

On sums of independent random variables without power moments

S. V. Nagaev, Vitali Wachtel

Angaben zur Veröffentlichung / Publication details:

Nagaev, S. V., and Vitali Wachtel. 2006. "On sums of independent random variables without power moments." *Doklady Mathematics* 74 (2): 683–85.
<https://doi.org/10.1134/s1064562406050176>.

Nutzungsbedingungen / Terms of use:

licgercopyright

Dieses Dokument wird unter folgenden Bedingungen zur Verfügung gestellt: / This document is made available under these conditions:

Deutsches Urheberrecht

Weitere Informationen finden Sie unter: / For more information see:

<https://www.uni-augsburg.de/de/organisation/bibliothek/publizieren-zitieren-archivieren/publiz/>



On Sums of Independent Random Variables without Power Moments

S. V. Nagaev^a and V. I. Vakhtel^b

1. Let X, X_1, \dots be independent identically distributed random variables. Assume that $V(x) = P(X \geq x)$ is a slowly varying function as $x \rightarrow \infty$; i.e.,

$$\lim_{x \rightarrow \infty} \frac{V(cx)}{V(x)} = 1 \quad (1)$$

for any $c > 0$. This condition implies that $E\{X^t; X > 0\} = \infty$ for any $t > 0$; i.e., all the power moments are infinite.

Define $S_n = X_1 + X_2 + \dots + X_n$ and $\bar{X}_n = \max_{k \leq n} X_k$.

Let X_n^* be the maximum term in absolute value; i.e., $|X_n^*| = \max_{k \leq n} |X_k|$.

It was P. Lévy who noted that, under condition (1), the absolute value of the difference $S_n - X_n^*$ is small compared to X_n^* ; i.e., X_n^* makes an overwhelming contribution to S_n [1] (see also [2, p. 212]). Assuming additionally that $X \geq 0$, it seems rather plausible that

$$P(S_n < x) \sim P(\bar{X}_n < x) = (1 - V(x))^n. \quad (2)$$

If n and x are related by $nV(x) = y$, where y is a fixed positive constant, we obtain the approximate equality

$$P(S_n < x) \sim e^{-y}.$$

Setting $x = V^{-1}\left(\frac{y}{n}\right)$, where V^{-1} is the inverse of V , we conclude that

$$\lim_{n \rightarrow \infty} P(nV(S_n) > y) = e^{-y}.$$

Thus, we have convergence to a nondegenerate distribution under the functional normalization in terms of $V(x)$. This approach was realized by Darling [3] with the constraint $X \geq 0$ not being required.

Theorem A (Darling). *If $X \geq 0$ or $P(X < -x) = o(V(x))$, then*

$$\lim_{n \rightarrow \infty} P(nV(S_n) < y; S_n \geq 0) = 1 - e^{-y}. \quad (3)$$

If the left tail is comparable with the right one, i.e.,

$$\frac{V(x)}{V(x) + P(X < -x)} \rightarrow p \in (0, 1), \quad (4)$$

then

$$\lim_{n \rightarrow \infty} P(nW(S_n) < y) = 1 - pe^{-y/p} - qe^{-y/q}, \quad (5)$$

where $q = 1 - p$ and $W(x) = V(x)I(x \geq 0) + P(X < x)I(x < 0)$.

In what follows, we need the following notation:

$$g^+(s) = E\{e^{-sX} | X \geq 0\}, \quad g^-(s) = E\{e^{sX} | X < 0\},$$

$$L^+(x) = \left[1 - g^+\left(\frac{1}{x}\right)\right]P(X \geq 0),$$

$$L^-(x) = \left[1 - g^-\left(\frac{1}{x}\right)\right]P(X < 0).$$

Let $R^\pm(x)$ denote the inverse of $L^\pm(x)$.

In these terms, Theorem A can be restated as follows.

Theorem 1. *Assume that $L^+(x)$ is a slowly varying function such that*

$$\lim_{x \rightarrow \infty} \frac{L^+(x)}{L^-(x)} = \frac{p^+}{p^-} \quad (6)$$

for some $p \in (0, 1)$, where $p^+ = p$ and $p^- = 1 - p$.

^a Sobolev Institute of Mathematics, Siberian Division, Russian Academy of Sciences, Universitetskii pr. 4, Novosibirsk, 630090 Russia
e-mail: nagaev@math.nsc.ru

^b Weierstrass Institute for Applied Analysis and Stochastics, Berlin, Germany
e-mail: vakhtel@wias-berlin.de

Then, for any $x > 0$,

$$\begin{aligned} \lim_{x \rightarrow \infty} P\left(\pm S_n > R^\pm\left(\frac{x}{n}\right)\right) &= \lim_{n \rightarrow \infty} P(nL^\pm(\pm S_n) < x; \pm S_n > 0) \\ &= p^\pm \left(1 - \exp\left\{-\frac{x}{p^\pm}\right\}\right). \end{aligned} \quad (7)$$

If $L^\pm(x)$ varies slowly and

$$L^\mp(x) = o(L^\pm(x)), \quad (8)$$

then

$$\begin{aligned} \lim_{x \rightarrow \infty} P\left(\pm S_n > R^\pm\left(\frac{x}{n}\right)\right) &= \lim_{n \rightarrow \infty} P(nL^\pm(\pm S_n) < x; \pm S_n > 0) \\ &= 1 - e^{-x}. \end{aligned} \quad (9)$$

