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Testing the executability of scenarios in general inhibitor nets

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Abstract

In this paper we introduce executions of place/transition Petri nets with weighted inhibitor arcs (PTI-net) as enabled labeled stratified order structures (LSOs) and present a polynomial algorithm to decide, whether a scenario given by an LSO is an execution of a given PTI-net.

The algorithm is based on an equivalent characterization of enabled LSOs called token flow property. Although the definition of the token flow property involves exponentially many objects in the size of the LSO, there is a nontrivial transformation into a flow optimization problem which can be solved in polynomial time.

1 Introduction

Specifications of concurrent systems are often formulated in terms of scenarios expressing causal dependencies and concurrency among events. In other words, it is often part of the specification, that some scenario should or should not be an execution of the system. Thus, given a system, a natural question is, whether a scenario is an execution of the system. In [7] we presented a polynomial algorithm to answer this question, when the system is given by a place/transition Petri net (p/t-net), and a scenario is given as a labeled partial order (LPO).

"Petri nets with inhibitor arcs are intuitively the most direct approach to increase the modeling power of Petri nets" [13] and have been found appropriate in various application areas [1, 2]. In fact, it is well known that such nets are even equivalent to Turing-machines (w.r.t. their sequential behavior), and thus several decision problems, such as the reachability problem, which are decidable for p/t-nets, are undecidable for nets with inhibitor arcs. Nevertheless, we can show in this paper that the results from [7] can be generalized to p/t-nets with weighted inhibitor arcs (PTI-nets), the most general notion of Petri nets with inhibitor arcs. As developed in [6, 10], for such nets, scenarios can be formally given as labeled stratified order structures (LSOs), a proper generalization of LPOs. Stratified order structures were originally introduced independently in [4] (under the name prossets) and in [5] (under the name composets).

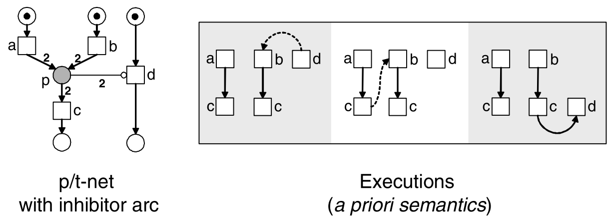


Figure 1. PTI-net with executions.

As an example, Figure 1 shows a p/t-net with inhibitor arc (p, d) having weight 2. This inhibitor arc specifies, that d is only enabled to occur, if p is not marked by more than two tokens. Thus, the enabledness of a transition depends on tests via weighted inhibitor arcs, whether the number of tokens in places does not exceed the weight of the inhibitor arc (beside the classical enabling conditions of p/t-nets). Throughout the paper, we consider the *a priori semantics* of PTI-nets (other semantics are briefly discussed in the conclusion). According to the a priori semantics, the test of markings of places via inhibitor arcs precedes the execution of transitions. In Figure 1, the behavior of the transitions a, b and c is not restricted by inhibitor arcs: In the initial marking, the transitions a and b can be executed concurrently (that means in any order and as well at the same time). Then, transition c can be executed twice and, for example, consumes once the tokens produced by a and once the tokens produced by b. That means, the transitions a and b are executed "earlier than" transition c (respectively). Consider now transition d: Since, after the occurrence of a and b, the place p is marked by four tokens, d cannot be executed concurrently to a and b (since then the occurrence of d is prohibited by the inhibitor arc (p, d)). But d can be executed concurrently to a, if it does not occur later than b (since then the number of tokens in place p does not exceed the inhibitor arc weight 2). In other words, the transitions b and d, when executed concurrently to a (independently from a), cannot occur concurrently or sequentially in order $b \to d$. But they still can occur at the same time (because of the occurrence rule "testing before execution") or sequentially in order $d \to b$ – this is exactly the behavior described by "d not later than b".

The described causal relations between transitions of the net are illustrated by the execution shown most left in Figure 1. The solid arcs represent the "earlier than" relation between events, and the dashed arc depicts the "not later than" relation explained above. There are also other possible executions of the PTI-net from Figure 1: If c is executed once "not later than" b, then the number of tokens in the place p cannot exceed the value 2, and thus d can be executed concurrently to the executions of a, b and (two times) c (see the second execution in Figure 1). Further (with the same argument), it is possible, that c is executed once "earlier than" d (see the third execution in Figure 1).

Of course, also symmetric "not later than" relations are in general possible between events, in which case these events can only occur at the same time, but not sequentially in any order. Such events are called *synchronous*. With partial orders, one can only model "earlier than" relations between events, but it is not possible to describe relationships, where synchronous occurrence is possible, but concurrency is not existent. In [6, 10], causal semantics based on LSOs, like the executions in Figure 1, consisting of a combination of "earlier than" and "not later than" relations between events, were proposed to cover such cases. Thus, we consider scenarios to be formally given through LSOs.

In this paper, we present a polynomial algorithm running in $O(|P|n^4)$ time to test, whether a given LSO is an execution of a given PTI-net, where n is the number of nodes of the LSO, and $|P|$ is the number of places of the PTI-net. Since up to now for PTI-nets there is no formal definition of executions, it is thereto necessary to lift this notion for p/t-nets to the PTI-net level. There are three equivalent characterizations (definitions) of executions of p/t-nets, namely (i) LPOs *enabled* w.r.t. a p/t-net, (ii) LPOs *executable* in a p/t-net, and (iii) LPOs fulfilling the *token flow property* w.r.t. a p/t-net. The first two characterizations do not lead to efficient tests. Their equivalence was shown in [9, 14]. In [7], we introduced the token flow property of LPOs, showed its equivalence to the other two characterizations and developed an efficient algorithm to test, whether a given LPO satisfies the token flow property w.r.t. a given p/t-net.

We propose a definition of *LSOs enabled w.r.t. a PTI-net*, which is a proper generalization of the definition of LPOs enabled w.r.t. a p/t-net and allows the representation of executions with minimal causal dependencies between transition occurrences of PTI-nets (Subsection 3.1). We also define the *token flow property* of LSOs w.r.t. PTI-nets as a generalization of the respective notion for LPOs and p/t-nets (Subsection 3.2) and show the equivalence of these two notions (Subsection 3.3).[1] The polynomial algorithm is then developed from the token flow property (Section 4). It is based on a nontrivial transformation of the token flow property into a flow optimization problem.

In Figure 2, the relationship between the different characterizations of executions is depicted for p/t-nets (left part) and PTI-nets (right part), thus illustrating the theorems shown in this paper in relation to analogue results, known for p/t-nets. We start with the introduction of LSOs and PTI-nets in Section 2.

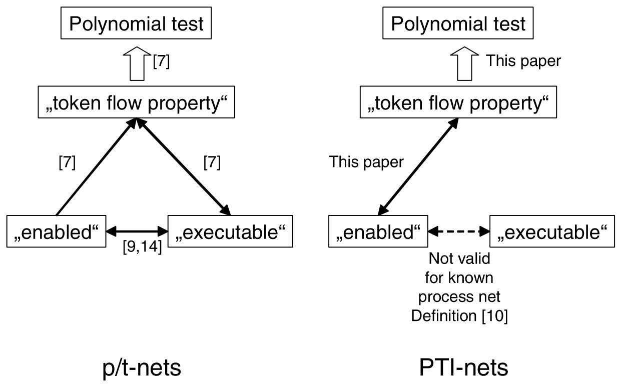


Figure 2. Theorems in this paper.

2 Preliminaries

In this section, we recall the basic definitions of *stratified order structures* and *p/t-nets with weighted inhibitor arcs*.

