ON SUMS OF INDEPENDENT RANDOM VARIABLES WITHOUT POWER MOMENTS

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Abstract: In 1952 Darling proved the limit theorem for the sums of independent identically distributed random variables without power moments under the functional normalization. This paper contains an alternative proof of Darling's theorem, using the Laplace transform. Moreover, the asymptotic behavior of probabilities of large deviations is studied in the pattern under consideration.

Keywords: slowly varying function, Laplace transform, binomial distribution, independent random variables, branching processes

§ 1. Introduction. Statement and Discussion of Results

Let $X, X_1, \dots$ be independent identically distributed random variables. Suppose that the function $V(x) = \mathbf{P}(X \geq x)$ is slowly varying as $x \to \infty$; i.e.,

$$\lim_{x\to\infty} \frac{V(cx)}{V(x)} = 1 \tag{1}$$

for every $c > 0$. It follows from this condition that $\mathbf{E}\{X^t; X > 0\} = \infty$ for every $t > 0$, i.e., all power moments are infinite.

Put $S_n = X_1 + \dots + X_n$, $\overline{X}_n = \max_{k\leq n} X_k$, and let X_n^* be the summand maximal in modulus; i.e., $|X_n^*| = \max_{k\leq n} |X_k|$.

It was Lévy ([1]; also see [2, p. 212]) who called attention to the fact that under condition (1) the absolute value of the difference $S_n - X_n^*$ is small as compared to X_n^*, i.e. X_n^* makes the overwhelming contribution to S_n. On assuming additionally that $X \geq 0$, it seems very likely that

$$\mathbf{P}(S_n < x) \sim \mathbf{P}(\overline{X}_n < x) = (1 - V(x))^n. \tag{2}$$

Supposing that $nV(x) = y$, where y is a fixed positive number, we arrive at the approximate equality

$$\mathbf{P}(S_n < x) \sim e^{-y}.$$

Letting $x = V^{-1}(y/n)$, where V^{-1} is the inverse function to V, we conclude that

$$\lim_{n\to\infty} \mathbf{P}(nV(S_n) > y) = e^{-y}.$$

Thus, the convergence to a nondegenerate distribution takes place under the functional normalization in terms of $V(x)$. This approach was realized in [3] without the restriction $X \geq 0$.

Theorem A (Darling). *If $X \geq 0$ or $\mathbf{P}(X < -x) = o(V(x))$, then*

$$\lim_{n\to\infty} \mathbf{P}(nV(S_n) < y; S_n \geq 0) = 1 - e^{-y}. \tag{3}$$

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If the left tail is comparable to the right, i.e.

$$\frac{V(x)}{V(x) + \mathbf{P}(X < -x)} \to p \in (0,1), \tag{4}$$

then

$$\lim_{n\to\infty} \mathbf{P}(nW(S_n) < y) = 1 - pe^{-y/p} - qe^{-y/q}, \tag{5}$$

where $q = 1 - p$ *and* $W(x) = V(x)I(x \geq 0) + \mathbf{P}(X < x)I(x < 0)$.

In contrast to Darling, we will give a direct proof of (3) and (5), bypassing relations of type (2). Notice that the latter follows from our results as a consequence. The range of the values of x such that $nV(x)$ remains bounded as $n \to \infty$ may be considered as the zone of normal deviations of S_n. Accordingly, the values of x hitting the supplementary domain may be considered as large deviations. The behavior of probabilities of large deviations will be the subject of special consideration (see Theorem 2 below).

We pass to the exact statement of results. We need the next notation:

$$g^+(s) = \mathbf{E}\{e^{-sX} \mid X \geq 0\}, \quad g^-(s) = \mathbf{E}\{e^{sX} \mid X < 0\},$$

$$L^+(x) = [1 - g^+(1/x)]\mathbf{P}(X \geq 0), \quad L^-(x) = [1 - g^-(1/x)]\mathbf{P}(X < 0).$$

Denote the inverse function to $L^{\pm}(x)$ by $R^{\pm}(x)$.

