LIMIT THEOREMS FOR PROBABILITIES OF LARGE DEVIATIONS OF A CRITICAL GALTON–WATSON PROCESS HAVING POWER TAILS[*]

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Abstract. Limit theorems are established for probabilities of large deviations of a critical Galton–Watson process given that the power moments are finite and the tail distribution of the offspring number of a single particle is regularly varying.

1. Introduction and statement of results. Let $\{Z_n, \ n \geqq 0\}$ be a critical Galton–Watson process. In what follows we assume (if the opposite is not stated) that $Z_0 = 1$. Let $\{p_k, \ k \geqq 0\}$ denote the offspring distribution of a particle and let $f(s)$ be the generating function of this distribution. Set $Q_n := \mathbf{P}\{Z_n > 0\}$, $B := f''(1)$ and denote by $\mathcal{A}_n(N)$ the event that each individual of the first n generations has at most N direct descendants.

The main goal of the present paper is to study the probabilities of large deviations of the random variables Z_n and $M_n := \max_{k \leqq n} Z_k$.

Papers [4], [5], [6], [7], and [8] investigate large deviations of Z_n under the Cramér conditions (meaning existence of an exponential moment of the distribution $\{p_k\}$). More precisely, articles [4], [5], and [7] are devoted to proving limit theorems for the probabilities of large deviations of the process Z_n. The most general results in this direction are given in [7], where asymptotic representations are deduced for the probabilities $\mathbf{P}\{Z_n = k\}$ and $\mathbf{P}\{Z_n \geqq k\}$ as $k = o(n^2)$. Probabilistic inequalities for $\mathbf{P}\{Z_n \geqq k\}$ and $\mathbf{P}\{M_n \geqq k\}$ were the subject of investigation in [6] and [8]. Paper [6] assumes the existence of an exponential moment, while in [8] a refinement of an estimate from [6] is obtained and inequalities are deduced under weaker moment hypotheses on the process.

In the present paper we prove limit theorems for probabilities of large deviations of a critical Galton–Watson process given that the power moments are finite and the tail distribution of the offspring number of a single particle is regularly varying.

THEOREM 1. *If* $\mathbf{E}Z_1^r < \infty$ *for some* $r \geqq 3$, *then*

$$(1) \qquad \mathbf{P}\{Z_n \geqq k\} = \frac{2}{Bn} \exp\left\{-\frac{2k}{Bn}\right\}(1 + o(1))$$

as $n \to \infty$ *and* $k \leqq B(r/2 - 1)\,n \log n - B(r/2 + \varepsilon)\,n \log \log n$, $\varepsilon > 0$. *If this equality holds true for* $k \leqq B(r/2 - 1)\,n \log n + B((r + 1)/2 + \varepsilon)\,n \log \log n$, *then* $\mathbf{E}Z_1^r < \infty$.

Recall that if the second moment is finite then, by the Yaglom theorem (see, for instance, [10, p. 39]), relation (1) is valid if the ratio k/n is bounded. If an exponential

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moment is finite then, according to [7], convergence to the exponential distribution takes place for $k = o(n^2/\log n)$.

If $r_0 := \sup\{r\colon \mathbf{E}Z_1^r < \infty\}$ is finite, then the conditions for the convergence to the exponential distribution given in Theorem 1 are close to the necessary and sufficient ones. Indeed, if $\overline{k}(n)$ denotes the upper boundary for the k meeting (1), then Theorem 1 implies

$$\lim_{n \to \infty} \frac{\overline{k}(n)}{n \log n} = \frac{B}{2}\,(r_0 - 2).$$

THEOREM 2. *Let*

$$(2) \qquad\qquad \mathbf{P}\{Z_1 \geqq x\} = x^{-t}\,L(x)$$

for some $t > 1$ and a slowly varying function $L(x)$. If $B = \infty$, then

$$(3) \qquad \mathbf{P}\{M_n \geqq k\} \sim \mathbf{P}\{Z_n \geqq k\} \sim \mathbf{P}\big(\overline{\mathcal{A}}_n(k)\big) \sim n\mathbf{P}\{Z_1 \geqq k\}$$

for k and $n = n_k$ such that $kQ_n \to \infty$ as $k \to \infty$.

If $B < \infty$, then relations (3) are valid for k and $n = n_k$, satisfying the condition $k/(n \log n) \to \infty$.

Remark. It is easy to see that if (2) is valid, then $B < \infty$ if and only if either $t > 2$ or $t = 2$ and $L(x)$ satisfies the condition $\int_1^\infty x^{-1}L(x)\,dx < \infty$.

If (2) is valid and $B = \infty$, then by (3),

$$(4) \qquad \mathbf{P}\{Z_n \geqq x_n Q_n^{-1}\} \sim n\mathbf{P}\{Z_1 \geqq x_n Q_n^{-1}\} \qquad \text{as} \quad n \to \infty$$

for any sequence $x_n \to \infty$. Using a Tauberian theorem (see, for instance, [9, Theorem XIII.5.5]) and a corollary from Lemma 5 in [2], it is easy to show that for $t \in (1, 2)$, relation (2) is equivalent to

$$(5) \qquad\qquad f(s) = s + (1 - s)^t\,L^*\big((1 - s)^{-1}\big),$$

where $L^*(x)$ is slowly varying and

$$(6) \qquad\qquad \frac{L^*(x)}{L(x)} \sim (t - 1)^{-1}\,\Gamma(2 - t) \qquad \text{as} \quad x \to \infty.$$

Further, if (2) is valid for $t = 2$ and $L(x)$ is such that $B = \infty$, then (5) is valid with $L^*(x)$ satisfying the relation

$$(7) \qquad\qquad L^*(x) \sim \int_1^x y^{-1}L(y)\,dy \qquad \text{as} \quad x \to \infty.$$

It is shown in [13] and [14] that condition (5) is necessary and sufficient for the equality

$$(8) \qquad\qquad \lim_{n \to \infty} Q_n^{-1}\mathbf{P}\{Z_n \geqq xQ_n^{-1}\} = 1 - F^{(t)}(x)$$

to be valid for any fixed $x > 0$, where $F^{(t)}(x)$ is a nondegenerate distribution function. Thus, if the variance is infinite and condition (2) is valid, equalities (8) and (4) describe the asymptotical behavior of the probabilities of all deviations.

