TRANSITION PHENOMENA FOR LADDER EPOCHS OF RANDOM WALKS WITH SMALL NEGATIVE DRIFT

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Abstract

For a family of random walks $\{S^{(a)}\}$ satisfying $\mathrm{E}\, S_1^{(a)} = -a < 0$, we consider ladder epochs $\tau^{(a)} = \min\{k \ge 1\colon S_k^{(a)} < 0\}$. We study the asymptotic behaviour, as $a \to 0$, of $\mathrm{P}(\tau^{(a)} > n)$ in the case when $n = n(a) \to \infty$. As a consequence, we also obtain the growth rates of the moments of $\tau^{(a)}$.

Keywords: Random walk; ladder epoch; transition phenomena

2000 Mathematics Subject Classification: Primary 60G50
Secondary 60G52

1. Introduction and statement of results

1.1. Background and purpose

Let $X, X_1, X_2, \ldots$ be independent, identically distributed random variables. Let $S = \{S_n,\ n \ge 0\}$ denote the random walk with increments X_i, that is,

$$S_0 := 0, \qquad S_n := \sum_{i=1}^{n} X_i.$$

Let us first recall what is known about the first descending ladder epoch τ of S, i.e.

$$\tau := \min\{k \ge 1\colon S_k < 0\}. \tag{1}$$

It is well known (see, for example, [17, Theorem 17.1]) that

$$\mathrm{P}(\tau < \infty) = 1 \quad \Longleftrightarrow \quad \sum_{k=1}^{\infty} k^{-1}\,\mathrm{P}(S_k < 0) = \infty.$$

Under the latter condition, Rogozin [15] studied the asymptotic behaviour, as $n \to \infty$, of the tail probability $\mathrm{P}(\tau > n)$. In particular,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathrm{P}(S_k \ge 0) = \rho \in (0, 1] \quad \Longleftrightarrow \quad \mathrm{P}(\tau > n) = n^{\rho-1}\ell(n), \tag{2}$$

where ℓ is slowly varying at infinity. Also, $\lim_{n\to\infty}(1/n)\sum_{k=1}^{n}\mathrm{P}(S_k \ge 0) = 0$ is equivalent to the relative stability of τ. The latter means that the function $x \mapsto \int_0^x \mathrm{P}(\tau > u)\,\mathrm{d}u$ is slowly

Received 11 May 2009; revision received 11 October 2009.
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varying at infinity. But this statement does not give any information about the asymptotic behaviour of $\mathrm{P}(\tau > n)$ in this case.

The situation when $\mathrm{E}\,\tau < \infty$, which is a particular case of the relative stability, was considered by Embrechts and Hawkes [5]. There it was shown that

$$\mathrm{P}(\tau > n) \sim n^{-1}\,\mathrm{P}(S_n > 0)\exp\bigg\{\sum_{j=1}^{\infty} j^{-1}\,\mathrm{P}(S_j \ge 0)\bigg\},$$

under certain conditions on the sequence $\{\mathrm{P}(S_n > 0),\ n \ge 1\}$.

If the expectation $\mathrm{E}\,X$ is finite then the condition $\sum_{k=1}^{\infty} k^{-1}\,\mathrm{P}(S_k \le 0) = \infty$ is equivalent to the inequality $\mathrm{E}\,X \le 0$; see again [17, Theorem 17.1]. If $\mathrm{E}\,X = 0$ and X belongs to the domain of attraction of a stable law of index $\alpha > 1$, then $\lim_{n\to\infty}\mathrm{P}(S_n \ge 0) \in (0,1)$. (For details, see the paragraph after (11).) This yields $\lim_{n\to\infty}(1/n)\sum_{k=1}^{n}\mathrm{P}(S_k \ge 0) = \rho \in (0,1)$. Then, using (2), we conclude that

$$\mathrm{P}(\tau > n) = n^{\rho-1}\ell(n). \tag{3}$$

If $\mathrm{E}\,X < 0$ then $\mathrm{E}\,\tau$ is finite; see [17, Proposition 18.1]. In this case of negative drift, Doney [4] applied the results from [5] to two special classes of random walks. He showed that if $\mathrm{E}\,X \in (-\infty, 0)$ and $\mathrm{P}(X > x)$ is regularly varying at ∞ with index $\alpha < -1$, then, as $n \to \infty$,

$$\mathrm{P}(\tau > n) \sim \mathrm{E}\,\tau\,\mathrm{P}(X > -n\,\mathrm{E}\,X) \quad \text{as } n \to \infty. \tag{4}$$

Besides this regularly varying tail case, Doney found the asymptotics of $\mathrm{P}(\tau > n)$ for random walks having negative drift and satisfying the following condition. If the equation $(\mathrm{d}/\mathrm{d}h)\,\mathrm{E}\,\mathrm{e}^{hX} = 0$ has a positive solution, say h_0, then

$$\mathrm{P}(\tau > n) \sim C\bigg(\frac{\mathrm{E}\,\mu^{\tau} - 1}{\mu - 1}\bigg)\mu^{-n} n^{-3/2} \quad \text{as } n \to \infty, \tag{5}$$

where $\mu = 1/\mathrm{E}\,\mathrm{e}^{h_0 X}$ and C is a constant depending on $\mathrm{E}\,\mathrm{e}^{hX}$. The latter relation was generalised by Bertoin and Doney [2] to the case where $(\mathrm{d}/\mathrm{d}h)\,\mathrm{E}\,\mathrm{e}^{hX} < 0$ for all $h > 0$ such that $\mathrm{E}\,\mathrm{e}^{hX} < \infty$.

It should be noted that [2] and [4] are devoted to the study of the asymptotic behaviour of $\mathrm{P}(\tau_x > n)$ for any fixed $x \ge 0$, where $\tau_x := \min\{k \ge 1\colon S_k < -x\}$. The main result can be stated as follows. If X satisfies the conditions stated before (4) or (5), then there exists a function U such that

$$\lim_{n\to\infty}\frac{\mathrm{P}(\tau_x > n)}{\mathrm{P}(\tau > n)} \to U(x).$$

By studying the asymptotic behaviour, as $n \to \infty$, of $\mathrm{P}(\tau > n)$, we hope to get a good approximation for large but finite values of n. The quality of such an approximation depends on different parameters of the random walk. It follows from the papers mentioned above that the asymptotic behaviour of $\mathrm{P}(\tau > n)$ depends crucially on whether $\mathrm{E}\,X = 0$ or $\mathrm{E}\,X < 0$. Therefore, it would be very useful to clarify the influence of $\mathrm{E}\,X$ on $\mathrm{P}(\tau > n)$ in the case when that expectation is quite small. We illustrate the problem with the following concrete example. Let S be a random walk with $\mathrm{E}\,X = 10^{-3}$; we want to calculate the quantity $\mathrm{P}(\tau > 10^5)$. Here we have two possibilities. On the one hand, we can say that the expectation is so small that we may apply asymptotic relations for zero-mean random walks. On the other hand, we can say that the expectation is negative and we should use (4) or (5), depending on the tail behaviour of X. But how do we decide which approximation is better for these values of $\mathrm{E}\,X$ and n? This question leads to the following mathematical problem. What can be said about the asymptotic behaviour of $\mathrm{P}(\tau > n)$ in the case when $\mathrm{E}\,X \to 0$ and $n \to \infty$ simultaneously?

In the present paper we consider this problem in the case when the random walk's increment belongs to the domain of attraction of a stable law. We shall show that there exists a function f such that

(a) if $n \ll f(\mathrm{E}\,X)$ then we have to use (3),

(b) if $n \gg f(\mathrm{E}\,X)$ then we have to use formulae for random walks with negative drift,

(c) if $n \sim vf(\mathrm{E}\,X)$, $v \in (0, \infty)$, then we have to use (3), but with a correction factor depending on v.

The last point seems to be the most interesting one. It describes *transition phenomena* for the ladder epoch τ, which appear in the case of small drift.

Our main result, Theorem 1, is devoted to the study of this transition. There it will be clarified what the function f and the correction factor look like. As a consequence, we shall obtain the claim in (a). Furthermore, Theorem 1 allows us to determine the asymptotic behaviour, as $\mathrm{E}\,X \to 0$, of some moments $\mathrm{E}\,\tau^r$; see Theorem 2. The expectation $\mathrm{E}\,\tau$ is of particular interest, since it appears in asymptotic relations connected to the claim in (b); see Theorems 3, 4, and 5.

1.2. Transition phenomena

We start with a more precise description of our model of random walks with asymptotically small drift. We shall consider a family of random walks $\{S^{(a)},\ a \in [0, a_0]\}$ with drift $-a$, that is, $\mathrm{E}\,S_1^{(a)} = -a$, and investigate the asymptotic behaviour, as $a \to 0$, of the probability $\mathrm{P}(\tau^{(a)} > n)$ for $n = n(a)$, where $\tau^{(a)}$ is the first descending ladder epoch of $S^{(a)}$, as in (1).

Let $X^{(a)}$ denote a random variable that is distributed as the increments of the random walk $S^{(a)}$. It is easy to see that if $X^{(a)}$ converges in distribution, as $a \to 0$, to $X^{(0)}$ then, for every fixed n,

$$\mathrm{P}(\tau^{(a)} > n) \sim \mathrm{P}(\tau^{(0)} > n) \quad \text{as } a \to 0. \tag{6}$$

A more interesting problem consists in investigating the asymptotic behaviour of the tail probability $\mathrm{P}(\tau^{(a)} > n)$ when $n = n(a) \to \infty$ as $a \to 0$. The solution to this problem depends on the structure of the family $\{S^{(a)},\ a \in [0, a_0]\}$.

In this paper we shall assume that there exists a random variable X with zero mean such that the random variables $X^{(a)}$ and $X - a$ have the same distribution for all $a \in [0, a_0]$. Then the random variables $S_n^{(a)}$ and $S_n^{(0)} - na$ are equal in distribution for all $a \in [0, a_0]$ and $n \geq 1$. Furthermore, we restrict ourselves from now on to so-called *asymptotically stable random walks*. Namely, we shall always assume that the distribution of X belongs to the domain of attraction of a stable law with characteristic function

$$G_{\alpha,\beta}(t) := \exp\left\{-|t|^\alpha \left(1 - \mathrm{i}\beta \frac{t}{|t|} \tan \frac{\pi\alpha}{2}\right)\right\} \tag{7}$$

with $\alpha \in (1, 2]$ and $|\beta| \leq 1$. In this case we write $X \in \mathcal{D}(\alpha, \beta)$.

Let $\{c_n,\ n \geq 1\}$ denote the sequence of positive integers specified by the relation

$$c_n := \inf\{u \geq 0\colon u^{-2}V(u) \leq n^{-1}\}, \tag{8}$$

where

$$V(u) := \int_{-u}^{u} x^2\, \mathrm{P}(X \in \mathrm{d}x), \qquad u > 0.$$

It is known (see, for instance, [6, Chapter XVII, Section 5]) that the function V is regularly varying at infinity with index $2-\alpha$ for every $X \in \mathcal{D}(\alpha, \beta)$. This implies that $\{c_n,\ n \geq 1\}$ is regularly varying with index α^{-1}, i.e. there exists a function l_1, slowly varying at infinity, such that

$$c_n = n^{1/\alpha} l_1(n). \tag{9}$$

In addition, the scaled sequence $\{S_n^{(0)}/c_n,\ n \geq 1\}$ converges in distribution, as $n \to \infty$, to the stable law corresponding to $G_{\alpha,\beta}$ in (7).

Let $\{Y_{\alpha,\beta}(t),\ t \geq 0\}$ denote a stable Lévy process such that $Y_{\alpha,\beta}(1)$ is distributed according to (7).

