Negativity and quantum discord in Davies environments

J Dajka[1], M Mierzejewski[1], J Łuczka[1], R Blattmann[2] and P Hänggi[2]

[1] Institute of Physics, University of Silesia, 40-007 Katowice, Poland
[2] Institute of Physics, Augsburg University, Germany

E-mail: jerzy.dajka@us.edu.pl

Abstract
We investigate the time evolution of negativity and quantum discord for a pair of non-interacting qubits with one being weakly coupled to a decohering Davies-type Markovian environment. At initial time of preparation, the qubits are prepared in one of the maximally entangled pure Bell states. In the limiting case of pure decoherence (i.e. pure dephasing), both the quantum discord and negativity decay to zero in the long time limit. In the presence of a manifest dissipative dynamics, the entanglement negativity undergoes a sudden death at finite time, while the quantum discord relaxes continuously to zero with increasing time. We find that in dephasing environments, the decay of the negativity is more propitious with increasing time; in contrast, the evolving decay of the quantum discord proceeds more weakly for dissipative environments. Particularly, the slowest decay of the quantum discord emerges when the energy relaxation time matches the dephasing time.

1. Introduction

Establishing correlations is a *sine qua non* condition in effective communication. If two parties are *quantum correlated*, one can attempt to conduct a quantum communication between them. Quantum communication protocols [1] make use of certain properties of states of quantum composite systems. The best known and most popular resource for quantum communication is quantum entanglement [2], a widely studied property of composite systems. Quantum entanglement [2], interesting *per se*, has been recognized as a powerful tool for quantum information processing [1]. For example, let us mention the use of quantum communication with dense coding and teleportation as most celebrated examples [1, 2].

If one presupposes that the history of quantum entanglement started out with the work by Schrödinger [3], it took almost a century to discover that there are quantum correlations which are essentially non-classical and these can exist even in the absence of entanglement [4, 5]. When compared to entanglement, these quantum correlations are, to some extent, 'more mysterious', since their mathematical setting has not been uniquely established. The well-known mathematical language for entanglement incorporating tensor products of state spaces, Schmidt decomposition, etc, not only allows one to pose many fundamental problems in purely algebraic context but also motivates the search for a variety of entanglement quantifiers. The physical context of entanglement encoded, e.g., in Bell-type inequalities [2] can be (sometimes incorrectly) considered as a consequence of mathematical theorems. Recent studies of quantum correlations have shown that a broad class of composite systems carries correlations which can be described in the context of information extracted from composite parties via a suitably formalized measurement procedure. For example, the value of the quantum discord studied in this paper depends on the results of the POVM measurements carried out on the system, see [5] for a recent review.

Among several quantifiers of quantum correlations, the *quantum discord* attained recent popularity [5]. Despite its fundamental meaning, exemplarily studied in the context of approximating a reduced quantum dynamics within trace preserving completely positive maps [6], quantum correlations quantified by the quantum discord can open new avenues for quantum computations [7] and quantum communication schemes [8–10].

A salient obstruction in the implementation of both quantum computation and communication protocols is quantum decoherence [1], induced by the influence of the environment on the relevant 'quantum hardware'. Studies of entanglement dynamics in the presence of noise enjoy a long history [11, 12]; in contrast, similar studies on quantum discord dynamics have been carried out only recently [13–16].

According to [17], the quantum discord and the entanglement present fundamentally different resources. Here, we investigate if this difference manifests itself in their robustness to environment-induced decoherence. In other words, we study whether it is possible to assign an environment to two general classes, one of them more suitable for quantum information processing using entanglement-based protocols, and the second preferring quantum correlations as being quantified by the quantum discord. With this study we obtain an affirmative answer to this question.

In doing so we consider a simple set-up: we investigate the time evolution of quantum correlations between a pair of qubits with only one of them being coupled to a decohering environment. We emphasize here that the *exact* reduced quantum dynamics for the two qubits is typically neither completely positive [18, 19], nor is it generally even linear [20], not to speak of undergoing a memoryless quantum Markovian dynamics [21]. Here, however, we restrict ourselves to a quantum Markovian dynamics, having in mind quantum optics with weak coupling at not too low temperatures. Then the time evolution can satisfactorily be approximated with a trace preserving completely positive Markovian map of Davies type [22].

The structure of the work is as follows. In section 2, we describe our model set-up and its dynamics in terms of a Davies map. Next, in section 3, we consider the time evolution of the correlation quantifier negativity while in section 4, we analyse the quantum discord. In section 5, we present our summary and conclusions.