If the variance is finite the results described do not cover the whole spectrum of deviations. For instance, if $t > 3$, then there is a gap between the zones covered by Theorems 1 and 2: the asymptotic behavior of the probability $\mathbf{P}\{Z_n \geqq k\}$ is not known for $k \in (cn \log n,\, a_n n \log n)$, where $c > B(t/2 - 1)$ and $a_n \to \infty$ arbitrary slowly.

2. Auxiliary results.

2.1. Properties of censors. Set $\tilde{f}(s) := \sum_{0 \leq k \leq N} p_k s^k$, denote by $x_1 = x_1(N)$ the maximal root of the equation $x = \tilde{f}(x)$.

LEMMA 1. *If $\tilde{f}''(1) > 0$, then*

(9)
$$0 \leq x_1 - 1 \leq \frac{1 - \tilde{f}'(1)}{\tilde{f}''(1)} + \left(\frac{(1 - \tilde{f}'(1))^2}{(\tilde{f}''(1))^2} + \frac{2(1 - \tilde{f}(1))}{\tilde{f}''(1)} \right)^{1/2}.$$

Proof. Put

$$\alpha(x) := \frac{\tilde{f}''(1)}{2}(x - 1)^2 + \tilde{f}'(1)(x - 1) + \tilde{f}(1).$$

It is easy to see that

$$\alpha(1) = \tilde{f}(1), \quad \alpha'(1) = \tilde{f}'(1), \quad \text{and} \quad \alpha''(x) \leq \tilde{f}''(x) \qquad \text{for} \quad x \geq 1.$$

Hence it follows that x_1 is less than the maximal root of the equation $x = \alpha(x)$. Finding the root, we obtain the upper bound in (9). To demonstrate the validity of the lower bound it is sufficient to observe that $\tilde{f}(1) \leq 1$ for all N.

Set $g(s) := \tilde{f}(sx_1)/x_1$, $A_g := g'(1)$, and $B_g := g''(1)$.

LEMMA 2. *Assume $\mathbf{E}Z_1^r < \infty$ for some $r \geq 3$. Then, as $N \to \infty$,*

(10)
$$A_g = 1 + O(N^{-r/2}),$$

(11)
$$B_g = B + O(N^{-\theta}),$$

where $\theta := \min\{r/2, r - 2\}$.

Proof. Note first of all that by the Markov inequality,

(12)
$$1 - \tilde{f}(1) = \mathbf{P}\{Z_1 > N\} \leq \frac{\mathbf{E}Z_1^r}{N^r}$$

and

(13)
$$1 - \tilde{f}'(1) = \mathbf{E}\{Z_1; \ Z_1 > N\} \leq \frac{\mathbf{E}Z_1^r}{N^{r-1}}.$$

Without loss of generality we may assume N to be so large that $\tilde{f}''(1) \geq B/2$. Applying this estimate and inequalities (12) and (13) to the right-hand side of (9), we have

(14)
$$0 \leq x_1 - 1 \leq cN^{-r/2}.$$

(Here and in what follows the symbol c stands for positive constants depending on only the distribution $\{p_k\}$.)

By the definition of $g(s)$ and the mean value theorem, we obtain

(15)
$$A_g = \tilde{f}'(x_1) \leq \tilde{f}'(1) + \tilde{f}''(x_1)(x_1 - 1) \leq 1 + Bx_1^N(x_1 - 1),$$

where the second inequality in the chain above follows from the estimates

$$\tilde{f}''(x_1) \leq x_1^N \tilde{f}''(1) \leq x_1^N B.$$

Substituting (14) in (15), we deduce for all sufficiently large N the estimate

$$(16) \qquad\qquad A_g \leqq 1 + cN^{-r/2}.$$

On the other hand, the definition of x_1 leads to $\tilde{f}'(x_1) \geqq 1$. Hence, $A_g > 1$ proves the first part of the lemma.

Clearly,

$$B_g = x_1 \tilde{f}''(x_1) > \tilde{f}''(1) > B - \mathbf{E}\{Z_1^2; Z_1 \geqq N\}.$$

Applying the Markov inequality to the expectation in the right-hand side, we have

$$(17) \qquad\qquad B_g > B - \frac{\mathbf{E}Z_1^r}{N^{r-2}}.$$

On the other hand, similarly to (16),

$$(18) \qquad\qquad B_g < B + cN^{-r/2}.$$

Combining (17) and (18) gives (11). Lemma 2 is proved.

From now on we consider the quantities y_0 and N involved in the subsequent arguments as functions of the variable n, i.e., $y_0 = y_0(n)$ and $N = N(n)$. In addition, we suppose that $f'''(1) < \infty$ in the remaining part of the point.

LEMMA 3. *Let y_j be a sequence specified by the equation*

$$y_{j+1} = g^{-1}(1 + y_j) - 1,$$

where y_0 is selected in such a way as to provide the boundedness of $g'''(1 + y_0)$ for all $n \geqq 1$. Then

$$\sup_{j \leqq n} \left| y_j \frac{y_0}{1 + Bjy_0/2} - 1 \right| = O(y_0 + nN^{-3/2} + N^{-1}) \qquad as \quad n \to \infty.$$

Proof. By definition,

$$y_j = g(1 + y_{j+1}) - 1.$$

Clearly, $g(1 + y) > 1 + A_g y > 1 + y$ for any $y > 0$. Consequently, the sequence y_j is decreasing. Expanding $g(1 + y)$ in a Taylor series, we obtain

$$y_j = A_g y_{j+1} + \frac{B_g}{2} y_{j+1}^2 + \frac{g'''(\theta_j)}{6} y_{j+1}^3, \qquad \theta_j \in (1, 1 + y_{j+1}).$$