It is known (see Proposition 17.5 of [17]) that the generating function of the sequence $\{\mathrm{P}(\tau^{(a)} > n),\ n \geq 0\}$ satisfies the identity

$$\sum_{n=0}^{\infty} \mathrm{P}(\tau^{(a)} > n) z^n = \exp\left\{\sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{P}(S_n^{(a)} \geq 0)\right\}, \qquad z \in (0,1). \tag{10}$$

Thus, for every $n \geq 1$, the probability $\mathrm{P}(\tau^{(a)} > n)$ is determined by $\{\mathrm{P}(S_k^{(a)} \geq 0),\ 1 \leq k \leq n\}$. From the definition of the family $S^{(a)}$ and from the asymptotic stability of $\{S_n^{(0)},\ n \geq 0\}$, we conclude that

$$\mathrm{P}(S_n^{(a)} \geq 0) \sim \mathrm{P}(S_n^{(0)} \geq 0) \sim \mathrm{P}(Y_{\alpha,\beta}(1) \geq 0) =: \rho \tag{11}$$

for $n = n(a) \to \infty$ satisfying $na/c_n \to 0$. It is known (see [18]) that

$$\rho = \frac{1}{2} + \frac{1}{\pi\alpha} \arctan\left(\beta \tan \frac{\pi\alpha}{2}\right)$$

for all $\alpha \in (1,2]$ and $|\beta| \leq 1$. We can easily verify that $\rho \in (0,1)$ for all $\alpha \in (1,2]$ and $|\beta| \leq 1$. Hence, we can expect that

$$\mathrm{P}(\tau^{(a)} > n) \sim \mathrm{P}(\tau^{(0)} > n) = n^{\rho-1}\ell(n), \tag{12}$$

where in the second step we have used (2). Furthermore, if $na/c_n \to u \in (0,\infty)$ then

$$\mathrm{P}(S_n^{(a)} \geq 0) \sim \mathrm{P}(Y_{\alpha,\beta}(1) \geq u) > 0.$$

In this case we expect, although this conjecture is not as obvious as (12), that

$$\mathrm{P}(\tau^{(a)} > n) \sim \mathrm{P}(\tau^{(0)} > n) G(u) \tag{13}$$

for some function G.

The following theorem confirms conjectures (12) and (13).

Theorem 1. *Suppose that $X \in \mathcal{D}(\alpha, \beta)$. If $n = n(a)$ is such that*

$$\lim_{a \to 0} \frac{an}{c_n} = u \in [0, \infty) \tag{14}$$

then

$$\lim_{a \to 0} \frac{\mathrm{P}(\tau^{(a)} > n)}{\mathrm{P}(\tau^{(0)} > n)} = 1 - F_{\alpha,\beta}(u), \tag{15}$$

where the distribution function $F_{\alpha,\beta}$ can be described by the equality

$$\int_0^\infty e^{-\lambda x} x^{\rho-1}(1 - F_{\alpha,\beta}(x^{1-1/\alpha}))\,dx = C\exp\biggl\{-\int_0^\infty \frac{1-e^{-\lambda t}}{t} P(Y_{\alpha,\beta}(t) - t > 0)\,dt\biggr\}, \qquad \lambda \ge 0 \tag{16}$$

with ρ defined as in (11) and C specified by the condition $F_{\alpha,\beta}(0) = 0$.

The existence of the limit in (15) is an easy consequence of the invariance principle for random walks conditioned to stay positive, which was proved in [3]. The most difficult part of the proof is the derivation of characterisation (16) of the limiting distribution $F_{\alpha,\beta}$; see Section 3.

It follows from (9) that (14) is equivalent to

$$n \sim u^{\alpha/(\alpha-1)}\biggl(\frac{1}{a}\biggr)^{\alpha/(\alpha-1)} l^*\biggl(\frac{1}{a}\biggr) \quad \text{as } a \to 0,$$

where l^* is slowly varying at infinity, which is determined by l_1. Therefore, the statement of Theorem 1 can be reformulated as follows. If $n = n(a)$ satisfies

$$n \sim v\biggl(\frac{1}{a}\biggr)^{\alpha/(\alpha-1)} l^*\biggl(\frac{1}{a}\biggr) \quad \text{as } a \to 0 \tag{17}$$

for some $v \ge 0$ then

$$\lim_{a\to 0} \frac{P(\tau^{(a)} > n)}{P(\tau^{(0)} > n)} = 1 - F_{\alpha,\beta}(v^{1-1/\alpha}). \tag{18}$$

In particular, if (17) holds with $v = 0$ then $P(\tau^{(a)} > n) \sim P(\tau^{(0)} > n)$. Roughly speaking, (3) gives a rather good approximation in the case when n is much smaller than $(1/a)^{\alpha/(\alpha-1)} l^*(1/a)$. But if $(1/a)^{\alpha/(\alpha-1)} l^*(1/a)$ and n are comparable, then we have to use a correction factor, given by the right-hand side of (18). To calculate this correction for concrete values of v, we need to know the form of the distribution function $F_{\alpha,\beta}$. We are able to give an explicit expression for $F_{\alpha,\beta}$ only in some special cases. We shall see in the proof of Theorem 1 that

$$1 - F_{\alpha,\beta}(u) = P\Bigl(\inf_{t\le 1}(M_{\alpha,\beta}(t) - ut) \ge 0\Bigr),$$

where $\{M_{\alpha,\beta}(t),\ t \in [0, 1]\}$ is the meander of $Y_{\alpha,\beta}$. Using the construction of the meander via the limit of conditioned distributions of the original process $Y_{\alpha,\beta}$, we shall show that

$$1 - F_{2,0}(u) = u\int_u^\infty v^{-2} e^{-v^2/2}\,dv$$

and

$$1 - F_{\alpha,1}(u) = \frac{u^{1/(\alpha-1)}}{(\alpha-1)g_{\alpha,1}(0)}\int_u^\infty v^{-\alpha/(\alpha-1)} g_{\alpha,1}(v)\,dv, \quad \alpha \in (1, 2),$$

where $g_{\alpha,\beta}$ denotes the density function of the random variable $Y_{\alpha,\beta}(1)$. For all other values of α and β, the explicit form of $F_{\alpha,\beta}$ remains unknown.

Remark 1. The expression on the right-hand side of (16) is known (see [1, p. 168]) to be the Laplace transform of the random variable

$$T_{\max} := \sup\Big\{t > 0\colon Y_{\alpha,\beta}(t) - t = \max_{u\ge 0}(Y_{\alpha,\beta}(u) - u)\Big\}.$$

Let $f_{\max}$ denote the density function of this random variable. Then from (16) we can obtain the equality

$$1 - F_{\alpha,\beta}(x) = Cx^{\alpha(1-\rho)/(\alpha-1)} f_{\max}(x^{\alpha/(\alpha-1)}), \qquad x > 0.$$

Having this relation we can obtain the explicit form of $f_{\max}$ in the case of Brownian motion ($\alpha = 2$ and $\beta = 0$) and in the case of spectrally positive Lévy processes ($\alpha \in (1, 2)$ and $\beta = 1$).

We now turn our attention to the moments of $\tau^{(a)}$.

It was shown by Gut [7] that the condition $\mathrm{E}(\max\{0, X\})^r < \infty$ for some $r > 0$ is necessary and sufficient for the finiteness of $\mathrm{E}(\tau^{(a)})^r$. Therefore, the condition $X \in \mathcal{D}(\alpha, \beta)$ yields the finiteness of $\mathrm{E}(\tau^{(a)})^r$ for all $r < \alpha$.

From the bound

$$\mathrm{P}(\tau^{(a)} > n) \le \mathrm{P}(\tau^{(0)} > n) \quad \text{for all } n \ge 0$$

and (6), using dominated convergence, we infer that

$$\lim_{a\to 0} \mathrm{E}(\tau^{(a)})^r = \mathrm{E}(\tau^{(0)})^r < \infty$$

for all $r \in (0, 1 - \rho)$. Furthermore, it easily follows from Theorem 1 and (12) that

$$\lim_{a\to 0} \mathrm{E}(\tau^{(a)})^r = \infty \quad \text{for all } r > 1 - \rho.$$

Theorem 1 allows us to determine the rate of growth as $a \to 0$ of $\mathrm{E}(\tau^{(a)})^r$ for $r \in (1 - \rho, \alpha)$.

Theorem 2. *Suppose that $X \in \mathcal{D}(\alpha, \beta)$. Then, for every $r \in (1-\rho, \alpha)$, there exists a function L_r slowly varying at infinity such that*

$$\mathrm{E}(\tau^{(a)})^r = L_r\left(\frac{1}{a}\right) a^{-\alpha(r+\rho-1)/(\alpha-1)}, \tag{19}$$

with ρ defined as in (11).

This is already known in some particular cases, which we now mention.

First of all we note that if the second moment of X is finite then, applying dominated convergence, we can show that $\mathrm{E}\, S^{(a)}_{\tau^{(a)}} \to \mathrm{E}\, S^{(0)}_{\tau^{(0)}}$ as $a \to 0$. Thus, using the Wald identity and the well-known equality (see [17, Proposition 18.5])

$$-\mathrm{E}\, S^{(0)}_{\tau^{(0)}} = \frac{(\mathrm{E}\, X^2)^{1/2}}{\sqrt{2}} \exp\left\{\sum_{k=1}^{\infty} k^{-1}\left(\mathrm{P}(S^{(0)}_k \ge 0) - \frac{1}{2}\right)\right\},$$

we obtain, as $a \to 0$,

$$\mathrm{E}\, \tau^{(a)} \sim \frac{-\mathrm{E}\, S^{(0)}_{\tau^{(0)}}}{a} = \frac{(\mathrm{E}\, X^2)^{1/2}}{a\sqrt{2}} \exp\left\{\sum_{k=1}^{\infty} k^{-1}\left(\mathrm{P}(S^{(0)}_k \ge 0) - \frac{1}{2}\right)\right\}.$$

Furthermore, the asymptotic behaviour of $\mathrm{E}\,\tau^{(a)}$ in the case of a non-Gaussian stable limit, that is, $\alpha < 2$, was recently studied by Lotov [9]. He proved that

$$\mathrm{E}\,\tau^{(a)} = a^{-\alpha\rho/(\alpha-1)+o(1)} \quad \text{as } a \downarrow 0$$

in this case. Moreover, he showed that (19) with $r = 1$ holds under the additional condition that

$$\sum_{k=1}^{\infty} \frac{1}{k} \sup_{x\in\mathbb{R}} |\mathrm{P}(S_k^{(0)} > c_k x) - \mathrm{P}(Y_{\alpha,\beta} > x)| < \infty.$$

Having expressions for the expectation $\mathrm{E}\,\tau^{(a)}$ we can describe the asymptotic behaviour of some further characteristics of the random walk $\{S_n^{(a)},\ n \geq 0\}$. First, from the Wald identity and Theorem 2, we obtain the equality

$$\mathrm{E}\,S_{\tau^{(a)}}^{(a)} = -a\,\mathrm{E}\,\tau^{(a)} = -L_1\left(\frac{1}{a}\right) a^{1-\alpha\rho/(\alpha-1)}.$$

Second, it is well known that the stopping time $\tau_+^{(a)} := \min\{k \geq 1\colon S_k^{(a)} \geq 0\}$ is infinite with positive probability and that $\mathrm{P}(\tau_+^{(a)} = \infty) = 1/\mathrm{E}\,\tau^{(a)}$. Then, using Theorem 2 once again, we obtain

$$\mathrm{P}(\tau_+^{(a)} = \infty) = \frac{a^{\alpha\rho/(\alpha-1)}}{L_1(1/a)}.$$

To conclude this subsection, we note that our assumption that the distributions of $X^{(a)}$ and $X - a$ are equal can be weakened. First of all we note that if $X^{(a)}$ satisfies the conditions

$$\mathrm{E}\,X^{(a)} = -a \quad \text{and} \quad \lim_{a\to 0} \mathrm{E}(X^{(a)})^2 = \sigma^2 \in (0, \infty),$$

then the results of the present subsection still hold. Moreover, in the case of an infinite second moment, the results of the present subsection remain valid if $X^{(a)} = X - a + Y^{(a)}$ in distribution, where $X \in \mathcal{D}(\alpha, \beta)$ for some $\alpha \in (1, 2)$ and $Y^{(a)}$ is such that

$$\mathrm{E}\,Y^{(a)} = 0, \qquad Y^{(a)} \to 0 \text{ in law and } \sup_{a\in[0,a_0]} \mathrm{E}\,|Y^{(a)}|^{\alpha+\delta} < \infty \text{ for some } \delta > 0.$$

We did not use these generalisations in the statements of our theorems because of results in the next subsection, where we need the assumption that $X^{(a)} = X - a$ in law.

