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CRITICAL GALTON–WATSON PROCESSES: THE MAXIMUM OF TOTAL PROGENIES WITHIN A LARGE WINDOW*

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Abstract. Consider a critical Galton–Watson process $Z = \{Z_n : n = 0, 1, \ldots\}$ of index $1 + \alpha$, $\alpha \in (0, 1]$. Let $S_k(j)$ denote the sum of the Z_n with n in the window $[k, \ldots, k+j)$ and let $M_m(j)$ be the maximum of the $S_k(j)$ with k moving in $[0, m-j]$. We describe the asymptotic behavior of the expectation $\mathbf{E}M_m(j)$ if the window width $j = j_m$ is such that $j/m \to \eta \in [0, 1]$ as $m \uparrow \infty$. This will be achieved via establishing the asymptotic behavior of the tail of the distribution of the random variable $M_\infty(j)$.

Key words. branching of index one plus alpha, limit theorem, conditional invariance principle, tail asymptotics, moving window, maximal total progeny, lower deviation probabilities

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1. Introduction and statement of results. Let $Z = \{Z_n \colon n \geqq 0\}$ denote a Galton–Watson process. As a rule, we start with a single ancestor: $Z_0 = 1$. It will be convenient to write ξ for the intrinsic number of offspring Z_1. We always assume that Z is *critical*, that is, $\mathbf{E}\xi = 1$. If not stated otherwise, we consider the case of *branching of index* $1 + \alpha$ for some $0 < \alpha \leqq 1$. With this we mean that the corresponding offspring generating function f satisfies

$$f(s) := \mathbf{E}s^{\xi} = s + (1-s)^{1+\alpha} L(1-s), \qquad 0 \leqq s \leqq 1, \tag{1}$$

where $x \mapsto L(x)$ is a function slowly varying as $x \downarrow 0$. For $k \geqq 0$ and $1 \leqq j \leqq m < \infty$, set

$$S_k(j) := \sum_{l=k}^{k+j-1} Z_l \quad \text{and} \quad M_m(j) := \max_{0 \leqq k \leqq m-j} S_k(j).$$

Extend these notations by monotone convergence to $m = \infty$ or even $j = \infty$, and put

$$M(j) := M_\infty(j), \qquad 1 \leqq j \leqq \infty.$$

Since any critical Galton–Watson process dies a.s. in finite time, $M(j)$ is a proper random variable for any j. In particular, $M(\infty)$ coincides with the total number $S_0(\infty) = Z_0 + Z_1 + \cdots$ of individuals of Z.

The *main purpose* of this paper is to study the asymptotic behavior of the expectation $\mathbf{E}M_m(j)$ when j might depend on m such that $j/m \to \eta \in [0, 1]$ as $m \uparrow \infty$.

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To get a feeling for this purpose, let us first discuss two special cases. If $j = m$, we have

$$\mathbf{E}M_m(m) = \mathbf{E}S_0(m) = \mathbf{E}\sum_{l=0}^{m-1} Z_l = m. \tag{2}$$

On the other hand, the case $j = 1$ reduces to the investigation of the asymptotic behavior of the expectation of $M_m(1) =: M_m = \max_{0\leqq k\leqq m-1} Z_k$ as $m \uparrow \infty$. The last issue has a rather long history. First, Weiner [18] demonstrated that if the critical process has a finite variance [which requires $\alpha = 1$ in our case (1)], then there exist constants $0 < \underline{c} \leqq \overline{c} < \infty$ such that $\underline{c} \leqq \mathbf{E}M_m/\log m \leqq \overline{c}$ for all $m > 1$. Then Kämmerle and Schuh [9] and Pakes [12] found explicit bounds for $\underline{c}$ from below and for $\overline{c}$ from above. Finally, Athreya [1] established (still under the condition $\operatorname{Var}\xi < \infty$) that

$$\mathbf{E}M_m(1) = \mathbf{E}M_m \sim \log m \qquad \text{as} \quad m \uparrow \infty. \tag{3}$$

In Borovkov and Vatutin [3] the validity of (3) was proved under condition (1). Moreover, in Topchii and Vatutin [17] and Bondarenko and Topchii [2] asymptotics (3) was established under much weaker conditions than (1), for instance, in [2] under $\mathbf{E}\xi\log^{\beta}(1+\xi) < \infty$ for any $\beta > 0$.

Comparing the difference of orders in the right-hand sides of (2) and (3) leads to the following natural question: What can be said about the behavior of $\mathbf{E}M_m(j)$ when the width j of the moving window, within which the total population size is calculated, may vary anyhow with m? For this purpose, we restrict our attention to processes satisfying (1). Here is our *main result.*

THEOREM 1. *Assume that $j_m \geqq 1$ satisfies $j_m/m \to \eta \in [0,1]$ as $m \uparrow \infty$.*

(a) *If $\eta = 0$, then*

$$\mathbf{E}M_m(j_m) \sim j_m \log\frac{m}{j_m} \qquad \textit{as} \quad m \uparrow \infty.$$

(b) *If $0 < \eta \leqq 1$, then*

$$\mathbf{E}M_m(j_m) \sim j_m\varphi(\eta) \qquad \textit{as} \quad m \uparrow \infty, \tag{4}$$

where φ is explicitly given in relation (89) to follow. In particular,

$$\varphi(1) = 1 \quad \textit{and} \quad \varphi(\eta) \sim \log\frac{1}{\eta} \qquad \textit{as} \quad \eta \downarrow 0. \tag{5}$$

Note that (5) yields a continuous transition between the cases (a) and (b).

We will deduce Theorem 1 via studying finer properties of $M(j) = M_\infty(j)$. In fact, we will establish the following asymptotic representation for tail probabilities of $M(j)$. As usual, we write $Q(n)$ for the survival probability $\mathbf{P}(Z_n > 0)$.

THEOREM 2. *Assume that $j_n \geqq 1$ satisfies $j_n/(Q(j_n)\,n) \to y \in [0,\infty]$ as $n \uparrow \infty$.*

(a) *If $y = \infty$, then as $n \uparrow \infty$,*

$$\mathbf{P}\big(M(j_n) \geqq n\big) \sim \mathbf{P}\big(M(\infty) \geqq n\big) \sim n^{-1/(1+\alpha)}\ell(n), \tag{6}$$

where ℓ is a function slowly varying at infinity.

(b) *If $0 < y < \infty$, then as $n \uparrow \infty$,*

$$\mathbf{P}\big(M(j_n) \geqq n\big) \sim Q(j_n)\,\psi(y),$$

where ψ *is explicitly given in formula* (74) *to follow.*

(c) *Finally, if* $y = 0$, *then as* $n \uparrow \infty$,

$$\mathbf{P}\big(M(j_n) \geqq n\big) \sim \mathbf{P}\big(M(1) \geqq n j_n^{-1}\big) \sim \frac{\alpha j_n}{n}. \tag{7}$$

The rest of the paper is organized as follows. In subsections 2.1 and 2.2, we state some (partially known) properties of critical Galton–Watson processes, preparing for the proof of parts (a) and (c) of Theorem 2, given in subsection 3.1. This is followed in subsection 2.3 by a conditional invariance principle for critical Galton–Watson processes of index $1+\alpha$; see Proposition 13, needed for the proof of Theorem 2(b) (also given in subsection 3.1). Properties of the limit process X^*, arising in the mentioned invariance principle, are studied in subsection 2.4 and applied as Proposition 15 in the proof of Theorem 1(b) in subsection 3.3.

2. Auxiliary tools.

2.1. Basic properties of critical processes of index $1 + \alpha$. We start with some further notational conventions. If the symbols L and ℓ (as in (1) and (6), respectively) have an index, they also denote functions slowly varying at zero or infinity, respectively. In this case, the index might refer to the first place of its occurrence, for instance, $\ell_{\#}$ occurring in Lemma #. Furthermore, the letter c will always denote a (positive and finite) constant, which might change from place to place, except if it has an index, which also might refer to the place of first occurrence. We will also use the following convention: If a mathematical expression (like Z_n) is defined only for an integer (here n), but we write a nonnegative number in it instead (such as Z_x), then we actually mean the integer part of that number (here $Z_{[x]}$).

Now we collect some basic properties of critical processes under our assumption (1). The first lemma is taken from [15, Lemma 1].

LEMMA 3. *For* f *from* (1) *we have*

$$1 - f'(s) \sim (1+\alpha)(1-s)^{\alpha} L(1-s) \qquad as \quad s \uparrow 1.$$

The following lemma is due to Slack [14].

LEMMA 4. *As* $n \uparrow \infty$,

$$\alpha Q^{\alpha}(n)\, L\big(Q(n)\big) \sim n^{-1}, \tag{8}$$

implying

$$Q(n) \sim n^{-1/\alpha} \ell_4(n) \tag{9}$$

for a function ℓ_4 (*slowly varying at infinity*).

Set $f_0(s) := s$ and $f_n(s) := f(f_{n-1}(s))$, $n \geqq 1$, for the iterations of f. The following lemma can be considered as a local limit statement.

LEMMA 5. *As* $n \uparrow \infty$,

$$d_n := \prod_{k=1}^{n-1} f'\big(f_k(0)\big) \sim n^{-1-1/\alpha} \ell_5(n). \tag{10}$$

Proof. It follows from Lemmas 3 and 4 that

$$1 - f'\big(1 - Q(k)\big) = \frac{1+\alpha}{\alpha k}\big(1 + \delta(k)\big),$$

where $\delta(k) \to 0$ as $k \uparrow \infty$. Hence,

$$d_n = \exp\left[\sum_{k=1}^{n-1} \log f'\big(f_k(0)\big)\right] = \exp\left[-\frac{1+\alpha}{\alpha}\sum_{k=1}^{n-1}\frac{1}{k}\big(1+\delta_1(k)\big)\right]$$

$$\sim n^{-(1+\alpha)/\alpha} e^{-(1+\alpha)\gamma/\alpha} \exp\left[-\frac{1+\alpha}{\alpha}\sum_{k=1}^{n-1}\frac{\delta_1(k)}{k}\right] \qquad \text{as} \quad n\uparrow\infty, \tag{11}$$

where γ is Euler's constant and also $\delta_1(k) \to 0$ as $k \uparrow \infty$. According to Seneta [13, section 1.5, Theorem 1.2], the function

$$n \longmapsto \exp\left[-\frac{1+\alpha}{\alpha}\sum_{k=1}^{n-1}\frac{\delta_1(k)}{k}\right]$$

is slowly varying at infinity. Combining this with (11) proves the lemma.

