MUON DECAY IN ORBIT ☆

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The decay spectrum of a muon bound in the $1s_{1/2}$ state of an atom is calculated using V-A theory. Accurate muon and electron wave functions including finite nuclear size effects and vacuum polarization are used for the evaluation of the electron emission spectra of various elements. The formalism to perform the angular integration of the matrix elements is developped for the V-A weak interaction.

About twenty years ago, Primakoff and Porter [1] found that the lifetime of bound muons is quite different from that observed in the free decay. Later discussions of this effect involved various approximations for the muon bound state wave function and the electron state [2–4]. It is our aim to recalculate the energy spectrum of the emitted electrons essentially without any approximation. The muon bound state wave function is obtained by numerical integration of the Dirac equation where the potential which includes vacuum polarization [5] is derived from a two parameter Fermi nuclear charge distribution. The finite size of the nucleus is introduced in the standard partial wave expansion for the final state of the electron with $m_e \neq 0$. In order to calculate accurately the spectrum for high electron energies the nuclear recoil is taken into account by a multiplicative factor $A(E_e)$ calculated in Born approximation. Higher order dispersive corrections between the initial bound muon and the final free electron are neglected. We use the convention $\hbar = c = 1$, and the γ-matrices are in the representation given by Källen [6].

Assuming V-A theory for the decay of a bound muon the transition probability $\delta P/\delta t$ is obtained after integration over the neutrino states and spin summations giving

$$\frac{\delta P}{\delta t} = \frac{G^2}{384\pi^7} \mathrm{d}^3 q_e \int_0^\omega \mathrm{d}^3 k \sum_{\alpha\beta} W_{\alpha\beta}(Q^2 \delta_{\alpha\beta} - Q_\alpha Q_\beta) , \tag{1}$$

where $Q_\alpha = (\boldsymbol{k}, \mathrm{i}\omega)$ is the 4-momentum transferred to the neutrinos with

$$\boldsymbol{k} = (q_{\nu_e} + q_{\nu_\mu}) , \qquad \omega = m_\mu - B_{1s} - E_e . \tag{2}$$

The transition matrix elements $W_{\alpha\beta}$ occuring in eq. (1) can be written as

$$W_{\alpha\beta} = \int \mathrm{d}^3 r \langle e^- | \Psi_e(\boldsymbol{r}) \gamma_\alpha (1+\gamma_5) \Psi_\mu(\boldsymbol{r}) | \mu^- \rangle \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}) \int \mathrm{d}^3 r \langle \mu^- | \bar{\Psi}_\mu(\boldsymbol{r}) \gamma_\beta (1+\gamma_5) \Psi_e(\boldsymbol{r}) | e^- \rangle \exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}) . \tag{3}$$

The integrals (3) contain an angular integration of the type

$$V_\alpha = \int \bar{\Psi}^{\mu_f}_{\kappa_f} \gamma_\alpha \Psi^{\mu_i}_{\kappa_i} \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}) \, \mathrm{d}\Omega . \tag{4}$$

These integrals can be expanded into multipoles giving

$$V_4 = \int \Psi^{\mu_f \dagger}_{\kappa_f} \Psi^{\mu_i}_{\kappa_i} \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}) \, \mathrm{d}\Omega = 4\pi \sum_{\lambda\mu} \mathrm{i}^\lambda j_\lambda(kr) Y^*_{\lambda\mu}(\hat{\boldsymbol{k}}) \int \Psi^{\mu_f \dagger}_{\kappa_f} Y_{\lambda\mu}(\hat{\boldsymbol{r}}) \Psi^{\mu_i}_{\kappa_i} \mathrm{d}\Omega , \tag{5}$$

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and

$$V = -\mathrm{i}\int \Psi^{\mu_f\dagger}_{\kappa_f}\alpha\Psi^{\mu_i}_{\kappa_i}\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\,\mathrm{d}\Omega = -4\pi\mathrm{i}\sum_{J\lambda\mu}\mathrm{i}^{\lambda}j_{\lambda}(kr)Y^{*}_{J\lambda\mu}(\hat{\boldsymbol{k}})\int \Psi^{\mu_f\dagger}_{\kappa_\tau}\alpha\cdot Y_{J\lambda\mu}(\hat{\boldsymbol{r}})\Psi^{\mu_i}_{\kappa_i}\,\mathrm{d}\Omega\,. \tag{6}$$