Theorem 1. *Assume that* $L^+(x)$ *is slowly varying and*

$$\lim_{x\to\infty} \frac{L^+(x)}{L^-(x)} = \frac{p}{q}, \quad q = 1 - p, \tag{6}$$

for some $p \in (0,1]$. *Then*

$$\lim_{n\to\infty} \mathbf{P}(S_n > R^+(x/n)) = \lim_{n\to\infty} \mathbf{P}(nL^+(S_n) < x; S_n > 0) = p(1 - e^{-x/p}) \tag{7}$$

for every $x > 0$. *Assume further that* $L^-(x)$ *is slowly varying and* (6) *holds for some* $q \in (0,1]$. *Then* (7) *remains valid with* S_n, $L^+(x)$, *and* p *replaced by* $-S_n$, $L^-(x)$, *and* q *respectively.*

We call the reader's attention to the fact that Theorem 1 is formulated not in terms of the distribution of the random variable X but in terms of the Laplace transforms of the positive and negative parts of this variable. The point is that according to the Tauberian theorem (e.g., see [4, Chapter XIII, formula (5.17)]), $V(x)$ is slowly varying iff so is $L^+(x)$ and

$$\lim_{x\to\infty} \frac{V(x)}{L^+(x)} = 1. \tag{8}$$

Hence, it is immediate that (4) and (6) are equivalent. We use the normalization $L^+(x)$ since $L^+(x)$ is a continuous strictly decreasing function. This allows us to avoid some difficulties that are associated with the inversion of $V(x)$.

It is easy to see that if $p = 1$ in (6) then (7) is equivalent to (3) since $L^+(x) \sim V(x)$. Moreover, (7) and the corresponding analog for the left tail of S_n imply (5).

Show now that Theorem 1 yields (2).

Indeed, if $X \geq 0$ then for $y = R^+(x/n)$

$$\mathbf{P}(\overline{X}_n < y) = (1 - V(y))^n = (1 - V(R^+(x/n)))^n.$$

By (8) we conclude that, for all positive x, the approximate equalities hold:

$$\mathbf{P}(\overline{X}_n < y) \sim (1 - L^+(R^+(x/n)))^n \sim (1 - x/n)^n \sim e^{-x}.$$

Comparing this with (7), we get the relation

$$\mathbf{P}(S_n > R^+(x/n)) \sim \mathbf{P}(\overline{X}_n > R^+(x/n)).$$

Theorem 2. *Suppose that $L^+(x)$ is slowly varying,* (6) *holds for $p > 0$, and $n, y \to \infty$ so that $nL^+(y) \to 0$. Then*

$$\lim_{n,y\to\infty} \frac{\mathbf{P}(S_n > y)}{nL^+(y)} = 1. \tag{9}$$

Under the conditions of the second part of Theorem 1 for the left tail of the distribution of S_n we have

$$\lim_{n,y\to\infty} \frac{\mathbf{P}(S_n < -y)}{nL^-(y)} = 1. \tag{10}$$

If the function $L(x) = L^+(x) + L^-(x)$ is slowly varying, then

$$\lim_{n,y\to\infty} \frac{\mathbf{P}(|S_n| > y)}{nL(y)} = 1 \quad \text{as } nL(y) \to 0. \tag{11}$$

If the functions $L^+(x)$ and $L^-(x)$ are simultaneously slowly varying, with p and q positive, then (11) is an obvious consequence of (9) and (10). From a formal standpoint, slow variation of $L(x)$ is a weaker condition as compared with slow variation of summands. However, under some additional restrictions these conditions become equivalent. Suppose for instance that

$$0 < c_1 \le \frac{L^+(x)}{L^-(x)} \le c_2 < \infty. \tag{12}$$

Since $L^+(x)$ is monotonic, the negation of slow variation of the function consists in assuming the existence of a constant $c > 1$ and a sequence x_n such that for some $\alpha < 1$

$$L^+(cx_n) < \alpha L^+(x_n). \tag{13}$$

Since $L(x)$ is slowly varying and $L^-(x)$ does not decrease, for all $\varepsilon > 0$ and all sufficiently large n, we have

$$(1-\varepsilon)L(x_n) < L(cx_n) < L^+(cx_n) + L^-(x_n).$$

Using (13), we obtain

$$(1-\varepsilon)L^+(x_n) + (1-\varepsilon)L^-(x_n) \le \alpha L^+(x_n) + L^-(x_n),$$

and, consequently,

$$(1-\alpha-\varepsilon)L^+(x_n) \le \varepsilon L^-(x_n).$$

Since $\varepsilon > 0$ is arbitrary,

$$\lim_{n\to\infty} \frac{L^+(x_n)}{L^-(x_n)} = 0$$

contradicting (12). Therefore, $L^+(x)$ is slowly varying. In the same way we can prove that $L^-(x)$ is slowly varying as well.

