha \in (1, 2)$  and  $Y^{(a)}$  is such that

$$E Y^{(a)} = 0, \quad Y^{(a)} \rightarrow 0 \text{ in law and } \sup_{a \in [0, a_0]} E |Y^{(a)}|^{\alpha+\delta} < \infty \text{ for some } \delta > 0.$$

We did not use these generalisations in the statements of our theorems because of results in the next subsection, where we need the assumption that  $X^{(a)} = X - a$  in law.

### 1.3. Results on large deviations

If  $na/c_n \rightarrow \infty$  then Theorem 1 says only that

$$P(\tau^{(a)} > n) = o(P(\tau^{(0)} > n)) \quad \text{as } a \rightarrow 0.$$

Our next purpose is to refine this relation and to find the rate of divergence of  $P(\tau^{(a)} > n)$  in the abovementioned domain of *large deviations* for  $\tau^{(a)}$ . To proceed in this situation, we need to know the asymptotic behaviour of  $P(S_n^{(a)} > 0)$  for  $na/c_n \rightarrow \infty$ . It follows from the definition of  $S_n^{(a)}$  that  $P(S_n^{(a)} > 0) = P(S_n^{(0)} > na)$ . Thus, the assumption that  $na/c_n \rightarrow \infty$  means that we are in the domain of large deviations for  $S_n^{(0)}$ . Since the behaviour of large deviation probabilities depends crucially on whether the limit of  $S_n^{(0)}/c_n$  is Gaussian or strictly stable, i.e.  $\alpha \in (1, 2)$ , we consider these two cases separately.



If  $S_n^{(0)}$  belongs to the domain of attraction of a strictly stable law then, as is well known,

$$P(S_n^{(0)} \geq x_n) \sim n P(X \geq x_n)$$

for any sequence  $x_n$  satisfying  $x_n/c_n \rightarrow \infty$ . This relation allows us to obtain the following result.

**Theorem 3.** *Suppose that  $X \in \mathcal{D}(\alpha, \beta)$  for some  $1 < \alpha < 2$  and  $\beta > -1$ . If  $n = n(a)$  is such that  $na/c_n \rightarrow \infty$  then*

$$P(\tau^{(a)} > n) \sim E \tau^{(a)} P(X \geq na) \quad \text{as } a \rightarrow 0. \quad (20)$$

The right-hand side of (20) coincides with that of (4). Roughly speaking, if  $n$  is very large then the asymptotic behaviour of  $P(\tau^{(a)} > n)$  is as in the case of the fixed negative drift. But there is one crucial difference between fixed and asymptotically small drift: the expectation  $E \tau^{(a)}$  grows unbounded if  $a \rightarrow 0$ , and is a constant when the drift is fixed. Therefore, (20) would be useless without Theorem 2.

We turn our attention to the case when  $\sigma^2 := E X^2$  is finite. Here we shall assume without loss of generality that  $\sigma^2 = 1$ . Under this condition, we have  $c_n = \sqrt{n}$ . Then the condition that  $an/c_n \rightarrow \infty$  reads as  $na^2 \rightarrow \infty$ . In this case of finite variance the asymptotic behaviour of  $P(S_n^{(0)} > x_n)$  depends not only on the tail behaviour of  $X$ , but also on the rate of the growth of  $x_n$ . If  $x_n$  does not grow very fast ( $x_n = o(r_1(n))$  for some  $r_1(n)$  depending on the distribution of  $X$ ) then we have an asymptotic expression for  $P(S_n^{(0)} > x_n)$  in terms of the so-called Cramér series (for the definition of the Cramér series see, for example, [14, Chapter VIII]). For this type of large deviation, we have the following result.

**Theorem 4.** *Assume that  $E X^2 = 1$ ,  $n = n(a)$  is such that  $na^2 \rightarrow \infty$ , and that*

$$P(S_j^{(0)} \geq ja) \sim \bar{\Phi}(\sqrt{j}a) \exp\{ja^3 \lambda_m(a)\} \quad \text{uniformly in } j \in [a^{-2}, n], \quad (21)$$

where  $\lambda_m(u)$  is the partial sum in the Cramér series containing the first  $m$  terms and  $\bar{\Phi}(x) := \int_x^\infty (1/\sqrt{2\pi}) e^{-u^2/2} du$ . Then

$$P(\tau^{(a)} > n) \sim 2 E \tau^{(a)} \frac{1}{n} \bar{\Phi}(\sqrt{na}) \exp\{na^3 \lambda_m(a)\}.$$

Condition (21) has one essential disadvantage: it involves the whole sequence  $\{S_k^{(0)}, k \geq 0\}$ . We now list some restrictions on the distribution of  $X$ , which imply the validity of (21).

Nagaev [11] proved that the condition  $E |X|^k < \infty$  with some  $k > 2$  implies that the relation

$$P(S_n^{(0)} \geq x) \sim \bar{\Phi}\left(\frac{x}{\sqrt{n}}\right) \quad \text{as } n \rightarrow \infty \quad (22)$$

holds uniformly in  $x \leq \sqrt{(k/2 - 1)n \log n}$ . Thus, the existence of  $E |X|^k$  for some  $k > 2$  yields (21) with  $m = 0$  for all  $n$  satisfying

$$n \leq \left(\frac{1}{2}k - 1\right)a^{-2} \log a^{-2}.$$

Furthermore, it has been proved by Nagaev [10] and Rozovskii [16] that if  $P(X > x)$  is regularly varying at infinity with index  $p < -2$  then, under some additional restrictions on the left tail,

$$P(S_n^{(0)} \geq x) \sim \bar{\Phi}\left(\frac{x}{\sqrt{n}}\right) + n P(X > x + \sqrt{n}) \quad \text{uniformly on } x > 0. \quad (23)$$

Thus, (22) holds for all  $x \leq C\sqrt{n \log n}$  for any  $C < (p-2)^{1/2}$ . Consequently, (21) with  $m = 0$  holds for

$$n \leq Ca^{-2} \log a^{-2}, \quad C < (p-2)^{1/2}.$$

Osipov [13] found necessary and sufficient conditions under which the relation

$$P(S_n^{(0)} \geq x) \sim \overline{\Phi}\left(\frac{x}{\sqrt{n}}\right) \exp\left\{\frac{x^3}{n^2} \lambda_{[1/(1-\gamma)]}\left(\frac{x}{n}\right)\right\}$$

holds uniformly in  $0 \leq x \leq n^\gamma$ ,  $\frac{1}{2} < \gamma < 1$ , where  $[t]$  denotes the integer part of  $t$ . If these conditions are fulfilled then, obviously, (21) holds with  $m = [1/(1-\gamma)]$  for all  $n \leq a^{1/(1-\gamma)}$ .

It is well known that if  $X$  satisfies the Cramér condition ( $E e^{h|X|} < \infty$  for some  $h > 0$ ) then (21) holds with  $m = \infty$  and for all  $n$  satisfying  $na^2 \rightarrow \infty$ . Thus, Theorems 1 and 4 describe the behaviour of  $P(\tau^{(a)} > n)$  for any choice of  $n = n(a)$  and any random walk satisfying the Cramér condition.

It is easy to see that the statement of Theorem 4 can be rewritten as follows. If (21) holds then

$$P(\tau^{(a)} > n) \sim \frac{2}{\sqrt{2\pi}} a^{-1} E \tau^{(a)} n^{-3/2} e^{-n\xi(a)},$$

where

$$\xi(a) := \frac{a^2}{2} - a^3 \lambda_m(a). \quad (24)$$

Furthermore, in the proof of Theorem 4 we shall see that

$$\frac{E(e^{\xi(a)\tau^{(a)}}, \tau^{(a)} \leq n) - 1}{e^{\xi(a)} - 1} \sim 2 E \tau^{(a)}.$$

Thus,

$$P(\tau^{(a)} > n) \sim \frac{1}{a\sqrt{2\pi}} \frac{E(e^{\xi(a)\tau^{(a)}}, \tau^{(a)} \leq n) - 1}{e^{\xi(a)} - 1} n^{-3/2} e^{-n\xi(a)},$$

which is rather close to relation (5). If, additionally,  $X$  satisfies the Cramér condition, implying (21) with  $m = \infty$ , then we can replace the truncated expectation  $E(e^{\xi(a)\tau^{(a)}}, \tau^{(a)} \leq n)$  by  $E(e^{\xi(a)\tau^{(a)}})$ :

$$P(\tau^{(a)} > n) \sim \frac{1}{a\sqrt{2\pi}} \frac{E(e^{\xi(a)\tau^{(a)}}) - 1}{e^{\xi(a)} - 1} n^{-3/2} e^{-n\xi(a)}. \quad (25)$$

It follows from the definition of the Cramér series that  $\xi(a)$ , defined in (24), is the unique positive solution to the equation  $(d/dh) E e^{hX^{(a)}} = 0$ . Therefore, (25) is an analog of (5) for random walks with vanishing drift.