Since y_j is decreasing and $g'''(1 + y_0)$ is bounded, we conclude that

$$(19) \qquad\qquad y_j = A_g y_{j+1} + \frac{B_g}{2} y_{j+1}^2 + O(y_{j+1}^3).$$

Therefore,

$$\frac{y_{j+1}}{y_j} = \left(A_g + \frac{B_g}{2} y_{j+1} + O(y_{j+1}^2) \right)^{-1}$$

$$(20) \qquad\qquad = \frac{1}{A_g} - \frac{B_g}{2A_g^2} y_{j+1} + O(y_{j+1}^2) = \frac{1}{A_g} - \frac{B_g}{2A_g^3} y_j + O(y_j^2).$$

Dividing both sides of (19) by $y_j y_{j+1}$ and using (20), we have

$$(21) \qquad \frac{1}{y_{j+1}} = \frac{A_g}{y_j} + \frac{B_g}{2A_g} + O(y_j) = \frac{A_g^{j+1}}{y_0} + \frac{B_g}{2A_g} \frac{A_g^{j+1} - 1}{A_g - 1} + O\left(\sum_{k=0}^{j} y_k A_g^{j-k}\right).$$

Using (10) for $r = 3$, we conclude that

$$|A_g^i - 1| \leqq c \frac{i}{N^{3/2}}, \qquad i \geqq 1,$$

and

$$\left| \frac{A_g^j - 1}{A_g - 1} - j \right| \leqq c \frac{j^2}{N^{3/2}}, \qquad j \geqq 1.$$

Applying these inequalities and (11) to the right-hand side of (21) and taking into account the relation $y_k < y_0$ being valid for all $k > 0$, we deduce for any $j \geqq 0$ the equality

$$\frac{1}{y_{j+1}} = \left(\frac{1}{y_0} + \frac{B}{2}(j+1) + O(jy_0) + O(N^{-1}) \right)\left(1 + O(N^{-1} + jN^{-3/2})\right)$$

$$(22) \qquad = \left(\frac{1}{y_0} + \frac{B}{2}(j+1) \right)\left(1 + O(y_0 + N^{-1} + jN^{-3/2})\right).$$

Hence the statement of the lemma follows.

Set $r_0 = 1 + y_n$ and, for each $j = 1, \ldots, n$, define the probability generating function

$$\rho_j(s) := \frac{g(g_{j-1}(r_0)\, s)}{g_j(r_0)}.$$

Introduce the notation

$$a_j := \rho_j'(1), \quad a(j) := \prod_{i=1}^{j} a_i, \quad b_j := \rho_j''(1),$$

$$T(n) := \sum_{i=0}^{n-1} \frac{b_{i+1}}{2a_{i+1}} a(i).$$

LEMMA 4.   *If $g'''(1 + y_0)$ is bounded, $y_0 \log n \to 0$, and $nN^{-3/2} \to 0$, then, as $n \to \infty$,*

$$(23) \qquad a(n) = \left(1 + \frac{Bny_0}{2} \right)^2 (1 + o(1)),$$

$$(24) \qquad T(n) = \frac{Bn}{2}\left(1 + \frac{Bny_0}{2} \right)(1 + o(1)).$$

Proof. In accordance with the definition of $a(j)$,

$$a(j) = \frac{r_0}{g_j(r_0)} \prod_{i=0}^{j-1} g'(1 + y_{n-i}) = \frac{r_0}{g_j(r_0)} \exp\left\{ \sum_{i=0}^{j-1} \log g'(1 + y_{n-i}) \right\}.$$

Using the equalities

$$g'(1+z) = A_g\left(1 + \frac{B_g z}{A_g} + O(z^2)\right), \quad \log(1+t) = t + O(t^2),$$

we have

$$(25) \qquad a(j) = \frac{r_0}{g_j(r_0)}\, A_g^j \exp\left\{\frac{B_g}{A_g}\sum_{i=0}^{j-1} y_{n-i} + O\left(\sum_{i=0}^{j-1} y_{n-i}^2\right)\right\}.$$

Applying Lemma 3 and estimates (23)–(25) of [7], we obtain

$$\sum_{i=0}^{j-1} y_{n-i} = \frac{2}{B}\,\log\left(\frac{1 + Bny_0/2}{1 + B(n-j)\,y_0/2}\right) + O\big(\log n\,(y_0 + nN^{-3/2} + N^{-1})\big),$$

$$\sum_{i=0}^{j-1} y_i^2 = O(y_0).$$

Substituting these equalities in (25) and applying Lemma 1, we obtain, for any $j \leqq n$, the equality

$$(26) \qquad a(j) = \left(\frac{1 + Bny_0/2}{1 + B(n-j)\,y_0/2}\right)^2\left(1 + O\big(\log n\,(y_0 + nN^{-3/2} + N^{-1})\big)\right).$$

Setting $j = n$ here gives (23).

Using the boundedness of $g'''(1 + y_0)$, equality (26), and Lemma 1, we see that

$$T(n) = \sum_{i=0}^{n-1}\left(\frac{1 + Bny_0/2}{1 + B(n-i)\,y_0/2}\right)^2\left(1 + O\big(\log n\,(y_0 + nN^{-3/2} + N^{-1})\big)\right).$$

By virtue of (30) in [7],

$$\sum_{i=0}^{n-1}\left(\frac{1 + Bny_0/2}{1 + B(n-i)\,y_0/2}\right)^2 = \frac{Bn}{2}\left(1 + \frac{Bny_0}{2}\right)\big(1 + O(y_0)\big).$$

The last two equalities imply (24). Lemma 4 is proved.

