THEORY PROBAB. APPL.
Vol. 50, No. 3, pp. 400–419

ON THE LOCAL LIMIT THEOREM FOR A CRITICAL GALTON–WATSON PROCESS*

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Abstract. The proof of the local limit theorem for a critical Galton–Watson process is given under minimal moment restrictions, i.e., under the condition that there exists the second moment of the number of direct offspring of one particle.

Key words. Galton–Watson process, Bellman–Harris process, concentration function, local theorem, bilinear generating function

DOI. 10.1137/S0040585X97981822

1. Introduction. Let Z_n be a Galton–Watson process. In what follows it is assumed, unless otherwise noted, that $Z_0 = 1$. Put $p_k = \mathbf{P}\{Z_1 = k\}$, $f(s) = \sum_{k=0}^{\infty} p_k s^k$. We shall consider only the critical case, i.e., $f'(1) = 1$. Suppose that Z_1 has the finite second moment and denote $B = f''(1) = \mathbf{D}Z_1$. Let d be the greatest common divisor of $\{k\colon\ p_k > 0\}$. Denote the kth iteration of the function $f(s)$ by $f_k(s)$. Obviously, $f_k(s)$ is the generating function of Z_k.

The main goal of the present work is to prove the local limit theorem for Z_n under minimal restrictions to moments of Z_1.

The first paper in which local limit theorems are proved for branching processes belongs apparently to Zolotarev [1], in which the asymptotic behavior of $\mathbf{P}\{Z_t = k\}$ with k given is studied for a Markov branching process with continuous time. For Galton–Watson processes this problem was investigated in [2], and for critical Bellman–Harris processes in Vatutin's paper [3].

Under the condition that there exists the fourth moment of the number of direct offspring, Chistyakov [4] gave an asymptotic formula for $\mathbf{P}\{Z_t = k\}$, while $t, k \to \infty$, where Z_t is the Markov branching process with continuous time. It is also mentioned in [4] that N. V. Smirnov obtained an analogous result for discrete time. However, neither the statement nor the proof of this result has been published since then.

In the joint paper of Kesten, Ney, and Spitzer [2] the following result is stated: If k and n tend to infinity in such a way that their ratio remains bounded, then

$$\lim_{n\to\infty} n^2 \exp\left(\frac{2kd}{Bn}\right) \mathbf{P}\{Z_n = kd\} = \frac{4d}{B^2}. \tag{1.1}$$

The authors of [2] note that this formula is valid without any excessive moment restrictions, i.e., the condition $B < \infty$ is sufficient. However, they accomplished their proof (1.1) only under the stronger condition

$$\mathbf{E}Z_1^2 \log(1 + Z_1) < \infty. \tag{1.2}$$

*Received by the editors April 25, 2003; revised January 30, 2004. This work was supported by Russian Foundation for Basic Research grants 02-01-01252, 02-01-00358 and by INTAS grants 99-01317, 00-265.

http://www.siam.org/journals/tvp/50-3/98182.html

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They also remark that this assumption is made only for simplicity of the presentation. However, in the monograph of Atreya and Ney [5] they write that up to the moment of the appearance of their book the proof of the local limit theorem without condition (1.2) had not been published anywhere. Almost at the same time as [2] the paper of Nagaev and Mukhamedkhanova [6] appeared in which the next equality is proved under condition $\mathbf{E}Z_1^4 < \infty$,

$$\frac{B^2n^2}{4}\,\mathbf{P}\{Z_n = k\} = \exp\biggl(-\frac{2k}{Bn}\biggr) + \alpha_{kn} + O(k^{-1}\log n), \tag{1.3}$$

where $\alpha_{kn} \to 0$ as $n \to \infty$ uniformly with respect to k. Equality (1.1) follows from this formula only if $k^{-1}\log n$ tends to zero. On the other hand, it follows from (1.3) that (1.1) remains valid if k/n tends to infinity slowly enough.

For the critical Bellman–Harris process Topchii [7] proved the analogue of (1.1). It was assumed in this paper that condition (1.2) holds for the embedded Galton–Watson process.

We formulate now the results which are proved in the present paper.

THEOREM. *Let $B < \infty$ and let k and n tend to infinity in such a way that the ratio k/n remains bounded*; *then*

$$\lim_{n\to\infty} \frac{B^2n^2}{4d}\left(1 + \frac{2d}{Bn}\right)^{k+1} \mathbf{P}\{Z_n = kd\} = 1. \tag{1.4}$$

Obviously, by replacing the factor $(1 + 2d/(Bn))^k$ in the left-hand side of this equality with the equivalent expression $\exp(2kd/(Bn))$ we obtain exactly (1.1). The reason for this formulation is that we approximate the distribution of the process Z_n by the geometric distribution with parameter $2/(Bn)$ instead of the exponential one. This approach looks more natural since the distribution of Z_n is concentrated on the set of nonnegative integers. In addition, approximating by the geometric distribution is, generally speaking, more precise. For example, for the bilinear generating function,

$$\mathbf{P}\{Z_n = k\} = \frac{4}{B^2n^2}\left(1 + \frac{2}{Bn}\right)^{-k-1}$$

for any $k \geqq 1$. Thereby (1.4) holds for every k, and (1.1) is valid only for $k = o(n^2)$.

The proof of the theorem is based on the next statement which is of independent interest as well.

PROPOSITION. *If $B < \infty$, then there exists the constant $C = C(f)$ such that*

$$\sup_{n,k\geqq 1} n^2\mathbf{P}\{Z_n = k\} \leqq C. \tag{1.5}$$

Our approach to proving the local theorem differs essentially from that of [2], though we apply some of their results.

It is shown in [2] that the next formula is valid in the case $d > 1$,

$$\mathbf{P}\{Z_n = kd\} = \frac{1}{d}\,\mathbf{P}\{Z_n^* = k\} + O(n^{-3}), \tag{1.6}$$

where Z_n^* is the auxiliary Galton–Watson process with generating function $[f(s^{1/d})]^d$. It follows from equality (1.6) that it is sufficient to prove the theorem and the proposition for $d = 1$. Therefore we suppose $d = 1$ in what follows.

We denote by $c, c_1, c_2, \dots$ constants which depend only on the distribution $\{p_k\}$. Since the remainder term in (1.4) is not estimated in our paper, it does not matter how exactly constants depend on $\{p_k\}$. In this connection, we provide constants with lower index only in the case when a misunderstanding is possible.

For every analytical at zero function $\rho(s)$ we will denote by $a_l[\rho(s)]$ the coefficient at s^l in the Taylor expansion of this function in the neighborhood of zero. Put $\|\rho(s)\|_1 = \sum_{l=0}^{\infty} |a_l[\rho(s)]|$.

2. Auxiliary statements.

LEMMA 1. *There exists a constant c such that for every n, $k \geqq 1$*

$$\mathbf{P}\{1 \leqq Z_n \leqq k\} \leqq c\,\frac{k}{n^2}. \tag{2.1}$$

Proof. For any $s \in (0,1)$ we have

$$s^k \mathbf{P}\{1 \leqq Z_n \leqq k\} \leqq \sum_{i=1}^{k} \mathbf{P}\{Z_n = i\}\, s^i \leqq \mathbf{E}\{s^{Z_n};\ Z_n > 0\} = f_n(s) - f_n(0).$$

Consequently

$$\mathbf{P}\{1 \leqq Z_n \leqq k\} \leqq s^{-k}\big(f_n(s) - f_n(0)\big).$$

Setting $s = f_k(0)$ in this inequality, we obtain

$$\mathbf{P}\{1 \leqq Z_n \leqq k\} \leqq \big(f_k(0)\big)^{-k}\big(f_{n+k}(0) - f_n(0)\big). \tag{2.2}$$

It is known (see, for instance, [2]) that

$$Q_k := \mathbf{P}\{Z_k > 0\} = 1 - f_k(0) = \frac{2}{Bk}\,\big(1 + o(1)\big) \tag{2.3}$$

if $\mathbf{E}Z_1^2 < \infty$.