Another type of large deviation behaviour appears in the case when  $x_n$  grows fast, i.e.  $x_n \gg r_2(n)$  and the tail of  $X$  varies in an appropriate way. (Recall that  $a_n \gg b_n$  means that  $a_n/b_n \rightarrow \infty$ .) Here, as in the case of the non-Gaussian stable limit, we have  $P(S_n^{(0)} \geq x_n) \sim n P(X \geq x_n)$ . We consider only the case when the tail of  $X$  is regularly varying.

**Theorem 5.** Assume that  $P(X \geq x)$  is regularly varying at infinity with index  $p < -2$  and

$$\int_{|x|>y} x^2 P(X \in dx) = o\left(\frac{1}{\log y}\right) \quad \text{as } y \rightarrow \infty. \quad (26)$$

Then, as  $a \rightarrow 0$ ,

$$P(\tau^{(a)} > n) \sim E \tau^{(a)} P(X \geq na)$$

for any  $n = n(a)$  satisfying the inequality  $n(a) \geq Ca^{-2} \log a^{-2}$  with some  $C > (p-2)^{1/2}$ .

After Theorem 4 we mentioned that, in the case of regularly varying tails, (21) holds for all  $n \leq Ca^{-2} \log a^{-2}$ ,  $C < (p-2)^{1/2}$ . Therefore, the behaviour of  $P(\tau^{(a)} > n)$  remains unclear only for  $n$  satisfying  $(na^2 / \log a^{-2}) \rightarrow (p-2)^{1/2}$ . We conjecture that if the conditions of Theorem 5 hold then, in agreement with (23),

$$P(\tau^{(a)} > n) \sim 2 E \tau^{(a)} \frac{1}{n} \overline{\Phi}(\sqrt{n}a) + E \tau^{(a)} P(X \geq \sqrt{n} + na)$$

for all  $n$  satisfying  $na^2 \rightarrow \infty$ .

The remaining part of the paper is organised as follows. In the next section we derive an upper bound for the probability  $P(\tau^{(a)} > n)$ , which is crucial for the proof of Theorem 2. This proof will be given in Section 4. Section 3 is devoted to the proof of Theorem 1. Finally, Theorems 3, 4, and 5 will be proved in Section 4.

## 2. Upper bounds for the tail of $\tau^{(a)}$

It follows from (10) that in order to obtain upper bounds for  $P(\tau^{(a)} > n)$  we need inequalities for  $P(S_n^{(a)} \geq 0) = P(S_n^{(0)} \geq na)$ . In the following lemma we adapt one of the well-known Fuk–Nagaev inequalities for our purposes.

**Lemma 1.** Assume that  $X \in \mathcal{D}(\alpha, \beta)$ . Then there exists a constant  $C$  such that the inequality

$$P(S_n^{(0)} \geq x) \leq n P\left(X \geq \frac{x}{3}\right) + C \left(\frac{nV(x)}{x^2}\right)^2$$

holds for all  $x > 0$  and  $n \geq 1$ .

*Proof.* Applying Theorem 1.2 of [12] with  $t = 2$  we have

$$P(S_n^{(0)} \geq x) \leq n P(X \geq y) + e^{x/y} \left(\frac{nV(y)}{xy}\right)^{x/y + nV(y)/y^2 - n\mu(y)/y}, \quad (27)$$

where  $\mu(y) := E(X, |X| \leq y)$ .

Since  $E X = 0$ ,

$$|\mu(y)| = \left| \int_{|x|>y} x P(X \in dx) \right| \leq \int_{x>y} x P(|X| \in dx) = y P(|X| > y) + \int_y^\infty P(|X| > x) dx.$$

It is well known that the assumption  $X \in \mathcal{D}(\alpha, \beta)$  yields

$$\lim_{x \rightarrow \infty} \frac{x^2 P(|X| > x)}{V(x)} = \frac{2 - \alpha}{\alpha}. \quad (28)$$

Therefore, as  $y \rightarrow \infty$ ,

$$|\mu(y)| \leq \left(\frac{2 - \alpha}{\alpha} + o(1)\right) \left(\frac{V(y)}{y} + \int_y^\infty \frac{V(x)}{x^2} dx\right) = \left(\frac{2 - \alpha}{\alpha - 1} + o(1)\right) \frac{V(y)}{y}.$$

In the last step we used the relation

$$\int_y^\infty \frac{V(x)}{x^2} dx \sim \frac{1}{\alpha-1} \frac{V(y)}{y} \quad \text{as } y \rightarrow \infty,$$

which follows from the fact that  $V(x)$  is regularly varying with index  $2 - \alpha$ . As a result, we have the bound

$$\frac{V(y)}{y^2} - \frac{\mu(y)}{y} \geq \left( \frac{2\alpha-3}{\alpha-1} + o(1) \right) \frac{V(y)}{y^2}.$$

It follows from definition (8) of the sequence  $\{c_n\}$  that  $V(c_n)/c_n^2 \sim n^{-1}$  as  $n \rightarrow \infty$ . Consequently, there exists a constant  $C(\alpha)$  such that

$$\frac{V(y)}{y^2} - \frac{\mu(y)}{y} \geq -\frac{1}{n}$$

for all  $y > C(\alpha)c_n$ . From this bound and (27) with  $y = x/3$  we obtain

$$\mathbb{P}(S_n^{(0)} \geq x) \leq n \mathbb{P}\left(X \geq \frac{x}{3}\right) + 27e^3 \left(\frac{nV(y)}{x^2}\right)^2, \quad x \geq 3C(\alpha)c_n.$$

This inequality, together with monotonicity of  $V$ , implies that the desired result holds for  $x > C(\alpha)c_n$ . Noting that

$$\min_{n \geq 1} \inf_{x \leq 3C(\alpha)c_n} \frac{nV(x)}{x^2} > 0,$$

we complete the proof of the lemma.

In order to ‘translate’ bounds for  $\mathbb{P}(S_n^{(0)} > na)$  into bounds for  $\mathbb{P}(\tau^{(a)} > n)$ , we shall use the recurrent relation

$$n \mathbb{P}(\tau^{(a)} > n) = \sum_{j=0}^{n-1} \mathbb{P}(\tau^{(a)} > j) \mathbb{P}(S_{n-j}^{(0)} > (n-j)a), \quad (29)$$

which can be obtained by differentiating (10).

**Proposition 1.** *The inequality  $\mathbb{P}(\tau^{(a)} > n) \leq C E \tau^{(a)} V(na)/(na)^2$  is valid for all  $a > 0$  and all  $n \geq n_a := \min\{n \geq 1 : an > c_n\}$ .*

*Proof.* Using Lemma 1, we have

$$\begin{aligned} & \sum_{0 \leq j < n/2} \mathbb{P}(\tau^{(a)} > j) \mathbb{P}(S_{n-j}^{(0)} > (n-j)a) \\ & \leq \sum_{0 \leq j < n/2} \mathbb{P}(\tau^{(a)} > j) \left( (n-j) \mathbb{P}\left(X \geq \frac{(n-j)a}{3}\right) + C \left( \frac{(n-j)V((n-j)a)}{((n-j)a)^2} \right)^2 \right) \\ & \leq \left( n \mathbb{P}\left(X \geq \frac{na}{6}\right) + C \left( \frac{nV(na)}{(na)^2} \right)^2 \right) \sum_{0 \leq j < n/2} \mathbb{P}(\tau^{(a)} > j) \\ & \leq E \tau^{(a)} \left( n \mathbb{P}\left(X \geq \frac{na}{6}\right) + C \left( \frac{nV(na)}{(na)^2} \right)^2 \right) \\ & \leq n E \tau^{(a)} \left( \mathbb{P}\left(X \geq \frac{na}{6}\right) + C \frac{V(na)}{(na)^2} \right). \end{aligned} \quad (30)$$

In the last step we used definition (8) of  $c_n$  and the bound  $an \geq c_n$ , which follows from the assumption that  $n \geq n_a$ .