We use $\mathbb{N}$ to denote the nonnegative integers. Given a function f from A to B and a subset C of A, we write $f|_C$ to denote the restriction of f to the set C. Given a finite set A, the symbol $|A|$ denotes the cardinality of A. The set of all multi-sets over a set A is denoted by $\mathbb{N}^A$. For $m \in \mathbb{N}^A$, $m(a)$ denotes the number of a's in the multi-set m. We write $a \in m$ if $m(a) > 0$. Finally, we denote the identity relation over A by id_A.

2.1 Stratified order structures

We start with some basic notions preparative to the definition of *stratified order structures* (*so-structures*). A *directed graph* is a pair $(V, \to)$, where V is a finite *set of nodes*, and $\to \subseteq V \times V$ is a binary relation over V, called the *set of arcs*. As usual, given a binary relation $\to$, we write $a \to b$ to denote $(a, b) \in \to$. Two nodes $a, b \in V$ are

[1]The definition of *executable* LPOs is strongly related to the definition of process nets of p/t-nets. Since the most general notion of process nets existent for PTI-nets [10] does not define minimal causal dependencies between transition occurrences, there are enabled LSOs, which are not executable, when lifting this notion to the PTI-net level. Therefore, we do not consider executable LSOs here. The exact relationship between the generalizations of the characterizations (i) and (ii) to the PTI-net level is examined in [8].

called *independent* w.r.t. the binary relation $\rightarrow$ if $a \not\rightarrow b$ and $b \not\rightarrow a$. We denote the set of all pairs of nodes independent w.r.t. $\rightarrow$ by $co_{\rightarrow} \subseteq V \times V$. A *partial order* is a directed graph $\mathrm{po} = (V, <)$, where $<$ is an irreflexive and transitive binary relation on V. If $co_{<} = id_V$, then $(V, <)$ is called *total*. Given two partial orders $\mathrm{po}_1 = (V, <_1)$ and $\mathrm{po}_2 = (V, <_2)$, we say that po_2 is a *sequentialization* (or *extension*) of po_1 if $<_1 \subseteq <_2$.

So-structures are, loosely speaking, combinations of two binary relations on a set of events, where one is a partial order representing an "earlier than" relation, and the other represents a "not later than" relation. Thus, so-structures describe finer causalities than partial orders. Formally, so-structures are *relational structures*, satisfying certain properties. A *relational structure* (rel-structure) is a triple $\mathcal{S} = (V, \prec, \sqsubset)$, where V is a set (of *events*), and $\prec \subseteq V \times V$ and $\sqsubset \subseteq V \times V$ are binary relations on V. A rel-structure $\mathcal{S}' = (V, \prec', \sqsubset')$ is an *extension* of another rel-structure $\mathcal{S} = (V, \prec, \sqsubset)$, written $\mathcal{S} \subseteq \mathcal{S}'$, if $\prec \subseteq \prec'$ and $\sqsubset \subseteq \sqsubset'$.

Definition 1 (Stratified order structure [6])**.** *A rel-structure $\mathcal{S} = (V, \prec, \sqsubset)$ is called* stratified order structure *(so-structure), if the following conditions are satisfied for all $u, v, w \in V$:*
(C1) $u \not\sqsubset u$.
(C2) $u \prec v \implies u \sqsubset v$.
(C3) $u \sqsubset v \sqsubset w \wedge u \neq w \implies u \sqsubset w$.
(C4) $u \sqsubset v \prec w \vee u \prec v \sqsubset w \implies u \prec w$.

In figures, $\prec$ is graphically expressed by solid arcs and $\sqsubset$ by dashed arcs. According to (C2), a dashed arc is omitted, if there is already a solid arc. Moreover, we omit arcs, which can be deduced by (C3) and (C4). It is shown in [6], that $(V, \prec)$ is a partial order. Therefore, so-structures are a generalization of partial orders, which turned out to be adequate to model the causal relations between events of PTI-nets under the a priori semantics. In this context, $\prec$ represents the ordinary "earlier than" relation (as for p/t-nets), while $\sqsubset$ models a "not later than" relation. According to [6], for nodes $u, v \in V$, there is an extension $\mathcal{S}' = (V, \prec', \sqsubset')$ of $\mathcal{S}$ with $u \prec' v$ if and only if $v \not\sqsubset u$ and $u \neq v$. In particular, there holds $u \prec v \implies v \not\sqsubset u$. These properties justify the described causal interpretation of $\prec$ and $\sqsubset$. The executions shown in Figure 1 are so-structures with labeled nodes.

We introduce a subclass of so-structures which turns out to be associated to sequences of (synchronous) transition steps of PTI-nets.

Definition 2 (Total linear so-structures)**.** *Let $\mathcal{S} = (V, \prec, \sqsubset)$ be an so-structure, then $\mathcal{S}$ is called* total linear *if $co_{\prec} = (\sqsubset \setminus \prec) \cup id_V$. The set of all* total linear extensions *(or* linearizations*) of an so-structure $\mathcal{S}$ is denoted by $lin(\mathcal{S})$.*

Figure 3 shows four total linear so-structures with labeled nodes (LSOs), where the first and second LSO are extensions of the second execution shown in Figure 1, and the third and fourth LSO are extensions of the first execution shown in Figure 1 (using the results from [6] about augmenting so-structures, one can conclude, that every so-structure is extendable to a total linear so-structure).

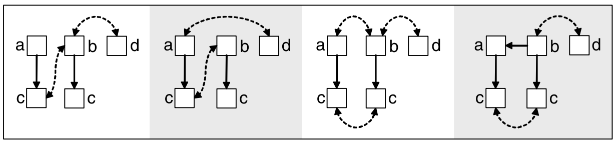


Figure 3. Total linear so-structures.

For the definition of the token flow property for LSOs, we need the notion of prefixes (of nodes) of so-structures. These are defined by subsets of nodes, which are downward closed w.r.t. the $\sqsubset$-relation:

Definition 3 (Prefix (of a node))**.** *Let $\mathcal{S} = (V, \prec, \sqsubset)$ be an so-structure, and let $V' \subseteq V$ be such that $(u' \in V' \wedge u \sqsubset u') \implies u \in V'$. Then $\mathcal{S}' = (V', \prec|_{V' \times V'}, \sqsubset|_{V' \times V'})$ is called* prefix *of $\mathcal{S}$. We say, that the prefix $\mathcal{S}'$ is* defined by *V'. If additionally $(u \prec v \implies u \in V')$ for some $v \in V \setminus V'$, then $\mathcal{S}'$ is called* prefix of *v* (w.r.t. $\mathcal{S}$).

This definition of prefixes is compatible with the set of linearizations of an so-structure in the following sense:

Lemma 4. *Let $\mathcal{S} = (V, \prec, \sqsubset)$ be an so-structure, $V' \subseteq V$ and $v \in V$. Then, V' defines a prefix of v w.r.t. $\mathcal{S}$, if and only if there is a linearization $\mathcal{S}' \in lin(\mathcal{S})$, such that V' defines a prefix of v w.r.t. $\mathcal{S}'$.*

Proof. **if**: Let $\mathcal{S}' = (V, \prec', \sqsubset') \in lin(\mathcal{S})$ and let $V' \subset V$ define a prefix of v w.r.t. $\mathcal{S}'$. Consider nodes $u' \in V'$ and $u \in V$ with $u \sqsubset u'$. Since $\mathcal{S}'$ is an extension of $\mathcal{S}$, this implies $u \sqsubset' u'$. Because V' defines a prefix of $\mathcal{S}'$, we get $u \in V'$. Thus, V' also defines a prefix of $\mathcal{S}$. Let further $v' \prec v$. Again, since $\mathcal{S}'$ is an extension of $\mathcal{S}$, this implies $v' \prec' v$, and therefore we have $v' \in V'$. Finally, if $v \sqsubset v'$, then $v \sqsubset' v'$ and therefore $v' \notin V'$. Thus, V' defines in fact a prefix of v.