It follows from (2.3) that there exists the constant c such that

$$1 - f_k(0) \leqq \frac{c}{k} \tag{2.4}$$

for every $k \geqq 1$. Thus,

$$\big(f_k(0)\big)^{-k} \leqq \left(1 - \frac{c}{k}\right)^{-k} < c_1. \tag{2.5}$$

Since $f_k''(s)$ increases,

$$f_k(s) - s = f_k(s) - 1 - f_k'(1)(s-1) < \frac{f_k''(1)}{2}\,(1-s)^2 = \frac{Bk}{2}\,(1-s)^2.$$

Letting $s = f_n(0)$ in this inequality and using bound (2.4), we conclude that

$$f_{n+k}(0) - f_n(0) \leqq \frac{Bk}{2}\,\big(1 - f_n(0)\big)^2 \leqq c\,\frac{k}{n^2}. \tag{2.6}$$

Combining (2.2), (2.5), and (2.6), we arrive at the required inequality.

Applying the identity $a = b - ab(a^{-1} - b^{-1})$, we obtain the representation

$$1 - f_n(s) = \left(\frac{1}{1-s} + \frac{Bn}{2}\right)^{-1} - h_n(s)\, g_n(s), \tag{2.7}$$

where

$$h_n(s) = \frac{1}{1 - f_n(s)} - \frac{1}{1-s} - \frac{Bn}{2},$$
$$g_n(s) = \bigl(1 - f_n(s)\bigr)\left(\frac{1}{1-s} + \frac{Bn}{2}\right)^{-1} = \sum_{i=0}^{\infty} g_{n,i} s^i.$$

Put

$$u(s) = \frac{1}{1 - f(s)} - \frac{1}{1-s} = \sum_{i=0}^{\infty} u_i s^i, \qquad \mu_k(s) = u\bigl(f_k(s)\bigr).$$

LEMMA 2. *As* $n \to \infty$

$$\bigl\|h_n(s)\bigr\|_1 = o(n). \tag{2.8}$$

Proof. It follows from Theorem 1 of [2] that

$$a_0\bigl[h_n(s)\bigr] = \frac{1}{1 - f_n(0)} - 1 - \frac{Bn}{2} = o(n). \tag{2.9}$$

Clearly,

$$a_l\bigl[h_n(s)\bigr] = a_l\left[\frac{1}{1 - f_n(s)} - \frac{1}{1-s}\right], \qquad l \geqq 1. \tag{2.10}$$

Observing that $\mu_k(s) = [1 - f(f_k(s))]^{-1} - [1 - f_k(s)]^{-1}$, we obtain

$$\frac{1}{1 - f_n(s)} - \frac{1}{1-s} = \sum_{k=0}^{n-1}\left(\frac{1}{1 - f_{k+1}(s)} - \frac{1}{1 - f_k(s)}\right)$$
$$= \sum_{k=0}^{n-1}\left(\frac{1}{1 - f(f_k(s))} - \frac{1}{1 - f_k(s)}\right) = \sum_{k=0}^{n-1} \mu_k(s). \tag{2.11}$$

By Lemma 6 of [2], $\sum_{j \geqq 0} |u_j| < \infty$. Consequently,

$$\sum_{l=1}^{\infty} \bigl|a_l[\mu_k(s)]\bigr| = \sum_{l=1}^{\infty} \left|\sum_{j=1}^{\infty} u_j a_l\bigl[f_k^j(s)\bigr]\right| \leqq \sum_{j=1}^{\infty} |u_j| \sum_{l=1}^{\infty} a_l\bigl[f_k^j(s)\bigr] = \sum_{j=1}^{\infty} |u_j| \bigl(1 - f_k^j(0)\bigr).$$

Here we used the inequalities

$$\sum_{l=1}^{\infty} a_l\bigl[f_k^j(s)\bigr] = \mathbf{P}(Z_k > 0 \mid Z_0 = j) = 1 - f_k^j(0).$$

Since $f_k(0)\to 1$ as $k\to\infty$ and $\sum_{j=1}^{\infty}|u_j|<\infty$, we conclude from the previous bound that

$$\sum_{l=1}^{\infty}\bigl|a_l[\mu_k(s)]\bigr|\longrightarrow 0 \qquad \text{as} \quad k\to\infty.$$

Consequently,

$$\sum_{k=0}^{n-1}\sum_{l=1}^{\infty}\bigl|a_l[\mu_k(s)]\bigr|=o(n). \tag{2.12}$$

Bound (2.8) follows from (2.9), (2.12), and the inequality

$$\bigl\|h_n(s)\bigr\|_1\leqq\bigl|a_0[h_n(s)]\bigr|+\sum_{k=0}^{n-1}\sum_{l=1}^{\infty}\bigl|a_l\bigl[\mu_k(s)\bigr]\bigr|.$$

LEMMA 3. *The equality*

$$1-f_n(s)=\left(\frac{1}{1-s}+\frac{Bn}{2}\right)^{-1}\bigl(1+o(1)\bigr) \tag{2.13}$$

holds uniformly in s from the unit disk.

Proof. It follows from representation (2.7) that

$$1-f_n(s)=\left(\frac{1}{1-s}+\frac{Bn}{2}\right)^{-1}\bigl(1-h_n(s)\bigl(1-f_n(s)\bigr)\bigr).$$

We conclude from the equality $\|1-f_n(s)\|=2Q_n$ and bounds (2.4), (2.8) that

$$\bigl\|h_n(s)(1-f_n(s))\bigr\|_1\leqq\bigl\|h_n(s)\bigr\|_1\bigl\|1-f_n(s)\bigr\|_1=o(1).$$

Noticing that the convergence to zero in the norm $\|\cdot\|_1$ implies the uniform convergence to zero in the unit disk, we obtain the desired result.

LEMMA 4. *Let $\rho(s)$ be a probability generating function and $\rho'(1)<\infty$. Then for every $a>0$*

$$a_l\bigl[\rho(s)\bigr]\leqq\left(\frac{96}{95}\right)^2\frac{1}{al}\int_{-a}^{a}\bigl|\rho'(e^{it})\bigr|\,dt. \tag{2.14}$$

Proof. Obviously, $\rho'(s)/\rho'(1)$ is a probabilistic generating function. The next bound for the concentration function is known (see, for example, [8])

$$\sup_x \mathbf{P}\{X=x\}\leqq\left(\frac{96}{95}\right)^2\frac{1}{a}\int_{-a}^{a}\bigl|\phi(t)\bigr|\,dt,$$

where $\phi(t)$ is the characteristic function of the random variable X, $a>0$. By applying this bound to the random variable with the generating function $\rho'(s)/\rho'(1)$, we get

$$\sup_l a_l\left[\frac{\rho'(s)}{\rho'(1)}\right]\leqq\left(\frac{96}{95}\right)^2\frac{1}{a}\int_{-a}^{a}\left|\frac{\rho'(e^{it})}{\rho'(1)}\right|dt.$$

Hence, noting that for any $l \geqq 1$

$$a_l\big[\rho(s)\big] = \frac{1}{l}\, a_{l-1}\big[\rho'(s)\big],$$

we obtain the assertion of the lemma.

LEMMA 5. *For $|s| \leqq 1$, $s \to 1$, the equality*

$$\log f'(s) = -B(1-s) + o(1-s) \tag{2.15}$$

holds.

Here and in what follows, $\log s$ denotes the principal branch of the logarithm.