Furthermore, using the Markov inequality, we obtain

$$\sum_{n/2 \leq j \leq n-1} \mathbb{P}(\tau^{(a)} > j) \mathbb{P}(S_{n-j}^{(0)} > (n-j)a) \leq \frac{2 \mathbb{E} \tau^{(a)}}{n} \sum_{k=1}^n \mathbb{P}(S_k^{(0)} \geq ka). \quad (31)$$

Applying Lemma 1, we obtain

$$\begin{aligned} \sum_{k=1}^n \mathbb{P}(S_k^{(0)} \geq ka) &\leq n_a + \sum_{k=n_a}^n \mathbb{P}(S_k^{(0)} \geq ka) \\ &\leq n_a + \sum_{k=n_a}^n k \mathbb{P}\left(X \geq \frac{ka}{3}\right) + C \sum_{k=n_a}^n \frac{V^2(ka)}{k^2 a^4}. \end{aligned} \quad (32)$$

Since  $V(x)$  is regularly varying with index  $2 - \alpha$ ,

$$\begin{aligned} \sum_{k=n_a}^n \frac{V^2(ka)}{k^2 a^4} &\leq C a^{-2} \sum_{k=n_a}^n \frac{V^2(ka)}{(ka)^2} \\ &\leq C a^{-3} \int_{an_a}^{an} \frac{V^2(x)}{x^2} dx \\ &\leq C a^{-3} V(an) \int_{an_a}^{an} \frac{V(x)}{x^2} dx \\ &\leq C a^{-3} V(an) \frac{V(an_a)}{an_a}. \end{aligned} \quad (33)$$

From the definitions of  $c_n$  and  $n_a$ , we infer that

$$V(an_a) \sim V(c_{n_a}) \sim \frac{c_{n_a}^2}{n_a} \sim a^2 n_a. \quad (34)$$

Applying this relation to the last line in (33), we obtain the bound

$$\sum_{k=n_a}^n \frac{V^2(ka)}{k^2 a^4} \leq C a^{-2} V(an). \quad (35)$$

Furthermore,

$$\begin{aligned} \sum_{k=n_a}^n k \mathbb{P}\left(X \geq \frac{ka}{3}\right) &\leq C a^{-2} \int_{an_a}^{an} x \mathbb{P}\left(X > \frac{x}{3}\right) dx \\ &\leq a^{-2} \int_0^{na} x \mathbb{P}(|X| > x) dx \\ &= \frac{a^{-2}}{2} (V(an) + (an)^2 \mathbb{P}(|X| > an)), \end{aligned} \quad (36)$$

where in the last step we used integration by parts. Combining (32), (35), and (36), we have

$$\sum_{k=1}^n \mathbb{P}(S_k^{(0)} \geq ka) \leq C n_a + C a^{-2} V(an) + n^2 \mathbb{P}(|X| > an). \quad (37)$$

It is easy to see that (34) yields  $n_a \sim a^{-2} V(an_a)$ . From this relation and monotonicity of  $V(x)$ , we conclude that  $n_a \leq C a^{-2} V(an)$  for all  $n \geq n_a$ . Applying this bound to (37) we obtain

$$\sum_{k=1}^n P(S_k^{(0)} \geq ka) \leq C a^{-2} V(an) + n^2 P(|X| > an). \quad (38)$$

Combining (30), (31), and (38), we arrive at the inequality

$$\sum_{j=0}^n P(\tau^{(a)} > j) P(S_{n-j}^{(0)} > (n-j)a) \leq C n E \tau^{(a)} \left( P\left(|X| \geq \frac{na}{6}\right) + \frac{V(na)}{(na)^2} \right). \quad (39)$$

It follows from (28) that

$$P(|X| > x) \leq C \frac{V(x)}{x^2}.$$

Therefore, the right-hand side of (39) is bounded by  $C n E \tau^{(a)} V(na)/(na)^2$ . Thus, the statement of the proposition follows from (29).

### 3. Proof of Theorem 1

From the definition of the first ladder epoch  $\tau^{(a)}$  we obtain

$$\begin{aligned} P(\tau^{(a)} > n) &= P\left(\min_{1 \leq k \leq n} (S_k^{(0)} - ka) > 0\right) \\ &= P\left(\min_{1 \leq k \leq n} S_k^{(0)} > 0\right) P\left(\min_{1 \leq k \leq n} (S_k^{(0)} - ka) > 0 \mid \min_{1 \leq k \leq n} S_k^{(0)} > 0\right) \\ &= P(\tau^{(0)} > n) P\left(\min_{1 \leq k \leq n} \left(\frac{S_k^{(0)}}{c_n} - \frac{k an}{n c_n}\right) > 0 \mid \min_{1 \leq k \leq n} S_k^{(0)} > 0\right). \end{aligned}$$

Doney [3] showed that  $\{S_{[tc_n]}^{(0)}/c_n, t \in [0, 1] \mid \min_{1 \leq k \leq n} S_k^{(0)} > 0\}$  converges weakly, as  $n \rightarrow \infty$ , to the Lévy meander  $\{M_{\alpha, \beta}(t), t \in [0, 1]\}$ . This yields

$$\begin{aligned} \lim_{n \rightarrow \infty} P\left(\min_{1 \leq k \leq n} \left(\frac{S_k^{(0)}}{c_n} - \frac{k an}{n c_n}\right) > 0 \mid \min_{1 \leq k \leq n} S_k^{(0)} > 0\right) &= P\left(\min_{0 \leq t \leq 1} (M_{\alpha, \beta}(t) - ut) > 0\right) \\ &=: 1 - F_{\alpha, \beta}(u). \end{aligned} \quad (40)$$

It is obvious that  $F_{\alpha, \beta}(u)$  is monotonously increasing and  $\lim_{u \rightarrow \infty} F_{\alpha, \beta}(u) = 1$ .

It is known that the corresponding meander  $M_{\alpha, \beta}$  can be defined as a weak limit, as  $\varepsilon \rightarrow 0$ , of  $Y_{\alpha, \beta}$  starting from  $\varepsilon > 0$  and conditioned to stay positive up to time 1:

$$\begin{aligned} \mathcal{L}\{M_{\alpha, \beta}(t), t \in [0, 1]\} \\ = \lim_{\varepsilon \rightarrow 0} \mathcal{L}\left\{Y_{\alpha, \beta}(t), t \in [0, 1] \mid \inf_{0 \leq t \leq 1} Y_{\alpha, \beta}(t) > 0, Y_{\alpha, \beta}(0) = \varepsilon\right\}. \end{aligned}$$

Therefore,

$$1 - F_{\alpha, \beta}(u) = \lim_{\varepsilon \rightarrow 0} \frac{P(\inf_{0 \leq t \leq 1} (Y_{\alpha, \beta}(t) - ut) > 0 \mid Y_{\alpha, \beta}(0) = \varepsilon)}{P(\inf_{0 \leq t \leq 1} Y_{\alpha, \beta}(t) > 0 \mid Y_{\alpha, \beta}(0) = \varepsilon)}.$$

Define  $H_{\alpha,\beta}^{(u)}(z) := \min\{t : Y_{\alpha,\beta}(t) - ut \leq z \mid Y_{\alpha,\beta}(0) = 0\}$ . Then

$$1 - F_{\alpha,\beta}(u) = \lim_{\varepsilon \rightarrow 0} \frac{\mathbb{P}(H_{\alpha,\beta}^{(u)}(-\varepsilon) > 1)}{\mathbb{P}(H_{\alpha,\beta}^{(0)}(-\varepsilon) > 1)}.$$

In the case of the Brownian motion, that is,  $\alpha = 2$  and  $\beta = 0$ , we can calculate the limit explicitly. Indeed, it is known that  $H_{2,0}^{(u)}(-\varepsilon)$  has the density

$$\frac{\varepsilon}{\sqrt{2\pi}t^{3/2}} \exp\left\{-\frac{(ut - \varepsilon)^2}{2t}\right\}, \quad t > 0.$$

