Dynamics of nonlinear dissipative oscillators

Peter Hanggi and Peter Riseborough
Department of Physics, Polytechnic Institute of New York, 333 Jay Street, Brooklyn, New York 11201

In this paper we review some of the important concepts and theorems dealing with nonlinear dissipative oscillators in the presence of random and deterministic periodic forces, with particular reference to the Van der Pol oscillator. We display simple techniques such as the Krylov–Bogoliubov averaging method and generalize this method to random oscillator systems exhibiting new noise-induced phenomena

I. PREFACE

Presently there is a great deal of interest in properties of nonlinear systems, since they have many applications in various fields of science as, e.g., nonlinear mechanics,[1,2] radio physics,[3] biology,[4] and nonlinear optics.[5] There are many examples of bizarre phenomena that can be modeled by various forms of nonlinear dissipative oscillators. The famous Van der Pol equation forms a canonical example of such nonlinear oscillators having the general form

$$m\ddot{x} + \dot{x}f(x) + g(x) = 0 \tag{1.1}$$

studied by Levinson and Smith.[6] In the above equation and the following text, differentiation with respect to time will be denoted by a dot. Typical examples of systems that can be represented by this general type of equation are the onset of coherent radiation in lasers and masers,[5,7] self-excitations in electric circuits,[3] self-organization in chemical reactions,[4,5] nonlinear mechanics,[1,2,8,9] etc.

In this tutorial paper we shall present some useful theorems and concepts applying to such systems. Throughout the paper we shall elucidate the concepts and theorems in the context of the Van der Pol oscillator[9] and derive a few new results.

In Sec. II we shall present a pedagogical treatment of limit cycles using the Krylov–Bogoliubov averaging method. In Sec. III we consider the effect of a periodic driving force on the Van der Pol oscillator. In Sec. IV we supplement the driven Van der Pol oscillator with a noise source, reflecting the coupling of the system to the environment. We introduce a generalization of the averaging method to stochastic systems and discuss the effect of fluctuations which generally may drastically alter the behavior of the system.

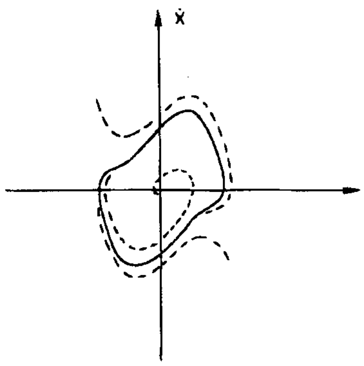

Fig. 1. Limit cycle of the Van der Pohl oscillator in the phase space $(x,\dot{x})$ corresponding to the parameters $\gamma = 1$, $\omega_0 = 1$.

II. LIMIT CYCLES AND FIRST APPROXIMATION OF KRYLOV AND BOGOLIUABOV

The instantaneous state of a nonlinear oscillator, at a particular time, can be described as a point in the phase space consisting of velocity $\dot{x}$ and position x. In the evolution of time, the points map out a trajectory in this phase space. As time increases, an arbitrary initial state will eventually settle into a stable pattern. One such stable pattern could be a point $\dot{x} = 0$, $x = \hat{x}$, which corresponds to completely damped out oscillations. Another more interesting possibility is the limit cycle: A limit cycle consists of a closed stable trajectory in the phase space (see Fig. 1).

There exists a practical theorem by Levinson and Smith[6] which gives the criteria for the existence of a stable limit cycle for a system described in Eq. (1.1) subject to the assumptions:

(i) $f(x)$ is even and $g(x)$ is an odd function, with the sign of $g(x)$ being the same as that of x.

(ii) The function $F(x) = \int_0^x f(y)dy$ possesses a single zero at a positive value of $x = x_1$. The function $F(x)$ is negative between $0 < x < x_1$ and increases monotonically for $x > x_1$ (Fig. 2).