Proof. Using the equality $\log(1+x) = x + O(x^2)$, we have

$$\log f'(s) = \log\big(1 + (f'(s)-1)\big) = \big(f'(s)-1\big) + O\big((f'(s)-1)^2\big). \tag{2.16}$$

Further, we conclude from the condition $B < \infty$ that

$$f'(s) - 1 = B(s-1) + o(s-1).$$

By applying this equality to both summands in the right-hand side of (2.16) we obtain

$$\log f'(s) = -B(1-s) + o(1-s) + O\big((1-s)^2\big).$$

Hence (2.15) follows.

LEMMA 6. *For every $\varepsilon > 0$, there exists $N = N(\varepsilon)$ such that for all s from the unity disk*

$$\big|f_n'(s)\big| \leqq \exp\Bigg(-B(1-\varepsilon)\sum_{j=N}^{n-1} \operatorname{Re}\big(1 - f_j(s)\big) + \varepsilon \sum_{j=N}^{n-1} \big|\operatorname{Im}\big(1 - f_j(s)\big)\big|\Bigg). \tag{2.17}$$

Proof. It is easily seen that

$$f_n'(s) = \prod_{j=0}^{n-1} f'\big(f_j(s)\big).$$

Since $|f'(f_j(s))| \leqq 1$, whatever $j \geqq 0$, the bound

$$\begin{aligned} \big|f_n'(s)\big| &\leqq \Bigg|\prod_{j=N}^{n-1} f'\big(f_j(s)\big)\Bigg| = \Bigg|\exp\Bigg(\sum_{j=N}^{n-1} \log f'\big(f_j(s)\big)\Bigg)\Bigg| \\ &= \exp\Bigg(\sum_{j=N}^{n-1} \operatorname{Re}\log f'\big(f_j(s)\big)\Bigg) \end{aligned} \tag{2.18}$$

is valid for every N. According to Lemma 5

$$\log f'(s) = -\big(B + \alpha(s)\big)(1-s), \qquad \alpha(s) \to 0 \quad \text{for} \quad s \to 1.$$

Hence, for $|s| \leqq 1$ the bound

$$\operatorname{Re}\log f'(s) \leqq -\big(B - \big|\operatorname{Re}\alpha(s)\big|\big)\ \operatorname{Re}(1-s) + \big|\operatorname{Im}\alpha(s)\big|\big|\operatorname{Im}(1-s)\big| \tag{2.19}$$

holds.

It follows from the inequalities $|1 - f_j(s)| \leqq 2(1 - f_j(0))$ and (2.4) that $f_j(s) \to 1$ as $j \to \infty$ uniformly in s from the unity disk. Consequently, we can choose N such that

$$\bigl|\operatorname{Re}\alpha\bigl(f_j(s)\bigr)\bigr| \leqq \varepsilon B, \qquad \bigl|\operatorname{Im}\alpha\bigl(f_j(s)\bigr)\bigr| \leqq \varepsilon$$

for all $j \geqq N$. Applying these inequalities to estimating the right-hand side in (2.19), we obtain

$$\operatorname{Re}\log f'\bigl(f_j(s)\bigr) \leqq -B(1-\varepsilon)\operatorname{Re}\bigl(1 - f_j(s)\bigr) + \varepsilon\bigl|\operatorname{Im}\bigl(1 - f_j(s)\bigr)\bigr|.$$

Hence, in view of (2.18) the assertion of the lemma follows easily.

Put

$$\varphi(x) = \frac{2(Bx+1)}{(Bx+1)^2 + c^2},$$

where c is an arbitrary constant.

LEMMA 7. *For every* $k \geqq 1$,

$$\left|\sum_{j=0}^{k-1}\varphi(j) - \frac{1}{B}\log\left(1 + \frac{(Bk+1)^2}{c^2}\right)\right| \leqq 1. \tag{2.20}$$

Proof. We will use the well-known Euler formula (see, for example, [9])

$$\sum_{j=0}^{k-1}\varphi(j) = \int_0^k \varphi(t)\,dt + \sum_{\nu=1}^{n-1}\frac{B_\nu}{\nu!}\left[\varphi^{(\nu-1)}(k) - \varphi^{(\nu-1)}(0)\right]$$
$$- \frac{B_n}{n!}\int_0^1 \left[B_n(t) - B_n\right]\sum_{j=0}^{n-1}\varphi^{(n)}(j+1-t)\,dt.$$

Here B_ν and $B_\nu(t)$ are, respectively, Bernoulli numbers and Bernoulli polynomials. Setting $n = 1$ in this equality, we get

$$\sum_{j=0}^{k-1}\varphi(j) = \int_0^k \varphi(t)\,dt - \int_0^1 t\sum_{j=0}^{k-1}\varphi'(j+1-t)\,dt. \tag{2.21}$$

It is easily seen that

$$\bigl|\varphi'(x)\bigr| = \left|\frac{2B(c^2 - (Bx+1)^2)}{(c^2 + (Bx+1)^2)^2}\right| \leqq \frac{2B}{(Bx+1)^2}.$$

With the aid of this inequality we get the bound

$$\left|\int_0^1 t\sum_{j=0}^{k-1}\varphi'(j+1-t)\,dt\right| \leqq \frac{1}{2}\sum_{j=0}^{k-1}\sup_{0\leqq t\leqq 1}\bigl|\varphi'(j+1-t)\bigr|$$
$$\leqq \sum_{j=0}^{k-1}\frac{B}{(Bj+1)^2} \leqq \int_1^\infty \frac{dx}{x^2} = 1. \tag{2.22}$$

We conclude from (2.21) and (2.22) that

$$\left|\sum_{j=0}^{k-1}\varphi(j)-\int_0^k\varphi(t)\,dt\right|\leqq 1.$$

Hence, noting that

$$\int_0^k\varphi(t)\,dt=\frac{1}{B}\,\log\bigl(c^2+(Bn+1)^2\bigr)-\frac{1}{B}\,\log c^2=\frac{1}{B}\,\log\biggl(1+\frac{(Bn+1)^2}{c^2}\biggr),$$

we obtain the desired result.

LEMMA 8. *For every $\varepsilon>0$, there exist N and the constants $a=a(\varepsilon,N)$, $b=b(\varepsilon,N)$ such that for all $|t|\leqq\pi/2$*

$$\sum_{j=N}^{n-1}\mathrm{Re}\,\bigl(1-f_j(e^{it})\bigr)\geqq\frac{(1-\varepsilon)}{B}\,\log\biggl(1+(Bn+1)^2\tan^2\frac{t}{2}\biggr)-a, \tag{2.23}$$

$$\sum_{j=N}^{n-1}\bigl|\,\mathrm{Im}\,\bigl(1-f_j(e^{it})\bigr)\bigr|\leqq\frac{\varepsilon}{B}\,\log\biggl(1+(Bn+1)^2\tan^2\frac{t}{2}\biggr)+b. \tag{2.24}$$

Proof. Fix an arbitrary $\varepsilon>0$. It follows from Lemma 3 that there exists N such that for any $j\geqq N$ the inequalities

$$\mathrm{Re}\,\bigl(1-f_j(s)\bigr)\geqq(1-\varepsilon)\,\mathrm{Re}\biggl(\frac{1}{1-s}+\frac{Bj}{2}\biggr)^{-1}-\varepsilon\left|\mathrm{Im}\biggl(\frac{1}{1-s}+\frac{Bj}{2}\biggr)^{-1}\right|, \tag{2.25}$$

$$\bigl|\,\mathrm{Im}\,\bigl(1-f_j(s)\bigr)\bigr|\leqq\varepsilon\,\mathrm{Re}\biggl(\frac{1}{1-s}+\frac{Bj}{2}\biggr)^{-1}+(1+\varepsilon)\left|\mathrm{Im}\biggl(\frac{1}{1-s}+\frac{Bj}{2}\biggr)^{-1}\right| \tag{2.26}$$

are valid.