The following statement might also be known from the literature. Recall that $M(\infty) = S_0(\infty)$.

Lemma 6. *As* $n \uparrow \infty$,

$$\mathbf{P}\big(M(\infty) \geqq n\big) = \mathbf{P}\big(S_0(\infty) \geqq n\big) \sim n^{-1/(1+\alpha)}\ell_6(n). \tag{12}$$

Proof. As is well known (see, for instance, [7, formula (1.13.3)]), $h(s) := \mathbf{E}s^{S_0(\infty)}$ solves the equation

$$h(s) = sf\big(h(s)\big), \qquad 0 \leqq s \leqq 1.$$

By assumption (1) we have

$$h(s) = sh(s) + \big(1-h(s)\big)^{1+\alpha} L\big(1-h(s)\big),$$

giving in view of $h(1) = 1$,

$$\big(1-h(s)\big)^{1+\alpha} L\big(1-h(s)\big) \sim 1-s \qquad \text{as} \quad s\uparrow 1. \tag{13}$$

Hence (cf. [13, section 1.5]), as $s \uparrow 1$,

$$1-h(s) \sim (1-s)^{1/(1+\alpha)} L_{(14)}(1-s) \tag{14}$$

and

$$\frac{1-h(s)}{1-s} = \sum_{n=0}^{\infty} \mathbf{P}\big(S_0(\infty) > n\big)\, s^n \sim \frac{L_{(14)}(1-s)}{(1-s)^{\alpha/(1+\alpha)}}$$

implying, by a Tauberian theorem (see, for instance, [5, section XIII]),

$$\mathbf{P}\big(S_0(\infty) \geqq n\big) \sim \frac{1}{\Gamma(\alpha/(1+\alpha))} n^{-1/(1+\alpha)}\, L_{(14)}\left(\frac{1}{n}\right) =: n^{-1/(1+\alpha)}\,\ell_6(n) \tag{15}$$

as $n \uparrow \infty$. This finishes the proof.

Lemma 7. *The following statements hold*:

(a) *As* $n \uparrow \infty$, *if* $j_n \geqq 1$ *satisfies* $j_n/(Q(j_n)\,n) \to \infty$, *then*

$$\frac{Q(j_n)}{\mathbf{P}(M(\infty) \geqq n)} \to 0.$$

(b) *As* $j \uparrow \infty$,

$$\mathbf{P}\left(M(\infty) \geqq \frac{j}{Q(j)}\right) \sim \frac{\alpha^{1/(1+\alpha)}}{\Gamma(\alpha/(1+\alpha))}\, Q(j).$$

Proof. (a) Recalling notation h introduced in the beginning of the proof of Lemma 6, set $b_x := 1 - h(1 - 1/x)$, $x \in [1, \infty)$. As $x \uparrow \infty$, it follows from (13) that

$$b_x^{1+\alpha} L(b_x) \sim x^{-1}, \tag{16}$$

and from (14) that

$$b_x \sim x^{-1/(1+\alpha)} L_{(14)}\, x^{-1}. \tag{17}$$

By our assumption, $j_n \to \infty$; hence, by Lemma 4,

$$Q^{1+\alpha}(j_n)\, L\big(Q(j_n)\big) \sim \frac{Q(j_n)}{\alpha j_n} \qquad \text{as} \quad n \uparrow \infty. \tag{18}$$

Thus, combined with (16),

$$\frac{Q^{1+\alpha}(j_n)\, L(Q(j_n))}{b_n^{1+\alpha} L(b_n)} \sim \frac{nQ(j_n)}{\alpha j_n} \longrightarrow 0 \qquad \text{as} \quad n \uparrow \infty. \tag{19}$$

Note that the function $s \mapsto (1-s)^{1+\alpha} L(1-s) = f(s) - s$ is monotone (its derivative $f'(s) - 1$ is negative for $s \in [0, 1)$ by criticality). Applying this to $1 - s = Q(j_n)$ and $1 - s = b_n$, it follows from (19) and the properties of slowly varying functions that

$$Q(j_n)\, b_n^{-1} \to 0 \qquad \text{as} \quad n j_n^{-1} Q(j_n) \to 0. \tag{20}$$

On the other hand, from (15) and (17) it follows that

$$\mathbf{P}\big(M(\infty) \geqq n\big) \sim \frac{1}{\Gamma(\alpha/(1+\alpha))}\, b_n \qquad \text{as} \quad n \uparrow \infty. \tag{21}$$

Combining this with (20) proves part (a) of the lemma.

(b) Observe that by (17) and (21),

$$\mathbf{P}\big(M(\infty)\big) \geqq \frac{j}{Q(j)} \sim \frac{1}{\Gamma(\alpha/(1+\alpha))}\, b_{j/Q(j)} \qquad \text{as} \quad j \uparrow \infty \tag{22}$$

and that

$$b_{j/Q(j)}^{1+\alpha} L(b_{j/Q(j)}) \sim \frac{Q(j)}{j} \qquad \text{as} \quad j \uparrow \infty.$$

This, combined with (18) and the properties of slowly varying functions, implies

$$b_{j/Q(j)} \sim \alpha^{1/(1+\alpha)}\, Q(j) \qquad \text{as} \quad j \uparrow \infty. \tag{23}$$

Substituting (23) into (22) finishes the proof.

2.2. Some properties of critical processes. We now discuss *arbitrary* critical Galton–Watson processes (i.e., we drop restriction (1)). For $R \geqq 2$, put

$$B_R := \mathbf{E}\big\{\xi(\xi - 1);\ \xi \leqq R\big\} \quad \text{and} \quad R_0 := \min\{R \geqq 2\colon\ B_R > 0\} < \infty$$

and set

$$\mathcal{F}(s) := \frac{1 - f(s)}{1 - s} = \sum_{j=0}^{\infty} \mathbf{P}(\xi > j)\, s^j, \qquad 0 \leqq s < 1.$$

LEMMA 8. *There exists a positive constant* c_8 *such that for any critical Galton–Watson process,*

$$B_R \leqq 2 \sum_{1 \leqq j \leqq R} j\, \mathbf{P}(\xi > j) \leqq c_8 R\big(1 - \mathcal{F}(1 - R^{-1})\big), \qquad R \geqq 2. \tag{24}$$

Proof. The first inequality in (24) essentially follows by integration by parts. For j satisfying $1 \leqq j \leqq R$ and $x \in (0,1)$, we have the following elementary inequality:

$$1 - (1-x)^j \geqq jx(1-x)^{j-1} \geqq jx(1-x)^R,$$

which can be rewritten as

$$j \leqq x^{-1}(1-x)^{-R}\big(1 - (1-x)^j\big).$$

Choosing $x = 1/R$ and using criticality $\sum_{j=0}^{\infty} \mathbf{P}(\xi > j) = 1$, we get from the first inequality in (24),

$$\begin{aligned} B_R &\leqq 2R\Big(1 - \frac{1}{R}\Big)^{-R} \sum_{0 \leqq j \leqq R} \bigg(1 - \Big(1 - \frac{1}{R}\Big)^j\bigg) \mathbf{P}(\xi > j) \\ &= 2R\Big(1 - \frac{1}{R}\Big)^{-R} \bigg(1 - \sum_{j > R} \mathbf{P}(\xi > j) - \sum_{0 \leqq j \leqq R} \Big(1 - \frac{1}{R}\Big)^j \mathbf{P}(\xi > j)\bigg) \\ &\leqq 2R\Big(1 - \frac{1}{R}\Big)^{-R} \bigg(1 - \sum_{j \geqq 0} \Big(1 - \frac{1}{R}\Big)^j \mathbf{P}(\xi > j)\bigg) \leqq cR\bigg(1 - \mathcal{F}\Big(1 - \frac{1}{R}\Big)\bigg), \end{aligned}$$

as desired.

The following statement is a particular case of [10, Theorem 3].

LEMMA 9. *For* $m \geqq 0$, $k \geqq 1$, $y_0 > 0$, *and* $R \geqq 2$,

$$\begin{aligned} \mathbf{P}(M_{m+1} \geqq k) \leqq \Big(y_0 + \frac{1}{R}\Big) \bigg[\bigg(1 + \frac{1}{1/y_0 + (e^2 + e^{y_0 R})\, m B_R/2}\bigg)^k - 1\bigg]^{-1} \\ + m\mathbf{P}(\xi > R). \end{aligned} \tag{25}$$

If the variance of ξ, for the moment denoted by B_∞, is finite and positive, then by Doob's inequality,

$$\mathbf{P}(M_{m+1} \geqq k) \leqq \frac{mB_\infty + 1}{k^2} \leqq \Big(1 + \frac{1}{B_\infty}\Big) \frac{mB_\infty}{k^2}. \tag{26}$$

Estimate (25) allows us to derive an analogous bound without imposing the finiteness of the variance of ξ, as follows.

LEMMA 10. *There exist finite constants* $c_{(27)}$ *and* c_{10} *such that*

$$\mathbf{P}(M_{m+1} \geqq k) \leqq c_{(27)} \frac{mB_k}{k^2} + m\mathbf{P}\Big(\xi > \frac{k}{2}\Big) \tag{27}$$

for all $k, m \geqq 1$ *satisfying* $k/(mB_k) > c_{10}$.

We see that, for k sufficiently large, the first term on the right-hand side of (27) coincides with (26) concerning the truncated variance B_k (except for the choice of the constant). The second term compensates for the truncation.