2. Open quantum dynamics: quantum Markovian–Davies map

In this work, somewhat stimulated by prior work done in [13], we consider a pair of non-interacting qubits A and B that initially are prepared in one of the maximally entangled pure

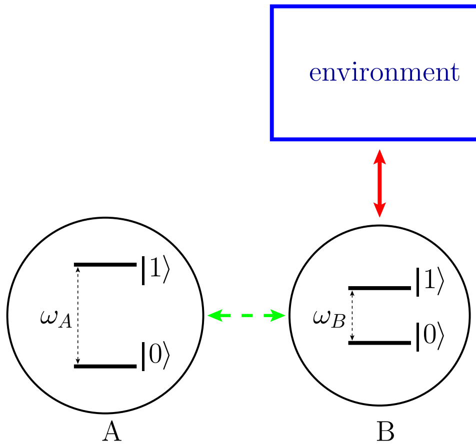

Figure 1. The sketch of the system set-up studied in this work. The qubits A and B with level separations ω_A and ω_B, respectively, do not physically interact; these are, however, initially prepared in one of the maximally entangled Bell states, as indicated by the broken (green) line. Here, only the qubit B is weakly coupled to an environment. The dynamics of the qubits is given by equation (6) in the text.

Bell states, i.e.

$$\rho_i = (\sigma_0 \otimes \sigma_i)\, \rho_0\, (\sigma_0 \otimes \sigma_i)\,, \quad i = 0, 1, 2, 3$$
$$\sigma_0 = \mathbf{1}, \ \ \sigma_{1,2,3} = \sigma_{x,y,z}, \tag{1}$$

where

$$\rho_0 = \tfrac{1}{2}\,(|01\rangle + |10\rangle)\,(\langle 01| + \langle 10|)\,. \tag{2}$$

The non-decohering part of the evolution of the qubits A and B is determined by the two two-level Hamiltonians, i.e.

$$H_A = \frac{\omega_A}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad H_B = \frac{\omega_B}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{3}$$

In the following, only qubit B is coupled at times $t > 0$ to an environment E at temperature T (see figure 1). Then, in the presence of this environment coupling, the dynamics of the two qubits, described by the reduced density matrix $\rho_{AB}(t)$, no longer proceeds unitary. Because the two qubits A and B are initially correlated, both the quantum entanglement and the quantum discord will evolve in the course of evolving time $t > 0$, as these become influenced by a non-zero system B-environment coupling.

A generic form of the Hamiltonian of the total system $A + B + E$ reads

$$H = H_A \otimes (\mathbb{I}_B \otimes \mathbb{I}_E) + \mathbb{I}_A \otimes (H_B \otimes \mathbb{I}_E) + \mathbb{I}_A \otimes H_{BE}^{\rm int} + \mathbb{I}_A \otimes (\mathbb{I}_B \otimes H_E), \tag{4}$$

where $\mathbb{I}_A, \mathbb{I}_B$ and $\mathbb{I}_E$ are identity operators (matrices) in the corresponding Hilbert spaces of the subsystems A, B and the environment E, respectively. The operator $H_{BE}^{\rm int}$ describes the interaction of the qubit B with its environment E and finally H_E is the Hamiltonian of the environment E.

In the presence of a very weak system–environment interaction $H_{BE}^{\rm int}$ and not extremely low temperatures, the reduced dynamics can satisfactorily be described [21] by a Markovian dynamics of the Davies type [22]. The main advantage of this approach is that it recovers the well-established steady-state properties at assumed weak coupling, namely stationarity and asymptotic stability in terms of the Gibbs state of the qubit B, reading

$$\rho_B(t) \to \begin{pmatrix} p & 0 \\ 0 & 1-p \end{pmatrix} \quad \text{for} \quad t \to \infty. \tag{5}$$

Here, the thermal weight is $p = \exp(-\beta\omega_{B/2})/\mathcal{Z}$, $\mathcal{Z} = \exp(-\beta\omega_{B/2}) + \exp(\beta\omega_{B/2})$ and $\beta = 1/kT$ denotes the inverse temperature. Using Davies theory for the Hamiltonian (4), one can explicitly construct the generator of a completely positive semigroup describing the reduced dynamics (with respect to the environment) of the open quantum system in terms of microscopic parameters in the Caldeira–Leggett-type Hamiltonian (4) of the full system. Such modelling has recently been applied to studies of entanglement dynamics [23], properties of dissipative geometric phases of qubits [24] and for some thermodynamic properties of nano-scale systems [25], to name but a few.

