NONLINEAR EFFECTS OF COLORED NONSTATIONARY NOISE: EXACT RESULTS

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The master equation is derived for random systems under nonlinear time-dependent conditions. The (non-Markov) process is of such a type that with a time-dependent state transformation the dynamics can be modelled by a nonlinear but drift-free Langevin equation. The focus is on the statistical content of resulting master equation. The existence of stationary solutions and the quality of approximative results is discussed.

Most of the work on noise-induced transitions [1–3] has been concerned with stationary processes occurring in systems subject to time-independent external conditions. The statistical features of such systems are independent of time, after the transients resulting from the preparation have died out. In contrast we focus here on systems driven by external forces varying in time. These systems are in contact with a generally nonstationary fluctuating environment. As a consequence we deal with time-inhomogeneous processes, i.e. time-translation symmetry for the macroscopic transition probabilities is broken. Examples of physical interest are electric networks containing time-dependent fluctuating dissipative elements (e.g. capacitance in a microphone), forced random oscillator systems, spin relaxation in a time-dependent fluctuating magnetic field, lasers driven in a time-dependent way, processing of speech signals etc.

Problems of this kind are appropriately studied in terms of stochastic differential equations of the Langevin type containing a colored-noise source $\xi(t)$; i.e. $\xi(t)$ is *not* idealized white noise. Because those Langevin equations generate non-Markov processes it is generally not an easy matter to extract the exact statistical information as e.g. the time rate of change of probability. Useful methods ‡1 for tackling those non-Markov Langevin equations (NMLE) are the cumulant expansion [5], the projector method [6], the method of functional derivatives [7,8] or retarding response functions [2,9]. In order to understand the nature and quality of various approximation schemes, put forward by the authors of the above mentioned techniques, it is important to derive exact results for special systems [1,2,8]. The goal we pursue here is to derive exact results and to clarify the relevant structures for colored-noise phenomena of generally nonstationary processes.

To start out we consider the example of *deterministically linear* but *time-dependent*, damped relaxation of a macrovariable x,

$$\mathrm{d}x/\mathrm{d}t = -\gamma(t)x\ . \qquad (1)$$

Representing the effect of a fluctuating environment by a fluctuating damping parameter (external random parameter relaxation) of the form

$$\gamma(t) \to \gamma_0(t) - \alpha(t)|x|^\rho \xi(t)\ , \qquad (2)$$

with ρ denoting some real number, $\xi(t)$ a nonstationary colored-noise source and $\alpha(t)$ an amplitude, we have the NMLE

$$\mathrm{d}x/\mathrm{d}t = -\gamma_0(t)x + \alpha(t)\,|x|^\rho x\xi(t)\ . \qquad (3)$$

This NMLE may model the charge fluctuations in a *RC*-circuit with a time-dependent capacitance. Another example of physical interest is the study of oscillations in a *RCL*-circuit with fluctuating capacitance. For small temperatures one may neglect the in-

‡1 For an illuminating discussion of the various methods see ref. [4].

fluence of the internal thermal noise by writing: $\omega^2(t) = [LC(t)]^{-1} = \omega_0^2(t) + \xi(t), \beta = R/L$,

$$\ddot{x} = -\omega_0^2(t)x - \beta\dot{x} - x\xi(t) . \quad (4)$$

In the overdamped limit we obtain upon an adiabatic elimination of $\dot{x}$ eq. (3) with $\rho = 0, \gamma_0(t) = \beta^{-1}\omega_0^2(t)$, $\alpha(t) = -\beta^{-1}$.