We shall illustrate the above concept of a limit cycle by applying the useful *first approximation of Krylov and Bogoliubov* for the case of the Van der Pol oscillator[9]:

$$\ddot{x} + \omega_0^2 x - \gamma(1 - x^2)\dot{x} = 0 \quad \gamma>0. \tag{2.1}$$

For $\gamma = 0$, Eq. (2.1) reduces to the simple linear oscillator equation with the solution

$$x(t) = a \sin(\omega_0 t + \phi), \tag{2.2a}$$

$$\dot{x}(t) = a\omega_0 \cos(\omega_0 t + \phi). \tag{2.2b}$$

For the case that γ is a small parameter, we look for a solution of Eq. (2.1) retaining the form of Eq. (2.2) but considering the quantities a (amplitude) and ϕ (phase) to be functions of time to be determined. In order for the ansatz (2.2) to be consistent, we find a condition by differentiating the coordinate $x(t)$ with respect to time

$$\dot{x}(t) = \dot{a}(t) \sin[\omega_0 t + \phi(t)] + a(t)\omega_0 \cos[\omega_0 t + \phi(t)] + a(t)\dot{\phi}(t) \cos[\omega_0 t + \phi(t)] \tag{2.3}$$

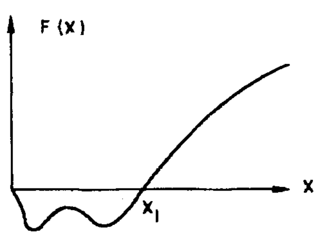

Fig. 2. Typical form of the function $F(x)$ required to have a limit cycle.

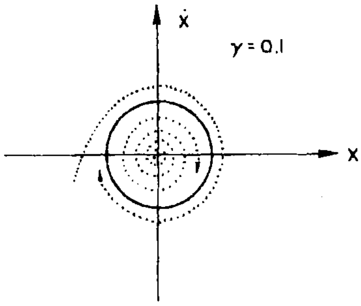

Fig. 3. Limit cycle, given by formulas (2.9) for the $\gamma = 0.1$ and $\omega_0 = 1$.

and by using Eq. (2.2b). This condition is

$$\dot{a}(t)\sin[\omega_0 t + \phi(t)] + a(t)\dot{\phi}(t)\cos[\omega_0 t + \phi(t)] = 0. \quad (2.4)$$

We shall, in the following supress the time dependence in the functions a and ϕ. After differentiating Eq. (2.2b) and substituting the values of x $\dot{x}$, and $\ddot{x}$ into Eq. (2.1), using the condition (2.4), we find the set of differential equations

$$\dot{a} = -\gamma[a^3\sin^2(\omega_0 t + \phi)\cos(\omega_0 t + \phi) - a\cos(\omega_0 t + \phi)]\cos(\omega_0 t + \phi), \quad (2.5a)$$

$$\dot{\phi} = (\gamma/a)[a^3\sin^2(\omega_0 t + \phi)\cos(\omega_0 t + \phi) - a\cos(\omega_0 t + \phi)]\sin(\omega_0 t + \phi). \quad (2.5b)$$

If γ is a small parameter, one can assume that both the amplitude a and phase ϕ are slowly varying quantities during one period $T = 2\pi/\omega_0$. Thus by *averaging* over a period (keeping a and ϕ as constant in this time interval) we obtain the approximation (first approximation of Krylov and Bogoliubov)

$$\dot{a} = (\gamma a/2)(1 - a^2/4), \quad (2.6a)$$

$$\dot{\phi} = 0. \quad (2.6b)$$

Multiplying Eq. (2.6a) by $2a$, we have

$$\frac{da^2}{dt} = \gamma a^2(1 - a^2/4), \quad (2.7)$$

which can be integrated to yield

$$a^2(t) = \hat{a}^2\exp\gamma t/[1 + (\hat{a}^2/4)(\exp\gamma t - 1)] \quad (2.8)$$

or

$$x(t) = \frac{\hat{a}\exp\frac{1}{2}\gamma t}{[1 + (\hat{a}^2/4)(\exp\gamma t - 1)]^{1/2}}\sin(\omega_0 t + \phi_0). \quad (2.9)$$

For any arbitrary initial state $(\hat{x},\dot{\hat{x}})$, that defines an amplitude

$$\hat{a} = [\hat{x}^2 + (\dot{\hat{x}}/\omega_0)^2]^{1/2} > 0, \quad (2.10)$$

the (approximate) trajectory of Eq. (2.1) spirals towards a circle of radius 2 (see Fig. 3). The approximation in Eq. (2.9) can be improved[10] by taking into account higher-order harmonics and calculating the fundamental frequency self-consistently. For large γ, the higher-order approximations do not lead to rapidly converging solutions. In this case, the method of adiabatic elimination of the velocity variable ($\ddot{x} = 0$) can be used to effect.

III. FORCED OSCILLATIONS

Nonlinear oscillators that are driven by an external periodic force can exhibit new phenomena such as generation of subharmonic and superharmonic oscillations with a transition to chaos,[8,11,12] frequency locking,[1,2,10] jumps of amplitude,[1,2,10] etc. the phenomenon of frequency locking plays an important role in technology such as synchronization of watches,[1] lasers, and electronic circuits.[3] Hence, we shall in the following focus our attention only on this latter phenomenon.