1.3. Results on large deviations

If $na/c_n \to \infty$ then Theorem 1 says only that

$$\mathrm{P}(\tau^{(a)} > n) = o(\mathrm{P}(\tau^{(0)} > n)) \quad \text{as } a \to 0.$$

Our next purpose is to refine this relation and to find the rate of divergence of $\mathrm{P}(\tau^{(a)} > n)$ in the abovementioned domain of *large deviations* for $\tau^{(a)}$. To proceed in this situation, we need to know the asymptotic behaviour of $\mathrm{P}(S_n^{(a)} > 0)$ for $na/c_n \to \infty$. It follows from the definition of $S_n^{(a)}$ that $\mathrm{P}(S_n^{(a)} > 0) = \mathrm{P}(S_n^{(0)} > na)$. Thus, the assumption that $na/c_n \to \infty$ means that we are in the domain of large deviations for $S_n^{(0)}$. Since the behaviour of large deviation probabilities depends crucially on whether the limit of $S_n^{(0)}/c_n$ is Gaussian or strictly stable, i.e. $\alpha \in (1, 2)$, we consider these two cases separately.

If $S_n^{(0)}$ belongs to the domain of attraction of a strictly stable law then, as is well known,

$$\mathrm{P}(S_n^{(0)} \geq x_n) \sim n\,\mathrm{P}(X \geq x_n)$$

for any sequence x_n satisfying $x_n/c_n \to \infty$. This relation allows us to obtain the following result.

Theorem 3. *Suppose that* $X \in \mathcal{D}(\alpha, \beta)$ *for some* $1 < \alpha < 2$ *and* $\beta > -1$. *If* $n = n(a)$ *is such that* $na/c_n \to \infty$ *then*

$$\mathrm{P}(\tau^{(a)} > n) \sim \mathrm{E}\,\tau^{(a)}\,\mathrm{P}(X \geq na) \quad \text{as } a \to 0. \tag{20}$$

The right-hand side of (20) coincides with that of (4). Roughly speaking, if n is very large then the asymptotic behaviour of $\mathrm{P}(\tau^{(a)} > n)$ is as in the case of the fixed negative drift. But there is one crucial difference between fixed and asymptotically small drift: the expectation $\mathrm{E}\,\tau^{(a)}$ grows unbounded if $a \to 0$, and is a constant when the drift is fixed. Therefore, (20) would be useless without Theorem 2.

We turn our attention to the case when $\sigma^2 := \mathrm{E}\,X^2$ is finite. Here we shall assume without loss of generality that $\sigma^2 = 1$. Under this condition, we have $c_n = \sqrt{n}$. Then the condition that $an/c_n \to \infty$ reads as $na^2 \to \infty$. In this case of finite variance the asymptotic behaviour of $\mathrm{P}(S_n^{(0)} > x_n)$ depends not only on the tail behaviour of X, but also on the rate of the growth of x_n. If x_n does not grow very fast ($x_n = o(r_1(n))$ for some $r_1(n)$ depending on the distribution of X) then we have an asymptotic expression for $\mathrm{P}(S_n^{(0)} > x_n)$ in terms of the so-called Cramér series (for the definition of the Cramér series see, for example, [14, Chapter VIII]). For this type of large deviation, we have the following result.

Theorem 4. *Assume that* $\mathrm{E}\,X^2 = 1$, $n = n(a)$ *is such that* $na^2 \to \infty$, *and that*

$$\mathrm{P}(S_j^{(0)} \geq ja) \sim \overline{\Phi}(\sqrt{j}a)\exp\{ja^3\lambda_m(a)\} \quad \text{uniformly in } j \in [a^{-2}, n], \tag{21}$$

where $\lambda_m(u)$ *is the partial sum in the Cramér series containing the first* m *terms and* $\overline{\Phi}(x) := \int_x^\infty (1/\sqrt{2\pi})\mathrm{e}^{-u^2/2}\,\mathrm{d}u$. *Then*

$$\mathrm{P}(\tau^{(a)} > n) \sim 2\,\mathrm{E}\,\tau^{(a)}\frac{1}{n}\overline{\Phi}(\sqrt{n}a)\exp\{na^3\lambda_m(a)\}.$$

Condition (21) has one essential disadvantage: it involves the whole sequence $\{S_k^{(0)},\ k \geq 0\}$. We now list some restrictions on the distribution of X, which imply the validity of (21).

Nagaev [11] proved that the condition $\mathrm{E}\,|X|^k < \infty$ with some $k > 2$ implies that the relation

$$\mathrm{P}(S_n^{(0)} \geq x) \sim \overline{\Phi}\left(\frac{x}{\sqrt{n}}\right) \quad \text{as } n \to \infty \tag{22}$$

holds uniformly in $x \leq \sqrt{(k/2 - 1)n\log n}$. Thus, the existence of $\mathrm{E}\,|X|^k$ for some $k > 2$ yields (21) with $m = 0$ for all n satisfying

$$n \leq \left(\tfrac{1}{2}k - 1\right)a^{-2}\log a^{-2}.$$

Furthermore, it has been proved by Nagaev [10] and Rozovskii [16] that if $\mathrm{P}(X > x)$ is regularly varying at infinity with index $p < -2$ then, under some additional restrictions on the left tail,

$$\mathrm{P}(S_n^{(0)} \geq x) \sim \overline{\Phi}\left(\frac{x}{\sqrt{n}}\right) + n\,\mathrm{P}(X > x + \sqrt{n}) \quad \text{uniformly on } x > 0. \tag{23}$$

Thus, (22) holds for all $x \le C\sqrt{n \log n}$ for any $C < (p-2)^{1/2}$. Consequently, (21) with $m = 0$ holds for

$$n \le Ca^{-2} \log a^{-2}, \qquad C < (p-2)^{1/2}.$$

Osipov [13] found necessary and sufficient conditions under which the relation

$$\mathrm{P}(S_n^{(0)} \ge x) \sim \overline{\Phi}\left(\frac{x}{\sqrt{n}}\right) \exp\left\{\frac{x^3}{n^2}\lambda_{[1/(1-\gamma)]}\left(\frac{x}{n}\right)\right\}$$

holds uniformly in $0 \le x \le n^\gamma$, $\frac{1}{2} < \gamma < 1$, where $[t]$ denotes the integer part of t. If these conditions are fulfilled then, obviously, (21) holds with $m = [1/(1-\gamma)]$ for all $n \le a^{1/(1-\gamma)}$.

It is well known that if X satisfies the Cramér condition ($\mathrm{E}\,\mathrm{e}^{h|X|} < \infty$ for some $h > 0$) then (21) holds with $m = \infty$ and for all n satisfying $na^2 \to \infty$. Thus, Theorems 1 and 4 describe the behaviour of $\mathrm{P}(\tau^{(a)} > n)$ for any choice of $n = n(a)$ and any random walk satisfying the Cramér condition.

It is easy to see that the statement of Theorem 4 can be rewritten as follows. If (21) holds then

$$\mathrm{P}(\tau^{(a)} > n) \sim \frac{2}{\sqrt{2\pi}} a^{-1}\,\mathrm{E}\,\tau^{(a)} n^{-3/2} \mathrm{e}^{-n\xi(a)},$$

where

$$\xi(a) := \frac{a^2}{2} - a^3 \lambda_m(a). \tag{24}$$

Furthermore, in the proof of Theorem 4 we shall see that

$$\frac{\mathrm{E}(\mathrm{e}^{\xi(a)\tau^{(a)}},\ \tau^{(a)} \le n) - 1}{\mathrm{e}^{\xi(a)} - 1} \sim 2\,\mathrm{E}\,\tau^{(a)}.$$

Thus,

$$\mathrm{P}(\tau^{(a)} > n) \sim \frac{1}{a\sqrt{2\pi}} \frac{\mathrm{E}(\mathrm{e}^{\xi(a)\tau^{(a)}},\ \tau^{(a)} \le n) - 1}{\mathrm{e}^{\xi(a)} - 1} n^{-3/2} \mathrm{e}^{-n\xi(a)},$$

which is rather close to relation (5). If, additionally, X satisfies the Cramér condition, implying (21) with $m = \infty$, then we can replace the truncated expectation $\mathrm{E}(\mathrm{e}^{\xi(a)\tau^{(a)}},\ \tau^{(a)} \le n)$ by $\mathrm{E}(\mathrm{e}^{\xi(a)\tau^{(a)}})$:

$$\mathrm{P}(\tau^{(a)} > n) \sim \frac{1}{a\sqrt{2\pi}} \frac{\mathrm{E}(\mathrm{e}^{\xi(a)\tau^{(a)}}) - 1}{\mathrm{e}^{\xi(a)} - 1} n^{-3/2} \mathrm{e}^{-n\xi(a)}. \tag{25}$$

It follows from the definition of the Cramér series that $\xi(a)$, defined in (24), is the unique positive solution to the equation $(\mathrm{d}/\mathrm{d}h)\,\mathrm{E}\,\mathrm{e}^{hX^{(a)}} = 0$. Therefore, (25) is an analog of (5) for random walks with vanishing drift.

Another type of large deviation behaviour appears in the case when x_n grows fast, i.e. $x_n \gg r_2(n)$ and the tail of X varies in an appropriate way. (Recall that $a_n \gg b_n$ means that $a_n/b_n \to \infty$.) Here, as in the case of the non-Gaussian stable limit, we have $\mathrm{P}(S_n^{(0)} \ge x_n) \sim n\,\mathrm{P}(X \ge x_n)$. We consider only the case when the tail of X is regularly varying.

Theorem 5. *Assume that* $\mathrm{P}(X \ge x)$ *is regularly varying at infinity with index* $p < -2$ *and*

$$\int_{|x|>y} x^2\,\mathrm{P}(X \in \mathrm{d}x) = o\left(\frac{1}{\log y}\right) \quad \textit{as } y \to \infty. \tag{26}$$

Then, as $a \to 0$,

$$\mathrm{P}(\tau^{(a)} > n) \sim \mathrm{E}\,\tau^{(a)}\,\mathrm{P}(X \ge na)$$

for any $n = n(a)$ *satisfying the inequality* $n(a) \ge Ca^{-2}\log a^{-2}$ *with some* $C > (p-2)^{1/2}$.

After Theorem 4 we mentioned that, in the case of regularly varying tails, (21) holds for all $n \le Ca^{-2}\log a^{-2}$, $C < (p-2)^{1/2}$. Therefore, the behaviour of $\mathrm{P}(\tau^{(a)} > n)$ remains unclear only for n satisfying $(na^2/\log a^{-2}) \to (p-2)^{1/2}$. We conjecture that if the conditions of Theorem 5 hold then, in agreement with (23),

$$\mathrm{P}(\tau^{(a)} > n) \sim 2\,\mathrm{E}\,\tau^{(a)}\frac{1}{n}\overline{\Phi}(\sqrt{n}a) + \mathrm{E}\,\tau^{(a)}\,\mathrm{P}(X \ge \sqrt{n} + na)$$

for all n satisfying $na^2 \to \infty$.