Here, instead of using the most general scenario of Davies semigroups, we restrict ourselves to a particular example of the completely positive Davies map [26]. The qubit–qubit reduced dynamics is given by the map Φ_t^B described in detail in [26]. It acts on the Hilbert space B and is completed with the tensorized unitary time evolution for the uncoupled qubit A, the latter acting solely on the Hilbert space of qubit A. Put differently, we have for the reduced dynamics a (super-operator)-time evolution, reading

$$\rho_{AB}(t) = \left(\mathbb{U}_t^A \otimes \Phi_t^B\right)\rho_i, \quad i = 0, 1, 2, 3. \tag{6}$$

This structure of evolution follows from the Hamiltonian (4) because the A-system Hamiltonian H_A commutes with all other remaining Hamiltonians of the system B and environment E. Stated explicitly, for any linear combination of matrices in the form $|i_A\rangle\langle j_A| \otimes |i'_B\rangle\langle j'_B|$, we have $\left(\mathbb{U}_t^A \otimes \Phi_t^B\right)|i_A\rangle\langle j_A| \otimes |i'_B\rangle\langle j'_B| = \mathbb{U}_t^A(|i_A\rangle\langle j_A|) \otimes \Phi_t^B(|i'_B\rangle\langle j'_B|)$. The part of the super-operator $\mathbb{U}_t^A$ is the unitary evolution operator generated by the Hamiltonian H_A, i.e. $\mathbb{U}_t^A(|i_A\rangle\langle j_A|) = \mathrm{e}^{-\mathrm{i}H_A t}|i_A\rangle\langle j_A|\mathrm{e}^{\mathrm{i}H_A t}$. Let us describe how the non-unitary map Φ_t^B acts on the density matrix ρ_B of the qubit B. Following [26], let us construct a super-operator $\mathbf{\Phi_t^B}$ corresponding to Φ_t^B acting on vectorized matrices

$$\rho_B = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \to ||\rho_B\rangle\rangle = \begin{pmatrix} \rho_{00} \\ \rho_{01} \\ \rho_{10} \\ \rho_{11} \end{pmatrix}, \tag{7}$$

$$\Phi_t^B\, \rho_B \to \mathbf{\Phi_t^B}\, ||\rho_B\rangle\rangle. \tag{8}$$

An explicit form of the super-operator $\mathbf{\Phi_t^B}$ reads [26]

$$\mathbf{\Phi_t^B} = \begin{pmatrix} 1-u(t) & 0 & 0 & r(t) \\ 0 & v(t) & 0 & 0 \\ 0 & 0 & v^*(t) & 0 \\ u(t) & 0 & 0 & 1-r(t) \end{pmatrix} \tag{9}$$

with the corresponding relaxation functions reading

$$u(t) = (1-p)[1-\exp(-Ft)], \qquad r(t) = \frac{p}{1-p}u(t) \tag{10}$$

$$v(t) = \exp(-\mathrm{i}\omega_B t - Gt). \tag{11}$$

The parameters $F = 1/\tau_1$ and $G = 1/\tau_2$ are related to the energy relaxation time τ_1 and the dephasing time τ_2, respectively. Given the fact that these relaxation times are subjected to obey the physical inequalities [27]

$$G \geqslant F/2 \geqslant 0, \quad \text{i.e.} \quad 2\tau_1 \geqslant \tau_2, \tag{12}$$

it then follows that this map indeed is a trace-preserving, completely positive map. If it does not hold true, the eigenvalues of the density matrix $\rho_B(t)$ can take negative values and $\rho_B(t)$ cannot describe a physical state. According to the nomenclature used in [26], the above-described map is named the Davies map. From the physical point of view, this map describes the Bloch-type relaxation of the spin with the longitudinal and transverse relaxation times τ_1 and τ_2, respectively. In other words, we take into account two mechanisms responsible for decoherence: dissipation (exchange of energy between the qubit B and its environment) and dephasing (exchange of information between the qubit and environment without energy exchange).

Let us consider the limiting case of an infinite energy relaxation time, yielding $F = 0$. This corresponds to pure dephasing without relaxation in energy taking place (no dissipation). Such dephasing scenarios, including also non-Markovian dephasing models, have been applied to studies of entanglement [28] and quantum discord [29] dynamics. The opposite case, i.e. $G = 0$ and $F \neq 0$, cannot be physically realized: if there is dissipation of energy, then necessarily finite dephasing accompanies this relaxation process. For $p = 1/2$ in equation (5), i.e. in the case of an infinitely large temperature, the single-qubit B state becomes maximally mixed: $\mathrm{Tr}_A(\rho_{AB}(t)) = \frac{1}{2}\mathbb{I}_{\mathbb{B}}$. Likewise, we have for qubit A that $\mathrm{Tr}_B(\rho_{AB}(t)) = \frac{1}{2}\mathbb{I}_{\mathbb{A}}$. Hence, with $p = 1/2$ the quantum discord, being evaluated below, is symmetric with respect to A and B labelling [5].

Because the explicit results below are *robust* with respect to any chosen value for the Boltzmann weight p, we shall depict in our figures the case with $p = 1/2$ only, yielding a symmetric quantum discord.