On general grounds, one would expect a forced nonlinear oscillator to exhibit (approximately) oscillations which occur at the frequencies corresponding to the natural frequency of the oscillator (ω_0) and the frequency of the driving force (ω). The two frequencies are not necessarily commensurate. The resulting motion of the oscillator would than have the appearance of beating. As the frequency of the driving force ω approaches the natural frequency of the oscillator the beating disappears suddenly at a certain value of the detuning $|\omega - \omega_0|$, and it is found that the system responds only at the frequency of the driving force ω. This phenomenon is called frequency locking, or entrainment of frequency. It has been first observed by Lord Raleigh[13] in connection with acoustic oscillations, and studied qualitatively by Vincent,[14] Moller,[15] Appleton[16] and Van der Pol.[17] For a more rigorous mathematical discussion see Ref. 18.

The Van der Pol oscillator in presence of a driving force is governed by the equation

$$\ddot{x} + \omega_0^2 x - \gamma(1 - x^2)\dot{x} = \gamma E\sin\omega t. \quad (3.1)$$

Applying the first approximation of Bogoliubov and Krylov, i.e.,

$$x = a(t)\cos[\omega t + \phi(t)],$$

we obtain the set of equations

$$\dot{a} = \frac{\gamma a}{2}\left(1 - \frac{a^2}{4}\right) + \frac{\gamma E}{2\omega}\sin\phi, \quad (3.2a)$$

$$\dot{\phi} = \omega_0 - \omega - \frac{\gamma E}{2\omega a}\cos\phi, \quad (3.2b)$$

where ω_0 is assumed to be close to ω. Stationary oscillations are obtained by setting $\dot{a} = 0$ and $\dot{\phi} = 0$ in Eq. (3.2). The singular points of these equations give us a solution with frequency ω; the frequency locked solution. Eliminating the phase ϕ yields the equation for the amplitudes:

$$a^2\left[\frac{4(\omega - \omega_0)^2}{\gamma^2} + \left(1 - \frac{a^2}{4}\right)^2\right] = \frac{E^2}{\omega^2} = \kappa^2 \quad (3.3)$$

for all given values of E, ω, ω_0, and γ. These amplitudes are plotted in Fig. 4 as a function of the detuning parameter $|\omega - \omega_0|/\gamma$, for various values of κ. Although there might exist up to three solutions (see Fig. 4), it can be shown,[18] that there is at most only one stable solution which corresponds to a physically realizable motion. The criteria for linear stability of the solution are calculated to be

$$a^2 > 2 \quad (3.4)$$

and

$$(3/16)a^4 - a^2 + 1 + 4(\omega - \omega_0)^2/\gamma^2 > 0. \quad (3.5)$$

The regions of stability in the $(a^2, |\omega - \omega_0|/\gamma)$ plane are depicted in Fig. 5. Solutions which lie above the dotted line are stable and correspond to the frequency-locked solutions. The solutions lying below the dotted line are unstable. In the latter case, one expects that the physical realizable solution is approximately described by a superposition of the forced oscillation and the natural oscillation, giving rise to a beating type of motion. The maximum amount of detuning $|\omega - \omega_0|$ for which a stable frequency locked so-

348

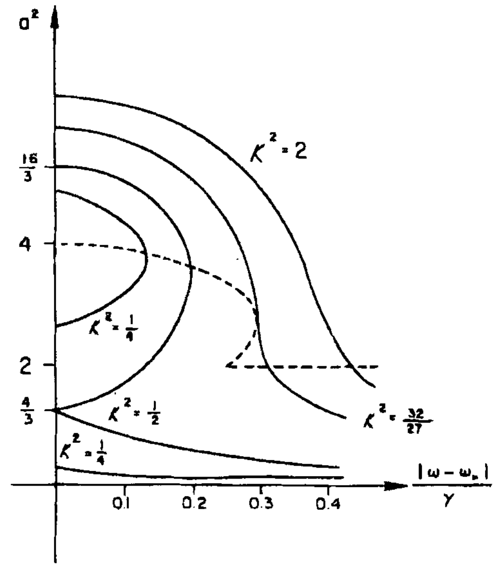

Fig. 4. Plot of amplitude squared versus the detuning for various strengths of the driving force.