Show now that we can combine the assertions on normal and large deviations. If n and y tend to infinity simultaneously and $nL^+(y) \to x \in (0, \infty)$, then $y = R^+((x + \varepsilon_n(y))/n)$ where $\varepsilon_n(y) \to 0$ as $n, y \to \infty$. In view of continuity of the right-hand side in (7) we have

$$\mathbf{P}(S_n > y) = p(1 - e^{-(x+\varepsilon_n(y))/p})(1 + o(1)).$$

Noticing that $x + \varepsilon_n(y) = nL^+(y)$ and applying (8), we arrive, as $nV(y) \to x$, at the equality

$$\mathbf{P}(S_n > y) = p(1 - e^{-nV(y)/p})(1 + o(1)). \tag{14}$$

If $nL^+(y) \to 0$ then $nL^+(y) \sim p(1 - e^{-nL^+(y)/p})$. Applying (8) once again, we conclude that (9) implies (14). Thus, for all y such that $nL^+(y)$ has finite limit, the tail of the distribution of S_n is approximated by the same function given on the right-hand side of (14).

Darling demonstrated (see the proof of Theorem 4.2 in [3]) that

$$\mathbf{P}(X_n^* > y) = p(1 - e^{-nV(y)/p})(1 + o(1)). \tag{15}$$

Comparing (14) and (15), we arrive at the approximate equality

$$\mathbf{P}(S_n > y) \sim \mathbf{P}(X_n^* > y).$$

Deducing this relation was the main point of Darling's arguments, whereas in our case it is the consequence of Theorem 1.

The proofs of Theorems 1 and 2 are given below in Sections 3 and 4. The proof of Theorem 1 is based on the Laplace transform. Section 2 contains the result about convergence of distributions and the corresponding Laplace transforms which is required in proving Theorem 1. Proving Theorem 2, we apply the upper bounds for $\mathbf{P}(S_n \ge x)$ and $\mathbf{P}(|S_n| \ge x)$ from [5] and the lower bounds for these probabilities from [6, 7]. Since $\mathbf{P}(X \ge x)$ and $\mathbf{P}(|X| \ge x)$ are slowly varying, these bounds approach one another, providing an asymptotically sharp result.

§ 2. The Criterion for Weak Convergence of Functional Transforms

Let ξ_n be an arbitrary sequence of nonnegative random variables, and let $L(x)$ be a continuous monotonic slowly varying function. Further, let $R(x)$ be the inverse function to $L(x)$, while a_n is an arbitrary sequence and $\varphi(x)$ is a nonnegative function.

Theorem 3. *If $L(x)$ increases and $a_n \to \infty$ then*

$$\lim_{n\to\infty} \mathbf{E}\exp\left\{-\frac{\xi_n}{R(a_n x)}\right\} = \varphi(x) \tag{16}$$

at every continuity point of φ iff

$$\lim_{n\to\infty} \mathbf{P}\big(a_n^{-1}L(\xi_n) < x\big) = \varphi(x) \tag{17}$$

at every continuity point of φ.

If $L(x)$ decreases and $a_n \to 0$ then (16) *holds iff*

$$\lim_{n\to\infty} \mathbf{P}(a_n^{-1}L(\xi_n) < x) = 1 - \varphi(x) \tag{18}$$

at every continuity point of φ.

The direct assertion of Theorem 3 in the case of $L(x)$ increasing was formulated (factually, in a somewhat different form) and proved by Hudson and Seneta [8].

We give two different proofs of Theorem 3.

The first proof. Start with the case of $L(x)$ increasing. Suppose that (6) is valid.