only if: Let V' define a prefix of v w.r.t. $\mathcal{S}$. We construct a linearization $\mathcal{S}' = (V, \prec', \sqsubset')$ of $\mathcal{S}$, such that V' also defines a prefix of v w.r.t. $\mathcal{S}'$. For this, denote $V_0 \subseteq V'$ the set of all nodes, which are minimal w.r.t. $\prec$ in $\mathcal{S}$. Then, consider the restriction of $\mathcal{S}$ onto the node set $V \setminus V_0$ and denote $V_1 \subseteq V'$ the set of all nodes, which are minimal w.r.t. $\prec$ in this new so-structure. Following this technique, we define inductively $V_n \subseteq V'$ as the set of nodes, which are minimal w.r.t. the restriction of $\prec$ onto the node set $V \setminus (\bigcup_{i=0}^{n-1} V_i)$, as long as $V' \setminus (\bigcup_{i=0}^{n-1} V_i) \neq \emptyset$. If $V' \setminus (\bigcup_{i=0}^{N-1} V_i) = \emptyset$ and $V \setminus (\bigcup_{i=0}^{N-1} V_i) \neq \emptyset$ for some N, then further define $V_N \subseteq V$ as the set of nodes, which are minimal w.r.t. the restriction of $\prec$ onto the node set $V \setminus (\bigcup_{i=0}^{N-1} V_i)$, and so on (note that $v \in V_N$, because V' defines a prefix of v).

We now can define $\mathcal{S}'$ through $\prec' = \bigcup_{i<j} V_i \times V_j$ and $\sqsubset' = ((\bigcup_i V_i \times V_i) \setminus id_{V_i}) \cup \prec'$. By construction, $\mathcal{S}'$ is a total linear so-structure. It remains to show that $\prec \subseteq \prec'$, $\sqsubset \subseteq \sqsubset'$, $\{v' \in V \mid v' \prec' v\} \subseteq V'$ and $\{v' \in V \mid v = v' \vee v \sqsubset' v'\} \cap V' = \emptyset$.

Let $u, v \in V$ with $u \prec v$: Since V' defines a prefix of $\mathcal{S}$, it is not possible that $v \in V'$ and $u \notin V'$. Suppose $u, v \in V'$, $u, v \in V \setminus V'$ or $u \in V'$ and $v \notin V'$: Then by construction, there must be $i < j$ with $u \in V_i$ and $v \in V_j$. This gives $u \prec' v$.

Let $u, v \in V$ with $u \sqsubset v$: Since V' defines a prefix of $\mathcal{S}$, it is not possible that $v \in V'$ and $u \notin V'$. Suppose $u, v \in V'$ or $u, v \in V \setminus V'$: Let $u \in V_i$ and $v \in V_j$. Assume, that v is minimal w.r.t. $\prec$ in an earlier step than u. Then in this step, there is $u' \prec u$ but $u' \not\prec v$. This contradicts (C4). Therefore either u and v are minimal in the same step or u is minimal in a step earlier than v. This gives $u \sqsubset' v$. Suppose $u \in V'$ and $v \notin V'$: Then by construction, there must be $i < j$ with $u \in V_i$ and $v \in V_j$. This gives $u \prec' v$.

Let $v' \in V$ with $v' \prec' v$: Since by construction $v \in V_N$, there is $n < N$ with $v' \in V_n \subseteq V'$. If finally $v \sqsubset' v'$, then $v' \in V_n$ for some $n \geqslant N$, i.e. $v' \notin V'$. $\square$

We will often use *labeled so-structures* (LSOs) in the following. These are so-structures $\mathcal{S} = (V, \prec, \sqsubset)$ together with a *set of labels* T and a *labeling function* $l : V \to T$. We use the notations defined for so-structures also for LSOs. If T is a set of labels of $\mathcal{S}$, i.e. $l : V \to T$, then for a set $U \subseteq V$, we define the multi-set $|U|_l \subseteq \mathbb{N}^T$ by $|U|_l(t) = |\{v \in V \mid v \in U \wedge l(v) = t\}|$. We will consider LSOs only up to isomorphism. Two LSOs $(V, \prec, \sqsubset, l)$ and $(V', \prec', \sqsubset', l')$ are called *isomorphic*, if there is a bijective mapping $\psi : V \to V'$ such that $l(v) = l'(\psi(v))$ for $v \in V$, $v \prec w \Leftrightarrow \psi(v) \prec' \psi(w)$ and $v \sqsubset w \Leftrightarrow \psi(v) \sqsubset' \psi(w)$ for $v, w \in V$. We will use the same notions for LPOs, too (since an LPO can be understood as an LSO with a "not later than"-relation, that equals the "earlier than"-relation).

2.2 PTI-nets

A *net* is a triple (P, T, F), where P is a finite set of *places*, T is a finite set of *transitions*, satisfying $P \cap T = \emptyset$, and $F \subseteq (P \cup T) \times (T \cup P)$ is a *flow relation*. Let (P, T, F) be a net, and $x \in P \cup T$ be an element. The *pre-set* $\bullet x$ is the set $\{y \in P \cup T \mid (y, x) \in F\}$, and the *post-set* $x\bullet$ is the set $\{y \in P \cup T \mid (x, y) \in F\}$. Given a (multi-) set $X \subseteq P \cup T$, this notation is extended by $\bullet X = \bigcup_{x \in X} \bullet x$ and $X \bullet = \bigcup_{x \in X} x \bullet$. A *place/transition net* (shortly *p/t-net*) N is a quadruple (P, T, F, W), where (P, T, F) is a net, and $W : F \to \mathbb{N}^+$ is a *weight function*. We extend the weight function W to pairs of net elements $(x, y) \in (P \times T) \cup (T \times P)$, satisfying $(x, y) \notin F$, by $W((x, y)) = 0$.

Definition 5 (PTI-net)**.** *A* PTI-net N *is a quadruple* (P, T, F, W, I)*, where* (P, T, F, W) *is a p/t-net, and* $I : P \times T \to \mathbb{N} \cup \{\omega\}$ *is the* weighted inhibitor relation*. If* $I(p, t) \neq \omega$*, then* $(p, t) \in P \times T$ *is called* (weighted) inhibitor arc*, and* p *is an* inhibitor place of t.

In the following, we denote $n < \omega$ for $n \in \mathbb{N}$. A *marking* of a PTI-net $N = (P, T, F, W, I)$ is a function $m : P \to \mathbb{N}$, i.e. a multi-set over P. A *marked PTI-net* is a pair (N, m_0), where N is a PTI-net, and m_0 is a marking of N, called *initial marking*. A transition t can only be executed, if p carries at most $I((p, t))$ tokens. In particular, if $I((p, t)) = 0$, then p must be empty. $I((p, t)) = \omega$ means, that t can never be prevented from occurring by the presence of tokens in p. In diagrams, inhibitor arcs have small circles as arrowheads. Just as normal arcs, inhibitor arcs are annotated with their weights. Now however, the weight 0 is not shown. Figure 1 shows a marked PTI-net.

According to the a priori semantics of PTI-nets, the inhibitor test for enabledness of a transition precedes the consumption and production of tokens in places. Thus, a multi-set (a step) of transitions is (synchronously) enabled in a marking, only if in this marking, each transition in the step obeys the inhibitor constraints before the step is executed.

Definition 6 (Occurrence rule)**.** *Let* $N = (P, T, F, W, I)$ *be a* PTI-net*. A multi-set of transitions* τ *(a step) is* enabled to occur in a marking m w.r.t. the a priori semantics *if* $m(p) \geq \sum_{t \in \tau} \tau(t) W((p, t))$ *for every place* $p \in \bullet\tau$ *and* $m(p) \leq I((p, t))$ *for each place* p *and each transition* $t \in \tau$.