lution may exist, can be expressed in terms of the strength of the driving force $\kappa = E/\omega$. For $\kappa^2 > 32/27$ the frequency locked solution is stable for detuning parameters which lie in the range

$$\frac{|\omega - \omega_0|}{\gamma} < \frac{1}{4}(2\kappa^2 - 1)^{1/2} \quad \kappa^2 > \frac{32}{27}. \tag{3.6}$$

For $\kappa^2 < 1$, the frequency locked solution is stable for all values ω up to a threshold value ω_c which is given by

$$\frac{|\omega_c - \omega_0|}{\gamma} = \frac{1}{\sqrt{12}}(1 - \theta_c)^{1/2} \tag{3.7}$$

with θ_c determined as the largest solution of the equation

$$\theta_c^{\,3} - 3\theta_c + 4 - (27/8)\kappa^2 = 0 \tag{3.8}$$

consistent with $\theta_c^2 < 1$. The three solutions are given by

$$\theta_c = 1 - 2\cos\left[\frac{1}{3}\tan^{-1}\left(\frac{\kappa(32/27 - \kappa^2)^{1/2}}{(16/27) - \kappa^2}\right)\right], \tag{3.9a}$$

$$\theta_c = 1 + 2\cos\left[\frac{1}{3}\tan^{-1}\left(\frac{\kappa(32/27 - \kappa^2)^{1/2}}{(16/27) - \kappa^2}\right) + \frac{\pi}{3}\right], \tag{3.9b}$$

$$\theta_c = 1 + 2\cos\left[\frac{1}{3}\tan^{-1}\left(\frac{\kappa(32/27 - \kappa^2)^{1/2}}{(16/27) - \kappa^2}\right) - \frac{\pi}{3}\right], \tag{3.9c}$$

Because of Eq. (3.7) we must have $\theta_c^2 < 1$, and since we must also satisfy Eq. (3.5), the relevant solution is Eq. (3.9a).

IV. DRIVEN NONLINEAR OSCILLATOR IN PRESENCE OF NOISE

In reality, oscillator systems are always subject to the action of environmental noise $\xi(t)$. Thus despite the externally applied oscillation "locking in" the solution, the noise is expected to bring about departures from completely locked-in behavior. There can occur (generally small but occasionally large) deviations of the phase from its "locked-in" value. These phase fluctuations in turn cause

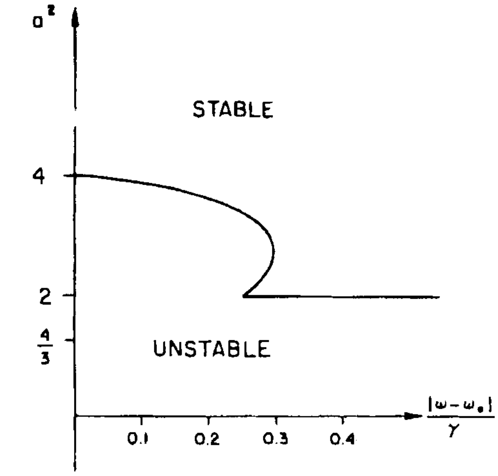

Fig. 5. Regions of stability.

the average frequency of the oscillation to differ from the locked-in value ω.

In presence of noise, the driven Van der Pol system is governed by the equation

$$\ddot{x} + \omega_0^2 x - \gamma(1 - x^2)\dot{x} = \gamma E \sin \omega t + \xi(t). \tag{4.1}$$

In what follows, we will assume the noise force $\xi(t)$ to be approximately independent of the state variables x, $\dot{x}$ (additive noise) and to obey Gaussian statistics with an infinitely short correlation time (Gaussian white noise):

$$\langle \xi(t)\xi(s)\rangle = 2C\delta(t - s). \tag{4.2}$$

With the noise present, we have in Eq. (4.1) a "stochastic" differential equation or Langevin equation.[3,5] This remark implies that $x(t)$ is no longer a deterministic but a stochastic trajectory. A most natural way to analyze Eq. (4.1) would be describe $x(t)$ in terms of a *stochastic* phase $\phi(t)$ and stochastic amplitude $a(t)$. Generally, however, this leads to complications, because under the nonlinear transformation $(x,\dot{x}) \to (a,\phi)$, into a polar coordinate description, the noise $\xi(t)$ in Eq. (4.1) transforms into state-dependent (multiplicative) noise. An approach which avoids this difficulty is the rotation of the phase plane $(x,\dot{x}/\omega)$ with angular velocity ω:

$$y_1 = x \cos \omega t - (\dot{x}/\omega)\sin \omega t, \tag{4.3a}$$

$$y_2 = x \sin \omega t + (\dot{x}/\omega)\cos \omega t, \tag{4.3b}$$

with inverse

$$x = y_1 \cos \omega t + y_2 \sin \omega t, \tag{4.4a}$$

$$\dot{x} = -\omega[y_1 \sin \omega t - y_2 \cos \omega t]. \tag{4.4b}$$

Differentiating Eq. (4.3) we obtain with Eqs. (4.4), (4.1):

$$\begin{aligned}\dot{y}_1 = (1/\omega)\{ &[\omega_0^2 - \omega^2](y_1 \cos \omega t + y_2 \sin \omega t) \\ &+ \omega\gamma[1 - (y_1 \cos \omega t + y_2 \sin \omega t)^2] \\ &\times (y_1 \sin \omega t - y_2 \cos \omega t) \\ &- \gamma E \sin \omega t - \xi(t)\} \sin \omega t,\end{aligned} \tag{4.5a}$$

$$\begin{aligned}\dot{y}_2 = (1/\omega)\{ &-[\omega_0^2 - \omega^2](y_1 \cos \omega t + y_2 \sin \omega t) \\ &- \omega\gamma[1 - (y_1 \cos \omega t + y_2 \sin \omega t)^2] \\ &\times (y_1 \sin \omega t - y_2 \cos \omega t) \\ &+ \gamma E \sin \omega t + \xi(t)\} \cos \omega t.\end{aligned} \tag{4.5b}$$

With ω close to ω_0, the variables (y_1, y_2) corresponding to the rotating phase plane will, with γ not too large a param-

349

eter, be slowly varying over a period $T = 2\pi/\omega$. Thus averaging Eq. (4.5) over a period T we obtain the approximate stochastic differential equations ($\omega_0 + \omega \sim 2\omega$):

$$\dot{y}_1 = (\omega_0 - \omega) y_2 + \frac{\gamma}{2} y_1 - \frac{\gamma}{8} y_1 (y^2{}_1 + y^2{}_2) - \frac{\gamma E}{2\omega} - \bar{\xi}_1(t), \quad (4.6a)$$

$$\dot{y}_2 = -(\omega_0 - \omega) y_1 + \frac{\gamma}{2} y_2 - \frac{\gamma}{8} y_2 (y^2{}_1 + y^2{}_2) + \bar{\xi}_2(t). \quad (4.6b)$$

Since the correlation time of $\langle \bar{\xi}(t) \bar{\xi}(s)\rangle$ is much smaller than the time scale over which the variables (y_1, y_2) change we may replace the noise $\bar{\xi}(t)$ by white noise:

$$\langle \bar{\xi}_1(t) \bar{\xi}_1(s)\rangle = \langle \bar{\xi}_2(t) \bar{\xi}_2(s)\rangle = (C/\omega^2)\delta(t-s) \equiv 2D\delta(t-s). \quad (4.7)$$

Instead of describing the system by stochastic trajectories, a more useful approach is the description by the probability

$$P_t(y_1, y_2; E) = \langle \delta(y_1(t) - y_1)\delta(y_2(t) - y_2)\rangle \quad (4.8)$$

obtained by averaging over all trajectories and initial distribution $p_{t=t_0}(y_1^{(0)}, y_2^{(0)}; E)$ of starting values $(y_1^{(0)}, y_2^{(0)})$. Taking into account the Gaussian nature of the white noise in Eq. (4.7), the rate of change of the probability Eq. (4.8) is governed by the Fokker–Planck equation (a summation over equal indices is implied)

$$\dot{p}_t = -\frac{\partial}{\partial y_i}(A_i p_t) + D_{ij}\frac{\partial^2}{\partial y_i \partial y_j} p_t, \quad (4.9a)$$

where

$$A_1(y_1, y_2; E) = (\omega_0 - \omega) y_2 + (\gamma/2) y_1 \times (1 - \tfrac{1}{4}(y_1^2 + y_2^2)) - \tfrac{1}{2}\gamma E/\omega, \quad (4.9b)$$

$$A_2(y_1, y_2; E) = -(\omega_0 - \omega) y_1 + \frac{\gamma}{2} y_2 (1 - \tfrac{1}{4}(y_1^2 + y_2^2)), \quad (4.9c)$$