Thus, as  $\varepsilon \rightarrow 0$ ,

$$\mathbb{P}(H_{2,0}^{(0)}(-\varepsilon) > 1) = \frac{\varepsilon}{\sqrt{2\pi}} \int_1^\infty t^{-3/2} e^{-\varepsilon^2/2t} dt \sim \frac{2\varepsilon}{\sqrt{2\pi}},$$

and, consequently,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{\mathbb{P}(H_{\alpha,\beta}^{(u)}(-\varepsilon) > 1)}{\mathbb{P}(H_{\alpha,\beta}^{(0)}(-\varepsilon) > 1)} &= \lim_{\varepsilon \rightarrow 0} \int_1^\infty \frac{1}{2t^{3/2}} \exp\left\{-\frac{(ut - \varepsilon)^2}{2t}\right\} dt \\ &= \int_1^\infty \frac{1}{2t^{3/2}} e^{-u^2t/2} dt \\ &= u \int_u^\infty v^{-2} e^{-v^2/2} dv. \end{aligned}$$

As a result, we have

$$1 - F_{2,0}(u) = u \int_u^\infty v^{-2} e^{-v^2/2} dv. \quad (41)$$

This equality can be generalised to stable Lévy processes without negative jumps, i.e.  $\{\alpha \in (1, 2), \beta = 1\}$  or  $\{\alpha = 2, \beta = 0\}$ . Indeed, using Kendall's equality (see [8]) and the scaling property of stable processes, we see that  $H_{\alpha,1}^{(u)}(-\varepsilon)$  has the density

$$u \mapsto \frac{\varepsilon}{t^{1+1/\alpha}} g_{\alpha,1}\left(\frac{-\varepsilon + ut}{t^{1/\alpha}}\right).$$

Then, analogously to the case of the Brownian motion,

$$1 - F_{\alpha,1}(u) = \frac{u^{1/(\alpha-1)}}{(\alpha-1)g_{\alpha,1}(0)} \int_u^\infty v^{-\alpha/(\alpha-1)} g_{\alpha,1}(v) dv.$$

Unfortunately, we cannot give an explicit expression for  $1 - F_{\alpha,\beta}$  for a process with positive jumps. But we can describe this function via the Laplace transform of  $x^{\rho-1}(1 - F_{\alpha,\beta}(x^{1-1/\alpha}))$ .

In order to prove (16), we show that  $1 - F_{\alpha,\beta}$  satisfies a certain integral equation. Dividing both parts of (29) by  $n \mathbb{P}(\tau^{(0)} > n)$ , we have

$$\frac{\mathbb{P}(\tau^{(a)} > n)}{\mathbb{P}(\tau^{(0)} > n)} = \sum_{j=0}^{n-1} \frac{\mathbb{P}(\tau^{(a)} > j)}{\mathbb{P}(\tau^{(0)} > j)} \frac{\mathbb{P}(\tau^{(0)} > j)}{\mathbb{P}(\tau^{(0)} > n)} \mathbb{P}(S_{n-j}^{(0)} \geq a(n-j)) \frac{1}{n}. \quad (42)$$

Fix any  $\varepsilon \in (0, \frac{1}{2})$ . We first note that

$$\sum_{0 \leq j \leq \varepsilon n} \frac{\mathbb{P}(\tau^{(a)} > j) \mathbb{P}(\tau^{(0)} > j)}{\mathbb{P}(\tau^{(0)} > j) \mathbb{P}(\tau^{(0)} > n)} \mathbb{P}(S_{n-j}^{(0)} \geq a(n-j)) \frac{1}{n} \leq \frac{\sum_{0 \leq j \leq \varepsilon n} \mathbb{P}(\tau^{(0)} > j)}{n \mathbb{P}(\tau^{(0)} > n)} \leq C\varepsilon^\rho \quad (43)$$

and

$$\sum_{(1-\varepsilon)n \leq j \leq n-1} \frac{\mathbb{P}(\tau^{(a)} > j) \mathbb{P}(\tau^{(0)} > j)}{\mathbb{P}(\tau^{(0)} > j) \mathbb{P}(\tau^{(0)} > n)} \mathbb{P}(S_{n-j}^{(0)} \geq a(n-j)) \frac{1}{n} \leq \frac{\mathbb{P}(\tau^{(0)} > n/2)}{n \mathbb{P}(\tau^{(0)} > n)} \varepsilon n \leq C\varepsilon. \quad (44)$$

In both bounds we have used the fact that  $\mathbb{P}(\tau^{(0)} > j)$  varies regularly with index  $\rho - 1$ .

It remains to consider the middle part of the sum on the right-hand side of (42). It is easy to see that the condition  $an/c_n \rightarrow u$  implies that

$$\frac{aj}{c_j} \rightarrow ut^{1-1/\alpha} \quad \text{as } a \rightarrow 0,$$

provided that  $j \sim tn$ . Then, in view of (40), for every  $t \in (0, 1)$ , the following is valid. As  $a \rightarrow 0$ ,

$$\begin{aligned} f_a(t) &:= \frac{\mathbb{P}(\tau^{(a)} > [tn]) \mathbb{P}(\tau^{(0)} > [tn])}{\mathbb{P}(\tau^{(0)} > [tn]) \mathbb{P}(\tau^{(0)} > n)} \mathbb{P}(S_{n-[tn]}^{(0)} \geq a(n-[tn])) \\ &\rightarrow (1 - F_{\alpha,\beta}(ut^{1-1/\alpha}))t^{\rho-1} \mathbb{P}(Y_{\alpha,\beta}(1) > u(1-t)^{1-1/\alpha}). \end{aligned}$$

Thus, by dominated convergence,

$$\begin{aligned} \lim_{a \rightarrow 0} \sum_{\varepsilon n < j < (1-\varepsilon)n} \frac{\mathbb{P}(\tau^{(a)} > j) \mathbb{P}(\tau^{(0)} > j)}{\mathbb{P}(\tau^{(0)} > j) \mathbb{P}(\tau^{(0)} > n)} \mathbb{P}(S_{n-j}^{(0)} \geq a(n-j)) \frac{1}{n} \\ = \int_{\varepsilon}^{1-\varepsilon} (1 - F_{\alpha,\beta}(ut^{1-1/\alpha}))t^{\rho-1} \mathbb{P}(Y_{\alpha,\beta}(1) > u(1-t)^{1-1/\alpha}) dt. \end{aligned}$$

Now using monotone convergence, we obtain

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \lim_{a \rightarrow 0} \sum_{\varepsilon n < j < (1-\varepsilon)n} \frac{\mathbb{P}(\tau^{(a)} > j) \mathbb{P}(\tau^{(0)} > j)}{\mathbb{P}(\tau^{(0)} > j) \mathbb{P}(\tau^{(0)} > n)} \mathbb{P}(S_{n-j}^{(0)} \geq a(n-j)) \frac{1}{n} \\ = \int_0^1 (1 - F_{\alpha,\beta}(ut^{1-1/\alpha}))t^{\rho-1} \mathbb{P}(Y_{\alpha,\beta}(1) > u(1-t)^{1-1/\alpha}) dt. \end{aligned} \quad (45)$$

Combining (42)–(45), and taking into account (40), we obtain

$$1 - F_{\alpha,\beta}(u) = \int_0^1 (1 - F_{\alpha,\beta}(ut^{1-1/\alpha}))t^{\rho-1} \mathbb{P}(Y_{\alpha,\beta}(1) > u(1-t)^{1-1/\alpha}) dt. \quad (46)$$

Setting

$$G_{\alpha,\beta}(u) := 1 - F_{\alpha,\beta}(u^{1-1/\alpha}) \quad \text{and} \quad \xi_{\alpha,\beta} := (Y_{\alpha,\beta}(1))^{\alpha/(\alpha-1)},$$

we can rewrite (46) as follows:

$$G_{\alpha,\beta}(u) = \int_0^1 G_{\alpha,\beta}(ut)t^{\rho-1} \mathbb{P}(\xi_{\alpha,\beta} > u(1-t)) dt.$$



Substituting  $t = y/u$ , we have

$$G_{\alpha,\beta}(u) = u^{-\rho} \int_0^u G_{\alpha,\beta}(y) y^{\rho-1} \mathbf{P}(\xi_{\alpha,\beta} > u - y) dy.$$

Therefore, the function  $Q_{\alpha,\beta}(u) := u^{\rho-1} G_{\alpha,\beta}(u)$  satisfies the equation

$$u Q_{\alpha,\beta}(u) = \int_0^u Q_{\alpha,\beta}(y) \mathbf{P}(\xi_{\alpha,\beta} > u - y) dy. \quad (47)$$