Thus to prove the lemma we need to estimate the sum of the real and imaginary parts of $((1-s)^{-1}+Bj/2)^{-1}$.

Put $s=e^{it}$. Using the equality

$$\frac{1}{1-s}=\frac{1}{(1-\cos t)-i\sin t}=\frac{1}{2}+\frac{i\sin t}{2(1-\cos t)}=\frac{1}{2}+\frac{i}{2}\,\cot\frac{t}{2}\,,$$

we obtain

$$\mathrm{Re}\biggl(\frac{1}{1-s}+\frac{Bj}{2}\biggr)^{-1}=\frac{2(Bj+1)}{(Bj+1)^2+\cot^2(t/2)},$$
$$\mathrm{Im}\biggl(\frac{1}{1-s}+\frac{Bj}{2}\biggr)^{-1}=-2\,\frac{\tan(t/2)}{1+(Bj+1)^2\tan^2(t/2)}.$$

Setting $c=\cot(t/2)$ in the previous lemma and noting that $0<\varphi(x)\leqq 2$ for all $x\geqq 0$, we have

$$\left|\sum_{j=N}^{n-1}\varphi(j)-\frac{1}{B}\,\log\biggl(1+(Bn+1)^2\tan^2\frac{t}{2}\biggr)\right|\leqq 1+\sum_{j=0}^{N-1}\varphi(j)\leqq 2N+1. \tag{2.27}$$

Obviously, the function

$$\psi(x) = \left|\operatorname{Im}\left(\frac{1}{1-s} + \frac{Bx}{2}\right)^{-1}\right| = \frac{2|\tan(t/2)|}{1+(Bx+1)^2\tan^2(t/2)}$$

decreases. Therefore

$$\sum_{j=N}^{n-1}\left|\operatorname{Im}\left(\frac{1}{1-s} + \frac{Bj}{2}\right)^{-1}\right| \leqq \int_0^\infty \psi(x)\,dx \leqq \frac{2}{B}\int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{B}. \tag{2.28}$$

Combining (2.25), (2.27), and (2.28), we obtain (2.23). Correspondingly inequality (2.24) follows from (2.26), (2.27), and (2.28). Lemma 8 is proved.

LEMMA 9. *For all* $n, k \geqq 1$, *the inequality*

$$\mathbf{P}\{Z_n = k\} \leqq \frac{c}{nk} \tag{2.29}$$

holds.

Proof. It follows from Lemmas 6 and 8 that

$$\left|f_n'(e^{it})\right| \leqq c(\varepsilon)\left(\frac{1}{n|\tan(t/2)|}\right)^{2((1-\varepsilon)^2-\varepsilon/B)}.$$

Applying the obvious inequality $|\tan x| > |x|$, we obtain

$$\left|f_n'(e^{it})\right| \leqq c(\varepsilon)\left(\frac{2}{n|t|}\right)^{2((1-\varepsilon)^2-\varepsilon/B)}.$$

Hence, choosing ε such that $(1-\varepsilon)^2 - \varepsilon/B = \frac{3}{4}$, we obtain the bound

$$\left|f_n'(e^{it})\right| \leqq c\big(n|t|\big)^{-3/2}.$$

Consequently,

$$\int_{1/n<|t|<\pi/2}\left|f_n'(e^{it})\right|dt \leqq \frac{c}{n^{3/2}}\int_{1/n<|t|<\pi/2}\frac{dt}{|t|^{3/2}} < \frac{c_1}{n}. \tag{2.30}$$

Obviously, $|f_n'(e^{it})| \leqq 1$. Therefore

$$\int_{|t|<1/n}\left|f_n'(e^{it})\right|dt \leqq \frac{2}{n}. \tag{2.31}$$

Putting $e_n = f_n$, $a = \pi/2$ in the inequality of Lemma 4 and taking into account (2.30) and (2.31), we arrive at the required inequality.

We need the following bound for the concentration function of a sum of independent identically distributed (i.i.d.) random variables (see, for example, [8]).

LEMMA 10. *Let* $S_k = X_1 + \cdots + X_k$ *be a sum of i.i.d. random variables. Then*

$$Q(S_k,\lambda) \leqq \frac{A}{\sqrt{k}}\,Q(X_1,\lambda)\big(1 - Q(X_1,\lambda)\big)^{-1}, \tag{2.32}$$

where A *is an absolute constant.*

Let $\{\xi_i^{(n)}\}_{i=1}^{\infty}$ be the sequence of i.i.d. random variables with the distribution which is defined by the equalities $\mathbf{P}\{\xi_i^{(n)} = j\} = \mathbf{P}(Z_n = j \,|\, Z_n > 0)$, $j \geqq 1$. Put $S_k^{(n)} = \sum_{i=1}^{k} \xi_i^{(n)}$.

LEMMA 11. *There exists a constant* c *such that for every* $k \geqq 2$ *the bound*

$$\sup_{l \geqq 1} \mathbf{P}\{S_k^{(n)} = l\} \leqq \frac{c}{n\sqrt{k}} \tag{2.33}$$

holds.

Note that in Lemma 9 in [2] the bound similar to (2.33) is obtained. Proving this result, the authors of [2] used the local limit theorem for a critical Galton–Watson process which was proved by them. We do not use this theorem. In contrast, the bound (2.33) is the important component in the proof of our main result.

Proof. We conclude from Lemma 9 and (2.3) that for $j \geqq l/2$ the bound

$$\mathbf{P}\{\xi_1^{(n)} = j\} = \frac{\mathbf{P}\{Z_n = j\}}{\mathbf{P}\{Z_n > 0\}} \leqq \frac{c}{j} \leqq \frac{c_1}{l}$$

is valid. Consequently,

$$\begin{aligned} \mathbf{P}\{S_2^{(n)} = l\} &= \sum_{j=1}^{l-1} \mathbf{P}\{\xi_1^{(n)} = j\}\,\mathbf{P}\{\xi_1^{(n)} = l-j\} \\ &\leqq 2 \sup_{i \geqq l/2} \mathbf{P}\{\xi_1^{(n)} = i\} \sum_{j \leqq l/2} \mathbf{P}\{\xi_1^{(n)} = j\} \leqq \frac{c}{l}\,\mathbf{P}\left\{\xi_1^{(n)} \leqq \frac{l}{2}\right\}. \end{aligned} \tag{2.34}$$

By the definition $\mathbf{P}\{\xi_1^{(n)} \leqq i\} = \mathbf{P}\{1 \leqq Z_n \leqq i\}/\mathbf{P}\{Z_n > 0\}$. Applying (2.1) and (2.3), we obtain

$$\mathbf{P}\{\xi_1^{(n)} \leqq i\} \leqq \frac{ci}{n}. \tag{2.35}$$

It follows from (2.34) and (2.35) that

$$\sup_{l \geqq 1} \mathbf{P}\{S_2^{(n)} = l\} \leqq \frac{c}{n}. \tag{2.36}$$

Suppose first that $k = 2m$, $m > 1$. Then the random variable $S_k^{(n)}$ can be represented as the sum of m i.i.d. random variables $X_i^{(n)}$ which coincide in distribution with $S_2^{(n)}$. Applying the previous lemma, we have

$$Q(S_k^{(n)}, \lambda) \leqq \frac{A}{\sqrt{m}}\, Q(S_2^{(n)}, \lambda)\big(1 - Q(S_2^{(n)}, \lambda)\big)^{-1}.$$

Taking into account (2.36), we conclude that for sufficiently large n

$$\sup_{l \geqq 1} \mathbf{P}\{S_{2m}^{(n)} = l\} \leqq \frac{c}{n\sqrt{2m}} \left(1 - \frac{2c_1}{n}\right)^{-1} \leqq \frac{c_2}{n\sqrt{2m}}, \tag{2.37}$$

which proves the lemma for even values of k. If $k = 2m+1$, one should use the obvious bound $Q(S_{2m+1}^{(n)}, \lambda) \leqq Q(S_{2m}^{(n)}, \lambda)$ and thereafter apply bound (2.37).