$$D_{ij} = D\delta_{ij} \quad i, j = 1, 2. \quad (4.9d)$$

Next we transform the Fokker–Planck equation (4.9a), into amplitude and phase variables,

$$y_1 = u_1 \sin u_2 \equiv a \sin \phi, \quad (4.10a)$$

$$y_2 = u_1 \cos u_2 \equiv a \cos \phi. \quad (4.10b)$$

In the new variables $(u_1, u_2) = (a, \phi)$ the drift $\bar{A}_i$ is calculated to be

$$\bar{A}_i = A_j \frac{\partial u_i}{\partial y_j} + D_{nm}\frac{\partial^2 u_i}{\partial y_n \partial y_m} \quad (4.11)$$

and the diffusion coefficients are

$$\bar{D}_{ij} = D_{nm}\left(\frac{\partial u_i}{\partial y_n}\right)\left(\frac{\partial u_j}{\partial y_m}\right), \quad (4.12)$$

which leads to

$$A_a = \frac{\gamma}{2} a \left(1 - \frac{a^2}{4}\right) - \gamma \frac{E}{2\omega}\sin\phi + \frac{D}{a}, \quad (4.13a)$$

$$A_\phi = (\omega_0 - \omega) - (\gamma E/2\omega a)\cos\phi, \quad (4.13b)$$

$$D_{aa} = D, \quad D_{\phi\phi} = D/a^2, \quad D_{a\phi} = D_{\phi a} = 0. \quad (4.13c)$$

Obviously, up to a phase change $\phi \rightarrow -\phi$, the phase drift A_ϕ, coincides with the deterministic drift in Eq. (3.2b) and the amplitude drift A_a coincides with Eq. (3.2a) supplemented by a noise-induced drift D/a.

We have not been able to solve the Fokker–Planck equation (4.9) or Eq. (4.13) for the stationary ($\dot{p}_t = 0$) distribution p_s,

$$p_s(y_1, y_2; E) dy_1\, dy_2 = p_s(a, \phi) a\, da\, d\phi. \quad (4.14)$$

However, if we assume the inequality

$$\gamma a \gg \gamma E/\omega a, \text{ i.e., } E/\omega a^2 \ll 1 \quad (4.15)$$

the amplitude a changes much more rapidly than the phase. Denoting the stable amplitude by a_0 and the phase by ϕ_0, respectively, the system will, in the limit of small noise,

$$D \ll \gamma a_0^2, \quad (4.16)$$

settle down on a fast time scale to a value near a_0, with small amplitude fluctutions [see Eq. (4.16)]

$$\langle (a - a_0)^2\rangle \ll a_0^2. \quad (4.17)$$

Therefore, we can to a good approximation replace a in Eq. (4.13a,b) by a_0, and by observing Eq. (4.15), neglect the second term in Eq. (4.13a). Thus the Fokker–Planck equation (4.13) decouples with these approximations yielding for the phase fluctuations the strongly simplified equation

$$\dot{p}_t(\phi) = -\frac{\partial}{\partial \phi}\{[\delta + \delta_E \cos\phi]\, p_t(\phi)\} + \frac{D}{a_0^2}\frac{\partial^2}{\partial \phi^2} p_t(\phi), \quad (4.18a)$$

where

$$\delta = \omega_0 - \omega, \quad \delta_E = -\gamma E/2\omega a_0. \quad (4.18b)$$

The drift term $A_\phi = -\partial V(\phi)/\partial\phi$, can be derived from the potential

$$V(\phi) = -\delta\phi - \delta_E \sin\phi. \quad (4.19)$$

The Fokker–Planck equation (4.18) appears in many applications.[3,19–21] It describes the motion of a "Brownian particle" in a tilted sinusoidal potential (Fig. 6). Equation (4.18) has first been studied in detail by Stratonovich while describing current oscillations in vacuum tubes.[22] Equation (4.18) possesses a stationary and periodic phase distribution $p_s(\phi)$ given by[22]

$$p_s(\phi) = \frac{1}{Z}\exp\left[-\frac{V(\phi)}{\bar{D}}\int_\phi^{\phi + 2\pi} \exp\frac{V(\psi)}{\bar{D}}\, d\psi\right] \quad (4.20a)$$

with

$$\bar{D} = D/a_0^2. \quad (4.20b)$$

This stationary phase distribution implies a frequency shift determined by

$$\langle \dot{\phi}\rangle \equiv \langle\omega\rangle - \omega = \delta + \delta_E\langle\cos\phi\rangle \quad (4.21)$$

in which the average is calculated by the use of Eq. (4.20) (see Fig. 7).

