On Sums of Independent Random Variables without Power Moments

S. V. Nagaev[a] and V. I. Vakhtel[b]

1\. Let $X, X_1, \ldots$ be independent identically distributed random variables. Assume that $V(x) = \mathrm{P}(X \geq x)$ is a slowly varying function as $x \to \infty$; i.e.,

$$\lim_{x \to \infty} \frac{V(cx)}{V(x)} = 1 \tag{1}$$

for any $c > 0$. This condition implies that $\mathrm{E}\{X^t; X > 0\} = \infty$ for any $t > 0$; i.e., all the power moments are infinite.

Define $S_n = X_1 + X_2 + \ldots + X_n$ and $\overline{X}_n = \max_{k \leq n} X_k$. Let X_n^* be the maximum term in absolute value; i.e., $|X_n^*| = \max_{k \leq n} |X_k|$.

It was P. Lévy who noted that, under condition (1), the absolute value of the difference $S_n - X_n^*$ is small compared to X_n^*; i.e., X_n^* makes an overwhelming contribution to S_n [1] (see also [2, p. 212]). Assuming additionally that $X \geq 0$, it is seems rather plausible that

$$\mathrm{P}(S_n < x) \sim \mathrm{P}(\overline{X}_n < x) = (1 - V(x))^n. \tag{2}$$

If n and x are related by $nV(x) = y$, where y is a fixed positive constant, we obtain the approximate equality

$$\mathrm{P}(S_n < x) \sim e^{-y}.$$

Setting $x = V^{-1}\left(\frac{y}{n}\right)$, where V^{-1} is the inverse of V, we conclude that

$$\lim_{n \to \infty} \mathrm{P}(nV(S_n) > y) = e^{-y}.$$

Thus, we have convergence to a nondegenerate distribution under the functional normalization in terms of $V(x)$. This approach was realized by Darling [3] with the constraint $X \geq 0$ not being required.

Theorem A (Darling). *If* $X \geq 0$ *or* $\mathrm{P}(X < -x) = o(V(x))$, *then*

$$\lim_{n \to \infty} \mathrm{P}(nV(S_n) < y;\ S_n \geq 0) = 1 - e^{-y}. \tag{3}$$

If the left tail is comparable with the right one, i.e.,

$$\frac{V(x)}{V(x) + \mathrm{P}(X < -x)} \to p \in (0, 1), \tag{4}$$

then

$$\lim_{n \to \infty} \mathrm{P}(nW(S_n) < y) = 1 - pe^{-y/p} - qe^{-y/q}, \tag{5}$$

where $q = 1 - p$ *and* $W(x) = V(x)I(x \geq 0) + \mathrm{P}(X < x)I(x < 0)$.

In what follows, we need the following notation:

$$g^+(s) = \mathrm{E}\{e^{-sX} | X \geq 0\}, \quad g^-(s) = \mathrm{E}\{e^{sX} | X < 0\},$$

$$L^+(x) = \left[1 - g^+\left(\frac{1}{x}\right)\right]\mathrm{P}(X \geq 0),$$

$$L^-(x) = \left[1 - g^-\left(\frac{1}{x}\right)\right]\mathrm{P}(X < 0).$$

Let $R^{\pm}(x)$ denote the inverse of $L^{\pm}(x)$.

In these terms, Theorem A can be restated as follows.

Theorem 1. *Assume that* $L^+(x)$ *is a slowly varying function such that*

$$\lim_{x \to \infty} \frac{L^+(x)}{L^-(x)} = \frac{p^+}{p^-} \tag{6}$$

for some $p \in (0, 1)$, *where* $p^+ = p$ *and* $p^- = 1 - p$.

---

[a] *Sobolev Institute of Mathematics, Siberian Division, Russian Academy of Sciences, Universitetskii pr. 4, Novosibirsk, 630090 Russia*
e-mail: nagaev@math.nsc.ru

[b] *Weierstrass Institute for Applied Analysis and Stochastics, Berlin, Germany*
e-mail: vakhtel@wias-berlin.de

Then, for any $x > 0$,

$$\lim_{x\to\infty} \mathrm{P}\left(\pm S_n > R^{\pm}\left(\frac{x}{n}\right)\right) = \lim_{n\to\infty} \mathrm{P}(nL^{\pm}(\pm S_n) < x;\ \pm S_n > 0)$$
$$= p^{\pm}\left(1 - \exp\left\{-\frac{x}{p^{\pm}}\right\}\right). \quad (7)$$

If $L^{\pm}(x)$ *varies slowly and*

$$L^{\mp}(x) = o(L^{\pm}(x)), \quad (8)$$

then

$$\lim_{x\to\infty} \mathrm{P}\left(\pm S_n > R^{\pm}\left(\frac{x}{n}\right)\right) = \lim_{n\to\infty} \mathrm{P}(nL^{\pm}(\pm S_n) < x;\ \pm S_n > 0)$$
$$= 1 - e^{-x}. \quad (9)$$

Note that Theorem 1 is stated in terms of the Laplace transforms of the positive and negative parts of X rather than in terms of its distribution function. The fact is that, according to the Tauberian theorem in [4, p. 503, formula (5.22)], the function $V^{\pm}(x) := \mathrm{P}(\pm X > x)$ is slowly varying if and only if $L^{\pm}(x)$ has the same property and

$$\lim_{x\to\infty} \frac{V^{\pm}(x)}{L^{\pm}(x)} = 1. \quad (10)$$

It follows that conditions (4) and (6) are equivalent. The use of the normalization of $L^{\pm}(x)$ is explained by the fact that $L^{\pm}(x)$ is a continuous strictly decreasing function, which prevents us from difficulties associated with the inversion of $V^{\pm}(x)$.