Put $l_0 = \min\{l \geqq 1\colon\ p_l > 0\}$.

LEMMA 12. *If $l_0 > 1$, then $\mathbf{P}\{Z_n = l\} = 0$ for all $1 \leqq l < l_0$ and $n \geqq 1$. In addition, for every l_0*

$$\mathbf{P}\{Z_n = l_0\} = p_{l_0} \prod_{i=1}^{n-1} f'\big(f_i(0)\big) > 0. \tag{2.38}$$

Proof. If the event $\{Z_n > 0\}$ occurs, then at least one particle in the $(n-1)$th generation had a nonzero number of offspring. By the definition of l_0 this number cannot be less than l_0. This means that the events $\{Z_n > 0\}$ and $\{Z_n \geqq l_0\}$ coincide and, consequently, $\mathbf{P}\{Z_n = l\} = 0$ for every $1 \leqq l < l_0$.

It follows from the definition of l_0 that for every $i \geqq l_0$ the equality

$$\mathbf{P}(Z_n = l_0 \mid Z_{n-1} = i) = i p_0^{i-1} p_{l_0} \tag{2.39}$$

is valid.

Consequently,

$$\mathbf{P}\{Z_n = l_0\} = \mathbf{E}\big(\mathbf{E}\big(I(Z_n = l_0) \mid Z_{n-1}\big)\big) = p_{l_0}\mathbf{E}\big(Z_{n-1} f^{Z_{n-1}-1}(0)\big) = p_{l_0} f'_{n-1}\big(f(0)\big)$$

which implies the second assertion of the lemma.

LEMMA 13. *The inequality*

$$\lim_{n\to\infty} n^2 \mathbf{P}\{Z_n = l_0\} > 0 \tag{2.40}$$

takes place.

Proof. It is proved in [2] that for every fixed j

$$\lim_{n\to\infty} \frac{B}{2}\, n^2 \mathbf{P}\{Z_n = j\} = \mu(j) < \infty, \tag{2.41}$$

where the sequence $\mu(j)$ satisfies the system of equations

$$\sum_{l=1}^{\infty} \mu(l)\, P(l,j) = \mu(j), \qquad j \geqq 1, \qquad \sum_{l=1}^{\infty} \mu(l)\, p_0^l = 1.$$

Here $P(l,j)$ are the transition probabilities of the process Z_n, i.e.,

$$P(l,j) = \mathbf{P}(Z_1 = j \mid Z_0 = l),$$

since in view of Lemma 12 $\mathbf{P}\{Z_n = l\} = 0$ for all n and $1 \leqq l < l_0$, $\mu(l) = 0$ for every $1 \leqq l < l_0$ as well.

Let us show that $\mu(l_0) > 0$. For this purpose we rewrite (2.39) in the following way:

$$P(i, l_0) = i p_0^{i-1} p_{l_0}.$$

Consequently,

$$\mu(l_0) = \sum_{i=l_0}^{\infty} \mu(i)\, i p_0^{i-1} p_{l_0} \geqq l_0 \,\frac{p_{l_0}}{p_0} \sum_{i=l_0}^{\infty} \mu(i)\, p_0^i = l_0\, \frac{p_{l_0}}{p_0} > 0.$$

Setting $j = l_0$ in (2.41), we obtain the desired result.

LEMMA 14. *For every* $1 \leqq j < k$, *the inequality*

$$\prod_{i=j}^{k-1} f'(f_i(0)) \leqq c\,\frac{j^2}{k^2} \tag{2.42}$$

is valid, where c *is a constant.*

Proof. It follows from (2.38) and (2.40) that as $j,\,k \to \infty$

$$\prod_{i=j}^{k-1} f'\big(f_i(0)\big) = \frac{\mathbf{P}\{Z_k = l_0\}}{\mathbf{P}\{Z_j = l_0\}} \sim \frac{j^2}{k^2},$$

whence the desired bound follows immediately.

LEMMA 15. *For every* $j \geqq 3$, $q \in (0,1)$, *the following inequality holds*:

$$\sum_{i=j}^{\infty} C_i^j q^{i-j} p_i = \frac{1}{j!}\, f^{(j)}(q) \leqq \frac{2B}{j^2 q(1-q)^{j-2}}. \tag{2.43}$$

Proof. Obviously,

$$\sum_{i=j}^{\infty} C_i^j q^{i-j} p_i = \frac{1}{j!}\sum_{i=j}^{\infty} \frac{i!}{(i-j)!}\, q^{i-j} p_i = \frac{1}{j!}\sum_{i=j}^{\infty}\left(\frac{(i-2)!}{(i-j)!}\, q^{i-j}\right) i(i-1)\, p_i.$$

Using the Abel identity, we have

$$j!\sum_{i=j}^{\infty} C_i^j q^{i-j} p_i = \sum_{i=j}^{\infty}\left(\sum_{k=1}^{i} k(k-1)\, p_k\right) q^{i-j}\left(\frac{(i-2)!}{(i-j)!} - \frac{(i-1)!}{(i+1-j)!}\, q\right) - (j-2)!\sum_{k=1}^{j-1} k(k-1)\, p_k.$$

Further,

$$\frac{(i-2)!}{(i-j)!}\, q^{i-j} - \frac{(i-1)!}{(i+1-j)!}\, q^{i+1-j} < \frac{(i-1)!}{(i+1-j)!}\, q^{i-j}(1-q).$$

On the other hand, for every i the bound

$$\sum_{k=1}^{i} k(k-1)\, p_k \leqq B$$

holds.

Consequently,

$$\sum_{i=j}^{\infty} C_i^j q^{i-j} p_i < \frac{B(1-q)}{j!}\sum_{i=j}^{\infty}\frac{(i-1)!}{(i+1-j)!}\, q^{i-j} < \frac{B(1-q)}{j!\,q}\sum_{i=j-1}^{\infty}\frac{(i-1)!}{(i+1-j)!}\, q^{i+1-j}.$$

Noticing that

$$\frac{d^{j-2}}{dq^{j-2}}\, q^{i-1} = \frac{(i-1)!}{(i+1-j)!}\, q^{i+1-j},$$

we arrive at the identity

$$\sum_{i=j-1}^{\infty} \frac{(i-1)!}{(i+1-j)!}\, q^{i+1-j} = \frac{d^{j-2}}{dq^{j-2}}(1-q)^{-1} = (j-2)!\,(1-q)^{-j+1}.$$

As a result, we have the inequality

$$\sum_{i=j}^{\infty} C_i^j q^{i-j} p_i < \frac{B}{j(j-1)\, q(1-q)^{j-2}}.$$

It remains to notice that $1/(j(j-1)) < 2/j^2$.