Let  $q_{\alpha,\beta}(\lambda)$  denote the Laplace transform of the function  $Q_{\alpha,\beta}$ , i.e.

$$q_{\alpha,\beta}(\lambda) = \int_0^\infty e^{-\lambda u} Q_{\alpha,\beta}(u) du, \quad \lambda > 0.$$

Now (47) implies that

$$\begin{aligned} \frac{d}{d\lambda} q_{\alpha,\beta}(\lambda) &= - \int_0^\infty u e^{-\lambda u} Q_{\alpha,\beta}(u) du \\ &= - \int_0^\infty e^{-\lambda u} \int_0^u Q_{\alpha,\beta}(y) \mathbf{P}(\xi_{\alpha,\beta} > u - y) dy \\ &= - \int_0^\infty e^{-\lambda u} Q_{\alpha,\beta}(u) du \int_0^\infty e^{-\lambda z} \mathbf{P}(\xi_{\alpha,\beta} > z) dz \\ &= -q_{\alpha,\beta}(\lambda) \int_0^\infty e^{-\lambda z} \mathbf{P}(\xi_{\alpha,\beta} > z) dz. \end{aligned}$$

Solving this differential equation, we see that

$$\begin{aligned} q_{\alpha,\beta}(\lambda) &= q_{\alpha,\beta}(\lambda_0) \exp \left\{ - \int_{\lambda_0}^\lambda \int_0^\infty e^{-\lambda z} \mathbf{P}(\xi_{\alpha,\beta} > z) dz \right\} \\ &= q_{\alpha,\beta}(\lambda_0) \exp \left\{ - \int_0^\infty \frac{e^{-\lambda_0 z} - e^{-\lambda z}}{z} \mathbf{P}(\xi_{\alpha,\beta} > z) dz \right\}. \end{aligned}$$

It follows from the definition of  $\xi_{\alpha,\beta}$  that

$$\mathbf{P}(\xi_{\alpha,\beta} > z) = \mathbf{P}(Y_{\alpha,\beta}(1) > z^{1-1/\alpha}) \sim \frac{C}{z^{\alpha-1}} \quad \text{as } z \rightarrow \infty.$$

This relation yields

$$\int_1^\infty \frac{1}{z} \mathbf{P}(\xi_{\alpha,\beta} > z) dz < \infty.$$

Therefore,

$$\begin{aligned} &\int_0^\infty \frac{e^{-\lambda_0 z} - e^{-\lambda z}}{z} \mathbf{P}(\xi_{\alpha,\beta} > z) dz \\ &= \int_0^\infty \frac{1 - e^{-\lambda z}}{z} \mathbf{P}(\xi_{\alpha,\beta} > z) dz - \int_0^\infty \frac{1 - e^{-\lambda_0 z}}{z} \mathbf{P}(\xi_{\alpha,\beta} > z) dz. \end{aligned}$$

Consequently,

$$q_{\alpha,\beta}(\lambda) = C \exp \left\{ - \int_0^\infty \frac{1 - e^{-\lambda z}}{z} \mathbf{P}(\xi_{\alpha,\beta} > z) dz \right\}.$$

To complete the proof of the theorem, it remains to note that, in view of the scaling property of  $Y_{\alpha,\beta}$ ,

$$\mathbf{P}(\xi_{\alpha,\beta} > z) = \mathbf{P}(Y_{\alpha,\beta}(1) > z^{1-1/\alpha}) = \mathbf{P}(Y_{\alpha,\beta}(z) - z > 0).$$

#### 4. Proof of Theorem 2

For every  $\varepsilon \in (0, 1)$ ,

$$\mathbb{E}(\tau^{(a)})^r = \sum_{n=0}^{\infty} [(n+1)^r - n^r] \mathbb{P}(\tau^{(a)} > n) = \Sigma_1 + \Sigma_2 + \Sigma_3, \quad (48)$$

where

$$\begin{aligned} \Sigma_1 &:= \sum_{0 \leq n \leq \varepsilon n_a} [(n+1)^r - n^r] \mathbb{P}(\tau^{(a)} > n), \\ \Sigma_2 &:= \sum_{\varepsilon n_a < n < n_a/\varepsilon} [(n+1)^r - n^r] \mathbb{P}(\tau^{(a)} > n), \\ \Sigma_3 &:= \sum_{n \geq n_a/\varepsilon} [(n+1)^r - n^r] \mathbb{P}(\tau^{(a)} > n). \end{aligned}$$

Since  $[(n+1)^r - n^r] \leq Cn^{r-1}$ ,

$$\Sigma_1 \leq C \sum_{0 \leq n \leq \varepsilon n_a} n^{r-1} \mathbb{P}(\tau^{(0)} > n) \leq C\varepsilon^{\rho+r-1} n_a^r \mathbb{P}(\tau^{(0)} > n_a). \quad (49)$$

In the last step we used the fact that  $\mathbb{P}(\tau^{(0)} > n)$  is regularly varying with index  $\rho - 1$ .

Furthermore, in view of (40),

$$\begin{aligned} \psi_a(r; x) &:= (([xn_a] + 1)^r - ([xn_a])^r) \frac{\mathbb{P}(\tau^{(a)} > [xn_a])}{n_a^{r-1} \mathbb{P}(\tau^{(0)} > n_a)} \\ &= \frac{([xn_a] + 1)^r - ([xn_a])^r}{n_a^{r-1}} \frac{\mathbb{P}(\tau^{(a)} > [xn_a])}{\mathbb{P}(\tau^{(0)} > [xn_a])} \frac{\mathbb{P}(\tau^{(0)} > [xn_a])}{\mathbb{P}(\tau^{(0)} > n_a)} \\ &\rightarrow rx^{r-1}(1 - F_{\alpha, \beta}(x^{1-1/\alpha}))x^{\rho-1} \quad \text{as } a \rightarrow 0. \end{aligned}$$

Then, by dominated convergence,

$$\begin{aligned} \lim_{a \rightarrow 0} \frac{\Sigma_2}{n_a^r \mathbb{P}(\tau^{(0)} > n_a)} &= \lim_{a \rightarrow 0} \int_{\varepsilon}^{1/\varepsilon} \psi_a(r; x) \, dx \\ &= \int_{\varepsilon}^{1/\varepsilon} x^{r-1}(1 - F_{\alpha, \beta}(x^{1-1/\alpha}))x^{\rho-1} \, dx. \end{aligned} \quad (50)$$

In view of Proposition 1,

$$\Sigma_3 \leq C \mathbb{E} \tau^{(a)} \sum_{n \geq n_a/\varepsilon} n^{r-1} \frac{V(na)}{(na)^2}.$$

Since  $V(x)$  varies regularly,

$$\begin{aligned} \sum_{n \geq n_a/\varepsilon} n^{r-1} \frac{V(na)}{(na)^2} &\sim a^{-r} \int_{an_a/\varepsilon}^{\infty} x^{r-3} V(x) \, dx \\ &\sim (\alpha - r)^{-1} \varepsilon^{\alpha-r} a^{-r} (an_a)^{r-2} V(an_a) \\ &\sim (\alpha - r)^{-1} \varepsilon^{\alpha-r} n_a^r \frac{V(an_a)}{(an_a)} \\ &\sim (\alpha - r)^{-1} \varepsilon^{\alpha-r} n_a^{r-1}. \end{aligned}$$

Here we used the relations

$$an_a \sim c_{n_a} \quad \text{as } a \rightarrow 0$$

and

$$c_n^{-2} V(c_n) \sim n^{-1} \quad \text{as } n \rightarrow \infty.$$

Consequently,

$$\Sigma_3 \leq C\varepsilon^{\alpha-r} \mathbb{E} \tau^{(a)} n_a^{r-1}. \quad (51)$$

Substituting (49)–(51) with  $r = 1$  into (48) with  $r = 1$ , we have

$$\limsup_{a \rightarrow 0} \frac{\mathbb{E} \tau^{(a)}}{n_a \mathbb{P}(\tau^{(0)} > n_a)} \leq \frac{1}{1 - C\varepsilon^{\alpha-1}} \left( \int_{\varepsilon}^{1/\varepsilon} (1 - F_{\alpha,\beta}(x^{1-1/\alpha})) x^{\rho-1} dx + C\varepsilon^{\rho} \right).$$

Thus,

$$\mathbb{E} \tau^{(a)} \leq C n_a \mathbb{P}(\tau^{(0)} > n_a).$$

Applying this inequality to (51) we find that

$$\Sigma_3 \leq C\varepsilon^{\alpha-r} n_a^r \mathbb{P}(\tau^{(0)} > n_a). \quad (52)$$