Note that Theorem 1 is stated in terms of the Laplace transforms of the positive and negative parts of X rather than in terms of its distribution function. The fact is that, according to the Tauberian theorem in [4, p. 503, formula (5.22)], the function $V^\pm(x) := P(\pm X > x)$ is slowly varying if and only if $L^\pm(x)$ has the same property and

$$\lim_{x \rightarrow \infty} \frac{V^\pm(x)}{L^\pm(x)} = 1. \quad (10)$$

It follows that conditions (4) and (6) are equivalent. The use of the normalization of $L^\pm(x)$ is explained by the fact that $L^\pm(x)$ is a continuous strictly decreasing function, which prevents us from difficulties associated with the inversion of $V^\pm(x)$.

For any fixed c , the domain $\{y: nL^\pm(y) > c\}$ can be viewed as a region of normal deviations. Accordingly, $\{y: nL^\pm(y) < \varepsilon_n\}$, where $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$, can be viewed as a region of large deviations.

The following assertion concerns the asymptotic behavior of large-deviation probabilities for S_n .

Theorem 2. Suppose that condition (6) or (8) holds and $n, y \rightarrow \infty$ so that $nL^\pm(y) \rightarrow 0$. Then

$$\lim_{n, y \rightarrow \infty} \frac{P(\pm S_n > y)}{nL^\pm(y)} = 1. \quad (11)$$

If $L(x) = L^+(x) + L^-(x)$ varies slowly and $nL(y) \rightarrow 0$, then

$$\lim_{n, y \rightarrow \infty} \frac{P(|S_n| > y)}{nL(y)} = 1. \quad (12)$$

Note that (11) can be written as

$$\frac{P(L^\pm(\pm S_n) < L^\pm(y))}{nL^\pm(y)} \rightarrow 1.$$

Furthermore,

$$\frac{p^\pm(1 - e^{-nL^\pm(y)/p^\pm})}{nL^\pm(y)} \rightarrow 1$$

if $nL^\pm(y) \rightarrow 0$.

Now setting $x = nL^\pm(y)$, we conclude that

$$P(nL^\pm(\pm S_n) < x) \sim p^\pm \left(1 - \exp\left\{-\frac{x}{p^\pm}\right\}\right)$$

as $x \rightarrow 0$. Thus, formula (12) also applies to large-deviation probabilities for S_n .

The proof of Theorem 1 is based on the following result, which relates the weak convergence of distributions to the convergence of the corresponding Laplace transforms.

Let ξ_n be an arbitrary sequence of nonnegative random variables, and let $L(x)$ be a continuous monotone slowly varying function. The inverse of $L(x)$ is denoted by $R(x)$. Suppose that a_n is a positive number sequence and $\varphi(x)$ is a nondecreasing function.

Theorem 3. If $L(x)$ increases and $a_n \rightarrow \infty$, then

$$\lim_{n \rightarrow \infty} E \exp\left\{-\frac{\xi_n}{R(a_n x)}\right\} = \varphi(x) \quad (13)$$

holds on the set continuity of φ if and only if

$$\lim_{n \rightarrow \infty} P(a_n^{-1} L(\xi_n) < x) = \varphi(x) \quad (14)$$

holds on the set continuity of φ .

If $L(x)$ decreases and $a_n \rightarrow 0$, then (13) holds if and only if

$$\lim_{n \rightarrow \infty} P(a_n^{-1} L(\xi_n) < x) = 1 - \varphi(x) \quad (15)$$

is true on the set continuity of φ .

The direct assertion in Theorem 3 for increasing $L(x)$ was stated (in a somewhat different form) and proved in [5].

Theorem 1 is deduced easily from Theorem 3 if $X \geq 0$. The general case is reduced to this special one. The proof of Theorem 2 makes use of the upper bounds for $P(S_n \geq x)$ and $P(|S_n| \geq x)$ derived in [6] and the lower bounds for these probabilities obtained in [7, 8]. Since $P(X \geq x)$ and $P(|X| \geq x)$ are slowly varying

functions, these bounds approach each other and give an asymptotically sharp result.

ACKNOWLEDGMENTS

This work was supported by the INTAS, project 03-51-5018.

REFERENCES

1. P. Lévy, *J. Math.* **14**, 347–402 (1935).
2. P. Lévy, *Théorie de l'addition des variables aléatoires* (Paris, 1937).
3. D. A. Darling, *Trans. Am. Math. Soc.* **73**, 95–107 (1952).
4. W. Feller, *An Introduction to Probability Theory and Its Applications* (Wiley, New York, 1968; Mir, Moscow, 1984), Vol. 2.
5. I. L. Hudson and E. Seneta, *J. Appl. Probab.* **14**, 836–842 (1977).
6. D. Kh. Fuk and S. V. Nagaev, *Teor. Veroyatn. Ee Primen.* **16**, 660–675 (1971).
7. A. A. Borovkov, *Sib. Mat. Zh.* **41**, 997–1038 (2000).
8. S. V. Nagaev, *Mat. Zametki* **34**, 309–313 (1983).