For any fixed c, the domain $\{y\colon nL^{\pm}(y) > c\}$ can be viewed as a region of normal deviations. Accordingly, $\{y\colon nL^{\pm}(y) < \varepsilon_n\}$, where $\varepsilon_n \to 0$ as $n \to 0$, can be viewed as a region of large deviations.

The following assertion concerns the asymptotic behavior of large-deviation probabilities for S_n.

Theorem 2. *Suppose that condition* (6) *or* (8) *holds and* $n, y \to \infty$ *so that* $nL^{\pm}(y) \to 0$. *Then*

$$\lim_{n, y\to\infty} \frac{\mathrm{P}(\pm S_n > y)}{nL^{\pm}(y)} = 1. \quad (11)$$

If $L(x) = L^{+}(x) + L^{-}(x)$ *varies slowly and* $nL(y) \to 0$, *then*

$$\lim_{n, y\to\infty} \frac{\mathrm{P}(|S_n| > y)}{nL(y)} = 1. \quad (12)$$

Note that (11) can be written as

$$\frac{\mathrm{P}(L^{\pm}(\pm S_n) < L^{\pm}(y))}{nL^{\pm}(y)} \to 1.$$

Furthermore,

$$\frac{p^{\pm}(1 - e^{-nL^{\pm}(y)/p^{\pm}})}{nL^{\pm}(y)} \to 1$$

if $nL^{\pm}(y) \to 0$.

Now setting $x = nL^{\pm}(y)$, we conclude that

$$\mathrm{P}(nL^{\pm}(\pm S_n) < x) \sim p^{\pm}\left(1 - \exp\left\{-\frac{x}{p^{\pm}}\right\}\right)$$

as $x \to 0$. Thus, formula (12) also applies to large-deviation probabilities for S_n.

The proof of Theorem 1 is based on the following result, which relates the weak convergence of distributions to the convergence of the corresponding Laplace transforms.

Let ξ_n be an arbitrary sequence of nonnegative random variables, and let $L(x)$ be a continuous monotone slowly varying function. The inverse of $L(x)$ is denoted by $R(x)$. Suppose that a_n is a positive number sequence and $\varphi(x)$ is a nondecreasing function.

Theorem 3. *If* $L(x)$ *increases and* $a_n \to \infty$, *then*

$$\lim_{n\to\infty} \mathrm{E}\exp\left\{-\frac{\xi_n}{R(a_n x)}\right\} = \varphi(x) \quad (13)$$

holds on the set continuity of φ *if and only if*

$$\lim_{n\to\infty} \mathrm{P}(a_n^{-1} L(\xi_n) < x) = \varphi(x) \quad (14)$$

holds on the set continuity of φ.

If $L(x)$ *decreases and* $a_n \to 0$, *then* (13) *holds if and only if*

$$\lim_{n\to\infty} \mathrm{P}(a_n^{-1} L(\xi_n) < x) = 1 - \varphi(x) \quad (15)$$

is true on the set continuity of φ.

The direct assertion in Theorem 3 for increasing $L(x)$ was stated (in a somewhat different form) and proved in [5].

Theorem 1 is deduced easily from Theorem 3 if $X \geq 0$. The general case is reduced to this special one. The proof of Theorem 2 makes use of the upper bounds for $\mathrm{P}(S_n \geq x)$ and $\mathrm{P}(|S_n| \geq x)$ derived in [6] and the lower bounds for these probabilities obtained in [7, 8]. Since $\mathrm{P}(X \geq x)$ and $\mathrm{P}(|X| \geq x)$ are slowly varying functions, these bounds approach each other and give an asymptotically sharp result.

ACKNOWLEDGMENTS

This work was supported by the INTAS, project 03-51-5018.

REFERENCES

1. P. Lévy, J. Math. **14**, 347–402 (1935).
2. P. Lévy, *Théorie de l'addition des variables aléatoires* (Paris, 1937).
3. D. A. Darling, Trans. Am. Math. Soc. **73**, 95–107 (1952).
4. W. Feller, *An Introduction to Probability Theory and Its Applications* (Wiley, New York, 1968; Mir, Moscow, 1984), Vol. 2.
5. I. L. Hudson and E. Seneta, J. Appl. Probab. **14**, 836–842 (1977).
6. D. Kh. Fuk and S. V. Nagaev, Teor. Veroyatn. Ee Primen. **16**, 660–675 (1971).
7. A. A. Borovkov, Sib. Mat. Zh. **41**, 997–1038 (2000).
8. S. V. Nagaev, Mat. Zametki **34**, 309–313 (1983).