Combining (48)–(50) and (52), we obtain

$$\liminf_{a \rightarrow 0} \frac{\mathbb{E}(\tau^{(a)})^r}{n_a^r \mathbb{P}(\tau^{(0)} > n_a)} \geq \int_{\varepsilon}^{1/\varepsilon} x^{r-1} (1 - F_{\alpha,\beta}(x^{1-1/\alpha})) x^{\rho-1} dx$$

and

$$\limsup_{a \rightarrow 0} \frac{\mathbb{E}(\tau^{(a)})^r}{n_a^r \mathbb{P}(\tau^{(0)} > n_a)} \leq \int_{\varepsilon}^{1/\varepsilon} x^{r-1} (1 - F_{\alpha,\beta}(x^{1-1/\alpha})) x^{\rho-1} dx + C\varepsilon^{\rho+r-1} + C\varepsilon^{\alpha-r}.$$

The latter inequality yields

$$\limsup_{a \rightarrow 0} \frac{\mathbb{E}(\tau^{(a)})^r}{n_a^r \mathbb{P}(\tau^{(0)} > n_a)} < \infty. \quad (53)$$

Hence, letting  $\varepsilon \rightarrow 0$ ,

$$\lim_{a \rightarrow 0} \frac{\mathbb{E}(\tau^{(a)})^r}{n_a^r \mathbb{P}(\tau^{(0)} > n_a)} = \int_0^{\infty} x^{r-1} (1 - F_{\alpha,\beta}(x^{1-1/\alpha})) x^{\rho-1} dx. \quad (54)$$

The integral  $\int_0^{\infty} x^{r-1} (1 - F_{\alpha,\beta}(x^{1-1/\alpha})) x^{\rho-1} dx$  is finite in view of (53). Noting now that  $n_a^r \mathbb{P}(\tau^{(0)} > n_a)$  is regularly varying with index  $-\alpha(\rho + r - 1)/(\alpha - 1)$ , we complete the proof of the theorem.

## 5. Proofs of large deviation results

### 5.1. Proof of Theorem 3

Since  $an/c_n \rightarrow \infty$ , there exists  $N(n)$  satisfying

$$\frac{aN(n)}{c_n} \rightarrow \infty \quad \text{and} \quad (n) = o(n).$$

We now split the sum in (29) into two parts:

$$\begin{aligned}\Sigma_1 &:= \sum_{k=0}^{N(n)} \mathbf{P}(\tau^{(a)} > k) \mathbf{P}(S_{n-k}^{(0)} > (n-k)a), \\ \Sigma_2 &:= \sum_{k=N(n)+1}^{n-1} \mathbf{P}(\tau^{(a)} > k) \mathbf{P}(S_{n-k}^{(0)} > (n-k)a).\end{aligned}$$

Since

$$\lim_{j \rightarrow \infty} \sup_{x > q_j c_j} \left| \frac{\mathbf{P}(S_j^{(0)} > x)}{j \mathbf{P}(X > x)} - 1 \right| = 0$$

for any sequence  $q_j \uparrow \infty$ , we obtain the relation

$$\begin{aligned}\Sigma_1 &= (1 + o(1))n \mathbf{P}(X > na) \sum_{k=0}^{N(n)} \mathbf{P}(\tau^{(a)} > k) \\ &= (1 + o(1))n \mathbf{P}(X > na) \left( \mathbf{E} \tau^{(a)} - \sum_{k=N(n)+1}^{n-1} \mathbf{P}(\tau^{(a)} > k) \right).\end{aligned}\quad (55)$$

Noting that  $N(n) \gg n_a$  and taking into account (51), we see that

$$\sum_{k=N(n)+1}^{n-1} \mathbf{P}(\tau^{(a)} > k) = o(\mathbf{E} \tau^{(a)}). \quad (56)$$

Combining (55) and (56), we have

$$\Sigma_1 = (1 + o(1))n \mathbf{E} \tau^{(a)} \mathbf{P}(X > na). \quad (57)$$

We now turn our attention to  $\Sigma_2$ . It follows from Proposition 1 that

$$\begin{aligned}\Sigma_2 &\leq \mathbf{P}(\tau^{(a)} > N(n)) \sum_{j=1}^n \mathbf{P}(S_j^{(0)} > aj) \\ &\leq C \mathbf{E} \tau^{(a)} \frac{V(aN(n))}{(aN(n))^2} \sum_{j=1}^n \mathbf{P}(S_j^{(0)} > aj).\end{aligned}$$

Furthermore, using (32), we obtain

$$\Sigma_2 \leq \mathbf{E} \tau^{(a)} \frac{V(aN(n))}{(aN(n))^2} n^2 \mathbf{P}(|X| \geq na). \quad (58)$$

From the definition of  $c_n$  and the relation  $aN(n) \gg c_n$ , we conclude that

$$\frac{V(aN(n))}{(aN(n))^2} = o\left(\frac{1}{n}\right).$$

Moreover,  $\mathbf{P}(|X| \geq na) \leq C \mathbf{P}(X \geq na)$  for every  $X \in \mathcal{D}(\alpha, \beta)$  with  $\alpha < 2$  and  $\beta > -1$ . Then, (58) implies that

$$\Sigma_2 = o(n \mathbf{E} \tau^{(a)} \mathbf{P}(X > na)). \quad (59)$$

Substituting (57) and (59) into (29) completes the proof.

## 5.2. Proof of Theorem 4

Recall definition (24) of  $\xi(a)$ . Set

$$\phi_j := e^{\xi(a)j} \mathbb{P}(\tau^{(a)} > j) \quad \text{and} \quad \theta_j := e^{\xi(a)j} \mathbb{P}(S_j^{(0)} > aj). \quad (60)$$

It is easily seen that

$$\theta_j \leq C \quad \text{for all } j \leq 1/a^2. \quad (61)$$

Furthermore, combining (21) with the relations

$$\overline{\Phi}(x) \leq \frac{1}{x\sqrt{2\pi}} e^{-x^2/2}$$

and

$$\overline{\Phi}(x) \sim \frac{1}{x\sqrt{2\pi}} e^{-x^2/2} \quad \text{as } x \rightarrow \infty,$$

we obtain

$$\theta_j \leq \frac{C}{a\sqrt{j}} \quad \text{for } j \leq n \quad (62)$$

and

$$\theta_j \sim \frac{1}{a\sqrt{2\pi j}} \quad \text{for } j \leq n \text{ and } ja^2 \rightarrow \infty, \quad (63)$$

respectively.

Multiplying both sides of (29) by  $e^{a^2 n/2}$ , we see that the sequence  $\phi_j$  satisfies the equation

$$k\phi_k = \sum_{j=0}^{k-1} \phi_j \theta_{k-j}, \quad k \geq 1. \quad (64)$$

If  $n$  satisfies the conditions of the theorem then, using (61) and (62), we have

$$\sup_{n \geq 1} \max_{j \leq n} \theta_j < \infty.$$

Consequently,

$$\phi_k \leq \frac{C}{k} \sum_{j=0}^k \phi_j \leq \frac{C}{k} \sum_{j=0}^n \phi_j$$

for all  $k \leq n$ . Setting  $\sigma_n := \sum_{j=0}^n \phi_j$ , we rewrite the latter bound as

$$\phi_k \leq \frac{C}{k} \sigma_n, \quad k \leq n.$$

Now, applying this bound and (62) to the terms on the right-hand side of (64), we obtain, for all  $k \leq n$ , the bound

$$\begin{aligned} \phi_k &= \frac{1}{k} \sum_{0 \leq j < k/2} \phi_j \theta_{k-j} + \frac{1}{k} \sum_{k/2 \leq j < k} \phi_j \theta_{k-j} \\ &\leq \frac{C}{ak^{3/2}} \sum_{0 \leq j < k/2} \phi_j + \frac{C\sigma_n}{k^2} \sum_{k/2 \leq j < k} \theta_{k-j} \\ &\leq \frac{C\sigma_n}{ak^{3/2}} + \frac{C\sigma_n}{k^2} \sum_{1 \leq j < k} \frac{1}{a\sqrt{j}} \\ &\leq \frac{C\sigma_n}{ak^{3/2}}. \end{aligned} \quad (65)$$