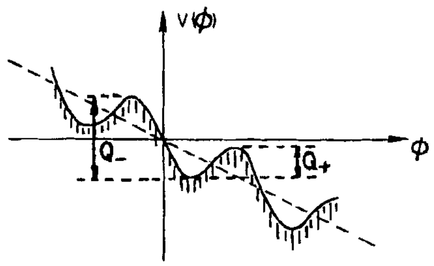

Fig. 6. Potential in Eq. (4.19) governing phase fluctuations.

350

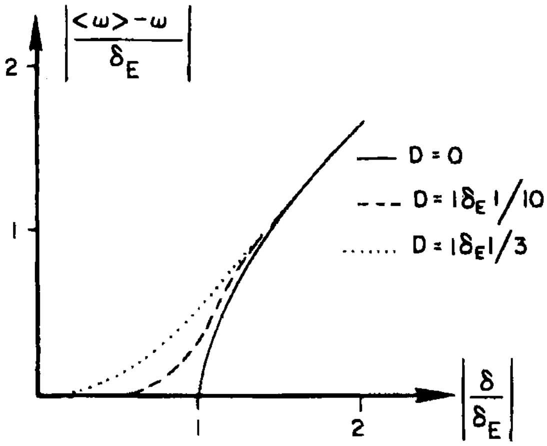

Fig. 7. Frequency shift versus the detuning for various noise levels (after Stratonovich).

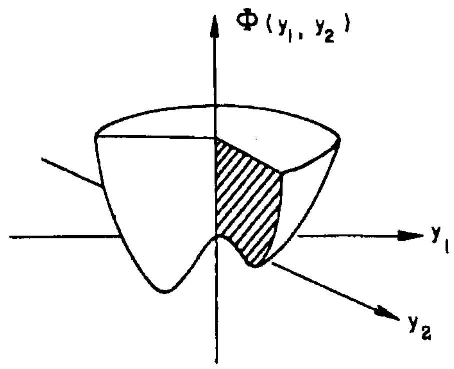

Fig. 8. Potential Φ corresponding to the probability distribution in Eq. (4.26a).

From Fig. 6 we have many points of stability (local minima), separated by 2π, where the phase of the oscillator is locked to the external oscillation. In the limit of small noise, the phase undergoes for most of the time only small perturbations around the value at its local minimum. Occasionally, there can occur fluctuation-induced large jumps over the local barrier. These large phase jumps occur with greater probability in the downhill direction and with less probability in the uphill direction. These give rise to a diffusion of the number $n(t)$ of phase jumps

$$\langle (n(t) - \langle n(t)\rangle)^2\rangle = (\lambda_+ + \lambda_-)t, \tag{4.22}$$

where $\lambda_\pm$ denote the rates for forward and backward jumps, respectively. If we denote the activation energy for forward and backward jumps by Q_+ and Q_-, respectively (Fig. 6), the rates $\lambda_\pm$ are in the limit of small noise (long lived locally stable states) approximately given by a Kramers[23]-type formula

$$\lambda_\pm = \frac{(V''(\phi_0)|V''(\phi_b)|)^{1/2}}{2\pi}\exp(-Q_\pm/\overline{D}), \tag{4.23}$$

where ϕ_0 denotes the phase value at the locally stable potential valley and ϕ_b the value at barrier top.

The case with

$$E/\omega \sim a^2 \tag{4.24}$$

cannot be treated by Eq. (4.18) and we must go back to Eq. (4.9). From a practical point of view, the optimal situation for a stable frequency locked regime is obtained for zero detuning. With $\omega_0 = \omega$, the drift terms can be derived again from a potential

$$A_j = -\frac{\partial}{\partial y_i}\Phi(y_1 y_2;E) \quad i = 1,2 \tag{4.25}$$

with

$$\Phi(y_1, y_2;E) = -\frac{\gamma}{4}(y_1^2 + y_2^2) + \frac{\gamma}{32}(y_1^2 + y_2^2)^2 + \frac{\gamma E}{2\omega}y_1 \tag{4.26a}$$

$$= -\frac{\gamma}{4}a^2\left(1 - \frac{a^2}{8}\right) + \frac{\gamma E}{2\omega}a\sin\phi. \tag{4.26b}$$