The *occurrence* of a (possibly empty) step of transitions τ leads to the new marking m', defined by $m'(p) = m(p) - \sum_{t \in \tau} \tau(t)(W((p, t)) - W((t, p)))$ for every $p \in P$. We write $m \xrightarrow{\tau} m'$ to denote, that τ is enabled to occur in m, and that its occurrence leads to m'. A finite sequence of steps $\sigma = \tau_1 \dots \tau_n,\ n \in \mathbb{N}$, is called a *step occurrence sequence enabled in a marking* m *and leading to* m_n, denoted by $m \xrightarrow{\sigma} m_n$, if there exists a sequence of markings $m_1, \dots, m_n$ such that $m \xrightarrow{\tau_1} m_1 \xrightarrow{\tau_2} \dots \xrightarrow{\tau_n} m_n$. The marking m_n is *reachable from the marking* m.

A step occurrence sequence can be understood as a possible single *observation* of the behavior of a PTI-net, where the occurrences of transitions in one step are observed *at the same time* or *synchronously*.

We will use the same notions for (marked) p/t-nets, too (since a p/t-net can be understood as a PTI-net with an inhibitor relation, which equals the constant ω).

3 Executions

In this Section, we lift the notions of "enabled LPOs" and "token flow property", known for LPOs w.r.t. p/t-nets, to the setting of PTI-nets w.r.t. the a priori semantics.

3.1 Enabled LSOs

We now introduce LSOs to model executions of PTI-nets. For this, the two relations of an LSO are interpreted as "earlier than" resp. "not later than" relations between transition occurrences. If two transition occurrences are in not later than relation, that means, they can be observed (are allowed to be executed) synchronously or sequentially in one specific order. If two transitions are neither in earlier than relation nor in not later than relation, they are concurrent and can be observed (are allowed to be executed) synchronously or sequentially in any order. In this sense one LSO "allows" many observations (step sequences). If all these observations are enabled step occurrence sequences, this LSO is called *enabled*.

Formally, the set of step sequences "allowed" by an LSO is defined as the set of step sequences extending (sequentializing) the LSO, where a step sequence can be easily interpreted itself as an LSO: Each step is represented by a set of events labeled by transitions (transition occurrences), which are in not later than relation with each other (representing synchronous transition occurrences), and transition occurrences in different steps are ordered in appropriate earlier than relation. Formally, for a sequence of transition steps $\sigma = \tau_1 \dots \tau_n$, define the LSO $\mathcal{S}_\sigma = (V, \prec, \sqsubset, l)$ *underlying* σ by: $V = \bigcup_{i=1}^n V_i$ and $l : V \to T$ with $|V_i|_l(t) = \tau_i(t)$, $\prec = \bigcup_{i<j} V_i \times V_j$ and $\sqsubset = ((\bigcup_i V_i \times V_i) \cup \prec) \setminus id_V$.

It is easy to see that LSOs underlying a sequence of transition steps are total linear. This is because $co_\prec = \bigcup_{i=1}^n V_i \times V_i$ (see Definition 2). For example, the LSOs shown in Figure 3 are associated to the sequences of transition steps (from left to right): $a(b+c+d)c$, $(a+d)(b+c)c$, $(a+b+d)(2c)$ and $(b+d)a(2c)$. Of course, also the reverse is valid, i.e. for each total linear LSO $\mathcal{S} = (V, \prec, \sqsubset, l)$, there is a step sequence σ, such that $\mathcal{S}$ and $\mathcal{S}_\sigma$ are isomorphic. Thus, for LSOs, which have transition names as labels, we can identify total linear LSOs with sequences of transition steps.

Definition 7 (enabled LSO)**.** *Let* (N, m_0)*,* $N = (P, T, F, W, I)$*, be a marked PTI-net.*

An LSO $\mathcal{S} = (V, \prec, \sqsubset, l)$ *with* $l : V \to T$ *is called* enabled (to occur) w.r.t. (N, m_0) (in the a priori semantics)*, if the following statement holds: Each finite step sequence* $\sigma = \tau_1 \dots \tau_n$ *with* $\mathcal{S}_\sigma \in lin(\mathcal{S})$ *is a step occurrence sequence of* (N, m_0)*. We say, that the occurrence of* $\mathcal{S}$ leads *to the marking* $m'(p)$*, given by* $m'(p) = m(p) + \sum_{v \in V}(W((l(v), p)) - W((p, l(v))))$.

This definition is consistent with and a proper generalization of the notion of enabled LPOs: An LPO $\mathrm{lpo} = (V, \prec, l)$ with $l : V \to T$ is *enabled to occur* w.r.t. a marked p/t-net (P, T, F, W, m_0), if each step sequence, which extends (sequentializes) lpo, is a step occurrence sequence enabled in m_0. Since in LPOs, concurrent and synchronous transition occurrences are not distinguished, here a step is considered as a set of events labeled by transitions (transition occurrences), which are concurrent.[2]

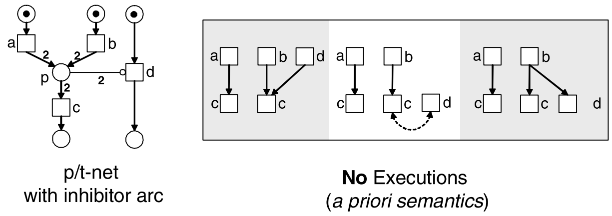


Figure 4. LSOs, not enabled.

It is easy to check, that the LSOs from Figure 1 are indeed enabled LSOs w.r.t. the shown PTI-net. As a further example, consider the three LSOs shown in Figure 4: In all three cases, the step sequence $(a + b)(c + d)c$ is a linearization of the LSO ($\mathcal{S}_{(a+b)(c+d)c} \in lin(\mathcal{S})$), but it is not an enabled step occurrence sequence. This is because after the execution of $(a + b)$, the place p carries four tokens, disabling the following step $(c + d)$. Therefore, all three LSOs are not enabled w.r.t. the shown marked PTI-net.

Observe, that there is no efficient test of Definition 7, since the number of step sequences σ with $\mathcal{S}_\sigma \in lin(\mathcal{S})$ may be exponential in the number of nodes of$\mathcal{S}$.

3.2 Token flow property

In this subsection, we extend the notions of token flow function and token flow property, known for LPOs and p/t-nets, to the setting of PTI-nets. In [7], it is shown, that LPOs are enabled, if and only if they fulfill the token flow property w.r.t. a p/t-net. Our aim is to show the same for LSOs and PTI-nets.

Fix a marked PTI-net (N, m_0), $N = (P, T, F, W, I)$, a place p of N and an LSO $\mathcal{S} = (V, \prec, \sqsubset, l)$ with $l : V \to T$. Assume, that $\mathcal{S}$ is enabled to occur w.r.t. (N, m_0) in the a priori semantics. Since the inhibitor relation I of (N, m_0) restricts the behavior of the underlying p/t-net $(N', m_0) = (P, T, F, W, m_0)$, $\mathcal{S}$ is then also enabled w.r.t. (N', m_0). In a p/t-net, transitions, which can be executed as one step, also can be executed in arbitrary order. Therefore, also the LPO $\mathrm{lpo}_\mathcal{S} = (V, \prec, l)$, underlying $\mathcal{S}$, is enabled w.r.t. the p/t-net (N', m_0). Altogether, we get, that the enabledness of the LPO underlying $\mathcal{S}$ w.r.t. the p/t-net underlying (N, m_0) is a necessary condition for the enabledness of $\mathcal{S}$ w.r.t. (N, m_0).

Lemma 8. *Let* $\mathcal{S} = (V, \prec, \sqsubset, l)$ *be an LSO enabled w.r.t. a marked PTI-net* (N, m_0)*,* $N = (P, T, F, W, I)$*, according to the a priori semantics. Then, the LPO* $(V, \prec, l)$ *is enabled w.r.t. the marked p/t-net* (P, T, F, W, m_0)*.*

[2] A discussion of further general semantical arguments, justifying this definition, can be found in [8].