LEMMA 16. *For every $k < n$, the identity*

$$\mathbf{P}\{Z_n = k\} = \mathbf{P}\{Z_k = k\} \prod_{i=k}^{n-1} f'\big(f_i(0)\big) + \sum_{j=k}^{n-1} r_j(k) \prod_{i=j+1}^{n-1} f'\big(f_i(0)\big) \tag{2.44}$$

is valid, where

$$r_j(k) = \sum_{i=2}^{\infty} Q_j^i \mathbf{P}\{S_i^{(j)} = k\} \sum_{l=i}^{\infty} C_l^i (1-Q_j)^{l-i}\, p_l.$$

Proof. Using the Markov property of Z_n, we have

$$\mathbf{P}\{Z_n = k\} = \sum_{l=1}^{\infty} p_l \mathbf{P}(Z_{n-1} = k \mid Z_0 = l).$$

The process beginning with l particles in zero generation can be represented as the sum of independent processes, each of which starts with one particle. Obviously the probability that $l-i$ processes will degenerate to the moment n is equal to $l-i$ $C_l^i Q_{n-1}^i (1-Q_{n-1})^{l-i}$. The distribution of every nondegenerated process coincides with that of $\xi_1^{(n-1)}$. Therefore

$$\mathbf{P}(Z_{n-1} = k \mid Z_0 = l) = \sum_{i=1}^{l} C_l^i Q_{n-1}^i (1-Q_{n-1})^{l-i} \mathbf{P}\{S_i^{(n-1)} = k\}. \tag{2.45}$$

Consequently,

$$\mathbf{P}\{Z_n = k\} = Q_{n-1}\mathbf{P}\{S_1^{(n-1)} = k\} \sum_{l=1}^{\infty} l(1-Q_{n-1})^{l-1}\, p_l + \sum_{i=2}^{\infty} Q_{n-1}^i \mathbf{P}\{S_i^{(n-1)} = k\} \sum_{l=i}^{\infty} C_l^i (1-Q_{n-1})^{l-i}\, p_l.$$

Since

$$Q_{n-1}\mathbf{P}\{S_1^{(n-1)} = k\} = \mathbf{P}\{Z_{n-1} = k\}$$

and

$$\sum_{l=1}^{\infty} l(1-Q_{n-1})^{l-1}\, p_l = f'(f_{n-1}(0)),$$

the first summand in this representation is equal to

$$\mathbf{P}\{Z_{n-1}=k\}\, f'(f_{n-1}(0)).$$

Therefore

$$\mathbf{P}\{Z_n=k\}=\mathbf{P}\{Z_{n-1}=k\}\, f'\big(f_{n-1}(0)\big)+r_{n-1}(k).$$

Repeating the procedure $n-k-1$ times we arrive at (2.44).

LEMMA 17. *For every* $1\leqq k\leqq j$, *the bound*

$$r_j(k)\leqq \frac{c}{j^2}\left(\mathbf{P}\{S_2^{(j)}=k\}+\frac{k}{j^2}\right) \tag{2.46}$$

holds.

Proof. According to definition

$$\begin{aligned} r_j(k)&=Q_j^2\mathbf{P}\{S_2^{(j)}=k\}\sum_{l=2}^{\infty}C_l^2(1-Q_j)^{l-2}\,p_l\\ &\quad+\sum_{i=3}^{\infty}Q_j^i\mathbf{P}\{S_i^{(j)}=k\}\sum_{l=i}^{\infty}C_l^i(1-Q_j)^{l-i}\,p_l. \end{aligned} \tag{2.47}$$

Setting $q=1-Q_j$ in Lemma 15, we obtain for every $i\geqq 3$ the inequality

$$\sum_{l=i}^{\infty}C_l^i(1-Q_j)^{l-i}\,p_l\leqq \frac{2B}{i^2Q_j^{i-2}(1-Q_j)}\,.$$

Hence, noticing that $\mathbf{P}\{S_i^{(j)}=k\}=0$ for $i>k$, we arrive at the bound

$$\sum_{i=3}^{\infty}Q_j^i\mathbf{P}\{S_i^{(j)}=k\}\sum_{l=i}^{\infty}C_l^i(1-Q_j)^{l-i}\,p_l\leqq \frac{2BQ_j^2}{(1-Q_j)}\sum_{i=3}^{k}\frac{\mathbf{P}\{S_i^{(j)}=k\}}{i^2}. \tag{2.48}$$

Applying (2.4) we conclude that

$$\frac{Q_j^2}{(1-Q_j)}\leqq \frac{c}{j^2}. \tag{2.49}$$

It follows from (2.33) and (2.35) that uniformly in $i\geqq 3$

$$\begin{aligned} \mathbf{P}\{S_i^{(j)}=k\}&=\sum_{l=1}^{k-1}\mathbf{P}\{S_{i-1}^{(j)}=l\}\,\mathbf{P}\{\xi_1^{(j)}=k-l\}\\ &\leqq \mathbf{P}\{\xi_1^{(j)}<k\}\sup_l \mathbf{P}\{S_{i-1}^{(j)}=l\}\leqq \frac{ck}{j^2}. \end{aligned} \tag{2.50}$$

By using (2.49) and (2.50) to estimate the right-hand side of (2.48), we obtain the inequality

$$\sum_{i=3}^{\infty}Q_j^i\mathbf{P}\{S_i^{(j)}=k\}\sum_{l=i}^{\infty}C_l^i(1-Q_j)^{l-i}\,p_l\leqq \frac{ck}{j^4}. \tag{2.51}$$

We now estimate the first summand in the right-hand side of equality (2.47). It is easily seen that

$$\sum_{l=2}^{\infty} C_l^2 (1-Q_j)^{l-2}\, p_l = \frac{f''(1-Q_j)}{2} < \frac{B}{2}.$$

Hence, applying (2.4), we obtain the bound

$$Q_j^2 \mathbf{P}\{S_2^{(j)} = k\} \sum_{l=2}^{\infty} C_l^2 (1-Q_j)^{l-2}\, p_l < \frac{c\mathbf{P}\{S_2^{(j)} = k\}}{j^2}. \tag{2.52}$$

The assertion of the lemma follows from (2.47), (2.51), and (2.52).

LEMMA 18. *Let ξ_n be a random variable having a binomial distribution with parameters n and p. Then the inequalities*

$$(np)^{-1/2} < \mathbf{E}\{\xi_n^{-1/2};\ \xi_n > 0\} < A(np)^{-1/2}, \tag{2.53}$$

$$np(1-p)^{n-1} < \mathbf{E}\{\xi_n^{-1/2};\ \xi_n > 0\} < np \tag{2.54}$$

are valid. The constant A does not exceed 2.73.

In Lemma 13 of [10] the following inequality is deduced:

$$\mathbf{E}\{\xi_n^{\beta};\ \xi_n > 0\} \leqq c(\beta)(np)^{\beta}, \qquad \beta \leqq 1.$$

The proof of the upper bound in (2.53) repeats almost word for word the proof of the latter. The only new element is the numerical bound for the constant $A = c(-\frac{1}{2})$.

Proof. The function $x^{-1/2}$ is convex. Hence, by the Jensen inequality,

$$\mathbf{E}\{\xi_n^{-1/2};\ \xi_n > 0\} > \big(\mathbf{E}\{\xi_n;\ \xi_n > 0\}\big)^{-1/2} = (np)^{-1/2}.$$

On the other hand,

$$\mathbf{E}\{\xi_n^{-1/2};\ \xi_n > 0\} < \mathbf{P}\left\{\xi_n < \frac{np}{2}\right\} + \left(\frac{np}{2}\right)^{-1/2}.$$

By the Bennet–Hoefding inequality [8, p. 77]

$$\mathbf{P}\left\{\xi_n < \frac{np}{2}\right\} < \exp\left\{-\frac{np}{2}\Big[(3-2p)\,\log\big(1+\big(2(1-p)\big)^{-1}\big) - 1\Big]\right\}.$$

It is easily seen that $\min_{0\leqq p\leqq 1}(3-2p)\,\log(1+(2(1-p))^{-1})$ is attained for $p=0$ and equals $3\,\log\frac{3}{2}$.

Since $\sup_{x\geqq 0}\sqrt{x}e^{-\alpha x} = (2e\alpha)^{-1/2}$, we have $e^{-\alpha x} \leqq (2e\alpha)^{-1/2}x^{-1/2}$. Letting in this bound $x = np$, $\alpha = (3\log\frac{3}{2} - 1)/2$, we get

$$\mathbf{P}\left\{\xi_n < \frac{np}{2}\right\} < \exp\left\{-\frac{np}{2}\left(3\log\frac{3}{2} - 1\right)\right\} < 1.31(np)^{-1/2}.$$

Thus, (2.53) is proved. The constant A in this bound does not exceed $1.31+\sqrt{2} < 2.73$.