Let $Z^* = \{Z_k^*;\ 0 \leqq k \leqq n\}$ be a time-inhomogeneous branching process whose transition probabilities are specified by the equalities

$$(27) \qquad \mathbf{E}\{s^{Z_k^*} \mid Z_{k-1}^* = 1\} = \rho_{n-k+1}(s), \qquad k = 1, \ldots, n.$$

Put

$$(28) \qquad F_n(x) := \mathbf{P}\big\{Z_n^* < xT(n) \mid Z_n^* > 0\big\}.$$

Lemma 5. *Under the conditions of the previous lemma,*

$$(29) \qquad \delta_n := \sup_x \left|F_n(x) - 1 + e^{-x}\right| \leqq c(y_0 + n^{-1})\,\log^2 n$$

and

$$(30) \qquad \mathbf{P}\{Z_n^* > 0\} = \left(\frac{2}{Bn} + y_0\right)\big(1 + o(1)\big) \qquad as \quad n \to \infty.$$

Proof. It is easy to see that $a_i > 1$ for all $i \geqq 1$. On the other hand,

$$\sup_{1 \leqq i \leqq n} a_i < \sup_{1 \leqq i \leqq n} g'(1 + y_{n-i+1}) < A_g + y_0 \, g''(1 + y_0).$$

Using Lemma 1 and observing that the boundedness of $g'''(1+y_0)$ implies the boundedness of $g''(1 + y_0)$, we obtain the estimate

$$\sup_{1 \leq i \leq n} a_i < 1 + c(N^{-3/2} + y_0).$$

As a result we have

$$(31) \qquad \sup_{1 \leq i \leq n} |a_i - 1| \to 0.$$

It is easy to see that

$$(32) \qquad \sup_{1 \leqq i \leqq n} b_i \geqq \frac{B}{2}$$

for all sufficiently large n and N. Using the boundedness of $g'''(1 + y_0)$ once again we obtain

$$(33) \qquad \sup_{1 \leqq i \leqq n} \rho_i'''(1) \leqq c.$$

Relations (31)–(33) mean that the process Z^* meets all the conditions of Theorem 3 in [1], according to which

$$\delta_n \leqq c \max_{i \leqq n} a(i) \, \frac{\log^2 T(n)}{T(n)} = ca(n) \frac{\log^2 T(n)}{T(n)}$$

and

$$\mathbf{P}\{Z_n^* > 0\} = \frac{a(n)}{T(n)} \left(1 + o(1)\right) \qquad \text{as} \quad n \to \infty.$$

These relations and Lemma 4 yield the desired statements. Lemma 5 is proved.

2.2. Estimates from below for large deviations.
LEMMA 6. *For any $k \geqq 2(B \vee 1)$ the following inequalities are valid:*

$$\mathbf{P}\{M_n \geqq k\} > \mathbf{P}\big(\overline{\mathcal{A}}_n(k-1)\big) > n\mathbf{P}\{Z_1 \geqq k\} \exp\left(-\frac{(2B+1)\,n}{k-1}\right).$$

Proof. Clearly, for any $j \geqq 1$ we have

$$\tilde{f}_j(1) - \tilde{f}_{j+1}(1) \geqq \tilde{f}'(x_0)\big(\tilde{f}_{j-1}(1) - \tilde{f}_j(1)\big) \geqq \big(\tilde{f}'(x_0)\big)^j\big(1 - \tilde{f}(1)\big),$$

where $x_0 = x_0(N)$ is the minimal positive solution of the equation $x = \tilde{f}(x)$. Therefore,

$$(34) \quad \mathbf{P}\big(\overline{\mathcal{A}}_n(N)\big) = 1 - \tilde{f}_n(1) = \sum_{j=0}^{n-1} \big(\tilde{f}_j(1) - \tilde{f}_{j+1}(1)\big) \geqq \frac{1 - (\tilde{f}'(x_0))^n}{1 - \tilde{f}'(x_0)}\big(1 - \tilde{f}(1)\big).$$

It is easy to see that

(35) $$\tilde{f}'(x_0) > \tilde{f}'(1) - \tilde{f}''(1)(1 - x_0).$$

Setting $r = 2$ in (13), we have

(36) $$\tilde{f}'(x_0) > 1 - (B+1)\,N^{-1} - \tilde{f}''(1)(1 - x_0).$$

According to (45) in [8],

(37) $$1 - x_0 < N^{-1}.$$

Substituting (37) in (36), we obtain

$$\tilde{f}'(x_0) > 1 - \frac{2B+1}{N}.$$

Applying this estimate to the right-hand side of (34), we get the inequality

$$\mathbf{P}\big(\overline{\mathcal{A}}_n(N)\big) \geqq \frac{N}{2B+1}\left(1 - \left(1 - \frac{2B+1}{N}\right)^n\right)\mathbf{P}\{Z_1 > N\}.$$

It is not difficult to see that

(38) $$1 - (1-x)^n \geqq 1 - e^{-nx} \geqq nxe^{-nx}$$

for any $x \in [0,1]$. Hence we conclude that

$$\mathbf{P}\big(\overline{\mathcal{A}}_n(N)\big) \geqq n\mathbf{P}\{Z_1 > N\}\,\exp\left(-\frac{(2B+1)\,n}{N}\right)$$

for any $N \geqq 2B+1$. To complete our arguments it remains to observe that $\overline{\mathcal{A}}_n(k-1) \subset \{M_n \geqq k\}$. Lemma 6 is proved.

LEMMA 7. *Assume $\mathbf{E}Z_1^3 < \infty$. If $n = n_k$ is such that $k/n \to \infty$ as $k \to \infty$, then*

$$\limsup_{k\to\infty} \frac{\mathbf{P}\{M_n \geqq k\}}{\mathbf{P}\{Z_n \geqq k\}} \leqq 2.$$

Proof. According to the von Bahr–Esseen inequality (see, for instance, [12, Chap. V, Theorem 4]),

$$\mathbf{P}\{Z_i \geqq k \mid Z_0 = k\} \geqq \frac{1}{2} - c_0 k^{-1/2}\frac{\mathbf{E}|Z_i - 1|^3}{(\mathbf{E}(Z_i - 1)^2)^{3/2}},$$

where c_0 is an absolute constant.