That means, the token flow property for $\mathcal{S}$ w.r.t. (N, m_0) should include the token flow property for $\mathrm{lpo}_{\mathcal{S}}$ w.r.t. (N', m_0). The token flow property for LPOs w.r.t. p/t-nets is based on the notion of *token flow functions*. For every place p, a token flow function x_p assigns non-negative integers to the edges of an LPO. The value $x_p((v, v'))$ of an edge (v, v') is interpreted as the number of tokens, which are produced by the transition $l(v)$ and consumed by the transition $l(v')$ in the place p. By this construction, we still cannot specify the number of tokens, which are consumed by a transition from the initial marking of a place, and the number of tokens, which are produced by some transition in a place p, but not consumed by further transitions (and thus remain in the final marking). Therefore, we extend a considered LPO by an *initial event*, which is interpreted as the occurrence of a transition, producing the initial marking, and a *final event*, which is interpreted as the occurrence of a transition, consuming the final marking.

Definition 9 ($\star$-extension of LPOs). *Let* $\mathrm{lpo} = (V, \prec, l)$ *be an LPO. Then, an LPO* $\mathrm{lpo}^\star = (V^\star, \prec^\star, l^\star)$*, where* $V^\star = (V \cup \{v_0, v_{\max}\})$*,* $v_0, v_{\max} \notin V$*,* $\prec^\star = \prec \cup (\{v_0\} \times V) \cup (V \times \{v_{\max}\}) \cup \{(v_0, v_{\max})\}$*,* $l^\star(v_0), l^\star(v_{\max}) \notin l(V)$*,* $l^\star(v_0) \neq l^\star(v_{\max})$ *and* $l^\star|_V = l$*, is called* $\star$-extension of lpo.

By assigning natural numbers to the arcs of a $\star$-extension of an LPO, we define a so called token flow function x of this LPO (with v_0 as its only smallest element and $v_{\max}$ as its only maximal element). It is clear, that equally (with the same transition) labeled events should produce and consume the same overall number of tokens in a place. The overall number of tokens produced by an event v of $\mathrm{lpo}^\star = (V^\star, \prec^\star, l^\star)$ is called the *outtoken flow of* v *w.r.t.* x and is denoted and defined by $Out(v, x) = \sum_{v <^\star w} x((v, w))$. The overall number of tokens consumed by an event v of $\mathrm{lpo}^\star$ is called the *intoken flow of* v *w.r.t.* x and is denoted and defined by $In(v, x) = \sum_{w \prec^\star v} x((w, v))$.

Definition 10 (Token flow function of LPOs). *Let* $\mathrm{lpo} = (V, \prec, l)$ *be an LPO, and* $\mathrm{lpo}^\star = (V^\star, \prec^\star, l^\star)$ *be a* $\star$*-extension of* lpo*. A function* $x : \prec^\star \to \mathbb{N}$ *is called* token flow function *of* lpo*, if equally labeled nodes have equal intoken and outtoken flow* $(*)\ \forall v, w \in V : \ l(v) = l(w) \implies (In(v, x) = In(w, x) \wedge Out(v, x) = Out(w, x))$.

An LPO $\mathrm{lpo} = (V, \prec, l)$ satisfies the *token flow property w.r.t. a marked p/t-net*, if for each place p of this net, there is a token flow function x_p compatible with p in the sense, that its intoken and outtoken flows respect the weight function and the initial marking of the net as follows:

Definition 11 (Token flow property of LPOs). *Let* (N', m_0)*,* $N' = (P, T, F, W)$*, be a marked p/t-net, and let* $\mathrm{lpo} = (V, \prec, l)$ *be an LPO with* $l(V) = T$*, and let* $lpo^\star = (V^\star, \prec^\star, l^\star)$ *be a* $\star$*-extension of* lpo*. Denote* $W((l(v_0), p)) = m_0(p)$ *for each place* p*. We say, that* lpo *fulfills the* token flow property w.r.t. (N, m_0)*, if the following statement holds: For every place* $p \in P$*, there exists a token flow function* $x_p : \prec^\star \to \mathbb{N}$ *such that*
(IN) $\forall v \in V : \ In(v, x_p) = W((p, l(v)))$
(OUT) $\forall v \in V \cup \{v_0\} : \ Out(v, x_p) = W((l(v), p))$.

In [7], we showed that an LPO fulfills the token flow property w.r.t. a marked p/t-net, if and only if it is enabled w.r.t. this net.[3]

We now come back to the consideration of $\mathcal{S}$. Since the "not later than"-relation of $\mathcal{S}$ does not describe the flow of tokens (since token flow always produces an earlier than relation between events), the token flow of $\mathcal{S}$ w.r.t. a place can be given by a token flow function of $\mathrm{lpo}_{\mathcal{S}}$. Clearly (as argued above), for each place, there must be a token flow function, satisfying (IN) and (OUT), if $\mathcal{S}$ is enabled.

The other way round the existence of such token flow functions x_p, satisfying (IN) and (OUT), is not enough to ensure that $\mathcal{S}$ is enabled. This is because the execution of a prefix of $\mathcal{S}$ still might produce too many tokens in a place p (according to x_p), disabling a subsequent transition, which tests this place via an inhibitor arc. Thus, we require, that token flow functions fulfill an additional property. This property should only allow token flow functions x_p, according to which for each event, the execution of one of its prefixes does not put too many tokens into p. In other words, each marking, which is reachable through the execution of a prefix of some event, should respect the inhibitor relations of the corresponding transition to all places.

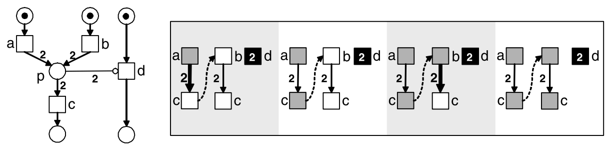
Assume, that we have given a token flow function x_p on the edges of $\mathrm{lpo}^\star_{\mathcal{S}}$, satisfying (IN) and (OUT) for some place p. We compute the number of tokens in this place after the execution of some prefix of $\mathcal{S}$. Let the prefix be defined by the set of events V'. On the one hand, by construction, the values of x_p on edges between events in V' correspond to tokens, which are produced and consumed by events in this prefix. On the other hand, the values of x_p on edges from events in V' to events in $V \setminus V'$ corresponds to tokens, which are produced by events in V' and remain in p after the execution of the prefix. Thus, the marking of the place after the execution of the prefix is given by the sum of the values of x_p on such edges. We define this sum for arbitrary token flow functions x and call its value the *final marking* of the prefix w.r.t. x. Formally, the initial event of $\mathrm{lpo}^\star_{\mathcal{S}}$ belongs to each prefix.

Definition 12 (Final marking). *Let* $\mathcal{S} = (V, \prec, \sqsubset, l)$ *be an LSO, and let* $\mathcal{S}' = (V', \prec', \sqsubset', l')$ *be a prefix of* $\mathcal{S}$*. Let further* $x : V^\star \to \mathbb{N}$ *be a token flow function of* $(V, \prec, l)$*, and*

[3] In particular, we showed, that a token flow function, satisfying (IN) and (OUT) w.r.t. a place, abstracts from the individuality of conditions of a process of the net and encodes the flow relation of this process by natural numbers.

let v_0 be the initial event of $(V^\star, \prec^\star, l^\star)$. The final marking of $\mathcal{S}'$ (w.r.t. x) *is denoted and defined by* $m_{\mathcal{S}'}(x) = \sum_{u \in V',\, v \notin V',\, u \prec^\star v} x((u,v)) + \sum_{v \notin V'} x((v_0, v))$.

We now state the token flow property for LSOs w.r.t. PTI-nets: For each event, the final marking of every of its prefixes must not exceed the weight of the inhibitor arc between the corresponding transition and place.



p/t-net with inhibitor arc

An execution (*a priori semantics*)

Figure 5. LSO with token flow function.