The stationary probability $p_s(y_1, y_2;E)$ is readily evaluated to be given by

$$p_s(y_1, y_2;E) = (1/Z)\exp - \Phi(y_1 y_2;E)/D. \tag{4.27}$$

Differentiating the potential Φ with respect to y_1, y_2, we observe that the extrema of p_s coincide with the deterministic steady state, and the deterministically stable frequency-locked state (see Sec. III) corresponds to the global minimum of p_s. The steady states have coordinate $y_2 = 0$, and for

$$\kappa^2 = E^2/\omega^2 < 16/27, \tag{4.28}$$

we have three real steady states with coordinates ($y_1^{(1)}, y_1^{(2)}, y_1^{(3)}$). Then, the potential $\Phi(y_1, y_2;E)$ is cup-shaped (see Fig. 8) with the lowest point corresponding to the frequency-locked solution for amplitude and phase. The stationary amplitude distribution is from Eq. (4.14) given by an integration over the phase ϕ

$$p_s(a) = \frac{a}{Z}I_0\left(\frac{\gamma Ea}{2\omega D}\right)\exp\frac{\gamma}{D}\left(\frac{a^2}{4} - \frac{a^4}{32}\right) \tag{4.29}$$

with I_0 being the modified Bessel function of the first kind and order zero.[24] The phase fluctuations are obtained by integrating over the amplitude

$$p_s(\phi) = \frac{1}{Z}\int_0^\infty a\exp\left\{\left[\frac{\gamma a^2}{4}\left(1 - \frac{a^2}{8}\right) - \frac{\gamma E}{2\omega}a\sin\phi\right]/D\right\}da, \tag{4.30}$$

which does not yield a simple closed expression.

In conclusion, the Krylov–Bogoliubov averaging method has proved to be rather fruitful in producing reliable results for stochastic oscillator dynamics.

[1]N. Minorski, *Nonlinear Oscillations* (Van Nostrand, Princeton, NJ, 1962).
[2]A. H. Nayfeh and D. T. Mook, *Nonlinear Oscillations, Pure and Applied Mathematics* (Wiley, New York, 1979).
[3]R. S. Stratonovich, *Topics in the Theory of Random Noise* (Gordon and Breach, New York, 1967), Vol. II.
[4]M. P. Volkenstein, *Obschchaya Biofizika* (Nauka, Moscow, 1978).
[5]H. Haken, Rev. Mod. Phys. 47, 67 (1975).
[6]N. Levinson and O. K. Smith, Duke Math. J. 9, 382 (1942).
[7]H. Risken, *Progress in Optics*, edited by E. Wolf (North-Holland, Amsterdam, 1970), Vol. 8, p. 240.
[8]P. Holmes, Proc. R. Soc. A 292, 419 (1979).
[9]B. Van der Pol, Philos. Mag. 2, (Nov. 1926).
[10]N. Krylov and N. Bogoliubov, *Introduction to Nonlinear Mechanics*, (Kiev, 1937, Chaps. 10, 11, 12 (Russian), translated by S. Lefschetz; *Annals Math. Studies No. 11* (Princeton University, Princeton, 1943).
[11]B. A. Huberman, J. P. Crutchfield, and N. H. Packard, Appl. Phys. Lett. 37, 750 (1980).

351

[12]P. S. Linsay, Phys. Rev. Lett. **47**, 1349 (1981); S. Novak and R. Prehlich, preprint 1981 (unpublished).
[13]Lord Raleigh, Philos. Mat. Ser. 5, **15**, (April 1883).
[14]J. H. Vincent, Proc. Phys. Soc. London **32**, Part II, (February 1919).
[15]H. G. Möller, Z. Telegraph. Telephon. 17, (April 1921).
[16]E. V. Appleton, Proc. Cambridge Philos. Soc. **21**, (Nov. 1922).
[17]B. Van der Pol, Philos. Mag. 3, 65 (1927).
[18]M. L. Cartwright, J. Inst. Elec. Eng. **95** III, 88 (1948); Ann. Math. Stud. No. 20, 202 (1950).
[19]W. C. Lindsey, Proc. IEEE **57**, 1705 (1969).
[20]M. Büttiker and R. Landauer, in *Nonlinear Phenomena at Phase Transitions and Instabilities*, edited by T. Riste (Plenum, New York, 1982), p. 111.
[21]A. R. Bishop and S. E. Trullinger, Phys. Rev. B **17**, 2175 (1978).
[22]Reference 3, p. 236–258.
[23]H. A. Kramers, Physica **7**, 284 (1940).
[24]M. Abramowitz and I. A. Stegun, *Handbook of Math. Functions* (Dover, New York 1970), Relation 9.6.16.

352