Since $\mathbf{E}|Z_i - 1|^3 \leqq ci^2$ and $\mathbf{E}(Z_i - 1)^2 = Bi$, it follows that

$$\mathbf{P}\{Z_i \geqq k \mid Z_0 = k\} \geqq \frac{1}{2} - c\,\frac{\sqrt{i}}{\sqrt{k}},$$

and, consequently,

(39) $$\min_{i<n}\mathbf{P}\{Z_i \geqq k \mid Z_0 = k\} \geqq \frac{1}{2} - c\,\frac{\sqrt{n}}{\sqrt{k}}.$$

As is shown in [3],

$$
(40) \qquad \mathbf{P}\{M_n \geqq k\} \leqq \frac{\mathbf{P}\{Z_n \geqq \nu k\}}{\min_{i<n} \mathbf{P}\{Z_i \geqq \nu k \mid Z_0 = k\}}, \qquad \nu \in [0,1].
$$

Letting $\nu = 1$ here and applying inequality (39) to the denominator, we deduce the required relation. Lemma 7 is proved.

3. Proof of the main results.

3.1. Proof of Theorem 1. Clearly,

$$
(41) \qquad \mathbf{P}\{Z_n \geqq k\} = \mathbf{P}\{Z_n \geqq k;\ \mathcal{A}_n(N)\} + \mathbf{P}\{Z_n \geqq k;\ \overline{\mathcal{A}}_n(N)\}
$$

for any $N \geqq 1$. According to the definition of $g(s)$,

$$
(42) \qquad \sum_{j=0}^{\infty} \mathbf{P}\{Z_n = j;\ \mathcal{A}_n(N)\} s^j = \tilde{f}_n(s) = x_1 g_n\left(\frac{s}{x_1}\right).
$$

Further, by virtue of (27),

$$
(43) \qquad \mathbf{E}s^{Z_n^*} = \frac{g_n(sr_0)}{g_n(r_0)} = \frac{g_n(sr_0)}{g_n(1+y_n)} = \frac{g_n(sr_0)}{1+y_0}.
$$

Combining (42) and (43), we conclude that

$$
\mathbf{P}\{Z_n \geqq k;\ \mathcal{A}_n(N)\} = x_1(1+y_0) \sum_{j=k}^{\infty} (x_1 r_0)^{-j} \mathbf{P}\{Z_n^* = j\}
$$
$$
= x_1(1+y_0)\, \mathbf{E}\{e^{-hZ_n^*};\ Z_n^* \geqq k\},
$$

where $h := \log(x_1 r_0)$. Recalling definition (28) of the function $F_n(x)$, we see that

$$
(44) \qquad \mathbf{P}\{Z_n \geqq k;\ \mathcal{A}_n(N)\} = x_1(1+y_0)\, \mathbf{P}\{Z_n^* > 0\} \int_{k/T(n)}^{\infty} e^{-hT(n)\,x}\, dF_n(x).
$$

Integration by parts gives

$$
\int_{k/T(n)}^{\infty} e^{-hT(n)\,x}\, dF_n(x) = hT(n) \int_{k/T(n)}^{\infty} F_n(x)\, e^{-hT(n)\,x}\, dx - F_n\left(\frac{k}{T(n)}\right) e^{-hk}
$$

and

$$
\frac{e^{-hk-k/T(n)}}{1+hT(n)} = \int_{k/T(n)}^{\infty} e^{-hT(n)\,x}\, d(1 - e^{-x})
$$
$$
= hT(n) \int_{k/T(n)}^{\infty} (1 - e^{-x})\, e^{-hT(n)\,x}\, dx - (1 - e^{-k/T(n)})\, e^{-hk}.
$$

Subtracting the first of these equalities from the second, we obtain

$$
\left| \int_{k/T(n)}^{\infty} e^{-hT(n)\,x}\, dF_n(x) - \frac{e^{-hk-k/T(n)}}{1+hT(n)} \right|
$$
$$
\leqq \sup_x \left| F_n(x) - 1 + e^{-x} \right| \left(e^{-hk} + hT(n) \int_{k/T(n)}^{\infty} e^{-hT(n)\,x}\, dx \right).
$$

Calculating the integral in the right-hand side of this inequality leads to the estimate

$$(45) \qquad \left| \int_{k/T(n)}^{\infty} e^{-hT(n)\,x}\, dF_n(x) - \frac{e^{-hk - k/T(n)}}{1 + hT(n)} \right| \leqq 2\delta_n e^{-hk}.$$

We set

$$N = \frac{n}{\log n}, \qquad y_0 = \frac{4k}{B^2 n^2} - \frac{2}{Bn}$$

and prove the boundedness of $g'''(1 + y_0)$ for the shown values of N and y_0.

Clearly,

$$(46) \qquad g'''(1 + y_0) = x_1^2 \tilde{f}'''\big(x_1(1 + y_0)\big) \leqq \tilde{f}'''(1)\, x_1^{N+2}(1 + y_0)^N.$$

According to (14) for $r = 3$,

$$(47) \qquad x_1^{N+2} \leqq \exp\big\{ c_1 N^{-3/2}(N + 2) \big\} \leqq c_2.$$

On the other hand, we are interested in the not too big values of k, namely, $k \leqq cn \log n$. This means that $y_0 \leqq cn^{-1} \log n$ and, consequently,

$$(48) \qquad (1 + y_0)^N \leqq c$$

for $N = n/\log n$. Combining (46)–(48), we see that $g'''(1 + y_0)$ is bounded. This fact allows us to use the earlier results established in Lemmas 3, 4, and 5. Setting $y_0 = 4k/(B^2 n^2) - 2/(Bn)$ in (24), (29), and (30), we see that, as $n \to \infty$,

$$(49) \qquad T(n) = k\big(1 + o(1)\big),$$

$$(50) \qquad \delta_n \leqq c\, \frac{\log^3 n}{n},$$

and

$$(51) \qquad \mathbf{P}\{Z_n^* > 0\} = \frac{4k}{B^2 n^2}\big(1 + o(1)\big)$$

(recall the definition of δ_n in (29)).