The remaining part of the paper is organised as follows. In the next section we derive an upper bound for the probability $\mathrm{P}(\tau^{(a)} > n)$, which is crucial for the proof of Theorem 2. This proof will be given in Section 4. Section 3 is devoted to the proof of Theorem 1. Finally, Theorems 3, 4, and 5 will be proved in Section 4.

2. Upper bounds for the tail of $\tau^{(a)}$

It follows from (10) that in order to obtain upper bounds for $\mathrm{P}(\tau^{(a)} > n)$ we need inequalities for $\mathrm{P}(S_n^{(a)} \ge 0) = \mathrm{P}(S_n^{(0)} \ge na)$. In the following lemma we adapt one of the well-known Fuk–Nagaev inequalities for our purposes.

Lemma 1. *Assume that* $X \in \mathcal{D}(\alpha, \beta)$. *Then there exists a constant* C *such that the inequality*

$$\mathrm{P}(S_n^{(0)} \ge x) \le n\,\mathrm{P}\left(X \ge \frac{x}{3}\right) + C\left(\frac{nV(x)}{x^2}\right)^2$$

holds for all $x > 0$ *and* $n \ge 1$.

Proof. Applying Theorem 1.2 of [12] with $t = 2$ we have

$$\mathrm{P}(S_n^{(0)} \ge x) \le n\,\mathrm{P}(X \ge y) + \mathrm{e}^{x/y}\left(\frac{nV(y)}{xy}\right)^{x/y + nV(y)/y^2 - n\mu(y)/y}, \tag{27}$$

where $\mu(y) := \mathrm{E}(X, |X| \le y)$.

Since $\mathrm{E}\,X = 0$,

$$|\mu(y)| = \left|\int_{|x|>y} x\,\mathrm{P}(X \in \mathrm{d}x)\right| \le \int_{x>y} x\,\mathrm{P}(|X| \in \mathrm{d}x) = y\,\mathrm{P}(|X| > y) + \int_y^\infty \mathrm{P}(|X| > x)\,\mathrm{d}x.$$

It is well known that the assumption $X \in \mathcal{D}(\alpha, \beta)$ yields

$$\lim_{x\to\infty} \frac{x^2\,\mathrm{P}(|X| > x)}{V(x)} = \frac{2-\alpha}{\alpha}. \tag{28}$$

Therefore, as $y \to \infty$,

$$|\mu(y)| \le \left(\frac{2-\alpha}{\alpha} + o(1)\right)\left(\frac{V(y)}{y} + \int_y^\infty \frac{V(x)}{x^2}\,\mathrm{d}x\right) = \left(\frac{2-\alpha}{\alpha-1} + o(1)\right)\frac{V(y)}{y}.$$

In the last step we used the relation

$$\int_y^\infty \frac{V(x)}{x^2}\,\mathrm{d}x \sim \frac{1}{\alpha-1}\frac{V(y)}{y} \quad \text{as } y\to\infty,$$

which follows from the fact that $V(x)$ is regularly varying with index $2-\alpha$. As a result, we have the bound

$$\frac{V(y)}{y^2} - \frac{\mu(y)}{y} \ge \left(\frac{2\alpha-3}{\alpha-1}+o(1)\right)\frac{V(y)}{y^2}.$$

It follows from definition (8) of the sequence $\{c_n\}$ that $V(c_n)/c_n^2 \sim n^{-1}$ as $n\to\infty$. Consequently, there exists a constant $C(\alpha)$ such that

$$\frac{V(y)}{y^2} - \frac{\mu(y)}{y} \ge -\frac{1}{n}$$

for all $y > C(\alpha)c_n$. From this bound and (27) with $y = x/3$ we obtain

$$\mathrm{P}(S_n^{(0)} \ge x) \le n\,\mathrm{P}\left(X \ge \frac{x}{3}\right) + 27\mathrm{e}^3\left(\frac{nV(y)}{x^2}\right)^2, \qquad x \ge 3C(\alpha)c_n.$$

This inequality, together with monotonicity of V, implies that the desired result holds for $x > C(\alpha)c_n$. Noting that

$$\min_{n\ge 1}\inf_{x\le 3C(\alpha)c_n}\frac{nV(x)}{x^2} > 0,$$

we complete the proof of the lemma.

In order to 'translate' bounds for $\mathrm{P}(S_n^{(0)} > na)$ into bounds for $\mathrm{P}(\tau^{(a)} > n)$, we shall use the recurrent relation

$$n\,\mathrm{P}(\tau^{(a)} > n) = \sum_{j=0}^{n-1}\mathrm{P}(\tau^{(a)} > j)\,\mathrm{P}(S_{n-j}^{(0)} > (n-j)a), \tag{29}$$

which can be obtained by differentiating (10).

Proposition 1. *The inequality* $\mathrm{P}(\tau^{(a)} > n) \le C\,\mathrm{E}\,\tau^{(a)}V(na)/(na)^2$ *is valid for all* $a>0$ *and all* $n \ge n_a := \min\{n\ge 1\colon an > c_n\}$.

Proof. Using Lemma 1, we have

$$\begin{aligned}
&\sum_{0\le j<n/2}\mathrm{P}(\tau^{(a)} > j)\,\mathrm{P}(S_{n-j}^{(0)} > (n-j)a)\\
&\quad\le \sum_{0\le j<n/2}\mathrm{P}(\tau^{(a)} > j)\left((n-j)\,\mathrm{P}\left(X \ge \frac{(n-j)a}{3}\right) + C\left(\frac{(n-j)V((n-j)a)}{((n-j)a)^2}\right)^2\right)\\
&\quad\le \left(n\,\mathrm{P}\left(X\ge\frac{na}{6}\right) + C\left(\frac{nV(na)}{(na)^2}\right)^2\right)\sum_{0\le j<n/2}\mathrm{P}(\tau^{(a)} > j)\\
&\quad\le \mathrm{E}\,\tau^{(a)}\left(n\,\mathrm{P}\left(X\ge\frac{na}{6}\right) + C\left(\frac{nV(na)}{(na)^2}\right)^2\right)\\
&\quad\le n\,\mathrm{E}\,\tau^{(a)}\left(\mathrm{P}\left(X\ge\frac{na}{6}\right) + C\frac{V(na)}{(na)^2}\right).
\end{aligned} \tag{30}$$

In the last step we used definition (8) of c_n and the bound $an \geq c_n$, which follows from the assumption that $n \geq n_a$.

Furthermore, using the Markov inequality, we obtain

$$\sum_{n/2\leq j\leq n-1} \mathrm{P}(\tau^{(a)} > j)\,\mathrm{P}(S^{(0)}_{n-j} > (n-j)a) \leq \frac{2\,\mathrm{E}\,\tau^{(a)}}{n}\sum_{k=1}^{n}\mathrm{P}(S^{(0)}_k \geq ka). \tag{31}$$

Applying Lemma 1, we obtain

$$\begin{aligned}\sum_{k=1}^{n}\mathrm{P}(S^{(0)}_k \geq ka) &\leq n_a + \sum_{k=n_a}^{n}\mathrm{P}(S^{(0)}_k \geq ka)\\ &\leq n_a + \sum_{k=n_a}^{n} k\,\mathrm{P}\left(X \geq \frac{ka}{3}\right) + C\sum_{k=n_a}^{n}\frac{V^2(ka)}{k^2a^4}.\end{aligned} \tag{32}$$

Since $V(x)$ is regularly varying with index $2-\alpha$,

$$\begin{aligned}\sum_{k=n_a}^{n}\frac{V^2(ka)}{k^2a^4} &\leq Ca^{-2}\sum_{k=n_a}^{n}\frac{V^2(ka)}{(ka)^2}\\ &\leq Ca^{-3}\int_{an_a}^{an}\frac{V^2(x)}{x^2}\,\mathrm{d}x\\ &\leq Ca^{-3}V(an)\int_{an_a}^{an}\frac{V(x)}{x^2}\,\mathrm{d}x\\ &\leq Ca^{-3}V(an)\frac{V(an_a)}{an_a}.\end{aligned} \tag{33}$$

From the definitions of c_n and n_a, we infer that

$$V(an_a) \sim V(c_{n_a}) \sim \frac{c^2_{n_a}}{n_a} \sim a^2 n_a. \tag{34}$$

Applying this relation to the last line in (33), we obtain the bound

$$\sum_{k=n_a}^{n}\frac{V^2(ka)}{k^2a^4} \leq Ca^{-2}V(an). \tag{35}$$

Furthermore,

$$\begin{aligned}\sum_{k=n_a}^{n} k\,\mathrm{P}\left(X \geq \frac{ka}{3}\right) &\leq Ca^{-2}\int_{an_a}^{an} x\,\mathrm{P}\left(X > \frac{x}{3}\right)\mathrm{d}x\\ &\leq a^{-2}\int_0^{na} x\,\mathrm{P}(|X| > x)\,\mathrm{d}x\\ &= \frac{a^{-2}}{2}(V(an) + (an)^2\,\mathrm{P}(|X| > an)),\end{aligned} \tag{36}$$

where in the last step we used integration by parts. Combining (32), (35), and (36), we have

$$\sum_{k=1}^{n}\mathrm{P}(S^{(0)}_k \geq ka) \leq Cn_a + Ca^{-2}V(an) + n^2\,\mathrm{P}(|X| > an). \tag{37}$$

It is easy to see that (34) yields $n_a \sim a^{-2}V(an_a)$. From this relation and monotonicity of $V(x)$, we conclude that $n_a \le Ca^{-2}V(an)$ for all $n \ge n_a$. Applying this bound to (37) we obtain

$$\sum_{k=1}^{n} \mathrm{P}(S_k^{(0)} \ge ka) \le Ca^{-2}V(an) + n^2\,\mathrm{P}(|X| > an). \tag{38}$$

Combining (30), (31), and (38), we arrive at the inequality

$$\sum_{j=0}^{n} \mathrm{P}(\tau^{(a)} > j)\,\mathrm{P}(S_{n-j}^{(0)} > (n-j)a) \le Cn\,\mathrm{E}\,\tau^{(a)}\biggl(\mathrm{P}\Bigl(|X| \ge \frac{na}{6}\Bigr) + \frac{V(na)}{(na)^2}\biggr). \tag{39}$$

It follows from (28) that

$$\mathrm{P}(|X| > x) \le C\frac{V(x)}{x^2}.$$

Therefore, the right-hand side of (39) is bounded by $Cn\,\mathrm{E}\,\tau^{(a)}V(na)/(na)^2$. Thus, the statement of the proposition follows from (29).

3. Proof of Theorem 1

From the definition of the first ladder epoch $\tau^{(a)}$ we obtain

$$\begin{aligned}
\mathrm{P}(\tau^{(a)} > n) &= \mathrm{P}\Bigl(\min_{1\le k\le n}(S_k^{(0)} - ka) > 0\Bigr)\\
&= \mathrm{P}\Bigl(\min_{1\le k\le n} S_k^{(0)} > 0\Bigr)\,\mathrm{P}\Bigl(\min_{1\le k\le n}(S_k^{(0)} - ka) > 0 \Bigm| \min_{1\le k\le n} S_k^{(0)} > 0\Bigr)\\
&= \mathrm{P}(\tau^{(0)} > n)\,\mathrm{P}\biggl(\min_{1\le k\le n}\Bigl(\frac{S_k^{(0)}}{c_n} - \frac{k}{n}\frac{an}{c_n}\Bigr) > 0 \biggm| \min_{1\le k\le n} S_k^{(0)} > 0\biggr).
\end{aligned}$$

Doney [3] showed that $\{S_{[tc_n]}^{(0)}/c_n,\ t \in [0,1] \mid \min_{1\le k\le n} S_k^{(0)} > 0\}$ converges weakly, as $n \to \infty$, to the Lévy meander $\{M_{\alpha,\beta}(t),\ t \in [0,1]\}$. This yields

$$\begin{aligned}
\lim_{n\to\infty} \mathrm{P}\biggl(\min_{1\le k\le n}\Bigl(\frac{S_k^{(0)}}{c_n} - \frac{k}{n}\frac{an}{c_n}\Bigr) > 0 \biggm| \min_{1\le k\le n} S_k^{(0)} > 0\biggr) &= \mathrm{P}\Bigl(\min_{0\le t\le 1}(M_{\alpha,\beta}(t) - ut) > 0\Bigr)\\
&=: 1 - F_{\alpha,\beta}(u).
\end{aligned} \tag{40}$$

It is obvious that $F_{\alpha,\beta}(u)$ is monotonously increasing and $\lim_{u\to\infty} F_{\alpha,\beta}(u) = 1$.