The changing of variables $y = R(a_n z)$ implies

$$\mathbf{E}\exp\left\{-\frac{\xi_n}{R(a_n x)}\right\} = \int_0^\infty \exp\left\{-\frac{y}{R(a_n x)}\right\} d\mathbf{P}(\xi_n < y)$$

$$= \int_0^\infty \exp\left\{-\frac{R(a_n z)}{R(a_n x)}\right\} d\mathbf{P}(\xi_n < R(a_n z)).$$

Since $R(x)$ is increasing for all $\varepsilon > 0$, we have

$$\mathbf{E}\exp\left\{-\frac{\xi_n}{R(a_n x)}\right\} \le \mathbf{P}(\xi_n < R(a_n(x+\varepsilon))) + \exp\left\{-\frac{R(a_n(x+\varepsilon))}{R(a_n x)}\right\}, \tag{19}$$

$$\mathbf{E}\exp\left\{-\frac{\xi_n}{R(a_n x)}\right\} \ge \exp\left\{-\frac{R(a_n(x-\varepsilon))}{R(a_n x)}\right\} \mathbf{P}(\xi_n < R(a_n(x-\varepsilon))). \tag{20}$$

Lemma 1. *If $L(x)$ increases then for all $c > 1$*

$$\lim_{x\to\infty}\frac{R(x)}{R(cx)} = 0. \tag{21}$$

If $L(x)$ decreases then for all $c < 1$

$$\lim_{x\to 0}\frac{R(x)}{R(cx)} = 0. \tag{22}$$

The proof of (21) is contained in Theorem 1.11 of [9], while to deduce (22) we need to make some minor changes in the proof of (21).

It follows from (21), (19), (20), and (16) that for all $\varepsilon > 0$ and sufficiently large n we have

$$\mathbf{P}(\xi_n < R(a_n(x-\varepsilon))) - \varepsilon < \varphi(x) < \mathbf{P}(\xi_n < R(a_n(x+\varepsilon))) + \varepsilon$$

or, which is the same,

$$\mathbf{P}\big(a_n^{-1}L(\xi_n) < x-\varepsilon\big) - \varepsilon < \varphi(x) < \mathbf{P}\big(a_n^{-1}L(\xi_n) < x+\varepsilon\big) + \varepsilon.$$

This means obviously that $\mathbf{P}\big(a_n^{-1}L(\xi_n) < x\big)$ tends to $\varphi(x)$ pointwise on the continuity set of $\varphi(x)$.

For proving the converse it suffices to reverse all arguments.

Suppose now that $L(x)$ decreases. The only distinction from the already considered case is that under the transformation $x \mapsto L(x)$ the inequality under the $\mathbf{P}$-symbol in (24) and (25) is reversed.

We may consider the expectation on the left-hand side of (16) as a generalized Laplace transform.

Call the reader's attention to the fact that the same function stands on the right-hand sides of (16) and (17). It means that (17) at once gives an explicit form of the limit distribution of the sequence $a_n^{-1}L(\xi_n)$, whereas we need another intermediate step to use the ordinary Laplace transform: the inversion of the limit transform which often is a difficult task.

The above proof bases on the same idea as that of Lemma 1 in [8] but is much simpler.

The method of the second proof is more classical as compared with the first: we start with finding the limit of the Laplace transform of the variables $\xi_n(x) := \xi_n/R(a_nx)$; then apply the continuity theorem. Some limit theorems for branching processes with migration were proved in this manner in [10, 11].

The second proof. Fix some $\varepsilon > 0$. It follows from (21) that for all $s > 1$ and all sufficiently large n we have

$$\frac{1}{R(a_nx)} < \frac{s}{R(a_nx)} < \frac{1}{R(a_n(x-\varepsilon))}.$$

Hence, applying (16), for $s > 1$ we obtain

$$\varphi(x-\varepsilon) \le \liminf_{n\to\infty}\mathbf{E}\exp\left\{-\frac{s\xi_n}{R(a_nx)}\right\} \le \limsup_{n\to\infty}\mathbf{E}\exp\left\{-\frac{s\xi_n}{R(a_nx)}\right\} \le \varphi(x).$$

Since ε is arbitrary,

$$\lim_{n\to\infty}\mathbf{E}\exp\left\{-\frac{s\xi_n}{R(a_nx)}\right\} = \varphi(x) \tag{23}$$

at every continuity point of $\varphi(x)$ for $s > 1$. The validity of (23) for $s < 1$ follows by analogy.