Using Lemma 3, we have

$$(52) \qquad y_n = \frac{2}{Bn} - \frac{1}{k} + O\left(\frac{\log^{3/2} n}{n^{3/2}}\right) \qquad \text{as} \quad n \to \infty.$$

On the other hand, (14) implies the estimate

$$(53) \qquad x_1 - 1 < c\, \frac{\log^{3/2} n}{n^{3/2}}.$$

By (52) and (53) we conclude that

$$h = \log\big(x_1(1 + y_n)\big) = (x_1 - 1) + y_n + O\big(y_n^2 + (x_1 - 1)^2\big)$$

$$(54) \qquad = \frac{2}{Bn} - \frac{1}{k} + O\left(\frac{\log^{3/2} n}{n^{3/2}}\right).$$

Combining (45), (49), (50), and (54) gives

$$
(55) \qquad \int_{k/T(n)}^{\infty} e^{-hT(n)\,x}\, dF_n(x) = \frac{Bn}{2k} \exp\!\left(-\frac{2k}{Bn}\right)\bigl(1 + o(1)\bigr).
$$

Substituting (51) and (55) in (44), we derive the following equality for $k \leqq cn \log n$:

$$
(56) \qquad \mathbf{P}\bigl\{Z_n \geqq k;\ \mathcal{A}_n(N)\bigr\} = \frac{2}{Bn} \exp\!\left(-\frac{2k}{Bn}\right)\bigl(1 + o(1)\bigr).
$$

Consider now the second summand in the right-hand side of (41). Obviously,

$$
\mathbf{P}\bigl\{Z_n \geqq k;\ \overline{\mathcal{A}}_n(N)\bigr\} \leqq \mathbf{P}\bigl(\overline{\mathcal{A}}_n(N)\bigr).
$$

On the other hand, in view of (43) in [8] and the Markov inequality,

$$
\mathbf{P}\bigl\{\overline{\mathcal{A}}_n(N)\bigr\} \leqq n\mathbf{P}\{Z_1 \geqq N\} \leqq n\mathbf{E}Z_1^r N^{-r}.
$$

Letting $N = n/\log n$, we have

$$
(57) \qquad \mathbf{P}\bigl\{Z_n \geqq k;\ \overline{\mathcal{A}}_n(N)\bigr\} \leqq \mathbf{E}Z_1^r \frac{\log^r n}{n^{r-1}}\,.
$$

Comparing the right-hand sides of (56) and (57), we conclude that

$$
\mathbf{P}\bigl\{Z_n \geqq k;\ \overline{\mathcal{A}}_n(N)\bigr\} = o\bigl(\mathbf{P}\bigl\{Z_n \geqq k;\ \mathcal{A}_n(N)\bigr\}\bigr)
$$

for $k \leqq B(r/2-1)\, n \log n - B(r/2+\varepsilon)\, n \log\log n$ and $\varepsilon > 0$. Thus, the first statement of Theorem 1 is proved.

By Lemmas 6 and 7 we deduce that for all sufficiently large n and k,

$$
(58) \qquad \mathbf{P}\{Z_n \geqq k\} \geqq cn\mathbf{P}\{Z_1 \geqq k\}.
$$

Put

$$
(59) \qquad k = \left[B\!\left(\frac{r}{2}-1\right) n \log n + B\!\left(\frac{r+1}{2}+\varepsilon\right) n \log\log n\right].
$$

If (1) is valid for the k shown in (59), it follows that

$$
(60) \qquad \mathbf{P}\{Z_1 \geqq k\} \leqq cn^{-r} \log^{-r-1-\varepsilon} n.
$$

In addition, we conclude by (59) that

$$
n = \frac{2}{B(r-2)} \frac{k}{\log k}\bigl(1 + o(1)\bigr) \qquad \text{as} \quad k \to \infty.
$$

Substituting this estimate in (60), we see that, for all sufficiently large k,

$$
\mathbf{P}\{Z_1 \geqq k\} \leqq ck^{-r} \log^{-1-\varepsilon} k.
$$

Consequently, $\sum_{k=1}^{\infty} k^{r-1}\mathbf{P}\{Z_1 \geqq k\} < \infty$, which is equivalent to the boundedness of $\mathbf{E}Z_1^r$.

3.2. Proof of Theorem 2 for the case of finite variance. Assume first that the equivalence

(61) $$\mathbf{P}\{M_n \geqq k\} \sim n\mathbf{P}\{Z_1 \geqq k\}$$

holds for $k/(n\log n) \to \infty$, and show that $\mathbf{P}\{Z_n \geqq k\}$ has the same asymptotic behavior. Since $\mathbf{P}\{Z_n \geqq k\} < \mathbf{P}\{M_n \geqq k\}$, it suffices to justify the estimate from below,

(62) $$\liminf_{k\to\infty} \frac{\mathbf{P}\{Z_n \geqq k\}}{\mathbf{P}\{M_n \geqq k\}} \geqq 1.$$

Fix an $\varepsilon > 0$. Setting $\nu = (1+\varepsilon)^{-1}$ in (40), we have

(63) $$\mathbf{P}\{Z_n \geqq k\} \geqq \mathbf{P}\{M_n \geqq (1+\varepsilon)\,k\} \min_{i<n} \mathbf{P}\{Z_i \geqq k \mid Z_0 = [(1+\varepsilon)\,k]\}.$$

By the Chebyshev inequality, we obtain

$$\mathbf{P}\{Z_i < k \mid Z_0 = [(1+\varepsilon)\,k]\} \leqq \frac{(1+\varepsilon)\,kBi}{\varepsilon^2 k^2} = \frac{B(1+\varepsilon)\,i}{\varepsilon^2 k}.$$

Therefore,

(64) $$\min_{i<n} \mathbf{P}\{Z_i \geqq k \mid Z_0 = [(1+\varepsilon)\,k]\} \geqq 1 - \frac{B(1+\varepsilon)\,n}{\varepsilon^2 k}.$$

Combining (61), (63), and (64) and recalling that $\mathbf{P}\{Z_1 \geqq k\}$ is regularly varying, we conclude that

$$\liminf_{k\to\infty} \frac{\mathbf{P}\{Z_n \geqq k\}}{\mathbf{P}\{M_n \geqq k\}} \geqq (1+\varepsilon)^{-t}.$$

This estimate and the arbitrariness of ε imply (62).