Proof of Lemma 10. In view of Lemma 8, $B_k/k \to 0$ as $k \uparrow \infty$. Hence, there is a constant $c_{(28)} \geqq \mathrm{e}$ such that for $k, m \geqq 1$ with $k/(mB_k) > c_{(28)}$,

$$\begin{aligned} y_0 := y_0(k,m) &:= \frac{2}{k}\,\log\frac{k}{mB_k} - \frac{3}{k}\,\log\log\frac{k}{mB_k} \\ &= \frac{1}{k}\,\log\left(\left(\frac{k}{mB_k}\right)^2 \log^{-3}\left(\frac{k}{mB_k}\right)\right) > 0. \end{aligned} \tag{28}$$

Hence, letting $R = k/2 \geqq 2$ in (25) and observing that B_R is nondecreasing in R, we get from Lemma 9 and our choice of R that

$$\begin{aligned} \mathbf{P}(M_{m+1} \geqq k) \leqq \left(y_0 + \frac{2}{k}\right)&\left[\left(1 + \frac{1}{1/y_0 + (e^2 + e^{y_0 k/2})\,mB_k/2}\right)^k - 1\right]^{-1} \\ &+ m\mathbf{P}\left(\xi > \frac{k}{2}\right). \end{aligned}$$

From the estimates

$$y_0 \leqq \frac{2}{k}\,\log\frac{k}{mB_k} \leqq \frac{2}{k}\,\log\frac{k}{B_{R_0}},$$

which are valid for all $k \geqq R_0$, it follows that $y_0 = y_0(k,m) \downarrow 0$ as $k \uparrow \infty$, and, in addition, there exists a constant $c_{(29)}$ such that for $k, m \geqq 1$ satisfying $k/(mB_k) \geqq c_{(29)}$,

$$\begin{aligned} \frac{1}{2}\,(e^2 + e^{y_0 k/2})\,mB_k &= \frac{1}{2}\left(e^2 + \frac{k}{mB_k}\,\log^{-3/2}\left(\frac{k}{mB_k}\right)\right) mB_k \\ &\leqq 2k\log^{-3/2}\left(\frac{k}{mB_k}\right) \leqq \frac{1}{4y_0}. \end{aligned} \tag{29}$$

Hence, for these k, m,

$$\frac{1}{y_0} + \frac{1}{2}\,(e^2 + e^{y_0 k/2})\,mB_k \leqq \frac{5}{4y_0}.$$

Clearly, for sufficiently small $y_0 > 0$,

$$\begin{aligned} \left(1 + \frac{1}{1/y_0 + (e^2 + e^{y_0 k/2})\,mB_k/2}\right)^k &\geqq \left(1 + \frac{4y_0}{5}\right)^k \\ = \exp\left[k\log\left(1 + \frac{4y_0}{5}\right)\right] &\geqq \exp\left[\frac{4ky_0}{5}\left(1 - \frac{y_0}{2}\right)\right]. \end{aligned}$$

By the definition of y_0 there exists $c_{(30)}$ such that

$$1 - \frac{y_0}{2} \geqq \frac{5}{6} \quad \text{and} \quad y_0 > \frac{6}{7}\,\frac{2}{k}\,\log\frac{k}{mB_k} \tag{30}$$

for $k/(mB_k) > c_{(30)}$. Thus, we get the bound

$$\left(1 + \frac{1}{1/y_0 + (e^2 + e^{y_0 k/2})\,mB_k/2}\right)^k \geqq \left(\frac{k}{mB_k}\right)^{8/7} \tag{31}$$

for $k/(mB_k) \geqq c_{(31)} := \max(c_{(28)}, c_{(29)}, c_{(30)})$. Moreover, if $k/(mB_k) > c_{10} := \max(c_{(31)}, 2)$, then

$$\left(\left(1+\frac{1}{1/y_0+(e^2+e^{y_0k/2})\,mB_k/2}\right)^k-1\right)^{-1} \leqq 2\left(\frac{mB_k}{k}\right)^{8/7}. \tag{32}$$

Combining (28)–(32) gives, for $k/(mB_k) > c_{10}$,

$$\mathbf{P}(M_{m+1} \geqq k) \leqq \frac{2}{k}\left(2+\log\frac{k}{mB_k}\right)\left(\frac{mB_k}{k}\right)^{8/7} + m\mathbf{P}\left(\xi > \frac{k}{2}\right).$$

The boundedness of the function $x \mapsto x^{-1/7}\log x$ for $x \geqq 2$ completes the proof of the lemma.

Now we return to the critical processes of index $1+\alpha$.

LEMMA 11. *Under condition* (1), *for* $\beta \in (1, 1+\alpha)$, *there is a constant* $c_{11} = c_{11}(\beta)$ *such that*

$$\mathbf{E}Z_m^\beta \leqq c_{11}Q^{1-\beta}(m), \qquad m \geqq 1.$$

Proof. According to assumption (1), for $0 \leqq s < 1$,

$$\frac{1-\mathcal{F}(s)}{1-s} = \frac{f(s)-s}{(1-s)^2} = \sum_{l=0}^{\infty} s^l \sum_{i=l+1}^{\infty} \mathbf{P}(\xi > i) = \frac{L(1-s)}{(1-s)^{1-\alpha}}. \tag{33}$$

Therefore, by Lemma 8, for all sufficiently large k,

$$B_k \leqq c_8 k^{1-\alpha} L(k^{-1}). \tag{34}$$

On the other hand, (33) and a Tauberian theorem (cf. [5, Theorem 13.5.5]) imply

$$\sum_{l=0}^{k-1}\sum_{i=l+1}^{\infty} \mathbf{P}(\xi > i) \sim \frac{1}{\Gamma(\alpha)}\, k^{1-\alpha} L\left(\frac{1}{k}\right) \qquad \text{as} \quad k \uparrow \infty.$$

Hence, for sufficiently large k,

$$\sum_{i=k}^{\infty} \mathbf{P}(\xi > i) \leqq \frac{2}{\Gamma(\alpha)}\, k^{-\alpha} L\left(\frac{1}{k}\right)$$

and

$$k\mathbf{P}(\xi > 2k) \leqq \sum_{i=k+1}^{2k} \mathbf{P}(\xi > i) \leqq \sum_{i=k}^{\infty} \mathbf{P}(\xi > i),$$

leading to

$$\mathbf{P}(\xi > k) \leqq ck^{-\alpha-1}L(k^{-1}). \tag{35}$$

Combining (27), (34), and (35), we see that there exist constants $c_{(36)}$ and $c'_{(36)}$ such that, for $m \geqq 1$ and all $k > c_{(36)}/Q(m)$,

$$\mathbf{P}(Z_m \geqq k) \leqq \mathbf{P}(M_{m+1} \geqq k) \leqq c'_{(36)} m k^{-1-\alpha} L(k^{-1}). \tag{36}$$

Clearly, for $\beta \in (1, 1+\alpha)$,

$$\mathbf{E} Z_m^\beta \leqq \sum_{k=0}^{\infty} \beta k^{\beta-1} \mathbf{P}(Z_m \geqq k).$$

In the range of the latter summation we distinguish between $k \leqq c_{(36)}/Q(m)$ and $k > c_{(36)}/Q(m)$. Then, by criticality, the sum restricted to the first case is bounded from above by $\beta\, c_{(36)}^{\beta-1} Q^{1-\beta}(m) = c\, Q^{1-\beta}(m)$ (with a constant c depending on β). On the other hand, by (36), the remaining restricted sum is bounded from above by

$$\beta c'_{(36)} m \sum_{k > c_{(36)}/Q(m)} k^{\beta-\alpha-2} L(k^{-1}) \leqq c m Q^{1+\alpha-\beta}(m)\, L\big(Q(m)\big)$$

(cf. [5, Theorem 8.9.1]), which by (8) also leads to $c\, Q^{1-\beta}(m)$, finishing the proof.

LEMMA 12. *Fix $1 < \beta < 1+\alpha$. Under condition* (1), *for $\varepsilon > 0$ there exists a constant $c_{12} = c_{12}(\beta, \varepsilon)$ such that for $j \geqq 1$ and all y satisfying $y \geqq 2/\varepsilon$,*

$$\mathbf{P}\Big\{ \min_{l<j} Z_l < y \;\Big|\; Z_0 = (1+\varepsilon)\, y \Big\} \leqq c_{12} \left(\frac{1}{yQ(j)} \right)^{\beta-1}. \tag{37}$$

Moreover, for all j and y satisfying $yj^{-1} \geqq 2/\varepsilon$,

$$\mathbf{P}\left\{ \sum_{l=0}^{j-1} Z_l < y \;\Big|\; Z_0 = (1+\varepsilon)\, y j^{-1} \right\} \leqq c_{12} \left(\frac{j}{yQ(j)} \right)^{\beta-1}. \tag{38}$$

Proof. Fix $j \geqq 1$ and $y \geqq 2/\varepsilon$. Clearly,

$$\begin{aligned} &\mathbf{P}\Big\{ \min_{l \leqq j-1} Z_l < y \;\Big|\; Z_0 = (1+\varepsilon)\, y \Big\} \\ &\quad = \mathbf{P}\Big\{ \min_{l \leqq j-1} (Z_l - Z_0) < y - Z_0 \;\Big|\; Z_0 = (1+\varepsilon)\, y \Big\}. \end{aligned} \tag{39}$$

Obviously, $y \geqq 2/\varepsilon$ implies that $Z_0 - y = [(1+\varepsilon)\, y] - y \geqq (\varepsilon/2)\, y$. Therefore, the right-hand side of (39) is bounded from above by

$$\mathbf{P}\Big\{ \max_{l \leqq j-1} |Z_l - Z_0| > \frac{\varepsilon}{2}\, y \;\Big|\; Z_0 = (1+\varepsilon)\, y \Big\}.$$

Using this, Doob's inequality gives

$$\mathbf{P}\Big\{ \min_{l \leqq j-1} Z_l < y \;\Big|\; Z_0 = (1+\varepsilon)\, y \Big\} \leqq \left(\frac{2}{\varepsilon} \right)^{\beta} \frac{\mathbf{E}\{ |Z_{j-1} - Z_0|^\beta \mid Z_0 = (1+\varepsilon)\, y \}}{y^\beta}. \tag{40}$$

For the fixed j, let $Z_{j-1}^{(k)},\ k \geqq 1$, denote independent copies of Z_{j-1} given $Z_0 = 1$. Then, by the von Bahr–Esseen inequality [16],

$$\begin{aligned} \mathbf{E}\{ |Z_{j-1} - Z_0|^\beta \mid Z_0 = (1+\varepsilon)\, y \} &= \mathbf{E} \left| \sum_{k=1}^{(1+\varepsilon)\, y} \big(Z_{j-1}^{(k)} - 1 \big) \right|^\beta \\ &\leqq (1+\varepsilon)\, y \mathbf{E}\big\{ |Z_{j-1} - 1|^\beta \mid Z_0 = 1 \big\}. \end{aligned}$$

Using now Lemma 11 we see that

$$\mathbf{E}\big\{|Z_{j-1}-1|^{\beta} \mid Z_0=1\big\} \leqq 1+c_{11}Q^{1-\beta}(j) \leqq (1+c_{11})\,Q^{1-\beta}(j). \tag{41}$$

Combining (40) and (41), we obtain (37).