Thus, we showed that Laplace transforms of the variables $\xi_n(x) = \xi_n/R(a_nx)$ converge to the function which is identically equal to $\varphi(x)$. The latter is the Laplace transform of the random variable $\xi(x)$ taking the values zero and infinity with probability $\varphi(x)$ and $1-\varphi(x)$. According to the continuity theorem for the Laplace transform (e.g., see [4, Chapter XIII, § 1, Theorem 2])

$$\lim_{n\to\infty}\mathbf{P}(\xi_n(x) < u) = \varphi(x)$$

for all $u > 0$. Letting $u = 1$ in this inequality, we have

$$\lim_{n\to\infty} \mathbf{P}(\xi_n < R(a_n x)) = \varphi(x). \tag{24}$$

Since $L(x)$ decreases, we arrive at (17).

Prove now that (17) implies (16).

Because of the slow variation of $L(x)$ for every fixed u

$$\begin{aligned}\mathbf{P}(\xi_n < uR(a_n x)) &= \mathbf{P}(L(\xi_n) < L(uR(a_n x))) \\ &= \mathbf{P}(L(\xi_n) < a_n x + \varepsilon_n(u)), \quad \varepsilon_n(u) \to 0.\end{aligned} \tag{25}$$

Hence, in view of (17) and the continuity of φ at x we conclude that

$$\lim_{n\to\infty} \mathbf{P}(\xi_n < uR(a_n x)) = \varphi(x)$$

for all $u > 0$. In other words, the sequence $\xi_n(x)$ weakly converges to $\xi(x)$. It means, in particular, that

$$\lim_{n\to\infty} \mathbf{E}e^{-s\xi_n(x)} = \mathbf{E}e^{-s\xi(x)} = \varphi(x)$$

for all $s > 0$. Letting $s = 1$, we come to (16).

Notice that the above-mentioned papers [8, 10, 11] are devoted to the study of distinct modifications of branching processes. A more complicated structure of branching processes as compared with the sums of independent random variables makes the deduction of (2) more difficult. It gave rise to the methods of proving the weak convergence which are in the current section.

§ 3. Proof of Theorem 1

Suppose first that the random variable X under study takes only nonnegative values. In this case $g^+(s)$ is the Laplace transform of X. It is easy to see that $L^+(x)$ is continuous and vanishes as x tends to infinity. Note now that

$$\begin{aligned}\mathbf{E}\exp\left\{-\frac{S_n}{R^+(x/n)}\right\} &= \left(\mathbf{E}\exp\left\{-\frac{X}{R^+(x/n)}\right\}\right)^n \\ &= \left(g^+\left(\frac{1}{R^+(x/n)}\right)\right)^n = (1 - L^+(R^+(x/n)))^n.\end{aligned}$$

Since $R^+(x)$ is inverse to $L^+(x)$, it follows from the last equality that

$$\mathbf{E}\exp\left\{-\frac{S_n}{R^+(x/n)}\right\} = \left(1 - \frac{x}{n}\right)^n \to e^{-x} \quad \text{as } n \to \infty.$$

Applying Theorem 2 to the case of $L(x)$ decreasing, we see

$$\lim_{n\to\infty} \mathbf{P}(nL^+(S_n) < x) = 1 - e^{-x}. \tag{26}$$

Let us abandon the condition that the random variables X_i are nonnegative.

Clearly, the distribution of X coincides with the distribution of the variable $\alpha X^+ - (1-\alpha)X^-$, where the random variables α, X^-, X^+ are independent, α has the Bernoulli distribution with parameter $r = \mathbf{P}(X \ge 0)$, $\mathbf{P}(X^+ > x) = \mathbf{P}(X > x \mid X \ge 0)$, and $\mathbf{P}(X^- > x) = \mathbf{P}(X < -x \mid X < 0)$. Hence,

$$S_n = S_\nu^+ - S_{n-\nu}^-, \tag{27}$$

where $S_k^+ = \sum_1^k X_j^+$, $S_k^- = \sum_1^k X_j^-$, and $\nu = \nu(n)$ is the random variable having the binomial distribution with parameters n and r.