Obviously,

$$\mathbf{E}\{\xi_n^{-1/2};\ \xi_n > 0\} < \mathbf{E}\xi_n, \qquad \mathbf{E}\{\xi_n^{-1/2};\ \xi_n > 0\} > \mathbf{P}\{\xi_n = 1\}.$$

The inequalities (2.54) follow easily from these bounds.

3. Proof of the proposition. If $k \geqq n$, then, applying (2.29), we obtain

$$\mathbf{P}\{Z_n = k\} \leqq \frac{c}{n^2}. \tag{3.1}$$

Now let $k < n$. It follows from (2.42) and (3.1) that

$$\mathbf{P}\{Z_k = k\} \prod_{i=k}^{n-1} f'\big(f_i(0)\big) \leqq \frac{c}{n^2}. \tag{3.2}$$

Using (2.42) and (2.46), we have

$$r_j(k) \prod_{i=j+1}^{n-1} f'(f_i(0)) \leqq \frac{c}{n^2}\left(\mathbf{P}\{S_2^{(j)} = k\} + \frac{k}{j^2}\right). \tag{3.3}$$

Combining (2.44), (3.2), and (3.3), we arrive at the inequality

$$\begin{aligned} \mathbf{P}\{Z_n = k\} &\leqq \frac{c}{n^2}\left(1 + \sum_{j=k}^{n} \mathbf{P}\{S_2^{(j)} = k\} + \sum_{j=k}^{n} \frac{k}{j^2}\right) \\ &\leqq \frac{c_1}{n^2}\left(1 + \sum_{j=k}^{n} \mathbf{P}\{S_2^{(j)} = k\}\right). \end{aligned} \tag{3.4}$$

Here we used the fact that for every $k \geqq 1$ the bound

$$\sum_{j=k}^{n} \frac{k}{j^2} \leqq \frac{1}{k} + k\int_k^\infty \frac{dx}{x^2} \leqq 2$$

holds. Estimating the quantities $\mathbf{P}\{S_2^{(j)} = k\}$ with the aid of (2.36), we have

$$\mathbf{P}\{Z_n = k\} \leqq \frac{c}{n^2}\left(1 + \sum_{k}^{n} j^{-1}\right) \leqq \frac{c_1(1 + \log(n/k))}{n^2}.$$

Hence, applying (2.3), we conclude that

$$\sup_{l \geqq k/2} \mathbf{P}\{\xi_1^{(j)} = l\} \leqq \frac{c\,(1 + \log(2j/k))}{n}. \tag{3.5}$$

It follows from (2.35) and (3.5) that

$$\begin{aligned} \mathbf{P}\{S_2^{(j)} = k\} &= \sum_{i=1}^{k-1} \mathbf{P}\{\xi_1^{(j)} = i\}\, \mathbf{P}\{\xi_1^{(j)} = k - i\} \\ &\leqq 2 \sup_{i \geqq k/2} \mathbf{P}\{\xi_1^{(j)} = i\}\, \mathbf{P}\left\{\xi_1^{(j)} \leqq \frac{k}{2}\right\} \leqq c\,\frac{\log(2j/k) + 1}{j^2}. \end{aligned}$$

Applying this inequality to the right-hand side of (3.4), we arrive at the bound

$$\mathbf{P}\{Z_n = k\} \leqq \frac{c}{n^2}\left(1 + \sum_{j=k}^{n} k\,\frac{\log(2j/k) + 1}{j^2}\right). \tag{3.6}$$

Since for $t > 0$ the function $t^{-2}(1 + \log(2t))$ decreases, we have

$$\frac{1 + \log(2j/k)}{(j/k)^2}\,\frac{1}{k} \leqq \int_{(j-1)/k}^{j/k} \frac{1 + \log(2t)}{t^2}\, dt.$$

Consequently, for every n and k

$$\sum_{j=k}^{n} k \frac{\log(2j/k) + 1}{j^2} = \sum_{j=k}^{n} \frac{1 + \log(2j/k)}{(j/k)^2}\,\frac{1}{k} \leqq \frac{1 + \log 2}{k} + \int_1^{n/k} \frac{1 + \log(2t)}{t^2}\, dt < \infty.$$

From (3.6) and the preceding inequality the desired result follows easily.

4. Proof of the theorem.

LEMMA 19. *There exists a constant c such that for every $n \geqq 1$*

$$g_{n,0} \leqq \frac{c}{n^2}, \tag{4.1}$$

$$\sup_{i \geqq 1} |g_{n,i}| \leqq \frac{c}{n^3}, \tag{4.2}$$

where $g_{n,i} = a_i(g_n(s))$.

Proof. We will use the identity

$$\left(\frac{1}{1-s} + \frac{Bn}{2}\right)^{-1} = \frac{1}{1 + Bn/2} - \frac{1}{(1 + Bn/2)^2} \sum_{j=1}^{\infty} \left(1 + \frac{2}{Bn}\right)^{-j+1} s^j.$$

Put $q = (1 + 2/(Bn))^{-1}$. Then the preceding equality will be rewritten as

$$\left(\frac{1}{1-s} + \frac{Bn}{2}\right)^{-1} = \frac{2}{Bn}\, q - \frac{4}{B^2 n^2} \sum_{j=1}^{\infty} q^{j-1} s^j. \tag{4.3}$$

According to (2.3),

$$g_{n,0} = \frac{\mathbf{P}\{Z_n > 0\}}{1 + Bn/2} \sim \frac{4}{B^2 n^2},$$

whence the inequality (4.1) follows immediately.

It follows from the definition of $g_n(s)$ and identity (4.3) that

$$|g_{n,i}| \leqq \mathbf{P}\{Z_n > 0\}\, \frac{4}{B^2 n^2}\, q^{i-1} + \mathbf{P}\{Z_n = i\}\, \frac{2}{Bn}\, q + \sum_{j=1}^{i-1} \mathbf{P}\{Z_n = j\}\, \frac{4}{B^2 n^2}\, q^{i-j-1}.$$

Estimating $\mathbf{P}\{Z_n > 0\}$ with the aid of (2.4) and $\mathbf{P}\{Z_n = j\}$ with the aid of the proposition, we obtain

$$|g_{n,i}| \leqq \frac{c_1}{n^3} + \frac{c_2}{n^4} \sum_{j=1}^{i-1} q^{i-j-1} \leqq \frac{c_1}{n^3} + \frac{c_2}{n^4(1-q)} \leqq c_3 n^{-3}.$$

This completes the proof of the lemma.

It follows from the identities (2.7) and (4.3) that

$$\mathbf{P}\{Z_n = l\} = \frac{4}{B^2 n^2}\, q^{l+1} + a_l\big[h_n(s)\, g_n(s)\big]. \tag{4.4}$$

Thus, to prove the theorem we have to show that the second summand in the right-hand side of (4.4) goes to zero as $l, n \to \infty$ faster than n^{-2}.

LEMMA 20. *There exists a constant c such that for every N and j*

$$\sum_{k=N}^{\infty} \sup_{l\geqq 1} \mathbf{P}(Z_k = l \mid Z_0 = j) \leqq c\left(I(j \geqq N) + \frac{j}{N}\, I(j < N)\right). \tag{4.5}$$

Here and in what follows, $I(A)$ denotes the indicator of the set A.