It is known that the corresponding meander $M_{\alpha,\beta}$ can be defined as a weak limit, as $\varepsilon \to 0$, of $Y_{\alpha,\beta}$ starting from $\varepsilon > 0$ and conditioned to stay positive up to time 1:

$$\begin{aligned}
&\mathcal{L}\{M_{\alpha,\beta}(t),\ t \in [0,1]\}\\
&\quad = \lim_{\varepsilon\to 0} \mathcal{L}\Bigl\{Y_{\alpha,\beta}(t),\ t \in [0,1] \Bigm| \inf_{0\le t\le 1} Y_{\alpha,\beta}(t) > 0,\ Y_{\alpha,\beta}(0) = \varepsilon\Bigr\}.
\end{aligned}$$

Therefore,

$$1 - F_{\alpha,\beta}(u) = \lim_{\varepsilon\to 0} \frac{\mathrm{P}(\inf_{0\le t\le 1}(Y_{\alpha,\beta}(t) - ut) > 0 \mid Y_{\alpha,\beta}(0) = \varepsilon)}{\mathrm{P}(\inf_{0\le t\le 1} Y_{\alpha,\beta}(t) > 0 \mid Y_{\alpha,\beta}(0) = \varepsilon)}.$$

Define $H^{(u)}_{\alpha,\beta}(z) := \min\{t\colon Y_{\alpha,\beta}(t) - ut \le z \mid Y_{\alpha,\beta}(0) = 0\}$. Then

$$1 - F_{\alpha,\beta}(u) = \lim_{\varepsilon\to 0} \frac{\mathrm{P}(H^{(u)}_{\alpha,\beta}(-\varepsilon) > 1)}{\mathrm{P}(H^{(0)}_{\alpha,\beta}(-\varepsilon) > 1)}.$$

In the case of the Brownian motion, that is, $\alpha = 2$ and $\beta = 0$, we can calculate the limit explicitly. Indeed, it is known that $H^{(u)}_{2,0}(-\varepsilon)$ has the density

$$\frac{\varepsilon}{\sqrt{2\pi}t^{3/2}} \exp\biggl\{-\frac{(ut-\varepsilon)^2}{2t}\biggr\}, \qquad t > 0.$$

Thus, as $\varepsilon \to 0$,

$$\mathrm{P}(H^{(0)}_{2,0}(-\varepsilon) > 1) = \frac{\varepsilon}{\sqrt{2\pi}} \int_1^\infty t^{-3/2} \mathrm{e}^{-\varepsilon^2/2t}\,\mathrm{d}t \sim \frac{2\varepsilon}{\sqrt{2\pi}},$$

and, consequently,

$$\begin{aligned}\lim_{\varepsilon\to 0} \frac{\mathrm{P}(H^{(u)}_{\alpha,\beta}(-\varepsilon) > 1)}{\mathrm{P}(H^{(0)}_{\alpha,\beta}(-\varepsilon) > 1)} &= \lim_{\varepsilon\to 0} \int_1^\infty \frac{1}{2t^{3/2}} \exp\biggl\{-\frac{(ut-\varepsilon)^2}{2t}\biggr\}\,\mathrm{d}t \\ &= \int_1^\infty \frac{1}{2t^{3/2}} \mathrm{e}^{-u^2t/2}\,\mathrm{d}t \\ &= u\int_u^\infty v^{-2}\mathrm{e}^{-v^2/2}\,\mathrm{d}v.\end{aligned}$$

As a result, we have

$$1 - F_{2,0}(u) = u\int_u^\infty v^{-2}\mathrm{e}^{-v^2/2}\,\mathrm{d}v. \tag{41}$$

This equality can be generalised to stable Lévy processes without negative jumps, i.e. $\{\alpha \in (1,2), \beta = 1\}$ or $\{\alpha = 2, \beta = 0\}$. Indeed, using Kendall's equality (see [8]) and the scaling property of stable processes, we see that $H^{(u)}_{\alpha,1}(-\varepsilon)$ has the density

$$u \mapsto \frac{\varepsilon}{t^{1+1/\alpha}} g_{\alpha,1}\biggl(\frac{-\varepsilon + ut}{t^{1/\alpha}}\biggr).$$

Then, analogously to the case of the Brownian motion,

$$1 - F_{\alpha,1}(u) = \frac{u^{1/(\alpha-1)}}{(\alpha-1)g_{\alpha,1}(0)} \int_u^\infty v^{-\alpha/(\alpha-1)} g_{\alpha,1}(v)\,\mathrm{d}v.$$

Unfortunately, we cannot give an explicit expression for $1 - F_{\alpha,\beta}$ for a process with positive jumps. But we can describe this function via the Laplace transform of $x^{\rho-1}(1 - F_{\alpha,\beta}(x^{1-1/\alpha}))$.

In order to prove (16), we show that $1 - F_{\alpha,\beta}$ satisfies a certain integral equation. Dividing both parts of (29) by $n\,\mathrm{P}(\tau^{(0)} > n)$, we have

$$\frac{\mathrm{P}(\tau^{(a)} > n)}{\mathrm{P}(\tau^{(0)} > n)} = \sum_{j=0}^{n-1} \frac{\mathrm{P}(\tau^{(a)} > j)}{\mathrm{P}(\tau^{(0)} > j)} \frac{\mathrm{P}(\tau^{(0)} > j)}{\mathrm{P}(\tau^{(0)} > n)} \mathrm{P}(S^{(0)}_{n-j} \ge a(n-j))\frac{1}{n}. \tag{42}$$

Fix any $\varepsilon \in (0, \frac{1}{2})$. We first note that

$$\sum_{0\le j\le \varepsilon n} \frac{\mathrm{P}(\tau^{(a)} > j)}{\mathrm{P}(\tau^{(0)} > j)} \frac{\mathrm{P}(\tau^{(0)} > j)}{\mathrm{P}(\tau^{(0)} > n)} \mathrm{P}(S^{(0)}_{n-j} \ge a(n-j))\frac{1}{n} \le \frac{\sum_{0\le j\le \varepsilon n} \mathrm{P}(\tau^{(0)} > j)}{n\,\mathrm{P}(\tau^{(0)} > n)} \le C\varepsilon^{\rho} \tag{43}$$

and

$$\sum_{(1-\varepsilon)n\le j\le n-1} \frac{\mathrm{P}(\tau^{(a)} > j)}{\mathrm{P}(\tau^{(0)} > j)} \frac{\mathrm{P}(\tau^{(0)} > j)}{\mathrm{P}(\tau^{(0)} > n)} \mathrm{P}(S^{(0)}_{n-j} \ge a(n-j))\frac{1}{n} \le \frac{\mathrm{P}(\tau^{(0)} > n/2)}{n\,\mathrm{P}(\tau^{(0)} > n)}\varepsilon n \le C\varepsilon. \tag{44}$$

In both bounds we have used the fact that $\mathrm{P}(\tau^{(0)} > j)$ varies regularly with index $\rho - 1$.

It remains to consider the middle part of the sum on the right-hand side of (42). It is easy to see that the condition $an/c_n \to u$ implies that

$$\frac{aj}{c_j} \to ut^{1-1/\alpha} \quad \text{as } a \to 0,$$

provided that $j \sim tn$. Then, in view of (40), for every $t \in (0, 1)$, the following is valid. As $a \to 0$,

$$\begin{aligned} f_a(t) &:= \frac{\mathrm{P}(\tau^{(a)} > [tn])}{\mathrm{P}(\tau^{(0)} > [tn])} \frac{\mathrm{P}(\tau^{(0)} > [tn])}{\mathrm{P}(\tau^{(0)} > n)} \mathrm{P}(S^{(0)}_{n-[tn]} \ge a(n-[tn])) \\ &\to (1 - F_{\alpha,\beta}(ut^{1-1/\alpha}))t^{\rho-1}\,\mathrm{P}(Y_{\alpha,\beta}(1) > u(1-t)^{1-1/\alpha}). \end{aligned}$$

Thus, by dominated convergence,

$$\begin{aligned} &\lim_{a\to 0} \sum_{\varepsilon n< j<(1-\varepsilon)n} \frac{\mathrm{P}(\tau^{(a)} > j)}{\mathrm{P}(\tau^{(0)} > j)} \frac{\mathrm{P}(\tau^{(0)} > j)}{\mathrm{P}(\tau^{(0)} > n)} \mathrm{P}(S^{(0)}_{n-j} \ge a(n-j))\frac{1}{n} \\ &\quad = \int_{\varepsilon}^{1-\varepsilon} (1 - F_{\alpha,\beta}(ut^{1-1/\alpha}))t^{\rho-1}\,\mathrm{P}(Y_{\alpha,\beta}(1) > u(1-t)^{1-1/\alpha})\,\mathrm{d}t. \end{aligned}$$

Now using monotone convergence, we obtain

$$\begin{aligned} &\lim_{\varepsilon\to 0}\lim_{a\to 0} \sum_{\varepsilon n< j<(1-\varepsilon)n} \frac{\mathrm{P}(\tau^{(a)} > j)}{\mathrm{P}(\tau^{(0)} > j)} \frac{\mathrm{P}(\tau^{(0)} > j)}{\mathrm{P}(\tau^{(0)} > n)} \mathrm{P}(S^{(0)}_{n-j} \ge a(n-j))\frac{1}{n} \\ &\quad = \int_0^1 (1 - F_{\alpha,\beta}(ut^{1-1/\alpha}))t^{\rho-1}\,\mathrm{P}(Y_{\alpha,\beta}(1) > u(1-t)^{1-1/\alpha})\,\mathrm{d}t. \end{aligned} \tag{45}$$

Combining (42)–(45), and taking into account (40), we obtain

$$1 - F_{\alpha,\beta}(u) = \int_0^1 (1 - F_{\alpha,\beta}(ut^{1-1/\alpha}))t^{\rho-1}\,\mathrm{P}(Y_{\alpha,\beta}(1) > u(1-t)^{1-1/\alpha})\,\mathrm{d}t. \tag{46}$$

Setting

$$G_{\alpha,\beta}(u) := 1 - F_{\alpha,\beta}(u^{1-1/\alpha}) \quad \text{and} \quad \xi_{\alpha,\beta} := (Y_{\alpha,\beta}(1))^{\alpha/(\alpha-1)},$$

we can rewrite (46) as follows:

$$G_{\alpha,\beta}(u) = \int_0^1 G_{\alpha,\beta}(ut)t^{\rho-1}\,\mathrm{P}(\xi_{\alpha,\beta} > u(1-t))\,\mathrm{d}t.$$

Substituting $t = y/u$, we have

$$G_{\alpha,\beta}(u) = u^{-\rho} \int_0^u G_{\alpha,\beta}(y) y^{\rho-1} \mathrm{P}(\xi_{\alpha,\beta} > u - y)\,\mathrm{d}y.$$

Therefore, the function $Q_{\alpha,\beta}(u) := u^{\rho-1} G_{\alpha,\beta}(u)$ satisfies the equation

$$u Q_{\alpha,\beta}(u) = \int_0^u Q_{\alpha,\beta}(y)\,\mathrm{P}(\xi_{\alpha,\beta} > u - y)\,\mathrm{d}y. \tag{47}$$