Noting that $\sum_{l=0}^{j-1} Z_l < y$ implies $\min_{l\leqq j-1} Z_l < yj^{-1}$, and using the verified inequality (37), claim (38) follows, and the proof is finished.

2.3. A conditional invariance principle. *From now on we always impose our basic assumption* (1). In this section, we establish convergence in law of the conditional scaled Galton–Watson processes

$$\big\{Q(n)\,Z_{nt}\colon\ 0 \leqq t \leqq t_0 \mid Z_n > 0\big\} \qquad \text{as} \quad n\uparrow\infty.$$

We start by describing of the desired limiting process X^*. First, we consider a continuous-state branching process $\{X(t)\colon\ 0 \leqq t < \infty\}$ of index $1+\alpha$; more precisely, X is a $[0,\infty)$-valued (time-homogeneous) Markov process with càdlàg paths and transition Laplace functions

$$\mathbf{E}\big\{e^{-\lambda X(t)} \mid X(0)=x\big\} = \exp\big[-x(t+\lambda^{-\alpha})^{-1/\alpha}\big], \qquad \lambda, t, x \geqq 0.$$

Introduce a random variable $\chi \geqq 0$ having the Laplace transform

$$\mathbf{E}e^{-\lambda\chi} = 1-(1+\lambda^{-\alpha})^{-1/\alpha}, \qquad \lambda \geqq 0 \tag{42}$$

(see, e.g., [14]). According to a general construction as in [4], we introduce a Markov process $\{X^+(t)\colon\ 0 \leqq t \leqq 1\}$ with càdlàg paths and with the following properties: For $y>0$ and $0<t\leqq 1$,

$$\mathbf{P}\big(X^+(t)\in dy\big) = t^{-1/\alpha}\mathbf{P}(t^{1/\alpha}\chi \in dy)\,\mathbf{P}\big\{X(1-t)>0 \mid X(0)=y\big\} \tag{43}$$

and, for $x>0$ and $0 \leqq s < t \leqq 1$,

$$\begin{aligned}\mathbf{P}\{X^+(t)\in dy \mid X^+(s)=x\} = {}& \frac{\mathbf{P}\{X(t-s)\in dy;\, X(t-s)>0 \mid X(0)=x\}}{\mathbf{P}\{X(t-s)>0 \mid X(0)=x\}}\\ & \times \mathbf{P}\{X(1-t)>0 \mid X(0)=x\}.\end{aligned}$$

Finally, we define the Markov process $\{X^*(t)\colon\ 0 \leqq t < \infty\}$ as a concatenation of processes X^+ and X; more precisely,

$$X^*(t) := \begin{cases} X^+(t) & \text{if} \quad 0 \leqq t \leqq 1,\\ X^{X^+(1)}(t-1) & \text{if} \quad t \geqq 1,\end{cases} \tag{44}$$

where X^x refers to X starting from $X(0)=x$, and this family $\{X^x\colon\ x>0\}$ is chosen independently of $\{X^+(t)\colon\ 0 \leqq t \leqq 1\}$.

PROPOSITION 13. *Let $0<t_0<\infty$. The following convergence in law on $D[0,t_0]$ holds*:

$$\big\{Q(n)\,Z_{nt}\colon\ 0 \leqq t \leqq t_0 \mid Z_n>0\big\} \underset{n\uparrow\infty}{\overset{\mathcal{L}}{\Longrightarrow}} \big\{X^*(t)\colon\ 0 \leqq t \leqq t_0\big\}. \tag{45}$$

Proof. It suffices to show that for $x>0$,

$$\left\{Q(n)\,Z_{nt}\colon\ 0 \leqq t \leqq t_0 \,\middle|\, Z_0 = \frac{x}{Q(n)}\right\} \underset{n\uparrow\infty}{\overset{\mathcal{L}}{\Longrightarrow}} \{X(t)\colon\ 0 \leqq t \leqq t_0 \mid X(0)=x\} \tag{46}$$

in $D[0,t_0]$, and that

$$\big\{Q(n)\,Z_{nt}\colon\ 0\leqq t\leqq 1\mid Z_n>0\big\}\ \underset{n\uparrow\infty}{\overset{\mathcal{L}}{\Longrightarrow}}\ \big\{X^+(t)\colon\ 0\leqq t\leqq 1\mid X^+(0)=0\big\} \tag{47}$$

in $D[0,1]$. In fact, from (46) and (47), the Markov properties of the processes X^+ and X, as well as the definition of X^*, the statement (45) follows.

From the conditional limit theorem in [14] it is easy to derive that for any $t,x>0$,

$$\bigg\{Q(n)\,Z_{nt}\,\bigg|\,Z_0=\frac{x}{Q(n)}\bigg\}\ \underset{n\uparrow\infty}{\overset{\mathcal{L}}{\Longrightarrow}}\ \big\{X(t)\mid X(0)=x\big\}. \tag{48}$$

By Theorem 3.4 in [6], the validity of (48) implies (46).

To demonstrate (47), we will use Theorem 3.9 from [4], according to which it is necessary to show in our situation that, besides (46), the following four statements hold:

$$\mathbf{P}\bigg\{\inf_{0\leqq s\leqq t}X(s)>0\,\bigg|\,X(0)=x\bigg\}>0,\qquad t,x>0; \tag{49}$$

$$\mathbf{P}\bigg\{Z_{nt_n}>0\,\bigg|\,Z_0=\frac{x_n}{Q(n)}\bigg\}\longrightarrow\mathbf{P}\big\{X(t)>0\mid X(0)=x\big\}, \tag{50}$$

whenever $t_n\to t>0$ and $x_n\to x>0$;

$$\mathbf{P}\bigg\{Z_{nt_n}>0\,\bigg|\,Z_0=\frac{x_n}{Q(n)}\bigg\}\longrightarrow 0, \tag{51}$$

whenever $t_n\to t>0$ and $x_n\to 0$; and finally,

$$X^+(t)\overset{\mathcal{L}}{\Longrightarrow}0\qquad\text{as}\quad t\downarrow 0. \tag{52}$$

Since the state 0 is absorbing for the branching process X, we have for $t,x>0$,

$$\begin{aligned}\mathbf{P}\bigg\{\inf_{0\leqq s\leqq t}X(s)>0\,\bigg|\,X(0)=x\bigg\}&=\mathbf{P}\big\{X(t)>0\mid X(0)=x\big\}\\ &=1-\lim_{\lambda\downarrow 0}\mathbf{E}\big\{e^{-\lambda X(t)}\mid X(0)=x\big\}=\ 1-\exp[-xt^{-1/\alpha}],\end{aligned} \tag{53}$$

proving (49). As $n\uparrow\infty$, if $t_n\to t>0$ and $x_n\to x>0$, then, in view of (9) and the properties of the slowly varying functions,

$$\frac{Q(nt_n)}{Q(n)}\to t^{-1/\alpha},$$

and, therefore,

$$\mathbf{P}\bigg\{Z_{nt_n}>0\,\bigg|\,Z_0=\frac{x_n}{Q(n)}\bigg\}=1-\big(1-Q(nt_n)\big)^{x_n/Q(n)}\longrightarrow 1-\exp\big[-xt^{-1/\alpha}\big]. \tag{54}$$

Combining (53) and (54), we get (50) and (51).

Finally, it follows from (42) that $\mathbf{E}\chi^\beta<\infty$ for any $\beta\in(1,1+\alpha)$. Using this fact and (43), we see that for such β and $\varepsilon>0$,

$$\mathbf{P}\big(X^+(t)\geqq\varepsilon\big)\leqq t^{-1/\alpha}\mathbf{P}(t^{1/\alpha}\chi\geqq\varepsilon)\leqq t^{(\beta-1)/\alpha}\varepsilon^{-\beta}\mathbf{E}\chi^\beta\longrightarrow 0$$

as $t\downarrow 0$. This justifies (52). Thus, (47) is proved, and the proof of the lemma is complete.

2.4. On the limiting process X^*. For convenience, we introduce the notation

$$V^*(T) := \sup_{0 \leqq s \leqq T-1} \int_s^{s+1} X^*(u)\,du, \qquad T \geqq 1, \tag{55}$$

and later we write $V(T)$ in the case of working with X instead of X^*.

PROPOSITION 14. *The following equality is valid*:

$$\mathbf{E}V^*(1) = \frac{\alpha+1}{2\alpha+1}.$$

Proof. The definitions of $X^*(u)$ and $V^*(T)$ imply

$$V^*(1) = \int_0^1 X^*(u)\,du = \int_0^1 X^+(u)\,du.$$

In view of (53),

$$\mathbf{P}\big\{X(1-t) > 0 \mid X(0) = y\big\} = 1 - \exp\big[-y(1-t)^{-1/\alpha}\big].$$

Combining this with (43) and setting $h(t) := t^{1/\alpha}(1-t)^{-1/\alpha}$, we obtain

$$\begin{aligned}\mathbf{E}X^+(t) &= \int_0^\infty t^{-1/\alpha} y \mathbf{P}(t^{1/\alpha}\chi \in dy)\big(1 - \exp[-y(1-t)^{-1/\alpha}]\big) \\ &= \int_0^\infty z\big(1 - \exp[-zh(t)]\big)\,\mathbf{P}(\chi \in dy) = \mathbf{E}\chi - \mathbf{E}\chi e^{-h(t)\chi}.\end{aligned}$$

In view of (42),

$$\mathbf{E}\chi e^{-u\chi} = (1+u^\alpha)^{-1/\alpha-1}.$$

Using this equality we find

$$\mathbf{E}X^+(t) = 1 - \big(1 + h^\alpha(t)\big)^{-1/\alpha-1} = 1 - (1-t)^{1/\alpha+1}.$$

Thus,

$$\mathbf{E}V^*(1) = \mathbf{E}\int_0^1 X^*(u)\,du = \int_0^1 \mathbf{E}X^+(u)\,du = \frac{\alpha+1}{2\alpha+1},$$

as required.