Assuming that at initial time the reduced two-qubit density matrix $\rho_{AB}(0) = \rho_0$, where ρ_0 is the Bell state given by equation (2), the density matrix $\rho_{AB}(t)$ evolving under the Davies map (9) takes the following form (see the appendix for details):

$$\rho_{AB}(t) = \frac{1}{4}\begin{pmatrix} 1-\mathrm{e}^{-Ft} & 0 & 0 & 0 \\ 0 & 1+\mathrm{e}^{-Ft} & 2\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{e}^{-Gt} & 0 \\ 0 & 2\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{e}^{-Gt} & 1+\mathrm{e}^{-Ft} & 0 \\ 0 & 0 & 0 & 1-\mathrm{e}^{-Ft} \end{pmatrix}, \tag{13}$$

where $\omega = \omega_A - \omega_B$ and $\rho_{AB}(t = 0) = \rho_0$. This density matrix assumes the form of a so termed X-state [30, 17]. Note that the X-structure of the reduced density matrix (13) is preserved during time evolution. This feature originates from both the symmetry of initial preparation (1), here assumed to be given by equation (2), and the character of dynamics given by a completely positive Davies map. Let us stress, however, that *none* of the physical results reported below depend on the choice of the specifically chosen initial Bell state in equation (2); put differently, the results remain robust for any of the four Bell states.

3. Entanglement dynamics

In this section, we investigate the entanglement of the two-qubit state $\rho_{AB}(t)$ given in (6). As a proper measure of entanglement, we use the quantifier of *negativity* $N(t)$ [2], being defined as

$$N(t) = \frac{1}{2}\sum_i (|\lambda_i| - \lambda_i), \tag{14}$$

where the λ_i are the eigenvalues of the partially transposed density matrix $\rho_{AB}(t)$, at fixed time t, of a composite system [2].

Let us next evaluate this negativity of the state (13). It explicitly reads

$$8N(t) = 2\,\mathrm{e}^{-Gt} + \mathrm{e}^{-Ft} - 1 + |2\,\mathrm{e}^{-Gt} + \mathrm{e}^{-Ft} - 1|. \tag{15}$$

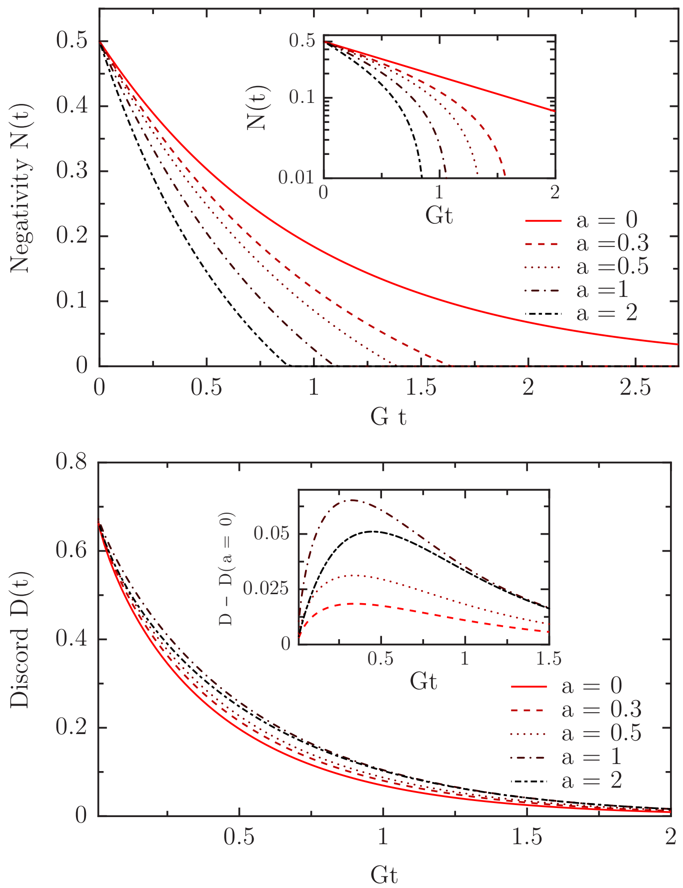

Figure 2. Upper panel: the entanglement quantifier for negativity $N(t)$ (see equation (15)) at temperature $T = \infty$, i.e. $p = 1/2$, as a function of the dimensionless time Gt for several values of the ratio $a = F/G = \tau_2/\tau_1$, where τ_2 and τ_1 are the dephasing and the energy relation times, respectively. The inset depicts the non-exponential decay of negativity $N(t)$ for a dissipative environment, implying that $F > 0$ (yielding $a > 0$). The negativity $N(t)$ versus the dimensionless time Gt is depicted on the semi-logarithmic plot with the vertical axis scaled logarithmically. Lower panel: time evolution of the quantum discord $D(t)$ (see equations (23) and (24)). The inset shows the difference between discord in the dissipative case (i.e. $F > 0$) and for pure dephasing (i.e. $F = 0$) for several values of the ratio a. The slowest decay of the quantum discord $D(t)$ is observed when $\tau_1 = \tau_2$.