Proof. Combining the assertions of Lemma 11 and the proposition we conclude that the inequality

$$\sup_{l\geqq 1} \mathbf{P}\{S_i^{(k)} = l\} \leqq \frac{c}{k\sqrt{i}}$$

is valid for all $i \geqq 1$ (in Lemma 11 this bound is deduced only for $i \geqq 2$). By using this inequality to estimate the right-hand side in (2.45), we obtain

$$\sup_{l\geqq 1} \mathbf{P}(Z_k = l \mid Z_0 = j) \leqq \frac{c}{k} \sum_{i=1}^{j} \frac{1}{\sqrt{i}}\, C_j^i Q_k^i (1-Q_k)^{j-i} = \frac{c}{k}\, \mathbf{E}\{\xi_{j,k}^{-1/2};\ \xi_{j,k} > 0\},$$

where $\xi_{j,k}$ is the number of successes in j Bernoulli trials with the probability of a success Q_k. Hence,

$$\begin{aligned}\sum_{k=N}^{\infty} \sup_{l\geqq 1} \mathbf{P}(Z_k = l \mid Z_0 = j) &\leqq c \sum_{k=N}^{\infty} \frac{1}{k} \mathbf{E}\{\xi_{j,k}^{-1/2};\ \xi_{j,k} > 0\} \\ &= c\left(\sum_{k=N}^{(N\vee j)-1} \frac{1}{k}\, \mathbf{E}\{\xi_{j,k}^{-1/2};\ \xi_{j,k} > 0\} + \sum_{k=N\vee j}^{\infty} \frac{1}{k}\, \mathbf{E}\{\xi_{j,k}^{-1/2};\ \xi_{j,k} > 0\}\right).\end{aligned}$$

At first we estimate the first sum in the right-hand side of this inequality. If $j > N$, then, applying the right inequality from (2.53), we arrive at the bound

$$\sum_{k=N}^{(N\vee j)-1} \frac{1}{k}\, \mathbf{E}\{\xi_{j,k}^{-1/2};\ \xi_{j,k} > 0\} \leqq c \sum_{k=N}^{j-1} \frac{1}{k(jQ_k)^{1/2}} \leqq c_1 j^{-1/2} \sum_{k=1}^{j} k^{-1/2} \leqq c_2.$$

Here we used (2.4) and the bound $\sum_{k=1}^{j} k^{-1/2} \leqq cj^{1/2}$. If $j \leqq N$, then the upper index in the considered sum becomes less than the lower one, and, consequently, its value is equal to zero. Combining these two cases, we obtain the inequality

$$\sum_{k=N}^{(N\vee j)-1} \frac{1}{k} \mathbf{E}\{\xi_{j,k}^{-1/2};\ \xi_{j,k} > 0\} \leqq cI(j > N).$$

Estimating the second sum with the aid of (2.54), we obtain

$$\begin{aligned}\sum_{k=N\vee j}^{\infty} \frac{1}{k}\, \mathbf{E}\{\xi_{j,k}^{-1/2};\ \xi_{j,k} > 0\} &\leqq j \sum_{k=N\vee j}^{\infty} \frac{Q_k}{k} \leqq cj \sum_{k=N\vee j}^{\infty} k^{-2} \leqq c_1 \frac{j}{N\vee j} \\ &= c_1 I(j > N) + c_1\, \frac{j}{N}\, I(j \leqq N).\end{aligned}$$

The assertion of the lemma follows from the last two inequalities.

LEMMA 21.

$$\lim_{l\to\infty} a_l\big[h_n(s)\big] = \lim_{l\to\infty} a_l\left[\frac{1}{1-f_n(s)} - \frac{1}{1-s}\right] = 0 \tag{4.6}$$

uniformly in $n \geqq 1$.

Proof. It follows from (2.10) that it is sufficient to prove that

$$\lim_{l\to\infty} a_l\left[\frac{1}{1-f_n(s)} - \frac{1}{1-s}\right] = 0.$$

In view of (2.11)

$$\frac{1}{1-f_n(s)} - \frac{1}{1-s} = \sum_{k=0}^{N-1}\mu_k(s) + \sum_{k=N}^{n-1}\mu_k(s) = \frac{1}{1-f_N(s)} - \frac{1}{1-s} + \sum_{k=N}^{n-1}\mu_k(s).$$

Consider first $\sum_{k=N}^{n-1}\mu_k(s)$. It is easily seen that for every $l \geqq 1$

$$\left|a_l\left[\sum_{k=N}^{n-1}\mu_k(s)\right]\right| \leqq \sum_{k=N}^{\infty}\left|a_l\big[\mu_k(s)\big]\right| \leqq \sum_{j=1}^{\infty}|u_j|\sum_{k=N}^{\infty}a_l\big[f_k^j(s)\big]$$
$$= \sum_{j=1}^{\infty}|u_j|\sum_{k=N}^{\infty}\mathbf{P}(Z_k = l \mid Z_0 = j).$$

Applying (4.5), we get the bound

$$\sup_{l\geqq 1}\left|a_l\left[\sum_{k=N}^{n-1}\mu_k(s)\right]\right| \leqq c\bigg(N^{-1}\sum_{j<N} j|u_j| + \sum_{j\geqq N}|u_j|\bigg).$$

It is easily seen that

$$N^{-1}\sum_{j<N} j|u_j| < N^{-1/2}\sum_{j\leqq\sqrt{N}}|u_j| + \sum_{\sqrt{N}<j<N}|u_j|.$$

Consequently,

$$\sup_{l\geqq 1}\left|a_l\left[\sum_{k=N}^{n-1}\mu_k(s)\right]\right| \leqq c\left(N^{-1/2}\sum_{j\leqq\sqrt{N}}|u_j| + \sum_{j>\sqrt{N}}|u_j|\right).$$

Hence, in view of the finiteness of $\sum|u_j|$ we conclude that for every $\varepsilon > 0$ one can select N such that for all $n \geqq 1$ the following inequality holds:

$$\sup_{l\geqq 1}\left|a_l\left[\sum_{k=N}^{n-1}\mu_k(s)\right]\right| \leqq \varepsilon. \tag{4.7}$$

Further, according to the renewal theorem (see, for example, [11]),

$$\lim_{l\to\infty} a_l\left[\frac{1}{1-f_N(s)} - \frac{1}{1-s}\right] = 0.$$

Hence, in view of (4.7), we conclude that uniformly in $n \geqq 1$

$$\limsup_{l\to\infty} \left| a_l \left[\frac{1}{1-f_n(s)} - \frac{1}{1-s} \right] \right| \leqq \varepsilon.$$

Because of the arbitrariness of ε this implies the desired result.

Now we are able to complete the proof of the theorem.

It is easily seen that

$$\left| a_l \left[h_n(s)\, g_n(s) \right] \right| \leqq \sum_{i=0}^{l} |g_{n,i}| \left| a_{l-i} \left[h_n(s) \right] \right| \leqq g_{n,0} \left| a_l \left[h_n(s) \right] \right| + \left\| h_n(s) \right\|_1 \sup_{i\geqq 1} |g_{n,i}|.$$

Using (4.1) and (4.6) for estimating the first summand, and bounding (4.2) and (2.8) with the aid of the second one, we arrive at the equality

$$\lim_{l,n\to\infty} n^2 \left| a_l \left[h_n(s)\, g_n(s) \right] \right| = 0. \tag{4.8}$$

Obviously,

$$q^{-l-1} = e^{2l/(Bn)}(1+o(1)) < c \tag{4.9}$$

if $l, n \to \infty$ and l/n is bounded. Combining (4.4), (4.8), and (4.9), we obtain the equality

$$\lim_{l,n\to\infty} \frac{B^2 n^2}{4} \left(1 + \frac{2}{Bn} \right)^{l+1} \mathbf{P}\{Z_n = l\} = 1.$$

This completes the proof of the theorem.

Acknowledgments. In conclusion we would like to thank the referee who called our attention to a number of inaccuracies and misprints and made useful remarks that improved the presentation.

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