Let $q_{\alpha,\beta}(\lambda)$ denote the Laplace transform of the function $Q_{\alpha,\beta}$, i.e.

$$q_{\alpha,\beta}(\lambda) = \int_0^\infty \mathrm{e}^{-\lambda u} Q_{\alpha,\beta}(u)\,\mathrm{d}u, \qquad \lambda > 0.$$

Now (47) implies that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\lambda} q_{\alpha,\beta}(\lambda) &= -\int_0^\infty u\mathrm{e}^{-\lambda u} Q_{\alpha,\beta}(u)\,\mathrm{d}u \\
&= -\int_0^\infty \mathrm{e}^{-\lambda u} \int_0^u Q_{\alpha,\beta}(y)\,\mathrm{P}(\xi_{\alpha,\beta} > u - y)\,\mathrm{d}y \\
&= -\int_0^\infty \mathrm{e}^{-\lambda u} Q_{\alpha,\beta}(u)\,\mathrm{d}u \int_0^\infty \mathrm{e}^{-\lambda z}\,\mathrm{P}(\xi_{\alpha,\beta} > z)\,\mathrm{d}z \\
&= -q_{\alpha,\beta}(\lambda) \int_0^\infty \mathrm{e}^{-\lambda z}\,\mathrm{P}(\xi_{\alpha,\beta} > z)\,\mathrm{d}z.
\end{aligned}$$

Solving this differential equation, we see that

$$\begin{aligned}
q_{\alpha,\beta}(\lambda) &= q_{\alpha,\beta}(\lambda_0) \exp\left\{-\int_{\lambda_0}^{\lambda} \int_0^\infty \mathrm{e}^{-\lambda z}\,\mathrm{P}(\xi_{\alpha,\beta} > z)\,\mathrm{d}z\right\} \\
&= q_{\alpha,\beta}(\lambda_0) \exp\left\{-\int_0^\infty \frac{\mathrm{e}^{-\lambda_0 z} - \mathrm{e}^{-\lambda z}}{z}\,\mathrm{P}(\xi_{\alpha,\beta} > z)\,\mathrm{d}z\right\}.
\end{aligned}$$

It follows from the definition of $\xi_{\alpha,\beta}$ that

$$\mathrm{P}(\xi_{\alpha,\beta} > z) = \mathrm{P}(Y_{\alpha,\beta}(1) > z^{1-1/\alpha}) \sim \frac{C}{z^{\alpha-1}} \quad \text{as } z \to \infty.$$

This relation yields

$$\int_1^\infty \frac{1}{z}\,\mathrm{P}(\xi_{\alpha,\beta} > z)\,\mathrm{d}z < \infty.$$

Therefore,

$$\begin{aligned}
&\int_0^\infty \frac{\mathrm{e}^{-\lambda_0 z} - \mathrm{e}^{-\lambda z}}{z}\,\mathrm{P}(\xi_{\alpha,\beta} > z)\,\mathrm{d}z \\
&\quad = \int_0^\infty \frac{1 - \mathrm{e}^{-\lambda z}}{z}\,\mathrm{P}(\xi_{\alpha,\beta} > z)\,\mathrm{d}z - \int_0^\infty \frac{1 - \mathrm{e}^{-\lambda_0 z}}{z}\,\mathrm{P}(\xi_{\alpha,\beta} > z)\,\mathrm{d}z.
\end{aligned}$$

Consequently,

$$q_{\alpha,\beta}(\lambda) = C \exp\left\{-\int_0^\infty \frac{1 - \mathrm{e}^{-\lambda z}}{z}\,\mathrm{P}(\xi_{\alpha,\beta} > z)\,\mathrm{d}z\right\}.$$

To complete the proof of the theorem, it remains to note that, in view of the scaling property of $Y_{\alpha,\beta}$,

$$\mathrm{P}(\xi_{\alpha,\beta} > z) = \mathrm{P}(Y_{\alpha,\beta}(1) > z^{1-1/\alpha}) = \mathrm{P}(Y_{\alpha,\beta}(z) - z > 0).$$

4. Proof of Theorem 2

For every $\varepsilon \in (0, 1)$,

$$\mathrm{E}(\tau^{(a)})^r = \sum_{n=0}^{\infty}[(n+1)^r - n^r]\,\mathrm{P}(\tau^{(a)} > n) = \Sigma_1 + \Sigma_2 + \Sigma_3, \tag{48}$$

where

$$\Sigma_1 := \sum_{0 \le n \le \varepsilon n_a} [(n+1)^r - n^r]\,\mathrm{P}(\tau^{(a)} > n),$$
$$\Sigma_2 := \sum_{\varepsilon n_a < n < n_a/\varepsilon} [(n+1)^r - n^r]\,\mathrm{P}(\tau^{(a)} > n),$$
$$\Sigma_3 := \sum_{n \ge n_a/\varepsilon} [(n+1)^r - n^r]\,\mathrm{P}(\tau^{(a)} > n).$$

Since $[(n+1)^r - n^r] \le Cn^{r-1}$,

$$\Sigma_1 \le C \sum_{0 \le n \le \varepsilon n_a} n^{r-1}\,\mathrm{P}(\tau^{(0)} > n) \le C\varepsilon^{\rho+r-1} n_a^r\,\mathrm{P}(\tau^{(0)} > n_a). \tag{49}$$

In the last step we used the fact that $\mathrm{P}(\tau^{(0)} > n)$ is regularly varying with index $\rho - 1$.

Furthermore, in view of (40),

$$\begin{aligned}
\psi_a(r; x) &:= (([xn_a] + 1)^r - ([xn_a])^r)\frac{\mathrm{P}(\tau^{(a)} > [xn_a])}{n_a^{r-1}\,\mathrm{P}(\tau^{(0)} > n_a)} \\
&= \frac{([xn_a] + 1)^r - ([xn_a])^r}{n_a^{r-1}} \frac{\mathrm{P}(\tau^{(a)} > [xn_a])}{\mathrm{P}(\tau^{(0)} > [xn_a])} \frac{\mathrm{P}(\tau^{(0)} > [xn_a])}{\mathrm{P}(\tau^{(0)} > n_a)} \\
&\to rx^{r-1}(1 - F_{\alpha,\beta}(x^{1-1/\alpha}))x^{\rho-1} \quad \text{as } a \to 0.
\end{aligned}$$

Then, by dominated convergence,

$$\begin{aligned}
\lim_{a\to 0} \frac{\Sigma_2}{n_a^r\,\mathrm{P}(\tau^{(0)} > n_a)} &= \lim_{a\to 0} \int_{\varepsilon}^{1/\varepsilon} \psi_a(r; x)\,\mathrm{d}x \\
&= \int_{\varepsilon}^{1/\varepsilon} x^{r-1}(1 - F_{\alpha,\beta}(x^{1-1/\alpha}))x^{\rho-1}\,\mathrm{d}x.
\end{aligned} \tag{50}$$

In view of Proposition 1,

$$\Sigma_3 \le C\,\mathrm{E}\,\tau^{(a)} \sum_{n \ge n_a/\varepsilon} n^{r-1}\frac{V(na)}{(na)^2}.$$

Since $V(x)$ varies regularly,

$$\begin{aligned}
\sum_{n \ge n_a/\varepsilon} n^{r-1}\frac{V(na)}{(na)^2} &\sim a^{-r} \int_{an_a/\varepsilon}^{\infty} x^{r-3} V(x)\,\mathrm{d}x \\
&\sim (\alpha - r)^{-1}\varepsilon^{\alpha-r} a^{-r}(an_a)^{r-2} V(an_a) \\
&\sim (\alpha - r)^{-1}\varepsilon^{\alpha-r} n_a^r \frac{V(an_a)}{(an_a)} \\
&\sim (\alpha - r)^{-1}\varepsilon^{\alpha-r} n_a^{r-1}.
\end{aligned}$$

Here we used the relations

$$an_a \sim c_{n_a} \quad \text{as } a \to 0$$

and

$$c_n^{-2} V(c_n) \sim n^{-1} \quad \text{as } n \to \infty.$$

Consequently,

$$\Sigma_3 \le C\varepsilon^{\alpha - r}\, \mathrm{E}\,\tau^{(a)} n_a^{r-1}. \tag{51}$$

Substituting (49)–(51) with $r = 1$ into (48) with $r = 1$, we have

$$\limsup_{a\to 0} \frac{\mathrm{E}\,\tau^{(a)}}{n_a\, \mathrm{P}(\tau^{(0)} > n_a)} \le \frac{1}{1 - C\varepsilon^{\alpha-1}} \left(\int_\varepsilon^{1/\varepsilon} (1 - F_{\alpha,\beta}(x^{1-1/\alpha}))x^{\rho-1}\, \mathrm{d}x + C\varepsilon^\rho \right).$$

Thus,

$$\mathrm{E}\,\tau^{(a)} \le Cn_a\, \mathrm{P}(\tau^{(0)} > n_a).$$

Applying this inequality to (51) we find that

$$\Sigma_3 \le C\varepsilon^{\alpha - r} n_a^r\, \mathrm{P}(\tau^{(0)} > n_a). \tag{52}$$

Combining (48)–(50) and (52), we obtain

$$\liminf_{a\to 0} \frac{\mathrm{E}(\tau^{(a)})^r}{n_a^r\, \mathrm{P}(\tau^{(0)} > n_a)} \ge \int_\varepsilon^{1/\varepsilon} x^{r-1}(1 - F_{\alpha,\beta}(x^{1-1/\alpha}))x^{\rho-1}\, \mathrm{d}x$$

and

$$\limsup_{a\to 0} \frac{\mathrm{E}(\tau^{(a)})^r}{n_a^r\, \mathrm{P}(\tau^{(0)} > n_a)} \le \int_\varepsilon^{1/\varepsilon} x^{r-1}(1 - F_{\alpha,\beta}(x^{1-1/\alpha}))x^{\rho-1}\, \mathrm{d}x + C\varepsilon^{\rho+r-1} + C\varepsilon^{\alpha-r}.$$

The latter inequality yields

$$\limsup_{a\to 0} \frac{\mathrm{E}(\tau^{(a)})^r}{n_a^r\, \mathrm{P}(\tau^{(0)} > n_a)} < \infty. \tag{53}$$

Hence, letting $\varepsilon \to 0$,

$$\lim_{a\to 0} \frac{\mathrm{E}(\tau^{(a)})^r}{n_a^r\, \mathrm{P}(\tau^{(0)} > n_a)} = \int_0^\infty x^{r-1}(1 - F_{\alpha,\beta}(x^{1-1/\alpha}))x^{\rho-1}\, \mathrm{d}x. \tag{54}$$

The integral $\int_0^\infty x^{r-1}(1 - F_{\alpha,\beta}(x^{1-1/\alpha}))x^{\rho-1}\, \mathrm{d}x$ is finite in view of (53). Noting now that $n_a^r\, \mathrm{P}(\tau^{(0)} > n_a)$ is regularly varying with index $-\alpha(\rho + r - 1)/(\alpha - 1)$, we complete the proof of the theorem.