$N(t)$ does not depend on ω, i.e. it is not affected by the individual single-qubit level spacings in equation (3). The time evolution of the negativity $N(t)$ is depicted with the upper panel in figure 2 for $T \to \infty$. For pure dephasing, i.e. for $F = 0$, the negativity exhibits a strictly exponential decay, reading

$$N(t) = \tfrac{1}{2}\,\mathrm{e}^{-Gt}, \qquad (16)$$

with the characteristic dephasing rate $G = 1/\tau_2$. For a dissipative environment with $F > 0$, the entanglement undergoes a sudden death [12] occurring at finite death time t_c. Put differently, for time $t > t_c$, the entanglement vanishes identically. From equation (15) it follows that t_c is determined by the relation

$$2\,\mathrm{e}^{-Gt_c} + \mathrm{e}^{-Ft_c} - 1 = 0. \qquad (17)$$

Upon increasing the relaxation rate F, the death time t_c monotonically decreases from $t_c = \infty$ for $F = 0$ to the minimal value $t_c^{\min} = \ln(\sqrt{2} + 1)/G$, which occurs when $F/G = \tau_2/\tau_1 = 2$. In the case of finite dissipation, $F > 0$, the negativity $N(t)$-decay proceeds faster than exponentially; see the inset in the upper panel of figure 2, where $N(t)$ versus the dimensionless time Gt is depicted on the semi-logarithmic plot with the vertical axis scaled logarithmically.

4. Quantum discord

Next let us investigate a quantum correlation measure as encoded with the quantum discord. We again consider a composite system consisting of the two qubits A and B. The full (classical and quantum) correlations in the composite system are encoded with quantum mutual information, defined as

$$\mathcal{I}(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}), \tag{18}$$

where $S(\rho) = -\mathrm{Tr}\,(\rho \ln \rho)$ denotes the von Neumann information entropy and ρ_{AB} is the density operator of the composite bipartite system AB. The part ρ_A refers to the reduced density operator of system part A, while, likewise, ρ_B is the reduced density operator for system part B.

The classical part of the total correlations is defined as the maximum information about one subsystem A that can be obtained by performing a measurement on the other subsystem B, as defined by a complete set of projectors $\{\Pi_k^B\}$. Let us recall that for $p = 1/2$, see section 2, an independence of the chosen A, B-labelling is granted. The label k distinguishes different outcomes of this measurement. The quantifier of the classical part of correlations is defined by the set of the following relations:

$$C(\rho_{AB}) = S(\rho_A) - \max_{\{\Pi_k^B\}} S(\rho_{AB}|\{\Pi_k^B\}), \tag{19}$$

$$\begin{aligned} S(\rho_{AB}|\{\Pi_k^B\}) &= \sum_{k=0}^{1} q_k S(\rho_A^k), \\ \rho_A^k &= \frac{1}{q_k} \mathrm{Tr}_B \left[\left(\mathbb{I}_A \otimes \Pi_k^B \right) \rho_{AB} \left(\mathbb{I}_A \otimes \Pi_k^B \right) \right], \\ q_k &= \mathrm{Tr}_{AB} \left[\left(\mathbb{I}_A \otimes \Pi_k^B \right) \rho_{AB} \right]. \end{aligned} \tag{20}$$

The quantity $C(\rho_{AB})$ characterizes the reduction in the entropy of the subsystem A after a measurement on the subsystem B, when maximized over a class of measurements $\{\Pi_k^B\}$. The difference between the total amount of correlation and the classical part of correlation thus reads

$$D(\rho_{AB}) = \mathcal{I}(\rho_{AB}) - C(\rho_{AB}). \tag{21}$$

This relation defines the *quantum discord*. Being so, it provides a measure for manifest quantum correlations [5]. Let us remark that, generally, this quantum discord (21) presents neither a unique nor the most optimal quantifier for quantum correlation [5]. However, for the case of bipartite systems one can summarize that the states can be divided into two groups [6], namely entangled (quantum correlated) and separable states. In turn, the separable states can either be classically correlated or quantum correlated (but not entangled). This classification is non-trivial, as e.g. classically correlated states always lead to completely positive maps while states with quantum correlations may give rise to non-completely positive maps [6]. For our set-up here, classically correlated states exhibit a zero quantum discord $D(t) = 0$, while states with quantum correlations exhibit a non-zero discord $D(t) \geqslant 0$.

There occurs a natural computational difficulty in evaluating the quantum discord, stemming from the maximization procedure in equation (19). Fortunately, this task becomes feasible for two-qubit systems: it is sufficient to consider projective measurements of the form [31, 5]

$$\Pi_0^B = \begin{pmatrix} \cos^2(\theta/2) & \sin(\theta)\exp(\mathrm{i}\phi) \\ \sin(\theta)\exp(-\mathrm{i}\phi) & \sin^2(\theta/2) \end{pmatrix},$$
$$\Pi_1^B = \mathbb{I}_B - \Pi_0^B, \tag{22}$$

where $\{\theta, \phi\}$ is a standard parameterization of a single-qubit Bloch sphere. This simplification is helpful in considering the quantum discord for models which can be effectively described in terms of two qubits [13, 16, 17].