Lemma 2. *The next relations are valid:*

$$\lim_{n\to\infty} \mathbf{P}\big(S_\nu^+ > R^+(x/n)\big) = 1 - e^{-x}, \tag{28}$$

$$\lim_{n\to\infty} \mathbf{P}\big(S_\nu^- < R^+(x/n)\big) = e^{-qx/p}. \tag{29}$$

PROOF. By definition $\mathbf{E}e^{-sX^\pm} = g^\pm(s)$. Put $L_*^\pm(x) = 1 - g^\pm(1/x)$. Since (7) is already proved for nonnegative variables,

$$\lim_{k\to\infty} \mathbf{P}\big(kL_*^+\big(S_k^+\big) < x\big) = 1 - e^{-x}.$$

By (6) we have $L_*^+(x) \sim L^+(x)/r$ as $x \to \infty$. Consequently,

$$\lim_{k\to\infty} \mathbf{P}\big(kL^+\big(S_k^+\big) < x\big) = 1 - e^{-x/r}.$$

It means in particular that for all $\varepsilon > 0$

$$\lim_{k\to\infty} \mathbf{P}\big(S_k^+ > R^+((1\pm\varepsilon)xr/k)\big) = 1 - e^{-x(1\pm\varepsilon)}.$$

Assume that $1-\varepsilon < \frac{k}{nr} < 1+\varepsilon$. Since $R^+(x)$ decreases,

$$\mathbf{P}\big(S_k^+ > R^+((1-\varepsilon)xr/k)\big) \le \mathbf{P}\big(S_k^+ > R^+(x/n)\big) \le \mathbf{P}\big(S_k^+ > R^+((1+\varepsilon)xr/k)\big)$$

for all $\varepsilon > 0$. It follows from the last two relations that for n sufficiently large

$$1 - e^{-x(1-\varepsilon)} - \varepsilon \le \mathbf{P}\big(S_\nu^+ > R^+(x/n) \mid |\nu/nr - 1| \le \varepsilon\big) \le 1 - e^{-x(1+\varepsilon)} + \varepsilon. \tag{30}$$

On the other hand, by the law of large numbers

$$\lim_{n\to\infty} \big[\mathbf{P}\big(S_\nu^+ > R^+(x/n)\big) - \mathbf{P}\big(S_\nu^+ > R^+(x/n) \mid |\nu/nr - 1| \le \varepsilon\big)\big] = 0. \tag{31}$$

Combining (30) and (31), we obtain (28).

Turn now to (29). In view of (28)

$$\lim_{n\to\infty} \mathbf{P}(S_\nu < R^-(x/n)) = e^{-x}.$$

In other words,

$$\lim_{n\to\infty} \mathbf{P}(nL^-(S_\nu^-(x/n)) > x) = e^{-x}.$$

Notice that by (6)

$$\lim_{n\to\infty} \frac{L^-(S_\nu^-(x/n))}{L^+(S_\nu^-(x/n))} = \frac{q}{p}$$

with probability 1. It follows from the last two relations that

$$\lim_{n\to\infty} \mathbf{P}(nL^+(S_\nu^-(x/n)) > x) = e^{-qx/p}. \quad \square$$

Lemma 3. *For all $x, y > 0$*

$$\lim\nolimits_{n\to\infty} \mathbf{P}\big(S_\nu^+ > R^+(x/n) + R^+(y/n)\big) = 1 - e^{-\min\{x,y\}}.$$

PROOF. Let $x < y$. Then by (22) for n sufficiently large

$$R(x/n) + R(y/n) < 2R(x/n) < R((1+\varepsilon)x/n)$$

for arbitrarily small $\varepsilon > 0$. Hence, by (28)

$$\liminf_{n\to\infty} \mathbf{P}(S_\nu^+ > R^+(x/n) + R^+(y/n)) \geq 1 - e^{-x}.$$

On the other hand, by (28)

$$\limsup_{n\to\infty} \mathbf{P}(S_\nu^+ > R^+(x/n) + R^+(y/n)) \leq \lim_{n\to\infty} \mathbf{P}(S_\nu^+ > R^+(x/n)) = 1 - e^{-x}.$$

Thus, for $x < y$

$$\lim_{n\to\infty} \mathbf{P}(S_\nu^+ > R^+(x/n) + R^+(y/n)) = 1 - e^{-x}.$$

Quite similarly, for $x > y$

$$\lim_{n\to\infty} \mathbf{P}(S_\nu^+ > R^+(x/n) + R^+(y/n)) = 1 - e^{-y}.$$

The proof of the lemma is complete.