PROPOSITION 15. *As* $T \uparrow \infty$,

$$\mathbf{E}V^*(T) \sim \log T.$$

The proof of this proposition will be obtained with the help of the following three lemmas.

LEMMA 16. *For* $\beta \in (1, 1+\alpha)$,

$$\mathbf{P}\Big(\sup_{0 \leqq t \leqq 1} X^+(t) \geqq x\Big) \leqq 1 \bigwedge \frac{c_{11}}{x^\beta}, \qquad x > 0. \tag{56}$$

Moreover, for all $T > 0$ *and* $0 < y \leqq x$,

$$\mathbf{P}\Big\{\sup_{0 \leqq t \leqq T} X(t) \geqq x \;\Big|\; X(0) = y\Big\} \leqq c_{11} T^{(\beta-1)/\alpha}\,\frac{y}{x^\beta}. \tag{57}$$

Proof. From the Donsker–Prokhorov invariance principle and (47) it follows that for $x > 0$,

$$\mathbf{P}\Big(\sup_{0\leqq t\leqq 1} X^{+}(t) \geqq x\Big) = \lim_{n\uparrow\infty} \mathbf{P}\big\{Q(n)\, M_n \geqq x \mid Z_n > 0\big\}. \tag{58}$$

Using Doob's inequality and Lemma 11, we obtain

$$\mathbf{P}\big\{Q(n)\, M_n \geqq x \mid Z_n > 0\big\} \leqq Q^{-1}(n)\, \mathbf{P}\big(Q(n)\, M_n \geqq x\big) \leqq Q^{\beta-1}(n)\, \frac{\mathbf{E}Z_n^{\beta}}{x^{\beta}} \leqq \frac{c_{11}}{x^{\beta}}.$$

From here and (58), estimate (56) follows.

To prove (57) observe that by the Doob and von Bahr–Esseen inequalities and Lemma 11,

$$\begin{aligned}\mathbf{P}\Big\{Q(n)\, M_{nT} \geqq x \,\Big|\, Z_0 = \frac{y}{Q(n)}\Big\} &\leqq \frac{y}{Q(n)}\, \frac{Q^{\beta}(n)\, \mathbf{E}\{Z_{nT}^{\beta} \mid Z_0 = 1\}}{x^{\beta}} \\ &\leqq \frac{c_{11} y}{x^{\beta}} \left(\frac{Q(n)}{Q(nT)}\right)^{\beta-1}.\end{aligned}$$

On the other hand, by (46) and the last estimate,

$$\begin{aligned}\mathbf{P}\Big\{\sup_{0\leqq t\leqq T} X(t) \geqq x \,\Big|\, X(0) = y\Big\} &= \lim_{n\uparrow\infty} \mathbf{P}\Big\{Q(n)\, M_{nT} \geqq x \,\Big|\, Z_0 = \frac{y}{Q(n)}\Big\} \\ &\leqq \frac{c_{11} y}{x^{\beta}} \lim_{n\uparrow\infty} \left(\frac{Q(n)}{Q(nT)}\right)^{\beta-1} = \frac{c_{11} y}{x^{\beta}}\, T^{(\beta-1)/\alpha}.\end{aligned}$$

The first needed lemma is proved.

LEMMA 17. *For* $0 < y \leqq x < \infty$,

$$\mathbf{P}\Big\{\sup_{0\leqq t<\infty} X(t) \geqq x \,\Big|\, X(0) = y\Big\} = 1 - \left(1 - \frac{y}{x}\right)^{\alpha}. \tag{59}$$

Proof. This follows from the Donsker–Prokhorov invariance principle, Lemma 1 in [3], and Theorem 2 in [11].

LEMMA 18. *For* $\varepsilon > 0$ *there exists a constant* $c_{18} = c_{18}(\varepsilon)$ *such that, for all* $\beta \in (1, 1+\alpha)$ *and* $x > 0$,

$$\mathbf{P}\Big\{\inf_{0\leqq t\leqq 1} X(t) \leqq x \,\Big|\, X(0) = (1+\varepsilon)\, x\Big\} \leqq 1 \bigwedge \frac{c_{18}}{x^{\beta-1}}.$$

Proof. Applying (37), we see that for $\varepsilon > 0$, $j \geqq 1$, and $y > 0$ satisfying $\varepsilon y \geqq 2$, the inequality

$$\mathbf{P}\Big\{\min_{l<j} Z_l < y \,\Big|\, Z_0 = (1+\varepsilon)\, y\Big\} \leqq c_{12} \left(\frac{1}{Q(j)\, y}\right)^{\beta-1}$$

is true. Choosing now $y = x/Q(j)$, we get for all sufficiently large j,

$$\mathbf{P}\Big\{\min_{l<j} Z_l < \frac{x}{Q(j)} \,\Big|\, Z_0 = \frac{x(1+\varepsilon)}{Q(j)}\Big\} \leqq \frac{c_{12}}{x^{\beta-1}}.$$

Hence, applying the Donsker–Prokhorov principle, (46), (50), and letting $j \uparrow \infty$, the desired estimate follows.

Having those three lemmas, the proof of Proposition 15 is now given by the following two lemmas.

LEMMA 19. *We have*

$$\limsup_{T\uparrow\infty} \frac{1}{\log T}\, \mathbf{E}V^*(T) \leqq 1.$$

Proof. Clearly,

$$V^*(T) \leqq \sup_{0\leqq s\leqq T} X^*(s).$$

From definition (44) of X^* it follows that for $x > 0$,

$$\mathbf{P}\Big(\sup_{0\leqq s\leqq T} X^*(s) \geqq x\Big) \leqq \mathbf{P}\Big(\sup_{0\leqq s\leqq 1} X^+(s) \geqq x\Big) + \int_0^x \mathbf{P}\big(X^+(1)\in dy\big)\,\mathbf{P}\Big\{\sup_{0\leqq t\leqq T} X(t)\geqq x \,\Big|\, X(0)=y\Big\}.$$

In view of (56),

$$\int_1^\infty \mathbf{P}\Big(\sup_{0\leqq s\leqq 1} X^+(s)\geqq x\Big)\,dx < \infty.$$

Fix any $0<\varepsilon<1$. By (57) and (59) we get for $x>0$, decomposing $(0,x)$,

$$\int_0^x \mathbf{P}(X^+(1)\in dy)\,\mathbf{P}\Big\{\sup_{0\leqq t\leqq T} X(t)\geqq x \,\Big|\, X(0)=y\Big\}$$
$$\leqq \int_0^{\varepsilon x} \mathbf{P}(X^+(1)\in dy)\min\Big(\frac{c_{11}}{x^\beta}T^{(\beta-1)/\alpha}y, 1-\Big(1-\frac{y}{x}\Big)^\alpha\Big) + \mathbf{P}\big(X^+(1)\geqq \varepsilon x\big).$$

Noting that by the mean value theorem, for all $y\leqq \varepsilon x$,

$$1-\Big(1-\frac{y}{x}\Big)^\alpha \leqq \frac{\alpha y}{x}\Big(1-\frac{y}{x}\Big)^{\alpha-1} \leqq \alpha(1-\varepsilon)^{\alpha-1}\,\frac{y}{x},$$

we have the bound

$$\int_0^{\varepsilon x} \mathbf{P}\big(X^+(1)\in dy\big)\min\Big(\frac{c_{11}}{x^\beta}\,T^{(\beta-1)/\alpha}y,\ 1-\Big(1-\frac{y}{x}\Big)^\alpha\Big)$$
$$\leqq \min\Big(\frac{c_{11}}{x^\beta}\,T^{(\beta-1)/\alpha},\ (1-\varepsilon)^{\alpha-1}\,\frac{\alpha}{x}\Big),$$

since $\mathbf{E}X^+(1)=\mathbf{E}\chi=1$. Therefore, decomposing $(1,\infty)$,

$$\int_1^\infty dx\int_0^x \mathbf{P}\big(X^+(1)\in dy\big)\,\mathbf{P}\Big\{\sup_{0\leqq t\leqq T} X(t)\geqq x \,\Big|\, X(0)=y\Big\}$$
$$\leqq \alpha(1-\varepsilon)^{\alpha-1}\int_1^{T^{1/\alpha}}\frac{dx}{x} + c_{11}T^{(\beta-1)/\alpha}\int_{T^{1/\alpha}}^\infty \frac{dx}{x^\beta}$$
$$+\int_1^\infty dx\,\mathbf{P}\big(X^+(1)\geqq\varepsilon x\big) \leqq (1-\varepsilon)^{\alpha-1}\log T + c + \frac{1}{\varepsilon},$$

where the last term follows by substitution and again by $\mathbf{E}X^+(1) = 1$. This implies the claim.

LEMMA 20. *We have*

$$\liminf_{T\uparrow\infty} \frac{1}{\log T}\, \mathbf{E}V^*(T) \geqq 1.$$

Proof. Recalling notation V introduced in (55), it is not difficult to check that for $T \geqq 2$ and $x > 0$,

$$\mathbf{P}\big(V^*(T) \geqq x\big) \geqq \int_{(0,\infty)} \mathbf{P}\big(X^+(1) \in dy\big)\, \mathbf{P}\big\{V(T-1) \geqq x \mid X(0) = y\big\}.$$

Fix $\varepsilon \in (0,1)$ and put $\rho := \inf\{u \geqq 0\colon\ X(u) \geqq (1+\varepsilon)\,x\}$ (being equal to infinity if $\sup_{u\geqq 0} X(u) < (1+\varepsilon)\,x$). Clearly, by the strong Markov property and the properties of continuous-state branching processes,

$$\begin{aligned}
\mathbf{P}\big\{V(T-1) \geqq x \mid X(0) = y\big\} &\geqq \int_0^{T-2} \mathbf{P}\big\{V(T-1) \geqq x,\ \rho \in dw \mid X(0) = y\big\}\\
&\geqq \int_0^{T-2} \mathbf{P}\bigg\{\int_w^{w+1} X(u)\,du \geqq x,\ \rho \in dw \,\bigg|\, X(0) = y\bigg\}\\
&\geqq \int_0^{T-2} \mathbf{P}\bigg\{\inf_{w\leqq u\leqq w+1} X(u) \geqq x,\ \rho \in dw \,\bigg|\, X(0) = y\bigg\}.
\end{aligned}$$