Let us deduce (61). According to Theorem 3 in [8], for any $r \geqq 2$, $N \geqq 1$, and $y_0 > 0$, the following estimate is valid:

$$\mathbf{P}\{M_n \geqq k\} \leqq \left(y_0 + \frac{1}{N}\right)\left[\left(1 + \frac{1}{1/y_0 + e^r \overline{B}n/2 + n\overline{\beta}_r e^{y_0 N}/N^{r-2}}\right)^k - 1\right]^{-1}$$

(65) $$\qquad\qquad + n\mathbf{P}\{Z_1 \geqq N\},$$

where $\overline{B} = \mathbf{E}\{Z_1(Z_1 - 1); Z_1 \leqq N\}$, $\overline{\beta}_r = \mathbf{E}\{Z_1^{r-1}(Z_1 - 1); Z_1 \leqq N\}/2$. Take $r = t$. It is not difficult to see that

(66) $$\overline{\beta}_t \leqq \frac{t}{2}\int_0^N x^{t-1}\mathbf{P}\{Z_1 \geqq x\}\,dx =: L_1(N).$$

Since $\mathbf{P}\{Z_1 \geqq x\}$ is a regularly varying function of order $-t$, it follows by Theorem VIII.9.1 in [9] that $L_1(x)$ is a slowly varying function. Besides, the finiteness of B implies the boundedness of $L_1(N)$ for $t = 2$. Hence it follows that, for all sufficiently large N, the quantity

$$y_0 := \frac{1}{N}\log\left(\frac{N^{t-1}}{n}\right) - \frac{2}{N}\log\log\left(\frac{N^{t-1}}{n}\right) - \frac{1}{N}\log L_1(N)$$

is positive. It is not difficult to see that if $N/n \to \infty$, then

(67) $$\frac{n\overline{\beta}_t e^{y_0 N}}{N^{t-2}} = \frac{N}{\log^2(N^{t-1}/n)} = o(y_0^{-1}).$$

Further, if $N/(n \log n) \to \infty$, then

$$(68) \qquad \frac{e^t \overline{B} n}{2} \leqq \frac{e^t B n}{2} = o(y_0^{-1}).$$

Selecting $N = (1 - \varepsilon) k$ in (65) and using (67) and (68), we conclude that for all k meeting the inequality $k > cn \log n$, the following estimate is valid:

$$\mathbf{P}\{M_n \geqq k\} < \left(y_0 + (1 - \varepsilon)^{-1} k^{-1}\right) \left[\left(1 + \frac{y_0}{1 + \varepsilon/2}\right)^k - 1\right]^{-1} + n \mathbf{P}\{Z_1 \geqq (1 - \varepsilon) k\}.$$

Substituting the selected value of y_0 gives the inequality

$$\mathbf{P}\{M_n \geqq k\} < c(\varepsilon) \frac{\log k}{k} \left(\frac{n \log L_1(k) \log^2 k}{k^{t-1}}\right)^{1/((1-\varepsilon)(1+\varepsilon/2))}$$

$$+ n \mathbf{P}\{Z_1 \geqq (1 - \varepsilon) k\}.$$

Observing that the first summand in the right-hand side of this inequality is $o(nk^{-t-\delta})$ for some $\delta = \delta(\varepsilon) > 0$, and recalling that $\mathbf{P}\{Z_1 \geqq x\}$ is regularly varying, we conclude that

$$\limsup_{k \to \infty} \frac{\mathbf{P}\{M_n \geqq k\}}{n \mathbf{P}\{Z_1 \geqq k\}} \leqq \limsup_{k \to \infty} \frac{n \mathbf{P}\{Z_1 \geqq (1 - \varepsilon) k\}}{n \mathbf{P}\{Z_1 \geqq k\}} \leqq (1 - \varepsilon)^{-t}.$$

Since ε is arbitrary, it follows that, as $k/(n \log n) \to \infty$,

$$(69) \qquad \limsup_{k \to \infty} \frac{\mathbf{P}\{M_n \geqq k\}}{n \mathbf{P}\{Z_1 \geqq k\}} \leqq 1.$$

On the other hand, by Lemma 6,

$$(70) \qquad \liminf_{k \to \infty} \frac{\mathbf{P}\{M_n \geqq k\}}{n \mathbf{P}\{Z_1 \geqq k\}} \geq \liminf_{k \to \infty} \frac{\mathbf{P}(\overline{\mathcal{A}}_n(k))}{n \mathbf{P}\{Z_1 \geqq k\}} \geq 1 \qquad \text{as} \quad \frac{k}{n} \to \infty.$$

Combining (69) and (70), we obtain (61) and, in addition, the equivalence

$$\mathbf{P}\{M_n \geqq k\} \sim \mathbf{P}(\overline{\mathcal{A}}_n(k)) \qquad \text{as} \quad \frac{k}{n \log n} \to \infty.$$

This completes the proof of the theorem for $t > 2$.

3.3. Proof of Theorem 2 for the case of infinite variance. Consider first the case $t < 2$. Under this condition we have the asymptotic relations

$$(71)$$

$$1 - \tilde{f}'(1) = \mathbf{E}\{Z_1; Z_1 \geqq N\} = N \mathbf{P}\{Z_1 \geqq N\} + \int_N^\infty \mathbf{P}\{Z_1 \geqq x\} \, dx \sim \frac{t}{t-1} N^{1-t} L(N)$$

and

$$(72) \qquad \tilde{f}''(1) = \overline{B} \leqq 2 \int_0^N x \mathbf{P}\{Z_1 \geqq x\} \, dx \sim \frac{2}{2-t} N^{2-t} L(N)$$

as $N \to \infty$.