This inequality allows us to determine the asymptotic behaviour of  $\phi_n$ . First of all we note that (63) yields

$$\sum_{0 \leq j \leq N(n)} \phi_j \theta_{n-j} \sim \frac{1}{a\sqrt{2\pi n}} \sum_{0 \leq j \leq N(n)} \phi_j \quad \text{as } a \rightarrow 0$$

for every  $N(n) = o(n)$ . Moreover, by (65),

$$0 \leq \sigma_n - \sum_{0 \leq j \leq N(n)} \phi_j = \sum_{N(n) < j \leq n} \phi_j \leq \frac{C\sigma_n}{aN(n)}. \quad (66)$$

Therefore, choosing  $N(n)$  satisfying

$$N(n) = o(n) \quad \text{and} \quad aN^2(n) \rightarrow \infty,$$

we have, as  $a \rightarrow 0$ ,

$$\sum_{0 \leq j \leq N(n)} \phi_j \theta_{n-j} \sim \frac{\sigma_n}{a\sqrt{2\pi n}}. \quad (67)$$

Furthermore, it follows from (62) and (65) that

$$\begin{aligned} \sum_{N(n) < j < n/2} \phi_j \theta_{n-j} &\leq \frac{C}{a\sqrt{n}} \sum_{N(n) < j < n/2} \phi_j \\ &\leq \frac{C}{a\sqrt{n}} \sum_{N(n) < j < n/2} \frac{\sigma_n}{aj^{3/2}} \\ &\leq \frac{C\sigma_n}{a^2\sqrt{nN(n)}} \end{aligned} \quad (68)$$

and

$$\sum_{n/2 \leq j < n} \phi_j \theta_{n-j} \leq \frac{C\sigma_n}{an^{3/2}} \sum_{j=1}^n \theta_j \leq \frac{C\sigma_n}{an^{3/2}} \sum_{j=1}^n \frac{1}{a\sqrt{j}} \leq \frac{C\sigma_n}{a^2n}. \quad (69)$$

Combining (67)–(69) and recalling that  $a^2N(n) \rightarrow \infty$ , we obtain

$$\sum_{j=0}^{n-1} \phi_j \theta_{n-j} \sim \frac{\sigma_n}{a\sqrt{2\pi n}} \quad \text{as } a \rightarrow 0.$$

Substituting this into (64) we have

$$\phi_n \sim \frac{\sigma_n}{a\sqrt{2\pi n^{3/2}}} \quad \text{as } a \rightarrow 0. \quad (70)$$

To complete the proof of the theorem, it remains to find the asymptotic behaviour of  $\sigma_n$ . First of all, (66) implies that the bounds

$$\sum_{j \leq 1/\varepsilon a^2} \phi_j \leq \sigma_n \leq (1 - C\sqrt{\varepsilon})^{-1} \sum_{j \leq 1/\varepsilon a^2} \phi_j \quad (71)$$

are valid for all sufficiently small values of  $\varepsilon$ . Applying Theorem 1 and recalling that  $P(\tau^{(0)} > j)$  is regularly varying with index  $-\frac{1}{2}$ , we see that

$$\begin{aligned} \lim_{a \rightarrow 0} \frac{\phi_{[xa^{-2}]}}{P(\tau^{(0)} > a^{-2})} &= \lim_{a \rightarrow 0} \frac{e^{[xa^{-2}]\xi(a)} P(\tau^{(a)} > [xa^{-2}]) P(\tau^{(0)} > [xa^{-2}])}{P(\tau^{(0)} > [xa^{-2}]) P(\tau^{(0)} > a^{-2})} \\ &= e^{x/2} (1 - F_{2,0}(\sqrt{x})) \frac{1}{\sqrt{x}} \end{aligned}$$

for every  $x > 0$ . Thus, by dominated convergence,

$$\begin{aligned} \lim_{a \rightarrow 0} \frac{\sum_{j \leq 1/\varepsilon a^2} \phi_j}{a^{-2} P(\tau^{(0)} > a^{-2})} &= \int_0^{1/\varepsilon} \frac{e^{x/2}}{\sqrt{x}} (1 - F_{2,0}(\sqrt{x})) dx \\ &=: I(\varepsilon). \end{aligned} \tag{72}$$

Using (41), we have

$$\begin{aligned} I(\varepsilon) &= \int_0^{1/\varepsilon} e^{x/2} \int_{\sqrt{x}}^{\infty} v^{-2} e^{-v^2/2} dv dx \\ &= \int_0^{\infty} e^{x/2} \int_{\sqrt{x}}^{\infty} v^{-2} e^{-v^2/2} dv dx - \int_{1/\varepsilon}^{\infty} e^{x/2} \int_{\sqrt{x}}^{\infty} v^{-2} e^{-v^2/2} dv dx. \end{aligned}$$

Noting that

$$\int_{\sqrt{x}}^{\infty} v^{-2} e^{-v^2/2} dv \leq \frac{e^{-x/2}}{x^{3/2}},$$

we have

$$0 \leq \int_0^{\infty} e^{x/2} \int_{\sqrt{x}}^{\infty} v^{-2} e^{-v^2/2} dv dx - I(\varepsilon) \leq \sqrt{\varepsilon}.$$

Changing the order of integration and substituting  $v^2/2 = u$ , we have

$$\begin{aligned} \int_0^{\infty} e^{x/2} \int_{\sqrt{x}}^{\infty} v^{-2} e^{-v^2/2} dv dx &= \int_0^{\infty} v^{-2} e^{-v^2/2} \int_0^{v^2} e^{x/2} dx dv \\ &= 2 \int_0^{\infty} v^{-2} e^{-v^2/2} (1 - e^{-v^2/2}) dv \\ &= \frac{1}{\sqrt{2}} \int_0^{\infty} u^{-3/2} (1 - e^{-u}) du. \end{aligned}$$

Integrating now by parts we obtain

$$\int_0^{\infty} u^{-3/2} (1 - e^{-u}) du = 2 \int_0^{\infty} u^{-1/2} e^{-u} du = 2\Gamma\left(\frac{1}{2}\right) = 2\sqrt{\pi}.$$

As a result, we have the bounds

$$\sqrt{2\pi} - \sqrt{\varepsilon} \leq I(\varepsilon) \leq \sqrt{2\pi}. \tag{73}$$

Substituting (72) and (73) into (71), we obtain

$$\sqrt{2\pi} - \sqrt{\varepsilon} \leq \liminf_{a \rightarrow 0} \frac{\sigma_n}{a^{-2} P(\tau^{(0)} > a^{-2})} \leq \limsup_{a \rightarrow 0} \frac{\sigma_n}{a^{-2} P(\tau^{(0)} > a^{-2})} \leq \frac{\sqrt{2\pi}}{1 - C\sqrt{\varepsilon}}.$$

Since  $\varepsilon$  can be chosen arbitrarily small,

$$\sigma_n \sim \sqrt{2\pi} a^{-2} \mathbf{P}(\tau^{(0)} > a^{-2}). \quad (74)$$

Combining (70) and (74), and recalling definition (60) of  $\phi_n$ , we have

$$\mathbf{P}(\tau^{(a)} > n) \sim a^{-3} n^{-3/2} e^{-\xi(a)n} \mathbf{P}(\tau^{(0)} > a^{-2}). \quad (75)$$

Furthermore, it follows from (54) that

$$\mathbf{E} \tau^{(a)} \sim a^{-2} \mathbf{P}(\tau^{(0)} > a^{-2}) \int_0^\infty (1 - F_{2,0}(\sqrt{x})) x^{-1/2} dx.$$

Substituting  $\sqrt{x} = y$  and using (41), we obtain

$$\begin{aligned} \int_0^\infty (1 - F_{2,0}(\sqrt{x})) x^{-1/2} dx &= 2 \int_0^\infty (1 - F_{2,0}(y)) dy \\ &= 2 \int_0^\infty y \left( \int_y^\infty v^{-2} e^{-v^2/2} dv \right) dy \\ &= 2 \int_0^\infty v^{-2} e^{-v^2/2} \left( \int_0^v y dy \right) dv \\ &= \int_0^\infty e^{-v^2/2} dv \\ &= \sqrt{\frac{\pi}{2}}. \end{aligned}$$

As a result, we have

$$a^{-2} \mathbf{P}(\tau^{(0)} > a^{-2}) \sim \sqrt{\frac{2}{\pi}} \mathbf{E} \tau^{(a)}. \quad (76)$$

Combining (75) and (76), and noting that

$$\frac{1}{a\sqrt{2\pi n}} e^{-\xi(a)n} \sim \overline{\Phi}(a\sqrt{n}) \exp\{na^3 \lambda_m(a)\},$$

completes the proof.