Introducing the variable

$$y = x \exp\int^t \gamma_0(s)\,\mathrm{d}s , \quad (5a)$$

we obtain the "driftless" NMLE

$$\mathrm{d}y/\mathrm{d}t = \alpha(t)\exp\left(-\rho\int^t \gamma_0(s)\,\mathrm{d}s\right)|y|^\rho y\xi(t) . \quad (5b)$$

More general, we consider the class of NMLE's which, generally after a *time-dependent* state transformation $y = T(x;t)$, can be cast into the drift-free form

$$\mathrm{d}y/\mathrm{d}t = c(t)g(y)\,\xi(t) . \quad (6)$$

The derivation of the master equation for the probability $p_t(y) = \langle\delta(y(t) - y)\rangle$ introduces the functional derivative [8]

$$\frac{\delta}{\delta\xi(\tau)}\delta(y(t) - y) = -\frac{\partial}{\partial y}\left[\delta(y(t) - y)\frac{\delta y(t)}{\delta\xi(\tau)}\right]. \quad (7)$$

Introducing formally the random time u,

$$u = \int^t c(s)\,\xi(s)\,\mathrm{d}s , \quad \text{i.e.} \quad \mathrm{d}y/\mathrm{d}u = g(y) , \quad (8)$$

we obtain on varying (7) with respect to $\xi(\tau)$:

$$\frac{\delta y(t)}{\delta\xi(\tau)} = \frac{\partial}{\partial\lambda}y([\xi(s) + \lambda\delta(\tau - s)])|_{\lambda=0}$$

$$= \frac{\partial}{\partial\lambda}y\left(\int^t c(s)[\xi(s) + \lambda\delta(\tau - s)]\,\mathrm{d}s\right)\Big|_{\lambda=0}$$

$$= c(\tau)g(y(t))\,\Theta(t - \tau) . \quad (9)$$

The fact that the functional derivative in (9) can be expressed as a functional of $y(t)$ enables us to write down a *closed* time-convolutionless master equation for the process in eq. (6) generated by any noise $\xi(t)$ whose cumulant averages are known a priori (see eq. (13.8) of ref. [8])! Sticking to nonstationary gaussian noise of vanishing mean and correlation,

$$\langle\xi(t)\,\xi(s)\rangle = \sigma(t,s) , \quad (10)$$

we readily find from (7) and (9) (making use of the Novikov theorem [7])

$$\dot{p}_t(y) = D(t)\,[\partial g(y)/\partial y]^2 p_t(y) , \quad (11)$$

where

$$D(t) = c(t)\int_0^t \sigma(t,s)c(s)\,\mathrm{d}s . \quad (12)$$

Focussing on the results (11), (12) we can state the following additional exact results:

(a) For the nonstationary colored gaussian noise in (10) we obtain for the class of NMLE in (6) a Fokker–Planck type operator, eq. (11), which exists for all times t. Generally this is not the case [10]. However, this Fokker–Planck operator is not necessarily a "markovian" Fokker–Planck operator. The diffusion coefficient $D(t)$ possibly can take on ***negative*** values for a whole range of parameter values t. In other words, for t fixed, the Fokker–Planck operator $\Gamma(t) \equiv \Gamma_t$ is not always a dissipative operator. Further note the (non-Markov) property

$$\dot{p}_{t=0^+}(y) = 0 . \quad (13)$$

(b) In terms of the function $D(t)$ we can introduce the new time scale $\tau = \int^t D(s)\,\mathrm{d}s$ yielding the *time-homogeneous* Fokker–Planck relaxation,

$$\dot{p}_\tau(y) = \frac{\partial}{\partial y}\left[-g(y)\left(\frac{\mathrm{d}g(y)}{\mathrm{d}y}\right)p_\tau(y)\right]$$

$$+ \frac{\partial^2}{\partial y^2}[g^2(y)p_\tau(y)] , \quad (14)$$

i.e. the (single-time) relaxation of the non-Markov process $y(\tau)$ equals the relaxation of the corresponding *time-homogeneous Markov* process $\tilde{y}(\tau)$ with the Fokker–Planck operator (14).

(c) Assuming $\int_{-\infty}^{\infty}\mathrm{d}y/|g(y)| = N$ being finite the process $y(t)$ has a *stationary* solution $\bar{p}(y)$ which is approached for $\tau(t) \to \infty$:

$$\bar{p}(y) = N^{-1}|g(y)|^{-1} , \quad (15)$$

$\bar{p}$ does not depend on the noise intensity, $\lim_{t\to\infty}\langle\xi^2(t)\rangle$, and not on an asymptotic amplitude, i.e. $\lim_{t\to\infty}c(t)$.