The angular integrals in eqs. (5) and (6) are well known from electron scattering and internal conversion processes [8, 9]. The functions Ψ^{μ}_{κ} simply denote the usual Dirac spinors [7]. The axial-vector matrix element can be calculated from eqs. (5) and (6) by replacing $\Psi^{\mu_i}_{\kappa_i}\to \mathrm{i}\gamma_5\Psi^{\mu_i}_{\kappa_i}$. The method outlined above can be generalized to all matrix elements of the form

$$M = \int \Psi^{\mu_f}_{\kappa_f}\hat{O}\Psi^{\mu_i}_{\kappa_i}\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\,\mathrm{d}\Omega\,, \tag{7}$$

where

$$\hat{O} = (1, \gamma_5, \gamma_\alpha, \mathrm{i}\gamma_\alpha\gamma_5, \sigma_{\alpha\beta})\ [10]\,. \tag{8}$$

To obtain the energy spectrum for the electrons we have to integrate over the directions of the asymptotic electron momentum $\boldsymbol{q}_e$ and of the transfer momentum $\boldsymbol{k}$. After the summation over the muon and electron spins we normalize the spectrum relative to a free muon decay rate. If we choose the $\boldsymbol{k}$-direction as both the z-axis and the quantization axis, the electron spectrum $N(E_e)$ can be written as

$$\begin{aligned} N(E_e)\mathrm{d}E_e = {} & \frac{4}{m_\mu^5\pi^2}A(E_e)E_e\sqrt{E_e^2-m_e^2}\,\mathrm{d}E_e\sum_{\kappa}\int_0^{\omega} k^2\mathrm{d}k\sum_{\mu\sigma\tau}|a_{\kappa\mu}(\tau)|^2\{(\omega^2-k^2)|J_{(\kappa\mu\sigma)}|^2 \\ & + k^2[|J_{4(\kappa\mu\sigma)}|^2 + |J_{z(\kappa\mu\sigma)}|^2] + 2\omega k\,\mathrm{Im}\,(J_{4(\kappa\mu\sigma)}J^{*}_{z(\kappa\mu\sigma)})\}, \end{aligned} \tag{9}$$

where $A(E_e)$ denotes the recoil factor defined below. The electron and muon wave functions are given by

$$\Psi_e(\tau) = \sum_{\kappa\mu}a_{\kappa\mu}(\tau)\begin{pmatrix} g\,\chi^{\mu}_{\kappa} \\ \mathrm{i}f_{\kappa}\,\chi^{\mu}_{\kappa}\end{pmatrix} \quad\text{and}\quad \Psi_\mu = \begin{pmatrix} g\ \chi^{\sigma}_{-1} \\ \mathrm{i}f\ \chi^{\sigma}_{+1}\end{pmatrix}.$$

For $J_\alpha = (J_{(\kappa\mu\sigma)}, J_{4(\kappa\mu\sigma)})$ we obtain

$$J_{4(\kappa\mu\sigma)} = \sum_{J=0}^{\infty}\mathrm{i}^J\hat{J}\int \mathrm{d}r\,r^2 j_J(kr)\,(\{(f_\kappa f + g_\kappa g)\langle\kappa\mu|Y_{J0}|-1\sigma\rangle\} + \text{subst.})\,,$$