The calculation of quantum discord requires, in general, a careful optimization with respect to (being sufficient for us) projective measurements (22). Fortunately, here the problem is even more tractable. Due to the symmetry of the system, finding the optimum in (19) does not involve the non-trivial ϕ-dependence, i.e. the measuring process (22) can be simplified to a single-parameter family of projectors. The quantum discord therefore is ϕ-independent. In addition, it exhibits extrema, depending on the value of the dissipation parameter F. For $F \leqslant G$, the maximum in equation (19) occurs at $\theta = 0$ in equation (22). The quantum discord $D(t)$ consequently reads

$$\begin{aligned} 4D(t) = {} & (\mathrm{e}^{-Ft} + 1 + 2\,\mathrm{e}^{-Gt})\ln(\mathrm{e}^{-Ft} + 1 + 2\,\mathrm{e}^{-Gt}) \\ & +(\mathrm{e}^{-Ft} + 1 - 2\,\mathrm{e}^{-Gt})\ln(\mathrm{e}^{-Ft} + 1 - 2\,\mathrm{e}^{-Gt}) \\ & -2(1+\mathrm{e}^{-Ft})\ln(1+\mathrm{e}^{-Ft}), \qquad F \leqslant G. \end{aligned} \tag{23}$$

To optimize quantum discord within the required regime $G < F \leqslant 2G$ (see (12)) in turn implies for consistency that $\theta = \pi/2$ in equation (19). The final result for $D(t)$ thus reads

$$\begin{aligned} 4D(t) = {} & (\mathrm{e}^{-Ft} + 1 + 2\,\mathrm{e}^{-Gt})\ln(\mathrm{e}^{-Ft} + 1 + 2\,\mathrm{e}^{-Gt}) + 2(1-\mathrm{e}^{-Ft})\ln(1-\mathrm{e}^{-Ft}) \\ & -2(1-\mathrm{e}^{-Gt})\ln(1-\mathrm{e}^{-Gt}) + (\mathrm{e}^{-Ft} + 1 - 2\,\mathrm{e}^{-Gt})\ln(\mathrm{e}^{-Ft} + 1 - 2\,\mathrm{e}^{-Gt}) \\ & -2(1+\mathrm{e}^{-Gt})\ln(1+\mathrm{e}^{-Gt}), \qquad G < F \leqslant 2G. \end{aligned} \tag{24}$$

It follows from equations (23) and (24) that the discord $D(t)$ monotonically approaches 0 as $t \to \infty$ and stays positive for finite time; see the lower panel in figure 2. In contrast to the negativity $N(t)$, which is most robust in the absence of energy relaxation $F = 0$, the slowest decay relaxation of the quantum discord is obtained for the case when $F = G$, i.e. when the two characteristic relaxation times match, i.e. $\tau_1 = \tau_2$. To clarify this behaviour, let us note that for pure dephasing $F = 0$ classical correlations (19) remain *constant* without further decay; put differently, Markovian dephasing does not affect the dynamics of classical correlations.

The difference between the quantum discord and the entanglement becomes best visible in the asymptotic long-time regime when $t \gg \tau_1$ or $t \gg \tau_2$. Here, one obtains the limiting behaviour

$$D(t) \simeq \frac{\mathrm{e}^{-2Gt}}{1+\mathrm{e}^{-Ft}}, \quad \text{for} \quad F \leqslant G;$$
$$D(t) \simeq \frac{1}{2}[\mathrm{e}^{-2Gt} + \mathrm{e}^{-2Ft}], \quad \text{for} \quad G < F \leqslant 2G. \tag{25}$$

Therefore, for $t \to \infty$ and $F \leqslant G$ the decay rate of quantum discord is determined solely by the dephasing time $1/G = \tau_2$ and is independent of the energy relaxation time τ_1. However, the entanglement $N(t)$ vanishes identically in this time regime if finite energy relaxation with $F > 0$ is at work. On the other hand, for a pure dephasing $F = 0$ one finds the asymptotic relation

$$D(t) \simeq \mathrm{e}^{-2Gt}/2 = 2N^2(t). \tag{26}$$

Although this very kind of relation between the quantum discord $D(t)$ and the entanglement $N(t)$ likely may not be generic [17, 32], it nicely illustrates the critical role of an environment for the asymptotic behaviour of both quantifiers. Notably, the ratio $N(t)/D(t)$ exhibits for $t \to \infty$ a divergence for a pure dephasing, while it vanishes identically in the presence of finite energy relaxation.