Using the strong Markov property at time ρ, the latter integral coincides with

$$\begin{aligned}
&\int_0^{T-2} \int_{(1+\varepsilon)\,x}^{\infty} \mathbf{P}\big\{\rho \in dw,\ X(w) \in dz \mid X(0) = y\big\}\, \mathbf{P}\bigg\{\inf_{0\leqq u\leqq 1} X(u) \geqq x \,\bigg|\, X(0) = z\bigg\}\\
&\quad\geqq \mathbf{P}\bigg\{\inf_{0\leqq u\leqq 1} X(u) \geqq x \,\bigg|\, X(0) = (1+\varepsilon)\,x\bigg\}\, \mathbf{P}\big\{\rho \leqq T-2 \mid X(0) = y\big\}.
\end{aligned}$$

Applying Lemma 18 we have, for all $x \geqq x_0(\varepsilon)$,

$$\mathbf{P}\big\{V(T-1) \geqq x \mid X(0) = y\big\} \geqq (1-\varepsilon)\, \mathbf{P}\bigg\{\sup_{0\leqq t\leqq T-2} X(t) \geqq (1+\varepsilon)\,x \,\bigg|\, X(0) = y\bigg\}.$$

On the other hand,

$$\begin{aligned}
&\mathbf{P}\bigg\{\sup_{0\leqq t\leqq T-2} X(t) \geqq (1+\varepsilon)\,x \,\bigg|\, X(0) = y\bigg\}\\
&\quad\geqq \mathbf{P}\bigg\{\sup_{0\leqq t<\infty} X(t) \geqq (1+\varepsilon)\,x \,\bigg|\, X(0) = y\bigg\} - \mathbf{P}\big\{X(T-2) > 0 \mid X(0) = y\big\}.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\mathbf{P}\big(V^*(T) \geqq x\big) \geqq (1-\varepsilon) &\int_0^{\infty} \mathbf{P}\big(X^+(1) \in dy\big)\, \mathbf{P}\bigg\{\sup_{0\leqq t<\infty} X(t) \geqq (1+\varepsilon)\,x \,\bigg|\, X(0) = y\bigg\}\\
&- \int_0^{\infty} \mathbf{P}\big(X^+(1) \in dy\big)\, \mathbf{P}\big\{X(T-2) > 0 \mid X(0) = y\big\}. \qquad (60)
\end{aligned}$$

From (42), (43), and (53), we see that

$$\int_0^\infty \mathbf{P}(X^+(1) \in dy)\,\mathbf{P}\big\{X(T-2) > 0 \mid X(0) = y\big\}$$
$$= 1 - \int_0^\infty \mathbf{P}\big(X^+(1) \in dy\big)\,\exp\big[-y(T-2)^{-1/\alpha}\big] = (T-1)^{-1/\alpha}. \tag{61}$$

It follows from (59) that for fixed $y > 0$,

$$\mathbf{P}\bigg\{\sup_{0 \leqq t < \infty} X(t) \geqq x \;\bigg|\; X(0) = y\bigg\} \sim \frac{\alpha y}{x} \qquad \text{as} \quad x \uparrow \infty.$$

Hence, by Fatou's lemma we see that for $A \geqq 1$,

$$\liminf_{x\uparrow\infty}\, x \int_0^\infty \mathbf{P}\bigg\{\sup_{0\leqq t<\infty} X(t) \geqq (1+\varepsilon)\,x \;\bigg|\; X(0) = y\bigg\}\mathbf{P}\big(X^+(1) \in dy\big)$$
$$\geqq \lim_{x\uparrow\infty}\, x \int_0^A \mathbf{P}\bigg\{\sup_{0\leqq t<\infty} X(t) \geqq (1+\varepsilon)\,x \;\bigg|\; X(0) = y\bigg\}\mathbf{P}\big(X^+(1) \in dy\big)$$
$$= \alpha(1+\varepsilon)^{-1} \int_0^A y\mathbf{P}\big(X^+(1) \in dy\big) \xrightarrow[A\uparrow\infty]{} \alpha(1+\varepsilon)^{-1}. \tag{62}$$

Substituting arrays (61) and (62) in (60) gives, for sufficiently large x, $\mathbf{P}(V^*(T) \geqq x) \geqq \alpha\,(1-2\varepsilon)/x - (T-1)^{-1/\alpha}$. Hence, for sufficiently large T,

$$\int_{T^\varepsilon}^{T^{1/\alpha}} \mathbf{P}\big(V^*(T) \geqq x\big)\,dx \geqq (1-2\varepsilon)\Big(\frac{1}{\alpha} - \varepsilon\Big)\,\alpha \log T - \Big(1 - \frac{1}{T}\Big)^{-1/\alpha}.$$

From here the statement of the lemma follows.

3. Proof of the main results.

3.1. Proof of Theorem 2.

(a) By monotonicity in $j \geqq 1$,

$$\mathbf{P}\big(M(j) \geqq n\big) \leqq \mathbf{P}\big(M(\infty) \geqq n\big). \tag{63}$$

On the other hand,

$$\mathbf{P}\big(M(j) \geqq n\big) \geqq \mathbf{P}\big(M(j) \geqq n,\ Z_j = 0\big) = \mathbf{P}\big(M(\infty) \geqq n,\ Z_j = 0\big)$$
$$\geqq \mathbf{P}\big(M(\infty) \geqq n\big) - \mathbf{P}(Z_j > 0). \tag{64}$$

Applying Lemmas 6 and 7(a) to (63) and (64) with $j = j_n$ justifies part (a) of the theorem.

(b) Recalling $M(\infty) = S_0(\infty)$, we see that since

$$\mathbf{P}\big(M(j_n) \geqq n,\ Z_{j_n} = 0\big) = \mathbf{P}\big(M(\infty) \geqq n,\ Z_{j_n} = 0\big)$$
$$= \mathbf{P}\big(S_0(\infty) \geqq n\big) - \mathbf{P}\big(S_0(\infty) \geqq n,\ Z_{j_n} > 0\big),$$

we have

$$\mathbf{P}\big(M(j_n) \geqq n\big) = \mathbf{P}\big(M(j_n) \geqq n,\ Z_{j_n} > 0\big) + \mathbf{P}\big(S_0(\infty) \geqq n\big)$$
$$- \mathbf{P}\big(S_0(\infty) \geqq n,\ Z_{j_n} > 0\big). \tag{65}$$

We investigate separately each term on the right-hand side of array (65). First, we deal with the second term. By (6), Lemma 7(b), and our conditions,

$$\mathbf{P}\big(S_0(\infty) \geqq n\big) \sim \mathbf{P}\bigg(S_0(\infty) \geqq \frac{j_n}{yQ(j_n)}\bigg) \sim \mathbf{P}\bigg(S_0(\infty) \geqq \frac{j_n y^{-\alpha/(1+\alpha)}}{Q(j_n y^{-\alpha/(1+\alpha)})}\bigg)$$

$$\sim \frac{\alpha^{1/(1+\alpha)}}{\Gamma(\alpha/(1+\alpha))}\, Q(j_n y^{-\alpha/(1+\alpha)}) \sim \frac{(\alpha y)^{1/(1+\alpha)}}{\Gamma(\alpha/(1+\alpha))}\, Q(j_n). \tag{66}$$

To study the asymptotic behavior of the last probability in array (65), note that, for any fixed $T \geqq 1$,

$$\begin{aligned}\mathbf{P}\big\{S_0(\infty) \geqq n \mid Z_{j_n} > 0\big\} &= \mathbf{P}\big\{S_0(\infty) \geqq n,\ Z_{Tj_n} = 0 \mid Z_{j_n} > 0\big\} \\ &\quad + \mathbf{P}\big\{S_0(\infty) \geqq n,\ Z_{Tj_n} > 0 \mid Z_{j_n} > 0\big\}.\end{aligned} \tag{67}$$

The first probability term on the right-hand side of decomposition (67) can be estimated from above as follows:

$$\mathbf{P}\big\{S_0(Tj_n) \geqq n,\ Z_{Tj_n} = 0 \mid Z_{j_n} > 0\big\} \leqq \mathbf{P}\big\{S_0(Tj_n) \geqq n \mid Z_{j_n} > 0\big\}.$$

Concerning the other probability term in decomposition (67), in view of (9) and the properties of slowly varying functions, there exists a constant $c_{(68)}$ such that for all $n \geqq 1$ and $j_n \geqq 1$,

$$\mathbf{P}\big\{S_0(\infty) \geqq n,\ Z_{Tj_n} > 0 \mid Z_{j_n} > 0\big\} \leqq \mathbf{P}\big\{Z_{Tj_n} > 0 \mid Z_{j_n} > 0\big\} = \frac{Q(Tj_n)}{Q(j_n)} \leqq \frac{c_{(68)}}{T^{1/\alpha}}. \tag{68}$$

Combining (67) and (68) we deduce

$$0 \leqq \mathbf{P}\big\{S_0(\infty) \geqq n \mid Z_{j_n} > 0\big\} - \mathbf{P}\big\{S_0(Tj_n) \geqq n \mid Z_{j_n} > 0\big\} \leqq c_{(68)} T^{-1/\alpha}. \tag{69}$$

Using the Donsker–Prokhorov invariance principle and Proposition 13 we see that

$$\begin{aligned}&\lim_{n\uparrow\infty} \mathbf{P}\big\{S_0(Tj_n) \geqq n \mid Z_{j_n} > 0\big\} \\ &\quad= \lim_{n\uparrow\infty} \mathbf{P}\bigg\{\int_0^{T-j_n^{-1}} Q(j_n)\, Z_{vj_n}\, dv \geqq \frac{nQ(j_n)}{j_n} \;\bigg|\; Z_{j_n} > 0\bigg\} \\ &\quad= \mathbf{P}\bigg(\int_0^T X^*(v)\, dv \geqq y^{-1}\bigg).\end{aligned} \tag{70}$$

Since T can be made arbitrarily large, (69) and (70) imply

$$\lim_{n\uparrow\infty} \mathbf{P}\big\{S_0(\infty) \geqq n \mid Z_{j_n} > 0\big\} = \mathbf{P}\bigg(\int_0^\infty X^*(v)\, dv \geqq y^{-1}\bigg).$$