5. Proofs of large deviation results

5.1. Proof of Theorem 3

Since $an/c_n \to \infty$, there exists $N(n)$ satisfying

$$\frac{aN(n)}{c_n} \to \infty \quad \text{and} \quad (n) = o(n).$$

We now split the sum in (29) into two parts:

$$\Sigma_1 := \sum_{k=0}^{N(n)} \mathrm{P}(\tau^{(a)} > k)\,\mathrm{P}(S^{(0)}_{n-k} > (n-k)a),$$
$$\Sigma_2 := \sum_{k=N(n)+1}^{n-1} \mathrm{P}(\tau^{(a)} > k)\,\mathrm{P}(S^{(0)}_{n-k} > (n-k)a).$$

Since

$$\lim_{j\to\infty} \sup_{x>q_j c_j} \left| \frac{\mathrm{P}(S^{(0)}_j > x)}{j\,\mathrm{P}(X > x)} - 1 \right| = 0$$

for any sequence $q_j \uparrow \infty$, we obtain the relation

$$\begin{aligned}\Sigma_1 &= (1+o(1))n\,\mathrm{P}(X > na) \sum_{k=0}^{N(n)} \mathrm{P}(\tau^{(a)} > k)\\ &= (1+o(1))n\,\mathrm{P}(X > na)\Bigl(\mathrm{E}\,\tau^{(a)} - \sum_{k=N(n)+1}^{n-1} \mathrm{P}(\tau^{(a)} > k)\Bigr).\end{aligned} \tag{55}$$

Noting that $N(n) \gg n_a$ and taking into account (51), we see that

$$\sum_{k=N(n)+1}^{n-1} \mathrm{P}(\tau^{(a)} > k) = o(\mathrm{E}\,\tau^{(a)}). \tag{56}$$

Combining (55) and (56), we have

$$\Sigma_1 = (1+o(1))n\,\mathrm{E}\,\tau^{(a)}\,\mathrm{P}(X > na). \tag{57}$$

We now turn our attention to Σ_2. It follows from Proposition 1 that

$$\begin{aligned}\Sigma_2 &\le \mathrm{P}(\tau^{(a)} > N(n)) \sum_{j=1}^{n} \mathrm{P}(S^{(0)}_j > aj)\\ &\le C\,\mathrm{E}\,\tau^{(a)} \frac{V(aN(n))}{(aN(n))^2} \sum_{j=1}^{n} \mathrm{P}(S^{(0)}_j > aj).\end{aligned}$$

Furthermore, using (32), we obtain

$$\Sigma_2 \le \mathrm{E}\,\tau^{(a)} \frac{V(aN(n))}{(aN(n))^2} n^2\,\mathrm{P}(|X| \ge na). \tag{58}$$

From the definition of c_n and the relation $aN(n) \gg c_n$, we conclude that

$$\frac{V(aN(n))}{(aN(n))^2} = o\Bigl(\frac{1}{n}\Bigr).$$

Moreover, $\mathrm{P}(|X| \ge na) \le C\,\mathrm{P}(X \ge na)$ for every $X \in \mathcal{D}(\alpha,\beta)$ with $\alpha < 2$ and $\beta > -1$. Then, (58) implies that

$$\Sigma_2 = o(n\,\mathrm{E}\,\tau^{(a)}\,\mathrm{P}(X > na)). \tag{59}$$

Substituting (57) and (59) into (29) completes the proof.

5.2. Proof of Theorem 4

Recall definition (24) of $\xi(a)$. Set

$$\phi_j := \mathrm{e}^{\xi(a)j}\,\mathrm{P}(\tau^{(a)} > j) \quad \text{and} \quad \theta_j := \mathrm{e}^{\xi(a)j}\,\mathrm{P}(S_j^{(0)} > aj). \tag{60}$$

It is easily seen that

$$\theta_j \le C \quad \text{for all } j \le 1/a^2. \tag{61}$$

Furthermore, combining (21) with the relations

$$\overline{\Phi}(x) \le \frac{1}{x\sqrt{2\pi}}\mathrm{e}^{-x^2/2}$$

and

$$\overline{\Phi}(x) \sim \frac{1}{x\sqrt{2\pi}}\mathrm{e}^{-x^2/2} \quad \text{as } x \to \infty,$$

we obtain

$$\theta_j \le \frac{C}{a\sqrt{j}} \quad \text{for } j \le n \tag{62}$$

and

$$\theta_j \sim \frac{1}{a\sqrt{2\pi j}} \quad \text{for } j \le n \text{ and } ja^2 \to \infty, \tag{63}$$

respectively.

Multiplying both sides of (29) by $\mathrm{e}^{a^2n/2}$, we see that the sequence ϕ_j satisfies the equation

$$k\phi_k = \sum_{j=0}^{k-1}\phi_j\theta_{k-j}, \qquad k \ge 1. \tag{64}$$

If n satisfies the conditions of the theorem then, using (61) and (62), we have

$$\sup_{n\ge 1}\max_{j\le n}\theta_j < \infty.$$

Consequently,

$$\phi_k \le \frac{C}{k}\sum_{j=o}^{k}\phi_j \le \frac{C}{k}\sum_{j=o}^{n}\phi_j$$

for all $k \le n$. Setting $\sigma_n := \sum_{j=0}^n \phi_j$, we rewrite the latter bound as

$$\phi_k \le \frac{C}{k}\sigma_n, \qquad k \le n.$$

Now, applying this bound and (62) to the terms on the right-hand side of (64), we obtain, for all $k \le n$, the bound

$$\begin{aligned}
\phi_k &= \frac{1}{k}\sum_{0\le j<k/2}\phi_j\theta_{k-j} + \frac{1}{k}\sum_{k/2\le j<k}\phi_j\theta_{k-j}\\
&\le \frac{C}{ak^{3/2}}\sum_{0\le j<k/2}\phi_j + \frac{C\sigma_n}{k^2}\sum_{k/2\le j<k}\theta_{k-j}\\
&\le \frac{C\sigma_n}{ak^{3/2}} + \frac{C\sigma_n}{k^2}\sum_{1\le j<k}\frac{1}{a\sqrt{j}}\\
&\le \frac{C\sigma_n}{ak^{3/2}}.
\end{aligned}\tag{65}$$

This inequality allows us to determine the asymptotic behaviour of ϕ_n. First of all we note that (63) yields

$$\sum_{0\le j\le N(n)} \phi_j\theta_{n-j} \sim \frac{1}{a\sqrt{2\pi n}} \sum_{0\le j\le N(n)} \phi_j \quad \text{as } a \to 0$$

for every $N(n) = o(n)$. Moreover, by (65),

$$0 \le \sigma_n - \sum_{0\le j\le N(n)} \phi_j = \sum_{N(n)<j\le n} \phi_j \le \frac{C\sigma_n}{aN(n)}. \tag{66}$$

Therefore, choosing $N(n)$ satisfying

$$N(n) = o(n) \quad \text{and} \quad aN^2(n) \to \infty,$$

we have, as $a \to 0$,

$$\sum_{0\le j\le N(n)} \phi_j\theta_{n-j} \sim \frac{\sigma_n}{a\sqrt{2\pi n}}. \tag{67}$$

Furthermore, it follows from (62) and (65) that

$$\begin{aligned}\sum_{N(n)<j<n/2} \phi_j\theta_{n-j} &\le \frac{C}{a\sqrt{n}} \sum_{N(n)<j<n/2} \phi_j \\ &\le \frac{C}{a\sqrt{n}} \sum_{N(n)<j<n/2} \frac{\sigma_n}{aj^{3/2}} \\ &\le \frac{C\sigma_n}{a^2\sqrt{nN(n)}} \end{aligned} \tag{68}$$

and

$$\sum_{n/2\le j<n} \phi_j\theta_{n-j} \le \frac{C\sigma_n}{an^{3/2}} \sum_{j=1}^{n} \theta_j \le \frac{C\sigma_n}{an^{3/2}} \sum_{j=1}^{n} \frac{1}{a\sqrt{j}} \le \frac{C\sigma_n}{a^2 n}. \tag{69}$$

Combining (67)–(69) and recalling that $a^2N(n) \to \infty$, we obtain

$$\sum_{j=0}^{n-1} \phi_j\theta_{n-j} \sim \frac{\sigma_n}{a\sqrt{2\pi n}} \quad \text{as } a \to 0.$$

Substituting this into (64) we have

$$\phi_n \sim \frac{\sigma_n}{a\sqrt{2\pi} n^{3/2}} \quad \text{as } a \to 0. \tag{70}$$

To complete the proof of the theorem, it remains to find the asymptotic behaviour of σ_n. First of all, (66) implies that the bounds

$$\sum_{j\le 1/\varepsilon a^2} \phi_j \le \sigma_n \le (1 - C\sqrt{\varepsilon})^{-1} \sum_{j\le 1/\varepsilon a^2} \phi_j \tag{71}$$

are valid for all sufficiently small values of ε. Applying Theorem 1 and recalling that $\mathrm{P}(\tau^{(0)} > j)$ is regularly varying with index $-\frac{1}{2}$, we see that

$$\lim_{a\to 0}\frac{\phi_{[xa^{-2}]}}{\mathrm{P}(\tau^{(0)} > a^{-2})} = \lim_{a\to 0}\frac{\mathrm{e}^{[xa^{-2}]\xi(a)}\,\mathrm{P}(\tau^{(a)} > [xa^{-2}])}{\mathrm{P}(\tau^{(0)} > [xa^{-2}])}\frac{\mathrm{P}(\tau^{(0)} > [xa^{-2}])}{\mathrm{P}(\tau^{(0)} > a^{-2})}$$
$$= \mathrm{e}^{x/2}(1 - F_{2,0}(\sqrt{x}))\frac{1}{\sqrt{x}}$$

for every $x > 0$. Thus, by dominated convergence,

$$\lim_{a\to 0}\frac{\sum_{j\le 1/\varepsilon a^2}\phi_j}{a^{-2}\,\mathrm{P}(\tau^{(0)} > a^{-2})} = \int_0^{1/\varepsilon}\frac{\mathrm{e}^{x/2}}{\sqrt{x}}(1 - F_{2,0}(\sqrt{x}))\,\mathrm{d}x$$
$$=: I(\varepsilon). \tag{72}$$

Using (41), we have

$$I(\varepsilon) = \int_0^{1/\varepsilon}\mathrm{e}^{x/2}\int_{\sqrt{x}}^{\infty}v^{-2}\mathrm{e}^{-v^2/2}\,\mathrm{d}v\,\mathrm{d}x$$
$$= \int_0^{\infty}\mathrm{e}^{x/2}\int_{\sqrt{x}}^{\infty}v^{-2}\mathrm{e}^{-v^2/2}\,\mathrm{d}v\,\mathrm{d}x - \int_{1/\varepsilon}^{\infty}\mathrm{e}^{x/2}\int_{\sqrt{x}}^{\infty}v^{-2}\mathrm{e}^{-v^2/2}\,\mathrm{d}v\,\mathrm{d}x.$$

Noting that

$$\int_{\sqrt{x}}^{\infty}v^{-2}\mathrm{e}^{-v^2/2}\,\mathrm{d}v \le \frac{\mathrm{e}^{-x/2}}{x^{3/2}},$$

we have

$$0 \le \int_0^{\infty}\mathrm{e}^{x/2}\int_{\sqrt{x}}^{\infty}v^{-2}\mathrm{e}^{-v^2/2}\,\mathrm{d}v\,\mathrm{d}x - I(\varepsilon) \le \sqrt{\varepsilon}.$$

Changing the order of integration and substituting $v^2/2 = u$, we have

$$\int_0^{\infty}\mathrm{e}^{x/2}\int_{\sqrt{x}}^{\infty}v^{-2}\mathrm{e}^{-v^2/2}\,\mathrm{d}v\,\mathrm{d}x = \int_0^{\infty}v^{-2}\mathrm{e}^{-v^2/2}\int_0^{v^2}\mathrm{e}^{x/2}\,\mathrm{d}x\,\mathrm{d}v$$
$$= 2\int_0^{\infty}v^{-2}\mathrm{e}^{-v^2/2}(1 - \mathrm{e}^{-v^2/2})\,\mathrm{d}v$$
$$= \frac{1}{\sqrt{2}}\int_0^{\infty}u^{-3/2}(1 - \mathrm{e}^{-u})\,\mathrm{d}u.$$