$$\begin{aligned} J_{(\kappa\mu\sigma)} = {} & -\sum_q\sum_{J=1}^{\infty}\langle J01q|Jq\rangle\frac{\hat{J}\cdot\mathrm{i}^J}{\sqrt{J(J+1)}}\int \mathrm{d}r\,r^2 j_J(kr)\,(\{(\kappa-1)C_+\langle-\kappa_f\mu_f|Y_{Jq}|-1\sigma\rangle\} + \text{subst.})\,e^{*}_q \\ & + \sum_q\sum_{J=0}^{\infty}\langle(J+1)01q|Jq\rangle\frac{\widehat{(J+1)}\,\mathrm{i}^{J+1}}{\hat{J}\sqrt{J+1}}\int \mathrm{d}r\,r^2 j_{J+1}(kr)\,(\{[(\kappa+1)C_+ + (J+1)C_-]\langle\kappa\mu|Y_{Jq}|-1\sigma\rangle\} + \text{subst.})e^{*}_q \\ & + \sum_q\sum_{J=1}^{\infty}\langle(J-1)01q|Jq\rangle\frac{\widehat{(J-1)}\,\mathrm{i}^{J-1}}{\hat{J}\sqrt{J}}\int \mathrm{d}r\ r^2 j_{J-1}(kr)\,(\{[(\kappa+1)C_+ - JC_-]\langle\kappa\mu|Y_{Jq}|-1\sigma\rangle\} + \text{subst.})e^{*}_q\,. \end{aligned} \tag{10}$$

In the above formula we have introduced the abbreviations $\hat{J} = \sqrt{2J+1}$ and $C_\pm = g_\kappa f \pm f_\kappa g$. The angular matrix element $\langle\kappa_f\mu_f|Y_{\lambda\mu}|\kappa_i\mu_i\rangle = \int\chi^{\mu_f}_{\kappa_f}Y_{\lambda\mu}\chi^{\mu_i}_{\kappa_i}\mathrm{d}\Omega$ is given explicitly in ref. [8]. Furthermore, the e_q denote the usual spherical unit vectors. The expression "subst." stands for the pseudo-vector term which is exactly analogous to the vector term in the curled brackets except for the substitution $g\to -\mathrm{i}f$, $f\to \mathrm{i}g$, and $\chi^{\sigma}_{-1}\to\chi^{\sigma}_{1}$. The recoil factor $A(E_e)$ introduced in the expression (9) is defined by

$$A(E_e) = N(E_e)^{\mathrm{R}}_{\mathrm{Born}}/N(E_e)_{\mathrm{Born}}\,, \tag{11}$$

where $N(E_e)^{R}_{Born}$ is the energy spectrum of the electron calculated in the plane wave Born approximation (PWBA) with inclusion of the nuclear recoil energy. For this case the muon bound state wave function $1s_{1/2}$ is assumed to have the simple hydrogenic form

$$\Psi_\mu = \frac{(Zm_\mu\alpha)^{3/2}}{\pi^{1/2}} \exp(-Zm_\mu\alpha \cdot r) . \tag{12}$$

The recoil effect becomes only important in the last third of the phase space available to the electron. With these approximations the integrals occuring in the expression for the energy spectrum $N(E_e)_{Born}$ without recoil correction can be evaluated analytically to yield [10]

$$N(E_e)_{Born}\,dE_e = \frac{1}{\tau_\mu} \cdot 256\pi^{-1}(Z\alpha)^5 E_e\sqrt{E_e^2 - m_e^2}\,dE_e \left\{ \left(\arctan\frac{2x-b}{\sqrt{\Delta}} + \arctan\frac{2x+b}{\sqrt{\Delta}} \right) \frac{1}{\Delta\sqrt{\Delta}} \left[\frac{2(c-a+2ea/b)}{\Delta} - \frac{e}{b} \right] \right.$$

$$+ \frac{e}{b}\frac{ax-x^3}{R_+R_-\Delta} + \frac{(c-a)/2+ea/b}{\Delta^2}\left(\frac{2x+b}{R_+} + \frac{2x-b}{R_-}\right) + \left(\frac{c/b+ea/b^2}{6\Delta}\right)\left(\frac{2a+bx}{R_+^2} + \frac{-2a+bx}{R_-^2}\right) \tag{13}$$

$$\left. + \left(\frac{e}{b^2} - \frac{1}{b}\right) \Big/ 6 \left(\frac{x^2 + (2a^2 + abx)/\Delta}{R_+^2} - \frac{x^2 + (2a^2 - qbx)/\Delta}{R_-^2} \right) \right\} \Bigg|_0^{W_0 - E_e}$$