Thus, as $n \uparrow \infty$,

$$\mathbf{P}\big(S_0(\infty) \geqq n,\ Z_{j_n} > 0\big) \sim Q(j_n)\, \mathbf{P}\bigg(\int_0^\infty X^*(v)\, dv \geqq y^{-1}\bigg). \tag{71}$$

Finally, to deal with the first probability term on the right-hand side of array (65), observe that

$$\begin{aligned}\mathbf{P}\big\{M(j_n) \geqq n \mid Z_{j_n} > 0\big\} &= \mathbf{P}\big\{M(j_n) \geqq n,\ Z_{Tj_n} = 0 \mid Z_{j_n} > 0\big\} \\ &\quad + \mathbf{P}\big\{M(j_n) \geqq n,\ Z_{Tj_n} > 0 \mid Z_{j_n} > 0\big\}.\end{aligned}$$

Here the first probability term can be written as

$$\mathbf{P}\big\{M_{Tj_n}(j_n) \geqq n,\ Z_{Tj_n} = 0 \mid Z_{j_n} > 0\big\} \leqq \mathbf{P}\big\{M_{Tj_n}(j_n) \geqq n \mid Z_{j_n} > 0\big\},$$

whereas for the other term we have the upper bound

$$\mathbf{P}\{Z_{Tj_n} > 0 \mid Z_{j_n} > 0\}.$$

Using both of these together and again applying (68) gives

$$0 \leqq \mathbf{P}\big\{M(j_n) \geqq n \mid Z_{j_n} > 0\big\} - \mathbf{P}\big\{M_{Tj_n}(j_n) \geqq n \mid Z_{j_n} > 0\big\} \leqq cT^{-1/\alpha}. \tag{72}$$

Using the representation

$$M_{Tj_n}(j_n) = \max_{0 \leqq k \leqq (T-1)j_n} \sum_{l=k}^{k+j_n-1} Z_l = j_n \max_{0 \leqq u \leqq T-1} \int_u^{u+1-j_n^{-1}} Z_{vj_n}\, dv$$

and applying the Donsker–Prokhorov invariance principle as well as Proposition 13 once again, we see that $j_n^{-1} Q(j_n)\, n \to y$ implies

$$\begin{aligned}\lim_{n\uparrow\infty} \mathbf{P}\big\{M_{Tj_n}(j_n) \geqq n \mid Z_{j_n} > 0\big\} &= \lim_{n\uparrow\infty} \mathbf{P}\left\{ j_n^{-1} Q(j_n)\, M(j_n) \geqq \frac{Q(j_n)\, n}{j_n} \,\Big|\, Z_{j_n} > 0 \right\} \\ &= \mathbf{P}\big(V^*(T) \geqq y^{-1}\big).\end{aligned}$$

Hence, letting $T \to \infty$ and taking into account (72) we obtain

$$\lim_{n\uparrow\infty} \mathbf{P}\big\{M(j_n) \geqq n \mid Z_{j_n} > 0\big\} = \mathbf{P}\big(V^*(\infty) \geqq y^{-1}\big). \tag{73}$$

Combining (73), (66), and (71) we see that $Q(j_n)\, n j_n^{-1} \to y \in (0, \infty)$ implies

$$\mathbf{P}\big(M(j_n) \geqq n\big) \sim \psi(y)\, Q(j_n),$$

where

$$\psi(y) := \mathbf{P}\big(V^*(\infty) \geqq y^{-1}\big) + \frac{(\alpha y)^{1/(1+\alpha)}}{\Gamma(\alpha/(1+\alpha))} - \mathbf{P}\left(\int_0^\infty X^*(v)\, dv \geqq y^{-1} \right). \tag{74}$$

Note that $\psi(y) > 0$ since the first term on the right-hand side of array (65) is of order $Q(j_n)$, while the difference of the second and third terms is nonnegative.

(c) To establish (7) observe that $M(j) \leqq jM(1)$, and therefore by Theorem 1 from [3], for any $\varepsilon > 0$ there exists $K = K(\varepsilon)$ such that for n and j satisfying $nj^{-1} > K$,

$$\mathbf{P}\big(M(j) \geqq n\big) \leqq \mathbf{P}\big(M(1) \geqq nj^{-1}\big) \leqq \frac{\alpha(1+\varepsilon)\, j}{n}. \tag{75}$$

To get a similar estimate from below note that for $\varepsilon \in (0,1)$,

$$\mathbf{P}\big(M(j) \geqq n\big) \geqq \mathbf{P}\big(M(j) \geqq n,\ M(1) \geqq (1+\varepsilon)\, nj^{-1}\big) = \sum_{l=1}^{\infty} \mathbf{P}\big(M(j) \geqq n,\ \varrho = l\big),$$

where $\varrho := \min\{l\colon\ Z_l \geqq (1+\varepsilon)\, nj^{-1}\}$ is the first moment when the generation size exceeds $(1+\varepsilon)\, nj^{-1}$. By the Markov property we get

$$\begin{aligned}\mathbf{P}\big(M(j) \geqq n,\ \varrho = l\big) &= \sum_{r \geqq (1+\varepsilon)\, nj^{-1}} \mathbf{P}\big(M(j) \geqq n,\ Z_l = r,\ \varrho = l\big)\\ &\geqq \sum_{r \geqq (1+\varepsilon)\, nj^{-1}} \mathbf{P}\bigg(\sum_{i=l}^{l+j-1} Z_{i+l} \geqq n,\ Z_l = r,\ \varrho = l\bigg)\\ &= \sum_{r \geqq (1+\varepsilon)\, nj^{-1}} \mathbf{P}\bigg\{\sum_{i=0}^{j-1} Z_i \geqq n \,\bigg|\, Z_0 = r\bigg\}\mathbf{P}\big(Z_l = r,\ \varrho = l\big)\\ &\geqq \mathbf{P}(\varrho = l)\,\mathbf{P}\bigg\{\sum_{i=0}^{j-1} Z_i \geqq n \,\bigg|\, Z_0 = (1+\varepsilon)\, nj^{-1}\bigg\}.\end{aligned}$$

Therefore,

$$\mathbf{P}\big(M(j) \geqq n\big) \geqq \mathbf{P}\big(M(1) \geqq (1+\varepsilon)\, nj^{-1}\big)\,\mathbf{P}\bigg\{\sum_{l=0}^{j-1} Z_l \geqq n \,\bigg|\, Z_0 = (1+\varepsilon)\, nj^{-1}\bigg\}. \tag{76}$$

Choosing $\beta \in (1, 1+\alpha)$ and using (38), we obtain for $nj^{-1} \geqq 2/\varepsilon$,

$$\mathbf{P}\big(M(j) \geqq n\big) \geqq \mathbf{P}\big(M(1) \geqq (1+\varepsilon)\, nj^{-1}\big)\bigg(1 - c_{12}\bigg(\frac{j}{nQ(j)}\bigg)^{\beta-1}\bigg). \tag{77}$$

Observing that $nj_n^{-1} \to \infty$ by our assumption in (c) and recalling that $\mathbf{P}(M(1) \geqq x) \sim \alpha/x$ as $x \uparrow \infty$, we see that estimates (75) and (77) together imply

$$\mathbf{P}\big(M(j_n) \geqq n\big) \sim \mathbf{P}\big(M(1) \geqq nj_n^{-1}\big) \sim \frac{\alpha j_n}{n} \qquad \text{if} \quad \frac{j_n}{nQ(j_n)} \to 0. \tag{78}$$

Theorem 2 is proved.

3.2. Proof of Theorem 1(a). Since $M_m(j) \leqq M(j)$ for all $j, m \geqq 1$, it follows from (63) that

$$\mathbf{P}\big(M_m(j) \geqq n\big) \leqq \mathbf{P}\big(M(j) \geqq n\big) \leqq \mathbf{P}\big(M(\infty) \geqq n\big).$$

From here, (12), and the properties of regularly varying functions we conclude that, for any $\varepsilon \in (0,1)$ and sufficiently large j,

$$\begin{aligned}\sum_{1 \leqq n \leqq j/(\varepsilon Q(j))} \mathbf{P}\big(M_m(j) \geqq n\big) &\leqq \sum_{1 \leqq n \leqq j/(\varepsilon Q(j))} \mathbf{P}\big(M(\infty) \geqq n\big)\\ &\leqq 2\frac{(1+\alpha)}{\alpha}\,\frac{j}{\varepsilon Q(j)}\,\mathbf{P}\bigg(M(\infty) \geqq \frac{j}{\varepsilon Q(j)}\bigg).\end{aligned}$$

By Lemma 7(b) we have for $\varepsilon \in (0,1)$ and for sufficiently large j,

$$\mathbf{P}\bigg(M(\infty) \geqq \frac{j}{\varepsilon Q(j)}\bigg) \leqq \mathbf{P}\bigg(M(\infty) \geqq \frac{j}{Q(j)}\bigg) \leqq cQ(j).$$

Hence,

$$\sum_{1 \leqq n \leqq j/(\varepsilon Q(j))} \mathbf{P}\big(M_m(j) \geqq n\big) \leqq \frac{c}{\varepsilon}\, j. \tag{79}$$

Moreover, for any $\beta \in (1, 1+\alpha)$,

$$\mathbf{P}\bigl(M_m(1) \geqq x\bigr) \leqq \frac{\mathbf{E}\{Z_m^\beta \mid Z_0 = 1\}}{x^\beta} \leqq \frac{cQ^{1-\beta}(m)}{x^\beta},$$

which, in view of $\mathbf{P}(M_m(j) \geqq n) \leqq \mathbf{P}(M_m(1) \geqq nj^{-1})$, implies

$$\sum_{n \geqq \varepsilon j/Q(m)} \mathbf{P}\bigl(M_m(j) \geqq n\bigr) \leqq \sum_{n \geqq \varepsilon j/Q(m)} \mathbf{P}\bigl(M_m(1) \geqq nj^{-1}\bigr)$$
$$\leqq cj^\beta Q^{1-\beta}(m) \sum_{n \geqq \varepsilon j/Q(m)} n^{-\beta} \leqq c\varepsilon^{1-\beta} j$$

for $j \geqq j_0$. Clearly,

$$\mathbf{P}\bigl(M(j) \geqq n\bigr) - \mathbf{P}(Z_m > 0) \leqq \mathbf{P}\bigl(M_m(j) \geqq n\bigr) \leqq \mathbf{P}\bigl(M(j) \geqq n\bigr).$$