Definition 13 (Token flow property of LSOs)**.** *Let $\mathcal{S} = (V, \prec, \sqsubset, l)$ be an LSO. An LSO $\mathcal{S}^\star = (V^\star, \prec^\star, \sqsubset^\star, l^\star)$, where $(V^\star, \prec^\star, l^\star)$ is a $\star$-extension of the LPO $(V, \prec, l)$ and $\sqsubset^\star = \sqsubset \cup \prec^\star$, is called* $\star$-extension of $\mathcal{S}$*. A function $x : \prec^\star \to \mathbb{N}$ is called* token flow function of $\mathcal{S}$*, if it is a token flow function of $(V, \prec, l)$.*

Let further (N, m_0), $N = (P, T, F, W, I)$, be a marked PTI-net and let $l(V) = T$. We say, that $\mathcal{S}$ fulfills the token flow property w.r.t. (N, m_0)*, if the following statement holds: For every place $p \in P$, there exists a token flow function $x_p : \prec^\star \to \mathbb{N}$, satisfying (IN), (OUT) and*
(FIN) For all nodes $v \in V$ and all prefixes $\mathcal{S}'$ of v, there holds: $m_{\mathcal{S}'}(x_p) \leqslant I((p, l(v)))$.

Figure 5 shows one of the executions from Figure 1 (four times) with annotated token flow function x_p w.r.t. the place p. Here, we omitted to draw the initial and maximal event and corresponding arcs, because there are no tokens in p in the initial marking and in the final marking after the execution of the LSO. The node labeled by a has intoken flow $0 = W((p,a))$ and outtoken flow $2 = W((a,p))$. The same holds for the node labeled by b. The nodes labeled by c have intoken flow $2 = W((p,c))$ and outtoken flow $0 = W((c,p))$. Finally, the node labeled by d has intoken flow $0 = W((p,d))$ and outtoken flow $0 = W((d,p))$. Therefore, x_p satisfies (IN) and (OUT) w.r.t. p. To examine condition (FIN), we must only consider the node labeled by d (node filled by black color): The execution is depicted four times, showing all prefixes (nodes filled by grey color) of this node. The arcs, which count for the final marking of a prefix, are highlighted: The first and third prefix have a final marking of $2 \leqslant 2 = I((p,d))$, the second and fourth prefix have a final marking of $0 \leqslant 2 = I((p,d))$. Thus, also (FIN) is satisfied. The maximum over all final markings of prefixes of the d-labeled node is displayed inside this node.

Observe that also the definition of the token flow property is inherent exponential in the size of the LSO, since it involves in general exponentially many prefixes of the LSO (condition (FIN)). Nonetheless, as will be explained in Section 4, the test of condition (FIN) can be transformed into a flow optimization problem, which can be solved in polynomial time.

3.3 Enabledness vs. token flow property

In this Subsection, we will prove the first main result of this paper given by the following Theorem. In the subsequent Subsection, we will finally present a polynomial test of the token flow property as the second main result.

Theorem 14. *Let (N, m_0), $N = (P, T, F, W, I)$, be a marked PTI-net, and let $\mathcal{S} = (V, \prec, \sqsubset, l)$ be an LSO with $l(V) = T$. Then, $\mathcal{S}$ is enabled w.r.t. (N, m_0), if and only if it fulfills the token flow property w.r.t. (N, m_0).*

Proof. **only if**: Let $\mathcal{S}$ be enabled w.r.t. (N, m_0). Then, by Lemma 8, $(V, \prec, l)$ is enabled w.r.t. (P, T, F, W, m_0), that means for each $p \in P$, there is a token flow function $x_p : \prec^\star \to \mathbb{N}$ of $(V, \prec, l)$, satisfying (IN) and (OUT).

We claim, that each x_p also fulfills (FIN). For this, let $v \in V$ and $\mathcal{S}'$ be a prefix of v, defined by V'. By Lemma 4, there is a linearization $\mathcal{S}_{lin}$ of $\mathcal{S}$, such that V' also defines a prefix $\mathcal{S}'_{lin}$ of v w.r.t. $\mathcal{S}_{lin}$. There is a step occurrence sequence $\sigma = \tau_1 \ldots \tau_n$ of (N, m_0), whose underlying LSO $\mathcal{S}_\sigma$ equals $\mathcal{S}_{lin}$. Since prefixes are downward $\sqsubset$-closed, a prefix $\sigma' = \tau_1 \ldots \tau_m$ $(m < n)$ of σ with $l(v) \in \tau_{m+1}$ must exist, which corresponds to $\mathcal{S}'_{lin}$. In other words, the LSO $\mathcal{S}_{\sigma'}$, underlying σ', equals $\mathcal{S}'_{lin}$. It is enough to show now that $m(p) = m_{\mathcal{S}'}(x_p)$ for the marking m, reached after the execution of σ', since $m(p) \leq I((p,t))$ for each place p and each transition $t \in \tau_{m+1}$ by Definition 6.

We finally compute: $m(p) = m_0(p) - \sum_{i=1}^{m} \sum_{t \in \tau_i} \tau_i(t)(W((p,t)) - W((t,p))) = Out(v_0, x_p) - \sum_{v \in V'}(In(v, x_p) - Out(v, x_p)) = \sum_{v \in V' \cup \{v_0\}}(\sum_{v \prec^\star w} x_p((v,w)) - \sum_{w \prec^\star v} x_p((w,v))) = m_{\mathcal{S}'}(x_p)$, since the values on edges within V' cancel each other out.

if: Let $\mathcal{S}$ fulfill the token flow property w.r.t. (N, m_0), and let x_p be a token flow function satisfying (IN), (OUT) and (FIN) w.r.t. the place p. Consider a sequence of transition steps $\sigma = \tau_1 \ldots \tau_n$, whose underlying LSO $\mathcal{S}_\sigma$ is a linearization of $\mathcal{S}$. We have to show, that σ is a step occurrence sequence of (N, m_0). For this, we show inductively, that, if $\sigma_k = \tau_1 \ldots \tau_k$ is a step occurrence sequence, then τ_{k+1} is a transition step, enabled in the marking m, reached after the execution of σ_k for $0 \leqslant k \leqslant n-1$.

First observe, that σ is a step occurrence sequence of (P, T, F, W, m_0), since $(V, \prec, l)$ satisfies the token flow property on the p/t-net level, and the *LPO* underlying σ clearly sequentializes $(V, \prec, l)$. That means, the first condition of Definition 6, that $m(p) \geq$

$\sum_{t\in\tau_{k+1}} \tau_{k+1}(t)W((p,t))$ for every place $p \in \bullet\tau_{k+1}$, is always satisfied. We still have to verify the condition of Definition 6, that $m(p) \le I((p,t))$ for each place p and each transition $t \in \tau_{k+1}$. If $\mathcal{S}_{\sigma_k}$ is the LSO underlying σ_k, then $\mathcal{S}_{\sigma_k}$ is a prefix of $\mathcal{S}_\sigma$. Denoting $\mathcal{S}_{\sigma_k} = (V_k, \prec_k, \sqsubset_k, l_k)$, by Lemma 4, V_k also defines a prefix $\mathcal{S}_k$ of $\mathcal{S}$. Fix $t \in \tau_{k+1}$ and $p \in P$ and let $v \in V$ with $l(v) = t$, such that $\mathcal{S}_{\sigma_k}$ is a prefix of v. Then, also (Lemma 4) $\mathcal{S}_k$ is a prefix of v. It is enough to show now, that $m(p) = m_{\mathcal{S}_k}(x_p)$, since $m_{\mathcal{S}_k}(x_p) \leqslant I((p,l(v))$ by (FIN). The necessary computation is as above. □

4 Testing the token flow property

In this section, we give a polynomial algorithm to test, whether an LSO $\mathcal{S} = (V, \prec, \sqsubset, l)$ with $l(V) = T$ fulfills the token flow property w.r.t. a marked PTI-net (N, m_0). In the case, that $\mathcal{S}$ fulfills the flow property, the algorithm constructs respective token flow functions for every place, satisfying (IN), (OUT), and (FIN).