Assuming in eq. (4) that $\xi(t)$ is gaussian noise, (10), we obtain from eqs. (11) and (5) for the overdamped motion the result

$$\dot{p}_t(x) = \beta^{-1}\omega_0^2(t)\left(\frac{\partial}{\partial x}x\right)p_t(x)$$

$$+\beta^{-2}\int_0^t \sigma(t,s)\,\mathrm{d}s\left(\frac{\partial}{\partial x}x\right)^2 p_t(x)\,. \qquad (16)$$

Eq. (16) does not possess a stationary solution. Nevertheless, the average of $x(t)$ satisfies

$$\frac{\mathrm{d}\langle x\rangle}{\mathrm{d}t} = -\beta^{-1}\omega_0^2(t)\langle x\rangle + \beta^{-2}\left(\int_0^t \sigma(t,s)\,\mathrm{d}s\right)\langle x\rangle\,, \quad (17)$$

i.e. $\lim_{t\to\infty}\langle x(t)\rangle$, depending on $\omega_0^2(t)$, can still be stable despite the nonexistence of $\bar{p}$ and despite the destabilizing effect of the second (noise) term. The higher-moment equations are closed also.

An important approximation scheme in the study of colored-noise phenomena is the small-relaxation-time approximation [2,9], i.e. if the correlation $\sigma(t,s)$ decays *rapidly*, the main contribution to (7) is from $\delta y(t)/\delta\xi(\tau)$ around $\tau = t^-$. By use of a Taylor expansion we obtain

$$\frac{\delta y(t)}{\delta\xi(\tau)} = \left.\frac{\delta y(t)}{\delta\xi(\tau)}\right|_{\tau=t^-} + \left.\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\delta y(t)}{\delta\xi(\tau)}\right)\right|_{\tau=t^-}(\tau - t) + \dots$$

$$= c(t)g(y(t)) + \dot{c}(t)g(y(t))(\tau - t) + \dots\,. \qquad (18)$$

Making use of (7) we ultimately find the small-relaxation-time approximation

$$\dot{p}_t(y) = c^2(t)\int_0^t \sigma(t,s)\,\mathrm{d}s\left(\frac{\partial}{\partial y}g(y)\right)^2 p_t(y)$$

$$+ c(t)\dot{c}(t)\int_0^t (s-t)\sigma(t,s)\,\mathrm{d}s\left(\frac{\partial}{\partial y}g(y)\right)^2 p_t(y)\,. \qquad (19)$$

Here transients [2,9] have not been neglected. In the white-noise limit, i.e.

$$\sigma(t,s) = \bar{D}(t)\delta(t-s)\,, \qquad \bar{D}(t) \geqslant 0\,,$$

one recovers the usual Fokker–Planck equation:

$$\dot{p}_t(y) = \tfrac{1}{2}c^2(t)\bar{D}(t)\,[\partial g(y)/\partial y]^2 p_t(y)\,. \qquad (20)$$

For a system, which with a transformation $y = T(x;t)$ can be cast into the form given in (6), we obtain from eq. (19) a useful criterion: If $c(t)$ is a constant [e.g. $\alpha(t) = \alpha$, $\rho = 0$ in eq. (5)], the small-relaxation-time approximation yields for gaussian noise, (10), the *exact* result. For $c(t)$ being smooth and slowly varying, the approximation made to arrive at eq. (19) consists with $\dot{c}(t) \approx 0$ in the substitution

$$D(t) \simeq c^2(t)\int_0^t \sigma(t,s)\,\mathrm{d}s\,. \qquad (21)$$

Further, we find that the exact results can be carried over to (transformed) vector processes y [e.g. spin relaxation in the time-dependent stochastic field $\xi(t)$] of the type

$$\mathrm{d}y_i/\mathrm{d}t = c_i(t)\,g_i(y)\,\xi(t)\,, \qquad (22)$$

for all components i.

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