This and (78) show that for any $\delta \in (0,1)$ there exists an $\varepsilon \in (0,1)$ such that

$$(1-\delta)\,\frac{\alpha j}{n} - \mathbf{P}(Z_m > 0) \leqq \mathbf{P}\bigl(M_m(j) \geqq n\bigr) \leqq (1+\delta)\,\frac{\alpha j}{n}$$

for all $n \geqq \varepsilon^{-1} j/Q(j)$. Denoting

$$D_\varepsilon(j,m) := \biggl\{n\colon\ \frac{\varepsilon^{-1} j}{Q(j)} \leqq n \leqq \frac{\varepsilon j}{Q(m)}\biggr\},$$

we conclude that

$$(1-\delta)\,\alpha j \sum_{n \in D_\varepsilon(j,m)} \frac{1}{n} - \varepsilon j \leqq \sum_{n \in D_\varepsilon(j,m)} \mathbf{P}\bigl(M_m(j) \geqq n\bigr) \leqq (1+\delta)\,\alpha j \sum_{n \in D_\varepsilon(j,m)} \frac{1}{n},$$

that is,

$$(1-\delta)\,\alpha j \log\frac{Q(j)}{Q(m)} - cj \leqq \sum_{n \in D_\varepsilon(j,m)} \mathbf{P}\bigl(M_m(j) \geqq n\bigr) \leqq (1+\delta)\,\alpha j \log\frac{Q(j)}{Q(m)} + cj.$$

Since the function ℓ_4 from (9) is slowly varying, there exists an $a > 0$ and functions σ and θ satisfying $\sigma(x) \to \sigma \in (0,\infty)$ and $\theta(x) \to 0$ as $x \uparrow \infty$, such that (see [13, section 1.5])

$$\ell_4(n) = \sigma(n) \exp\biggl[\int_a^n \frac{\theta(x)}{x}\,dx\biggr].$$

Hence, it follows easily that for any $\mu > 0$, there exists $w = w(\mu)$ such that

$$\Bigl(\frac{m}{j}\Bigr)^{-\mu/\alpha} \leqq \frac{\ell_4(j)}{\ell_4(m)} \leqq \Bigl(\frac{m}{j}\Bigr)^{\mu/\alpha} \qquad \text{as} \quad \frac{j}{m} \leqq w.$$

Therefore, for $j/m < w$,

$$(1-\delta)(1-\mu)\,j \log\frac{m}{j} \leqq \sum_{n \in D_\varepsilon(j,m)} \mathbf{P}\bigl(M_m(j) \geqq n\bigr) \leqq (1+\delta)(1+\mu)\,j \log\frac{m}{j}. \tag{80}$$

Combining (79)–(80) and taking into account that δ and μ can be made arbitrarily small, we get

$$\mathbf{E}M_m(j) \sim j \log\frac{m}{j} \qquad \text{as} \quad \frac{j}{m} \to 0,$$

completing the proof of Theorem 1(a).

3.3. Proof of Theorem 1(b). Clearly,

$$\mathbf{E}M_{Tj}(j) = \mathbf{E}\big\{M_{Tj}(j);\ Z_j = 0\big\} + \mathbf{E}\big\{M_{Tj}(j);\ Z_j > 0\big\}$$
$$= \mathbf{E}\big\{S_0(j);\ Z_j = 0\big\} + \mathbf{E}\big\{M_{Tj}(j);\ Z_j > 0\big\}, \tag{81}$$

since

$$M_{Tj}(j)1_{\{Z_j=0\}} = \max_{0\leqq k\leqq Tj-j} \sum_{l=k}^{k+j-1} Z_l\, 1_{\{Z_j=0\}} = \max_{0\leqq k\leqq Tj-j} \sum_{l=k}^{j-1} Z_l\, 1_{\{Z_j=0\}}$$
$$= \sum_{l=0}^{j-1} Z_l\, 1_{\{Z_j=0\}} = S_0(j)\, 1_{\{Z_j=0\}}.$$

We study separately each term in (81), namely in Lemmas 21 and 22 below.

LEMMA 21. *As* $j \uparrow \infty$,

$$a_j\colon = \mathbf{E}\big\{S_0(j); Z_j = 0\big\} \sim \frac{\alpha j}{2\alpha+1}.$$

Remark. Under $\operatorname{Var}\xi < \infty$ this result was obtained by Karpenko and Nagaev in [8].

Proof of Lemma 21. Set

$$h_j(s_1, s_2) := \mathbf{E}\big\{s_1^{Z_j} s_2^{S_0(j)} \,\big|\, Z_0 = 1\big\}, \qquad h_0(s_1, s_2) := s_1 s_2.$$

Clearly,

$$h_j(s_1, s_2) = \mathbf{E}\Big\{\mathbf{E}\{s_1^{Z_j} s_2^{S_0(j)} \mid Z_1\} \,\Big|\, Z_0 = 1\Big\}$$
$$= \mathbf{E}\bigg\{s_2\Big(\mathbf{E}\big\{s_1^{Z_{j-1}} s_2^{S_0(j-1)} \,\big|\, Z_0 = 1\big\}\Big)^{Z_1} \,\bigg|\, Z_0 = 1\bigg\} = s_2 f\big(h_{j-1}(s_1, s_2)\big).$$

Hence,

$$\mathbf{E}\Big\{s_2^{S_0(j)},\ Z_j = 0 \,\big|\, Z_0 = 1\Big\} = h_j(0, s_2) = s_2 f\big(h_{j-1}(0, s_2)\big). \tag{82}$$

Note, that $h_j(0,1) = f_j(0) = \mathbf{P}\{Z_j = 0 \mid Z_0 = 1\}$ and $a_1 = f_1(0)$. Differentiating (82) at $s_2 = 1-$ gives, for $j \geqq 2$,

$$a_j = f_j(0) + f'\big(f_{j-1}(0)\big)\, a_{j-1}$$
$$= f_j(0) + f'\big(f_{j-1}(0)\big)\, f_{j-1}(0) + f'\big(f_{j-1}(0)\big)\, f'\big(f_{j-2}(0)\big)\, a_{j-2},$$

leading to

$$a_j = f_j(0) + \sum_{k=1}^{j-1} f_k(0) \prod_{r=k}^{j-1} f'\big(f_r(0)\big) = f_j(0) + d_j \sum_{k=1}^{j-1} f_k(0)\, \frac{1}{d_k}, \tag{83}$$

where the d_k are as in Lemma 5. Recalling Lemma 5 and observing that $f_k(0) \uparrow 1$ as $k \uparrow \infty$, we get

$$\sum_{k=1}^{j-1} f_k(0)\, \frac{1}{d_k} \sim \sum_{k=1}^{j-1} \frac{k^{1+1/\alpha}}{l_3(k)} \sim \frac{\alpha}{2\alpha+1}\, \frac{j^{2+1/\alpha}}{l_3(j)} \qquad \text{as} \quad j \uparrow \infty.$$

From here, (83), and (10), the statement of the lemma follows easily.

LEMMA 22. *For* $T \geqq 1$,

$$\lim_{j\to\infty} \mathbf{E})\big\{j^{-1}Q(j)\, M_{Tj}(j) \mid Z_j > 0\big\} = \mathbf{E}V^*(T). \tag{84}$$

Proof. It follows from Proposition 13 and the Donsker–Prokhorov invariance principle that

$$\big\{j^{-1}Q(j)\, M_{Tj}(j) \mid Z_j > 0\big\} \underset{n\uparrow\infty}{\overset{\mathcal{L}}{\Longrightarrow}} V^*(T). \tag{85}$$

To prove that convergence of the expectations occurs, recall that

$$M_{Tj}(j) \leqq M_{Tj}(Tj) = \sum_{l=0}^{Tj-1} Z_l.$$

Hence,

$$\mathbf{P}\big\{j^{-1}Q(j)\, M_{Tj}(j) > y \mid Z_j > 0\big\} \leqq \mathbf{P}\big\{j^{-1}Q(j)\, M_{Tj}(Tj) > y \mid Z_j > 0\big\}$$

$$\leqq \mathbf{P}\left\{Q(j) \max_{0\leqq k\leqq Tj} Z_k > y \,\middle|\, Z_j > 0\right\} \leqq \frac{Q^\beta(j)}{Q(j)}\, \frac{\mathbf{E}Z_{Tj}^\beta}{y^\beta}, \tag{86}$$

the last step by Doob's inequality. By Lemma 11 and (9), we can continue with

$$\leqq \frac{c}{y^\beta}\, \frac{Q^{\beta-1}(j)}{Q^{\beta-1}(Tj)} \leqq \frac{c}{y^\beta}\, T^{(\beta-1)/\alpha}. \tag{87}$$

Therefore,

$$\mathbf{P}\big(V^*(T) > y\big) \leqq \frac{c}{y^\beta}\, T^{(\beta-1)/\alpha}. \tag{88}$$

In order to complete the proof, note that since $\beta > 1$, the derived chain of estimates (86)–(87) and inequality (88) provide the uniform integrability of the prelimiting and limiting variables in (85). Hence, the claimed convergence (84) of moments follows.

Now we are ready to complete the proof of Theorem 1. Clearly,

$$j_m^{-1}\mathbf{E}M_m(j_m) = j_m^{-1}\mathbf{E}\big\{M_m(j_m),\ Z_{j_m} = 0\big\} + j_m^{-1}\mathbf{E}\big\{M_m(j_m),\ Z_{j_m} > 0\big\}$$

$$= j_m^{-1}\mathbf{E}\big\{S_0(j_m), Z_{j_m} = 0\big\} + \mathbf{E}\big\{j_m^{-1}Q(j_m)\, M_m(j_m) \mid Z_{j_m} > 0\big\}.$$

Applying Lemmas 21 and 22 with $T = 1/\eta$, we obtain

$$\lim_{m\uparrow\infty} j_m^{-1}\mathbf{E}M_m(j_m) = \frac{\alpha}{2\alpha+1} + \mathbf{E}V^*\left(\frac{1}{\eta}\right) =:\ \varphi(\eta), \tag{89}$$

that is, (4). Recalling Propositions 14 and 15, we see that (5) is valid as well.

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