Combining (35), (37), (71), and (72), we see that $\tilde{f}'(x_0) > 1 - c_0 N^{1-t} L(N)$. Applying this inequality to the right-hand side of (34) and using (38), we obtain the estimate

$$\mathbf{P}\{M_n \geqq k\} > \mathbf{P}\{\overline{\mathcal{A}}_n(k-1)\} \geqq n\mathbf{P}\{Z_1 \geqq k\} \exp\bigl(-c_0 n k^{1-t} L(k)\bigr)$$

for the k such that $c_0 k^{1-t} L(k) \leqq 1$. Hence (70) follows for $nk^{1-t} L(k) \to 0$.

Now we deduce an upper estimate for $\mathbf{P}\{M_n \geqq k\}$. Letting $r = 2$ in (65) and using (72) gives

(73)
$$\mathbf{P}\{M_n \geqq k\} < \left(y_0 + \frac{1}{N}\right)\left[\left(1 + \frac{1}{1/y_0 + cne^{y_0 N} N^{2-t} L(N)}\right)^k - 1\right]^{-1} + n\mathbf{P}\{Z_1 \geqq N\},$$

where $c = c(t)$ is a positive constant. Putting

$$N = (1-\varepsilon)\, k, \quad y_0 = \frac{1}{N} \log\left(\frac{N^{t-1}}{nL(N)}\right) - \frac{2}{N} \log\log\left(\frac{N^{t-1}}{nL(N)}\right)$$

in this estimate and proceeding similarly to the case $t > 2$, we conclude that (69) is valid as $nk^{1-t} L(k) \to 0$. Thus,

(74) $\qquad \mathbf{P}\{M_n \geqq k\} \sim \mathbf{P}\bigl(\overline{\mathcal{A}}_n(k)\bigr) \sim n\mathbf{P}\{Z_1 \geqq k\} \qquad$ as $\quad nk^{1-t} L(k) \to 0.$

If $t = 2$, estimates (72) are replaced by

$$\tilde{f}''(1) = \overline{B} \leqq 2 \int_0^N x\mathbf{P}\{Z_1 \geqq x\}\, dx \sim \widetilde{L}(N) \qquad \text{as} \quad N \to \infty,$$

where $\widetilde{L}(N) := \int_1^N x^{-1} L(x)\, dx$. It is easy to check that the arguments we have used to derive (74) remain valid for $t = 2$ as well by substituting $\widetilde{L}(x)$ for $L(x)$. Therefore,

(75) $\qquad \mathbf{P}\{M_n \geqq k\} \sim \mathbf{P}\bigl(\overline{\mathcal{A}}_n(k)\bigr) \sim n\mathbf{P}\{Z_1 \geqq k\} \qquad$ as $\quad nk^{-1} \widetilde{L}(k) \to 0.$

It is shown in [15] that for any critical Galton–Watson process,

(76)
$$\int_0^{1-Q_n} \frac{ds}{f(s) - s} \sim n \qquad \text{as} \quad n \to \infty.$$

Since condition (2) implies (5) we let $s = 1 - y^{-1}$ in the integral in (76) and obtain

$$\int_0^{1-Q_n} \frac{ds}{f(s) - s} = \int_0^{Q_n^{-1}} \frac{y^{t-2}}{L^*(y)}\, dy = \frac{(1+o(1))}{t-1} \frac{Q_n^{1-t}}{L^*(Q_n^{-1})} \qquad \text{as} \quad n \to \infty.$$

Combining this representation with (76) and recalling (6) and (7) we obtain, as $n \to \infty$,

$$\frac{Q_n^{1-t}}{L(Q_n^{-1})} \sim \Gamma(2-t)\, n \qquad \text{for} \quad t < 2, \qquad \frac{Q_n^{-1}}{\widetilde{L}(Q_n^{-1})} \sim n \qquad \text{for} \quad t = 2.$$

Using these relations it is easy to check that if $kQ_n \to \infty$, then $nk^{1-t} L(k) \to 0$ for $t < 2$ and $nk^{-1}\widetilde{L}(k) \to 0$ for $t = 2$. Thus, we may combine (74) and (75) as follows:

(77) $\qquad \mathbf{P}\{M_n \geqq k\} \sim \mathbf{P}\bigl(\overline{\mathcal{A}}_n(k)\bigr) \sim n\mathbf{P}\{Z_1 \geqq k\} \qquad$ as $\quad kQ_n \to \infty.$

We now find the asymptotics of the probability $\mathbf{P}\{Z_n \geqq k\}$. According to the inequality established in Theorem 2 of [11],

$$\mathbf{E}\left|\sum_1^n X_i\right|^r \leqq 2\sum_1^n \mathbf{E}|X_i|^r, \qquad 1 \leqq r \leqq 2,$$

where $\{\sum_1^n X_i; \ n \geqq 1\}$ is a martingale. Applying this inequality to the process Z_n starting by k particles in the zero generation, we have for $i \leqq n$ the estimate

$$\mathbf{E}\{|Z_i - k|^r \mid Z_0 = k\} \leqq 2k\mathbf{E}|Z_i - 1|^r < 2k(1 + \mathbf{E}Z_i^r) \leqq 4k\mathbf{E}Z_i^r.$$

Therefore, by the Markov inequality,

(78) $$\min_{i<n}\mathbf{P}\{Z_i > (1-\delta)\,k \mid Z_0 = k\} \geq 1 - \frac{4k\max_{i<n}\mathbf{E}Z_i^r}{\delta^r k^r}.$$

Evaluating the expectation in the right-hand side of (78) by means of the estimate

(79) $$\mathbf{E}Z_i^r < cQ_i^{1-r} \qquad \text{for all} \quad r \in (1,t) \quad \text{and} \quad i \geqq 1,$$

established in [16], we obtain

$$\min_{i<n}\mathbf{P}\{Z_i > (1-\delta)\,k \mid Z_0 = k\} \geq 1 - c(kQ_n)^{1-r}.$$

Applying this estimate and (77) to the right-hand side of (63) shows that (62) is valid as $kQ_n \to \infty$. This completes the proof of the theorem.

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