Let us turn now to the final stage of the proof. It is easy to see that

$$P_\nu(x) := \mathbf{P}\big(S_\nu^+ - S_{n-\nu}^- > x\big) = \mathbf{E}\int_0^\infty F_\nu^-(y)dF_{n-\nu}^+(x+y),$$

where $F_k^\pm(x) = \mathbf{P}(S_k^\pm < x)$. Making the change of variables $x = R^+(u/n)$, $y = R^+(v/n)$, we have

$$P_\nu(R^+(u/n)) = -\mathbf{E}\int_0^\infty F_\nu^-(R^-(v/n))dF_{n-\nu}^-(R^+(u/n) + R^+(v/n)).$$

Put $k_1 = nr - n^{2/3}$, $k_2 = nr + n^{2/3}$. Notice that

$$\lim_{n\to\infty} \mathbf{P}(k_1 \leq \nu \leq k_2) = 1. \tag{32}$$

Consequently, as $n \to \infty$,

$$P_\nu(R^+(u/n)) = -\mathbf{E}\bigg(\int_0^\infty F_{n-\nu}^-(R^+(u/n))dF_\nu^+(R^+(u/n) + R^+(v/n)); k_1 \leq \nu \leq k_2\bigg) + o(1). \tag{33}$$

It is easy that

$$\lim_{n\to\infty} F_{n-k_1}^-(R^+(v/n)) = \lim_{n\to\infty} F_{n-k_2}^-(R^+(v/n)) = e^{-qv/p}, \tag{34}$$

where the convergence is uniform in v. Further, $F_{k+1}^-(x) \leq F_k^-(x)$. Therefore, for $k_1 \leq \nu \leq k_2$

$$F_{n-k_1}^-(R^+(v/n)) \leq F_{n-\nu}^-(R^+(v/n)) \leq F_{n-k_2}^-(R^+(v/n)). \tag{35}$$

It follows from (32)–(35) that

$$P_\nu(R^+(u/n)) = -\int_0^\infty e^{-qv/p} d\mathbf{E}F_\nu^+(R^+(u/n) + R^+(v/n)) + o(1).$$

Applying now Lemma 3, we see that

$$\lim_{n\to\infty} P_\nu(R^+(u/n)) = -\int_0^\infty e^{-qv/p} de^{-\min(u,v)} = \int_0^u e^{-v/p}\, dv = p(1 - e^{-u/p}).$$

Hence, returning to (27), we obtain (7).

§ 4. Proof of Theorem 2

According to Theorem 1 from [5] for all $t \in (0,1]$ and $y > 0$ we have

$$\mathbf{P}(S_n > x) \le e^{x/y} \left(1 + \frac{xy^{t-1}}{n\mathbf{E}\{X^t; 0 < X \le y\}}\right)^{-x/y} + n\mathbf{P}(X > y).$$

Letting $t = 1$, $y = x$ in this inequality, we arrive at the bound

$$\mathbf{P}(S_n > x) \le n\mathbf{P}(X > x) + e\frac{n\mathbf{E}\{X; 0 < X \le x\}}{x}. \tag{36}$$

Obviously,

$$\mathbf{E}\{X; 0 < X \le x\} = -x\mathbf{P}(X \ge x) + \int_0^x \mathbf{P}(X \ge z)\, dz.$$

Since $\mathbf{P}(X \ge z)$ is slowly varying, by Theorem 2.1 from [9]

$$\int_0^x \mathbf{P}(X \ge z) dz = x\mathbf{P}(X \ge x)(1 + o(1))$$

and, consequently, $\mathbf{E}\{X; 0 < X \le x\} = o(x\mathbf{P}(X \ge x))$. Hence, we conclude that

$$\frac{n\mathbf{E}\{X; 0 < X \le x\}}{x} = o(n\mathbf{P}(X \ge x)). \tag{37}$$

Using (37) to estimate the right-hand side in (36), we get for $nV(x) \to 0$ the inequality

$$\limsup_{n\to\infty} \frac{\mathbf{P}(S_n > x)}{n\mathbf{P}(X > x)} \le 1. \tag{38}$$