5. Conclusions

In this work, we have investigated the time evolution of the entanglement negativity $N(t)$ and quantum discord $D(t)$ for a pair of qubits, with one qubit (B) weakly coupled to a decohering environment. The decoherence dynamics for this subsystem has been approximated by a Markovian, completely positive Davies semigroup dynamics. We identified two classes of environments that impact differently the quantum correlations as quantified by the entanglement and quantum discord. (i) In the limiting case of the pure decoherence (i.e. strict dephasing), both the quantum discord $D(t)$ and the negativity $N(t)$ decay exponentially toward zero in the asymptotic long-time regime. Moreover, there exists an appealing functional relation between these two measures, being detailed in (26). (ii) In the case of a dissipative dynamics, the entanglement undergoes a sudden death at a finite time t_c, while the quantum discord $D(t)$ smoothly relaxes toward zero at long times. The slowest decay of the quantum discord occurs when the energy relaxation time τ_1 matches the dephasing time τ_2. Our findings may serve as a potential guideline for the implementation of quantum information, e.g. communication protocols. We have elucidated which of the two types of correlation measures, namely the quantum entanglement $N(t)$ or the one encoded by the quantum discord $D(t)$, can provide an advantageous and/or more suitable quantifier for quantum information processing occurring in an open quantum system undergoing an ubiquitous decoherence dynamics.

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Appendix

In this appendix, we want to present a way to obtain the reduced density matrix (13). Any density matrix of two qubits can be presented in the form

$$\rho_{AB} = \sum_{i,j,k,n} a_{ijkn} |i_A\rangle\langle j_A| \otimes |k_B\rangle\langle n_B|, \tag{A.1}$$

where $i, j, k, n = 0, 1$. Equation (6) takes the form

$$\rho_{AB}(t) = \left(\mathbb{U}_t^A \otimes \Phi_t^B\right) \rho_{AB} = \sum_{i,j,k,n} a_{ijkn}\, \mathbb{U}_t^A(|i_A\rangle\langle j_A|) \otimes \Phi_t^B(|k_B\rangle\langle n_B|). \tag{A.2}$$

In particular, we rewrite the Bell state (2) in the similar form:

$$\rho_0 = \tfrac{1}{2}(|0_A\rangle\langle 0_A| \otimes |1_B\rangle\langle 1_B| + |0_A\rangle\langle 1_A| \otimes |1_B\rangle\langle 0_B| \\ + |1_A\rangle\langle 0_A| \otimes |0_B\rangle\langle 1_B| + |1_A\rangle\langle 1_A| \otimes |0_B\rangle\langle 0_B|). \tag{A.3}$$

Then the unitary evolution of the qubit A is determined by the relations

$$\mathbb{U}_t^A|0_A\rangle\langle 0_A| = |0_A\rangle\langle 0_A|, \qquad \mathbb{U}_t^A|1_A\rangle\langle 1_A| = |1_A\rangle\langle 1_A|,$$
$$\mathbb{U}_t^A|0_A\rangle\langle 1_A| = \mathrm{e}^{\mathrm{i}\omega_A t}|0_A\rangle\langle 1_A|, \qquad \mathbb{U}_t^A|1_A\rangle\langle 0_A| = \mathrm{e}^{-\mathrm{i}\omega_A t}|1_A\rangle\langle 0_A|. \tag{A.4}$$

From equations (7)–(9), it follows that

$$\Phi_t^B|0_B\rangle\langle 0_B| = [1 - u(t)]|0_B\rangle\langle 0_B| + r(t)|1_B\rangle\langle 1_B|,$$
$$\Phi_t^B|0_B\rangle\langle 1_B| = v(t)|0_B\rangle\langle 1_B|,$$
$$\Phi_t^B|1_B\rangle\langle 0_B| = v^*(t)|1_B\rangle\langle 0_B|,$$
$$\Phi_t^B|1_B\rangle\langle 1_B| = u(t)|0_B\rangle\langle 0_B| + [1 - r(t)]|1_B\rangle\langle 1_B|. \tag{A.5}$$

Therefore, the time evolution of the Bell state ρ_0 reads

$$\rho_{AB}(t) = \left(\mathbb{U}_t^A \otimes \Phi_t^B\right)\rho_0 = u(t)|A\rangle\langle A| + [1 - r(t)]|B\rangle\langle B| + v^*(t)\,\mathrm{e}^{\mathrm{i}\omega_A t}|B\rangle\langle C|$$
$$+v(t)\mathrm{e}^{-\mathrm{i}\omega_A t}|C\rangle\langle B| + [1 - u(t)]|C\rangle\langle C| + r(t)|D\rangle\langle D|, \tag{A.6}$$

where the two-qubit basis is denoted in the following way:

$$|A\rangle = |0_A\rangle \otimes |0_B\rangle, \quad |B\rangle = |0_A\rangle \otimes |1_B\rangle,$$
$$|C\rangle = |1_A\rangle \otimes |0_B\rangle, \quad |D\rangle = |1_A\rangle \otimes |1_B\rangle. \tag{A.7}$$

In this basis, the density matrix $\rho_{AB}(t)$ for the considered case $p = 1/2$ takes the form presented by equation (13).