### 5.3. Proof of Theorem 5

It is easy to see that there exist a constant  $C$  and a regularly varying function  $N(a)$  such that

$$\lim_{a \rightarrow 0} \frac{N(a)}{a^{-2} \log a^{-2}} = (p-2)^{1/2}, \quad (77)$$

and

$$\sup_{n \leq N(a)} \frac{n \mathbf{P}(X \geq na + \sqrt{n})}{\overline{\Phi}(a\sqrt{n})} \leq C \quad \text{and} \quad \sup_{n \geq N(a)} \frac{\overline{\Phi}(a\sqrt{n})}{n \mathbf{P}(X \geq na + \sqrt{n})} \leq C. \quad (78)$$

We now split the right-hand side of (10) into the product of two exponentials:

$$\begin{aligned} \exp \left\{ \sum_{n=1}^\infty \frac{z^n}{n} \mathbf{P}(S_n^{(a)} > 0) \right\} &= \exp \left\{ \sum_{n=1}^{N(a)} \frac{z^n}{n} \mathbf{P}(S_n^{(a)} > 0) \right\} \exp \left\{ \sum_{n=N(a)+1}^\infty \frac{z^n}{n} \mathbf{P}(S_n^{(a)} > 0) \right\} \\ &=: \left( \sum_{n=0}^\infty \psi_{1,n} z^n \right) \left( 1 + \sum_{n=N(a)+1}^\infty \psi_{2,n} z^n \right). \end{aligned}$$



Therefore,

$$P(\tau^{(a)} > n) = \psi_{1,n} + \sum_{k=N(a)+1}^n \psi_{1,n-k} \psi_{2,k}, \quad n \geq 1. \quad (79)$$

We first want to find the asymptotic behaviour of  $\psi_{2,n}$ . We start by noting that

$$\psi_{2,n} = \sum_{j=1}^{\infty} \frac{1}{j!} q_n^{*j}, \quad n > N(a), \quad (80)$$

where  $\{q_n^{*j}, n \geq 1\}$  is the  $j$ th convolution of  $\{n^{-1} P(S_n^{(a)} > 0) \mathbf{1}\{n > N(a)\}, n \geq 1\}$ . It follows from the second inequality in (78) that

$$\begin{aligned} q_n^{*2} &= \sum_{k=N(a)+1}^{n-N-1} \frac{1}{k} P(S_k^{(a)} > 0) \frac{1}{n-k} P(S_{n-k}^{(a)} > 0) \\ &\leq C \sum_{k=N(a)+1}^{n-N-1} P(X \geq ak) P(X \geq a(n-k)) \\ &\leq C P\left(X \geq \frac{an}{2}\right) \sum_{N(a)+1}^{n-N-1} P(X \geq ak) \\ &\leq C P(X \geq an) \int_{N(a)}^{\infty} P(X \geq ay) dy. \end{aligned}$$

Since  $P(X \geq y)$  is regularly varying, we have

$$\int_{N(a)}^{\infty} P(X \geq ay) dy = \frac{1}{a} \int_{aN(a)}^{\infty} P(X \geq y) dy \leq C N(a) P(X \geq aN(a)).$$

From this bound and (77) we obtain

$$q_n^{*2} \leq G(a) P(X \geq an),$$

where  $G$  is regularly varying with index  $p-2 > 0$ . Then, by induction,

$$q_n^{*j} \leq G(a) P(X \geq an) \quad \text{for all } j \geq 2. \quad (81)$$

Combining (80) and (81), and using (23) and (78), we obtain the bound

$$\begin{aligned} \psi_{2,n} &= P(S_n^{(a)} > 0) + \sum_{j=2}^{\infty} q_n^{*j} \\ &\leq C \left( \frac{1}{n} \overline{\Phi}(a\sqrt{n}) + P(X \geq an) + G(a) P(X \geq an) \right) \\ &\leq C P(X \geq an) \end{aligned} \quad (82)$$

and, for  $n \geq Ca^{-2} \log a^{-2}$  with some  $C > (p-2)^{1/2}$ , the relation

$$\psi_{2,n} = P(S_n^{(a)} > 0) + O(G(a) P(X \geq an)) \sim \frac{1}{n} \overline{\Phi}(a\sqrt{n}) + P(X \geq an) \sim P(X \geq an). \quad (83)$$

In the last step we have used the fact that  $\bar{\Phi}(a\sqrt{n}) = o(\mathbb{P}(X \geq an))$  for  $n \geq Ca^{-2} \log a^{-2}$ ,  $C > (p-2)^{1/2}$ .

From the first inequality in (78) and (23), which is valid under condition (26), we conclude that

$$\mathbb{P}(S_n^{(a)} > 0) \leq C\bar{\Phi}(a\sqrt{n})$$

for all  $n \leq N(a)$ . Using arguments from the proof of Theorem 4, we see that

$$\psi_{1,k} \leq \frac{C}{k} \bar{\Phi}(a\sqrt{k}), \quad k \geq 1. \quad (84)$$

Combining (82) and (84), and applying the second inequality in (78), we obtain

$$\begin{aligned} \sum_{k=N(a)}^{n-N(a)} \psi_{1,n-k} \psi_{2,k} &\leq C \sum_{k=N(a)}^{n-N(a)} \frac{1}{n-k} \bar{\Phi}(a\sqrt{n-k}) \mathbb{P}(X \geq ak) \\ &\leq C \sum_{k=N(a)}^{n-N(a)} \mathbb{P}(X \geq a(n-k)) \mathbb{P}(X \geq ak). \end{aligned}$$

In the derivation of (81) we showed that the sum in the last line is bounded by  $G(a) \mathbb{P}(X \geq an)$ . Hence,

$$\sum_{k=N(a)}^{n-N(a)} \psi_{1,n-k} \psi_{2,k} = O(G(a) \mathbb{P}(X \geq an)). \quad (85)$$

It follows from (10) and the definition of  $\{\psi_{1,n}, n \geq 1\}$  that  $\psi_{1,k} = \mathbb{P}(\tau^{(a)} > k)$  for all  $k \leq N(a)$ . Consequently,

$$\begin{aligned} \sum_{k=n-N(a)+1}^n \psi_{1,n-k} \psi_{2,k} &= \sum_{k=0}^{N(a)-1} \mathbb{P}(\tau^{(a)} > k) \psi_{2,n-k} \\ &= \sum_{k=0}^{\tilde{N}(a)-1} \mathbb{P}(\tau^{(a)} > k) \psi_{2,n-k} + \sum_{k=\tilde{N}(a)}^{N(a)-1} \mathbb{P}(\tau^{(a)} > k) \psi_{2,n-k}, \end{aligned}$$

where  $\tilde{N}(a)$  is such that  $a^{-2} \ll \tilde{N}(a) \ll a^{-2} \log a^{-2}$ . Applying (82) to the first sum and (83) to the second sum, we obtain

$$\begin{aligned} \sum_{k=n-N(a)+1}^n \psi_{1,n-k} \psi_{2,k} &= (1 + o(1)) \mathbb{P}(X \geq an) \sum_{k=0}^{\tilde{N}(a)-1} \mathbb{P}(\tau^{(a)} > k) \\ &\quad + O\left(\mathbb{P}(X \geq an) \sum_{k=\tilde{N}(a)}^{N(a)-1} \mathbb{P}(\tau^{(a)} > k)\right). \end{aligned}$$

Note that (51) implies that

$$\sum_{k=\tilde{N}(a)}^{\infty} \mathbb{P}(\tau^{(a)} > k) = o(\mathbb{E} \tau^{(a)}).$$

Hence, we finally obtain

$$\sum_{k=n-N(a)+1}^n \psi_{1,n-k} \psi_{2,k} \sim E \tau^{(a)} P(X \geq an). \quad (86)$$

Combining (79), (85), and (86), we have

$$P(\tau^{(a)} > n) = (1 + o(1)) E \tau^{(a)} P(X \geq an) + \psi_{1,n}.$$

In order to complete the proof, it remains to apply (84) and to note that

$$n^{-1} \overline{\Phi}(a\sqrt{n}) = o(P(X \geq an)).$$

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