To obtain a corresponding lower bound we apply the inequality from [6] (Theorem 6.1) which can be written as

$$\mathbf{P}(S_n > x) \ge n\mathbf{P}(X \ge \alpha x)[1 - \mathbf{P}(S_{n-1} < -(\alpha - 1)x) - ((n-1)/2)\mathbf{P}(X \ge \alpha x)],$$

where α is an arbitrary positive number.

Letting $\alpha = 2$ in this inequality, we obtain

$$\mathbf{P}(S_n > x) \ge \mathbf{P}(X_1 > 2x)[1 - \mathbf{P}(S_{n-1} < -x) - ((n-1)/2)\mathbf{P}(X \ge 2x)]. \tag{39}$$

Notice that by (6) and $nL^+(x) \to 0$ we have $\lim_{n\to\infty} n\mathbf{P}(X < -x) = 0$. Applying now (38) to $-S_n$, we see that

$$\lim_{n\to\infty} \mathbf{P}(S_{n-1} < -x) = 0 \quad \text{as } nL^+(x) \to 0. \tag{40}$$

Combining (39) and (40), we get

$$\liminf_{n\to\infty} \frac{\mathbf{P}(S_n > x)}{n\mathbf{P}(X > 2x)} \ge 1 \quad \text{as } nL^+(x) \to 0.$$

Since $\mathbf{P}(X > x)$ is slowly varying, we have

$$\liminf_{n\to\infty} \frac{\mathbf{P}(S_n > x)}{n\mathbf{P}(X > x)} \ge 1. \tag{41}$$

From (38) and (41) we have (9). Applying (9) to $-S_n$ yields (10).

It remains to prove (11). To estimate $\mathbf{P}(|S_n| > x)$ we apply Theorem 5 of [5]. According to this theorem

$$\mathbf{P}(|S_n| > x) \le e^{x/y}\left(1 + \frac{xy^{t-1}}{n\mathbf{E}\{|X|^t; 0 < |X| \le y\}}\right)^{-x/y} + n\mathbf{P}(|X| > y)$$

for $0 < t \le 1$, $y > 0$. Taking into account that $\mathbf{P}(|X| > x) \sim L(x)$ as $x \to \infty$ and word for word repeating the deduction of (38), we infer

$$\limsup_{n\to\infty} \frac{\mathbf{P}(|S_n| > y)}{nL(y)} \le 1. \tag{42}$$

To get a lower bound for $\mathbf{P}(|S_n| > x)$, we use (1) from [7], which, as applied to our case, becomes as follows:

$$\mathbf{P}(|S_n| > x) \ge n\mathbf{P}(|X| \ge \alpha x)[\min\{\mathbf{P}(S_{n-1} \ge -(\alpha - 1)x), \mathbf{P}(S_{n-1} \le (\alpha - 1)x)\} - n\mathbf{P}(|X| \ge \alpha x)],$$

with α an arbitrary positive number. Putting here $\alpha = 2$ and noticing that

$$\min\{\mathbf{P}(S_{n-1} \ge -x),\ \mathbf{P}(S_{n-1} \le x)\} \ge \mathbf{P}(|S_{n-1}| \le x),$$

we arrive at the inequality

$$\mathbf{P}(|S_n| > x) \ge n\mathbf{P}(|X| \ge 2x)[1 - \mathbf{P}(|S_{n-1}| > x) - n\mathbf{P}(|X| \ge 2x)].$$

Estimating $\mathbf{P}(|S_{n-1}| > x)$ with the aid of (42), we conclude that

$$\liminf_{n\to\infty} \frac{\mathbf{P}(|S_n| > y)}{nL(2y)} \ge 1 \quad \text{as } nL(y) \to 0. \tag{43}$$

Since $L(x)$ is slowly varying, (42) and (43) imply (11).

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