The abbreviations used in (13) are defined by

$$\Delta = 4(Zm_\mu\alpha)^2 , \qquad b = 2\sqrt{E_e^2 - m_e} , \qquad c = 3(W_0^2 + E_e^2 - 2W_0E_e), \qquad R_\pm = a \pm bx + x^2 ,$$

$a = 4(Zm_\mu\alpha)^2 + E_e^2 - m_e^2$, $W_0 = m_\mu - m_\mu(Z\alpha)^2/2$, $e = b(W_0 - E_e)/E_e$, $1/\tau_\mu = m_\mu^5 G^2/192\pi^3$ free decay rate. In fig. 1 we show a comparison between various approximations for the energy spectrum $N(E_e)$ of ^{205}Tl. The dotted line in fig. 1 shows the energy spectrum calculated by means of eq. (13), where the bound state of the muon was approximated by a nonrelativistiv point nucleus wave function and the electron by a plane wave. If one uses distorted outgoing electron wave functions which include the finite size of the nucleus, the dashed-dotted line is obtained, which shows the expected improvement upon the previous approximation. The muon binding energy was still assumed to be the Schrödinger eigenvalue corresponding to the simple bound state wave function (12). Finally, the solid line denotes the exact energy spectrum $N(E_e)$ calculated by means of eq. (9). In this case we used the numerical muon binding energy B_{1s}, including the vacuum polarization and several higher

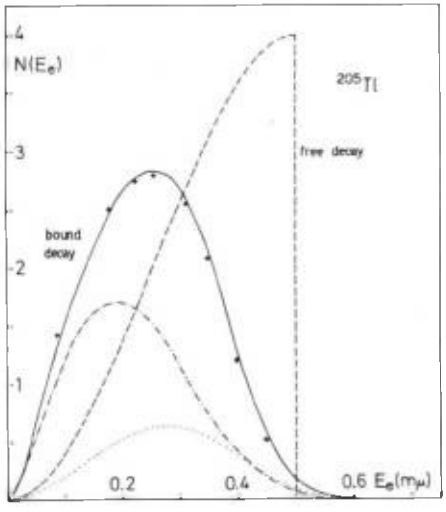

Fig. 1. The calculated electron spectra $N(E_e)$ for bound muon decay in $^{205}_{81}$Tl is plotted for various approximations versus the total energy E_e of the electron in the CM system of the muonic atom. The dotted line shows $N(E_e)$ in the PWBA, the dashed-dotted line gives the spectrum calculated with distorted electron wave functions, and the solid line shows the spectrum evaluated by means of eq. (9). The crosses correspond to calculations of Huff [4] but for the charge number $Z = 82$. The free muon decay is included for comparison. The energy of the electron is given in units of the free muon rest mass.

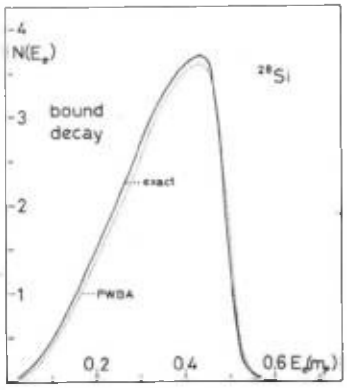

Fig. 2. The calculated energy spectrum $N(E_e)$ for bound muon decay in ^{28}Si is plotted versus the total energy E_e given in units of the free muon rest mass. The solid line shows the exact spectrum calculated with eq. (9), whereas the dotted line shows $N(E_e)$ in the PWBA.

order corrections. The improvement upon other calculations [4] are essentially obtained for low and high free electron energies. This will give the important background for further experiments in the field of weak interactions. To study the dependence of $N(E_e)$ on the nuclear charge number Z as well as on the different approximations discussed above we investigated the following elements [10]: ^{16}O, ^{24}Mg, ^{28}Si, ^{64}Cu and ^{205}Tl. We found a good agreement between our calculation and PWBA theory for charge numbers $Z \lesssim 20$. This can be verified in fig. 2 where we compare the exact spectrum calculated by means of eq. (9) with the PWBA result (eq. (13)) for ^{28}Si.

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