From [7], we have a polynomial test, whether for each place, there is a token flow function, satisfying (IN) and (OUT). If such token flow functions do not exist, then clearly, the LSO does not fulfill the token flow property. In the positive case, the algorithm from [7] generates such token flow functions. We claim, that either these token flow functions also satisfy (FIN), or the LSO does not fulfill the token flow property (i.e. there are no such token flow functions). This observation is based on the following lemma, stating that the final marking of a prefix w.r.t. a token flow function x_p, satisfying (IN) and (OUT) for p, only depends on the initial making $m_0(p)$ and the arc weights $W((p,t))$ and $W((t,p))$ for $t \in T$, but not on the concrete distribution of the token flow given by x_p. This follows directly from the fact, that the final marking can be computed as the marking reached after the execution of the prefix (the corresponding computation can be found in the proof of Theorem 14).

Lemma 15. *Let p be a place, and let x_p be a token flow function, satisfying (IN) and (OUT). Then, it holds for each prefix $\mathcal{S}' = (V', \prec', \sqsubset', l')$ of $\mathcal{S}$: $m_{\mathcal{S}'}(x_p) = m_0(p) + \sum_{t\in T} |V'|_l(t)(W((t,p)) - W((p,t)))$.*

Thus, for different token flow functions x_p and x'_p, satisfying (IN) and (OUT) for a place p, the values $m_{\mathcal{S}'}(x_p)$ and $m_{\mathcal{S}'}(x'_p)$ coincide, and thus either both fulfill (FIN), or both do not fulfill (FIN). It remains to test property (FIN) for the computed token flow functions x_p satisfying (IN) and (OUT). For this, it is enough to compute for each node v the maximum of the values $m_{\mathcal{S}'}(x_p)$ over all prefixes $\mathcal{S}'$ of v and to compare this maximum with the value $I((p, l(v)))$.

Definition 16 (Inhibitor value). *The* inhibitor value $Inh(v, x)$ *of an event v w.r.t. a token flow function x is defined by $Inh(v, x) = \max\{m_{\mathcal{S}'}(x) \mid \mathcal{S}'$ is prefix of $v\}$.*

A straightforward way to compute the inhibitor value of some node v is to enumerate all prefixes of this node and compute the final markings of all these prefixes according to Lemma 15. Unfortunately, this is not efficient, since there may be exponentially many prefixes in the number of nodes.

Another possible formalization of the problem is as follows: The final marking of a prefix is defined as the sum over the values of the token flow function on edges leaving the prefix. These edges separate the node set of the prefix from the subsequent nodes. Formally, this separation can be seen as a cut through $\mathcal{S}$ (resp. $\mathrm{lpo}_{\mathcal{S}}$), partitioning the set of nodes of $\mathcal{S}$ into two node sets. Such cuts are considered in flow theory, and to avoid confusion, we use the term *flow cuts* for this kind of cuts from now on. In flow theory, one searches for maximal or minimal flows through flow networks with upper and/or lower capacities on edges. Thereto, one considers capacities of flow cuts. Interpreting $\mathrm{lpo}_{\mathcal{S}}$ as a flow network and the values of the token flow function as lower capacity bounds for flows through this network, the final marking of a prefix is given as the capacity of some flow cut, and the inhibitor value of some node can be seen as the maximum capacity of flow cuts of the network. This maximum then can be computed efficiently through its correspondence to minimal flows.

Definition 17 (Flow network, flow, flow cut, capacity). *A* flow network (with lower capacities) *is a tuple (G, c, s, t), where $G = (W, E)$ is a directed graph, $c : E \to \mathbb{N}_0$ is a capacity function, $s \in W$ is a node with $\forall v \in W : (v, s) \notin E$, called* source, *and $t \in W$ is a node with $\forall v \in W : (t, v) \notin E$, called* sink.

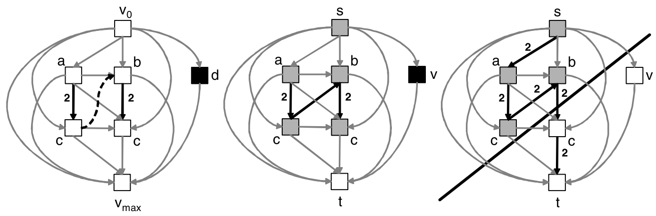
The capacity is interpreted as a lower bound for flows, that means a flow *is a function $f : E \to \mathbb{N}_0$ such that (a) $\forall (v, v') \in E : f((v, v')) \geqslant c((v, v'))$ and (b) $\forall v \in W \setminus \{s, t\} : \sum_{(w,v)\in E} f((w, v)) = \sum_{(v,w)\in E} f((v, w))$. The* value *$|f|$ of a flow f is defined as the outgoing flow of the source (or equivalently the ingoing flow of the sink) $\sum_{(s,v)\in E} f((s, v))$. A* minimal flow *is the flow with minimal value among all flows.*

A flow cut *is a tuple $(S, T) \subseteq W \times W$ such that $s \in S$, $t \in T$, $S \cap T = \emptyset$ and $S \cup T = W$. The capacity of a flow cut is defined by $c((S, T)) = \sum_{v\in S,\, w\in T,\, (v,w)\in E} c((v, w))$ if $(T \times S) \cap E = \emptyset$ and $c((S, T)) = 0$ else.*

In the following, we describe, how the inhibitor value of a node v can be interpreted as the maximal capacity of some flow cut in a flow network. For this, we interpret, loosely speaking, $\mathcal{S}$ as a flow network. We first omit the "not later than"-relation: We can glue events of $\mathcal{S}$, which are in a symmetric "not later than"-relation. If $u \sqsubset v$ but $v \not\sqsubset u$, then there might be prefixes containing u but not v, and there might be prefixes, which contain or do not contain both events u and v together. Since the same holds if $u \prec v$, we replace remaining "not later"-than relations by "earlier than”- relations. We do not want to consider all flow cuts of this flow network, but only those, corresponding to prefixes of v. Therefore, we only define (lower) capacity constraints on edges leaving a prefix of v.

Definition 18 (Associated flow network)**.** *Let $\mathcal{S} = (V, \prec, \sqsubset, l)$ be an LSO, $v \in V$, $\mathcal{S}^\star = (V^\star, \prec^\star, \sqsubset^\star, l^\star)$ be a $\star$-extension of $\mathcal{S}$ with initial event v_0 and maximal event $v_{\max}$, and x be a token flow function of $\mathcal{S}$. Let further U be the set of all the nodes occurring in the prefixes of v. Define the* flow network (G, c, s, t), $G = (W, E)$, associated to x and v, *by*

- *For $u \in V^\star$, denote $[u] = [u]_\sqsubset = \{w \in V^\star \mid w = u \vee (w \sqsubset^\star u \wedge u \sqsubset^\star w)\}$. Define $W = \{[u] \mid u \in V^\star\}$, $s = [v_0]$ ($= \{v_0\}$) and $t = [v_{\max}]$ ($= \{v_{\max}\}$).*
- *Set $E = \{([u], [w]) \mid u \sqsubset^\star w\}$.*
- *Set $c(([u], [w])) = \sum_{u' \in [u],\, w' \in [w],\, u' \prec^\star w'} x((u', w'))$ if $u \in U \wedge w \not\prec v$ and $c(([u], [w])) = 0$ else.*



Execution with token flow function | Associated flow network | Minimal flow and maximal cut

Figure 6. Associated flow network.