Integrating now by parts we obtain

$$\int_0^{\infty}u^{-3/2}(1 - \mathrm{e}^{-u})\,\mathrm{d}u = 2\int_0^{\infty}u^{-1/2}\mathrm{e}^{-u}\,\mathrm{d}u = 2\Gamma\big(\tfrac{1}{2}\big) = 2\sqrt{\pi}.$$

As a result, we have the bounds

$$\sqrt{2\pi} - \sqrt{\varepsilon} \le I(\varepsilon) \le \sqrt{2\pi}. \tag{73}$$

Substituting (72) and (73) into (71), we obtain

$$\sqrt{2\pi} - \sqrt{\varepsilon} \le \liminf_{a\to 0}\frac{\sigma_n}{a^{-2}\,\mathrm{P}(\tau^{(0)} > a^{-2})} \le \limsup_{a\to 0}\frac{\sigma_n}{a^{-2}\,\mathrm{P}(\tau^{(0)} > a^{-2})} \le \frac{\sqrt{2\pi}}{1 - C\sqrt{\varepsilon}}.$$

Since ε can be chosen arbitrarily small,

$$\sigma_n \sim \sqrt{2\pi} a^{-2} \mathrm{P}(\tau^{(0)} > a^{-2}). \tag{74}$$

Combining (70) and (74), and recalling definition (60) of ϕ_n, we have

$$\mathrm{P}(\tau^{(a)} > n) \sim a^{-3} n^{-3/2} \mathrm{e}^{-\xi(a)n} \mathrm{P}(\tau^{(0)} > a^{-2}). \tag{75}$$

Furthermore, it follows from (54) that

$$\mathrm{E}\, \tau^{(a)} \sim a^{-2} \mathrm{P}(\tau^{(0)} > a^{-2}) \int_0^\infty (1 - F_{2,0}(\sqrt{x})) x^{-1/2} \, \mathrm{d}x.$$

Substituting $\sqrt{x} = y$ and using (41), we obtain

$$\begin{aligned}
\int_0^\infty (1 - F_{2,0}(\sqrt{x})) x^{-1/2} \, \mathrm{d}x &= 2 \int_0^\infty (1 - F_{2,0}(y)) \, \mathrm{d}y \\
&= 2 \int_0^\infty y \left(\int_y^\infty v^{-2} \mathrm{e}^{-v^2/2} \, \mathrm{d}v \right) \mathrm{d}y \\
&= 2 \int_0^\infty v^{-2} \mathrm{e}^{-v^2/2} \left(\int_0^v y \, \mathrm{d}y \right) \mathrm{d}v \\
&= \int_0^\infty \mathrm{e}^{-v^2/2} \, \mathrm{d}v \\
&= \sqrt{\frac{\pi}{2}}.
\end{aligned}$$

As a result, we have

$$a^{-2} \mathrm{P}(\tau^{(0)} > a^{-2}) \sim \sqrt{\frac{2}{\pi}} \, \mathrm{E}\, \tau^{(a)}. \tag{76}$$

Combining (75) and (76), and noting that

$$\frac{1}{a\sqrt{2\pi n}} \mathrm{e}^{-\xi(a)n} \sim \overline{\Phi}(a\sqrt{n}) \exp\{na^3 \lambda_m(a)\},$$

completes the proof.

5.3. Proof of Theorem 5

It is easy to see that there exist a constant C and a regularly varying function $N(a)$ such that

$$\lim_{a \to 0} \frac{N(a)}{a^{-2} \log a^{-2}} = (p-2)^{1/2}, \tag{77}$$

and

$$\sup_{n \le N(a)} \frac{n \, \mathrm{P}(X \ge na + \sqrt{n})}{\overline{\Phi}(a\sqrt{n})} \le C \quad \text{and} \quad \sup_{n \ge N(a)} \frac{\overline{\Phi}(a\sqrt{n})}{n \, \mathrm{P}(X \ge na + \sqrt{n})} \le C. \tag{78}$$

We now split the right-hand side of (10) into the product of two exponentials:

$$\begin{aligned}
\exp\left\{ \sum_{n=1}^\infty \frac{z^n}{n} \mathrm{P}(S_n^{(a)} > 0) \right\} &= \exp\left\{ \sum_{n=1}^{N(a)} \frac{z^n}{n} \mathrm{P}(S_n^{(a)} > 0) \right\} \exp\left\{ \sum_{n=N(a)+1}^{\infty} \frac{z^n}{n} \mathrm{P}(S_n^{(a)} > 0) \right\} \\
&=: \left(\sum_{n=0}^\infty \psi_{1,n} z^n \right) \left(1 + \sum_{n=N(a)+1}^\infty \psi_{2,n} z^n \right).
\end{aligned}$$

Therefore,

$$\mathrm{P}(\tau^{(a)} > n) = \psi_{1,n} + \sum_{k=N(a)+1}^{n} \psi_{1,n-k}\psi_{2,k}, \qquad n \ge 1. \tag{79}$$

We first want to find the asymptotic behaviour of $\psi_{2,n}$. We start by noting that

$$\psi_{2,n} = \sum_{j=1}^{\infty} \frac{1}{j!} q_n^{*j}, \qquad n > N(a), \tag{80}$$

where $\{q_n^{*j},\ n \ge 1\}$ is the jth convolution of $\{n^{-1}\,\mathrm{P}(S_n^{(a)} > 0)\,\mathbf{1}\{n > N(a)\},\ n \ge 1\}$. It follows from the second inequality in (78) that

$$\begin{aligned} q_n^{*2} &= \sum_{k=N(a)+1}^{n-N-1} \frac{1}{k}\,\mathrm{P}(S_k^{(a)} > 0)\frac{1}{n-k}\,\mathrm{P}(S_{n-k}^{(a)} > 0) \\ &\le C \sum_{k=N(a)+1}^{n-N-1} \mathrm{P}(X \ge ak)\,\mathrm{P}(X \ge a(n-k)) \\ &\le C\,\mathrm{P}\biggl(X \ge \frac{an}{2}\biggr) \sum_{N(a)+1} \mathrm{P}(X \ge ak) \\ &\le C\,\mathrm{P}(X \ge an) \int_{N(a)}^{\infty} \mathrm{P}(X \ge ay)\,\mathrm{d}y. \end{aligned}$$

Since $\mathrm{P}(X \ge y)$ is regularly varying, we have

$$\int_{N(a)}^{\infty} \mathrm{P}(X \ge ay)\,\mathrm{d}y = \frac{1}{a}\int_{aN(a)}^{\infty} \mathrm{P}(X \ge y)\,\mathrm{d}y \le CN(a)\,\mathrm{P}(X \ge aN(a)).$$

From this bound and (77) we obtain

$$q_n^{*2} \le G(a)\,\mathrm{P}(X \ge an),$$

where G is regularly varying with index $p-2 > 0$. Then, by induction,

$$q_n^{*j} \le G(a)\,\mathrm{P}(X \ge an) \quad \text{for all } j \ge 2. \tag{81}$$

Combining (80) and (81), and using (23) and (78), we obtain the bound

$$\begin{aligned} \psi_{2,n} &= \mathrm{P}(S_n^{(a)} > 0) + \sum_{j=2}^{\infty} q_n^{*j} \\ &\le C\biggl(\frac{1}{n}\overline{\Phi}(a\sqrt{n}) + \mathrm{P}(X \ge an) + G(a)\,\mathrm{P}(X \ge an)\biggr) \\ &\le C\,\mathrm{P}(X \ge an) \end{aligned} \tag{82}$$

and, for $n \ge Ca^{-2}\log a^{-2}$ with some $C > (p-2)^{1/2}$, the relation

$$\psi_{2,n} = \mathrm{P}(S_n^{(a)} > 0) + O(G(a)\,\mathrm{P}(X \ge an)) \sim \frac{1}{n}\overline{\Phi}(a\sqrt{n}) + \mathrm{P}(X \ge an) \sim \mathrm{P}(X \ge an). \tag{83}$$

In the last step we have used the fact that $\overline{\Phi}(a\sqrt{n}) = o(\mathrm{P}(X \ge an))$ for $n \ge Ca^{-2}\log a^{-2}$, $C > (p-2)^{1/2}$.

From the first inequality in (78) and (23), which is valid under condition (26), we conclude that

$$\mathrm{P}(S_n^{(a)} > 0) \le C\overline{\Phi}(a\sqrt{n})$$

for all $n \le N(a)$. Using arguments from the proof of Theorem 4, we see that

$$\psi_{1,k} \le \frac{C}{k}\overline{\Phi}(a\sqrt{k}), \qquad k \ge 1. \tag{84}$$

Combining (82) and (84), and applying the second inequality in (78), we obtain

$$\begin{aligned}\sum_{k=N(a)}^{n-N(a)} \psi_{1,n-k}\psi_{2,k} &\le C \sum_{k=N(a)}^{n-N(a)} \frac{1}{n-k}\overline{\Phi}(a\sqrt{n-k})\,\mathrm{P}(X \ge ak)\\ &\le C \sum_{k=N(a)}^{n-N(a)} \mathrm{P}(X \ge a(n-k))\,\mathrm{P}(X \ge ak).\end{aligned}$$

In the derivation of (81) we showed that the sum in the last line is bounded by $G(a)\,\mathrm{P}(X \ge an)$. Hence,

$$\sum_{k=N(a)}^{n-N(a)} \psi_{1,n-k}\psi_{2,k} = O(G(a)\,\mathrm{P}(X \ge an)). \tag{85}$$

It follows from (10) and the definition of $\{\psi_{1,n},\ n \ge 1\}$ that $\psi_{1,k} = \mathrm{P}(\tau^{(a)} > k)$ for all $k \le N(a)$. Consequently,

$$\begin{aligned}\sum_{k=n-N(a)+1}^{n} \psi_{1,n-k}\psi_{2,k} &= \sum_{k=0}^{N(a)-1} \mathrm{P}(\tau^{(a)} > k)\psi_{2,n-k}\\ &= \sum_{k=0}^{\tilde{N}(a)-1} \mathrm{P}(\tau^{(a)} > k)\psi_{2,n-k} + \sum_{k=\tilde{N}(a)}^{N(a)-1} \mathrm{P}(\tau^{(a)} > k)\psi_{2,n-k},\end{aligned}$$

where $\tilde{N}(a)$ is such that $a^{-2} \ll \tilde{N}(a) \ll a^{-2}\log a^{-2}$. Applying (82) to the fist sum and (83) to the second sum, we obtain

$$\begin{aligned}\sum_{k=n-N(a)+1}^{n} \psi_{1,n-k}\psi_{2,k} &= (1+o(1))\,\mathrm{P}(X \ge an) \sum_{k=0}^{\tilde{N}(a)-1} \mathrm{P}(\tau^{(a)} > k)\\ &\quad + O\Bigg(\mathrm{P}(X \ge an) \sum_{k=\tilde{N}(a)}^{N(a)-1} \mathrm{P}(\tau^{(a)} > k)\Bigg).\end{aligned}$$

Note that (51) implies that

$$\sum_{k=\tilde{N}(a)}^{\infty} \mathrm{P}(\tau^{(a)} > k) = o(\mathrm{E}\,\tau^{(a)}).$$

Hence, we finally obtain

$$\sum_{k=n-N(a)+1}^{n} \psi_{1,n-k}\psi_{2,k} \sim \mathrm{E}\,\tau^{(a)}\,\mathrm{P}(X \geq an). \tag{86}$$

Combining (79), (85), and (86), we have

$$\mathrm{P}(\tau^{(a)} > n) = (1 + o(1))\,\mathrm{E}\,\tau^{(a)}\,\mathrm{P}(X \geq an) + \psi_{1,n}.$$

In order to complete the proof, it remains to apply (84) and to note that

$$n^{-1}\overline{\Phi}(a\sqrt{n}) = o(\mathrm{P}(X \geq an)).$$

Acknowledgement

The author would like to thank Klaus Fleischmann for remarks which led to a better presentation of the results.

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