References

[1] Nielsen M A and Chuang I L 2000 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)
[2] Horodecki R, Horodecki P, Horodecki M and Horodecki K 2009 *Rev. Mod. Phys.* **81** 565
[3] Schrödinger E 1935 *Naturwissenschaften* **23** 844
[4] Żurek W H 2000 *Ann. Phys., Berlin* **9** 855
Henderson L and Vedral V 2001 *J. Phys. A: Math. Gen.* **34** 6899
Ollivier H and Żurek W H 2001 *Phys. Rev. Lett.* **88** 017901
[5] Modi K, Brodutch A, Paterek C H T and Vedral V 2012 *Rev. Mod. Phys.* at press (arXiv:1112.6238v1)
[6] Rodrguez-Rosario C A, Modi K, Kuah A, Shaji A and Sudarshan E C G 2008 *J. Phys. A: Math. Theor.* **41** 205301
Shabani A and Lidar D A 2009 *Phys. Rev. Lett.* **102** 100402
[7] Datta A, Flammia S T and Caves C M 2005 *Phys. Rev.* A **72** 042316
[8] Madhok V and Datta A 2011 *Phys. Rev.* A **83** 032323
[9] Piani M, Horodecki P and Horodecki R 2008 *Phys. Rev. Lett.* **100** 090502
[10] Dakić B *et al* 2012 arXiv:1203.1629v1
Dakić B *et al* 2012 *Nature Phys.* **8** 666
Madhok V and Datta A 2012 arXiv:1204.6042v1
Brodutch A *et al* 2011 *J. Phys.: Conf. Ser.* **306** 012030
[11] Benatti F and Floreanini R 2005 *Int. J. Mod. Phys.* B **19** 3063
[12] Yu T and Eberly J R 2009 *Science* **323** 598
[13] Mazziero J, Werlang T, Fanchini F F, Céleri L C and Serra R M 2010 *Phys. Rev.* A **81** 022116
[14] Auccaise R *et al* 2011 *Phys. Rev. Lett.* **107** 140403
Mazzola L *et al* 2010 *Phys. Rev. Lett.* **104** 200401
[15] Groisman B, Popescu S and Winter A 2005 *Phys. Rev.* A **72** 032317
[16] Luo S 2008 *Phys. Rev.* A **77** 042303
[17] Mazhar A, Rau A R P and Alber G 2010 *Phys. Rev.* A **81** 042105
[18] Pechukas P 1994 *Phys. Rev. Lett.* **73** 1060
Pechukas P 1995 *Phys. Rev. Lett.* **75** 3021
[19] Shaji A and Sudarshan E C G 2005 *Phys. Lett.* **341** 48
[20] Fonseca Romero K M, Talkner P and Hänggi P 2004 *Phys. Rev.* A **69** 052109

[21] Breuer H P 2012 *J. Phys. B: At. Mol. Opt. Phys.* **45** 154001
Breuer H P, Laine E M and Piilo J 2009 *Phys. Rev. Lett.* **103** 210401
Hänggi P and Ingold G L 2005 *Chaos* **15** 026105
[22] Alicki R and Lendi K 2007 *Dynamical Semigroups and Applications (Lecture Notes in Physics* vol 717*)* (Berlin: Springer) p 1
[23] Lendi K and van Wonderen J 2007 *J. Phys. A: Math. Theor.* **40** 279
[24] Dajka J, Łuczka J and Hänggi P 2011 *Quantum Inform. Process.* **10** 85
[25] Szelag M, Dajka J, Zipper E and Łuczka J 2008 *Acta Phys. Pol.* B **39** 1177
[26] Roga W, Fannes M and Życzkowski K 2010 *Rep. Math. Phys.* **66** 311
[27] Levitt M H 2008 *Spin Dynamics: Basics of Nuclear Magnetic Resonance* (New York: Wiley) section 11.9.2
[28] Dajka J, Mierzejewski M and Łuczka J 2008 *Phys. Rev.* A **77** 042316
[29] Haikka P and Maniscalco S 2012 arXiv:1203.6469v1
[30] Yu T and Eberly J R 2007 *Quantum Inform. Comput.* **7** 459
[31] Hamieh S, Kobes R and Zaraket H 2004 *Phys. Rev.* A **70** 052325
Galve F, Giorgi G L and Zambrini R 2011 *Europhys. Lett.* **96** 40005
D'Ariano G M, Lo Presti P and Perinotti P 2005 *J. Phys. A: Math. Gen.* **38** 5979
[32] Brodutch A and Terno D R 2011 *Phys. Rev.* A **83** 010301
Debarba T *et al* 2012 arXiv:1207.1298v1