Observe, that the associated flow network is well-defined, that means for $u' \in [u]$ and $w' \in [w]$, we have $u \sqsubset^\star w \Longrightarrow u' \sqsubset^\star w'$ and $c(([u], [w])) = c(([u'], [w']))$. Figure 6 shows the flow network associated the ($\star$-extension of the) LSO with token flow function from Figure 5, the associated flow network (the nodes in U are filled with grey color), a flow cut (S, T) with maximal capacity (which equal 2) through this flow network (the nodes in S and T are filled with different colors) and a minimal flow in this flow network. Capacity and flow values which equal 0 are not shown. The following lemma states, that the final marking of prefixes can be computed by capacities of flow cuts in the associated flow network.

Lemma 19. *Let $\mathcal{S}' = (V', \prec', \sqsubset', l')$ be a prefix of a node v. Let further x be a token flow function of $\mathcal{S}$, and (G, c, s, t), $G = (W, E)$, be the flow network associated to x and v. Denote $S = \{[v] \mid v = v_0 \vee v \in V'\}$ and $T = W \setminus S$. Then $m_{\mathcal{S}'}(x) = c((S, T))$.*

Proof. Since $V' \subseteq U$ for the set U of all the nodes, occurring in the prefixes of v, we have for each $u \in V' \cup \{v_0\}$ and $w \notin V' \cup \{v_0\}$ that $c(([u], [w])) = \sum_{u' \in [u],\, w' \in [w],\, u' \prec^\star w'} x((u', w'))$. The statement is now an easy computation. Just observe, that $(T \times S) \cap E = \emptyset$ since $w \not\sqsubset^\star u$ for $[u] \in S$, $[w] \in T$. $\square$

Since flow cuts, which do not correspond to prefixes of v, do not have bigger capacities than flow cuts, corresponding to such prefixes, we get:

Theorem 20. *Let v be a node, and x be a token flow function of an LSO $\mathcal{S}$. Let further (G, c, s, t), $G = (W, E)$, be the flow network associated to x and v. Then $Inh(v, x) = \max\{c((S, T)) \mid (S, T)$ flow cut of $(G, c, s, t)\}$.*

Proof. Let (S, T) be a flow cut of (G, c, s, t), which does not correspond to a prefix of v, in the sense that $S \neq \{[u] \mid u = v_0 \vee u \in V'\}$ for each prefix $\mathcal{S}' = (V', \prec', \sqsubset', l')$ of v.

We first claim, that if (S, T) does not correspond to a prefix of $\mathcal{S} = (V, \prec, \sqsubset, l)$, then $c((S, T)) = 0$, since there is $[u] \in S$ and $[w] \in T$ with $([w], [u]) \in E$. Indeed, in this case $V_S = \bigcup_{[u] \in S \setminus [v_0]} [u]$ does not define a prefix of $\mathcal{S}$. That means, that there is $u \in V_S$ and $w \notin V_S$ with $w \sqsubset u$. By the definition of V_S, it is not possible, that also $u \sqsubset w$ (because then $[w] = [u]$). Therefore, by the definition of E, we get $([w], [u]) \in E$.

Let finally $\mathcal{S}' = (V', \prec', \sqsubset', l')$ be a prefix of $\mathcal{S} = (V, \prec, \sqsubset, l)$, but not a prefix of v. Let further (S, T) correspond to $\mathcal{S}'$. We claim, that then there is a prefix $\mathcal{S}'' = (V'', \prec'', \sqsubset'', l'')$ of v such that $c((S, T)) \leqslant c((S'', T''))$ for the flow cut $S'' = \{[u] \mid u = v_0 \vee u \in V''\}$ and $T'' = W \setminus S''$.

Observe, that the intersection and the union of two node sets, defining two prefixes, always defines a prefix again. This implies, that there is a maximal prefix of v, which is defined exactly by the set U of all the nodes, occurring in the prefixes of v, and that there is also a minimal prefix of v, defined by the set $U' = \{u \in V \mid u \prec v\}$.

In particular, the intersection $V'' = V' \cap U$ defines a prefix $\mathcal{S}''$. Let $S'' = \{[v] \mid v = v_0 \vee v \in V''\}$ and $T'' = W \setminus S''$ be the corresponding flow cut. Then clearly $c((S, T)) \leqslant c((S'', T''))$, since $c(([u], [w])) = 0$ if $u \notin U$, and there may be edges $([u], [w]) \in E$ with $u \in V''$ and $w \in V' \setminus V''$, which only count in the second case. Thus, if $\mathcal{S}''$ is a prefix of v, we are done. Assume, that $\mathcal{S}''$ is not a prefix of v. Then, $V''' = V'' \cup U'$ defines a prefix of v and $c((S'', T'')) \leqslant c((S''', T'''))$ for $S''' = \{[v] \mid v = v_0 \vee v \in V'''\}$ and $T''' = W \setminus S'''$ (since $c(([u], [w])) = 0$ if $w \prec v$, and there may be edges $([u], [w]) \in E$ with $u \in V''' \setminus V''$, which only count in the second case). $\square$

Thus, inhibitor values can be computed through the maximal capacity of a flow cut in the associated flow network. This maximal capacity equals the minimal flow through this network. The proof for this statement is analogous to the proof of the better known *maximal flow equals minimal cut* theorem of Ford/Fulkerson [3] in flow networks with upper capacities (see the technical report [11] for details). As for maximal flows in flow networks with upper capacities [3, 12], there are polynomial algorithms to compute minimal flows in flow networks with lower capacities running in $O(n^3)$ time, where n is the number of nodes of the flow network resp. the given LSO (an explanation of the main arguments can be found in the technical report [11]).

If p is a place, for which there is a token flow function of the given LSO satisfying (IN) and (OUT) (which can be computed in $O(|P|n^4)$ time), then the inhibitor value w.r.t. this token flow function must be computed for each node of the LSO. Thus, the polynomial test of the token flow property takes $O(|P|n^4)$ time and looks formally as follows:

$test \leftarrow true$
for all $(p \in P)$ **do**
  if $((V, \prec, l)$ does not fulfill token flow property w.r.t. (P, T, F, W, m_0) and $p)$ **then**
    $test \leftarrow false$
  else
    $x_p \leftarrow$ token flow function of $\mathcal{S}$ satisfying (IN) and (OUT) w.r.t. p
    for all $(v \in V)$ **do**
      $(G, c, s, t) \leftarrow$ flow network associated to x_p and v
      $M \leftarrow$ value of a minimal flow in (G, c, s, t)
      if $(M > I((p, l(v))))$ **then**
        $test \leftarrow false$
      end if
    end for
  end if
end for
return test

Algorithm 1: Tests, whether $\mathcal{S} = (V, \prec, \sqsubset, l)$ fulfills the token flow property w.r.t. (N, m_0), $N = (P, T, F, W, I)$.

5 Conclusion

We defined executions of PTI-nets w.r.t. the a priori semantics as enabled LSOs. The definition of enabled LSOs is a proper generalization of the definition of enabled LPOs and allows the representation of executions of PTI-nets with minimal causal dependencies between transition occurrences. As the first main result, we showed, that enabled LSOs can be characterized through the so called token flow property, which we lifted from LPOs to LSOs, thus establishing a part of the semantical framework of p/t-net-executions to PTI-nets (Figure 2). As the second main result we developed a polynomial test of the token flow property.

These results are also valid for the a posteriori semantics of PTI-nets. In this case, the test of inhibitor conditions need not precede the execution of transitions, therefore synchronicity and concurrency are not distinguished, and executions are represented by enabled LPOs. Nonetheless, the generalized notion of the token flow property (Definition 13) can be used for such LPOs. In condition (FIN) classical prefixes of LPOs and a modified notion of final markings, corresponding to the a posteriori occurrence rule, must be considered. Then, the equivalence of enabledness and token flow property follows by construction. For the efficient test of the token flow property, one needs to consider a modified flow network to compute the inhibitor value of nodes.

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