Topological Computation of Stokes Data of Weighted Projective Lines

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Topological Computation of Stokes Data of Weighted Projective Lines

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Abstract

In this thesis, we compute the Stokes data of some differential equations arising from small quantum cohomology of a Fano variety (resp. a stack) $\mathcal{X}$. It is known from mirror symmetry that these connections—being irregular singular at ∞—are essentially given by the localized Fourier–Laplace transform of the regular singular Gauß–Manin system of the Landau–Ginzburg model of $\mathcal{X}$. A. D'Agnolo, M. Hien, G. Morando, and C. Sabbah in [**9**] compute the Stokes data at ∞ for the Fourier–Laplace transform of a regular singular holonomic $\mathcal{D}$-module on the affine line in a purely topological way. Fitting perfectly into this situation, we compute the Stokes data of the Fourier–Laplace transform of the Gauß–Manin system of the Landau–Ginzburg model of some weighted projective lines purely topologically. B. Dubrovin conjectured that, under appropriate choices, the Stokes matrix of the quantum connection can be obtained as the Gram matrix of the Euler–Poincaré pairing χ on $\mathrm{D}^{\mathrm{b}}(\mathrm{Coh}\,(\mathcal{X}))$. We explicitly give the transformations that deform the Gram matrix of χ into the topologically computed Stokes matrices.

Zusammenfassung

In dieser Arbeit berechnen wir die Stokes-Daten einiger Differentialgleichungen, die der kleinen Quantenkohomologie einer Fano-Varietät (oder allgemeiner eines Stacks) $\mathcal{X}$ entstammen. Diese Zusammenhänge haben eine irreguläre Singularität bei ∞. Aus der Spiegelsymmetrie ist bekannt, dass sie im Wesentlichen der lokalisierten Fourier–Laplace-Transformierten des Gauß–Manin-Systems eines Spiegelpartners von $\mathcal{X}$, des sogenannten Landau–Ginzburg-Modells, entsprechen. In [**9**] berechnen A. D'Agnolo, M. Hien, G. Morando und C. Sabbah die Stokes-Daten der Fourier–Laplace-Transformation eines regulär singulären holonomen $\mathcal{D}$-Moduls auf der affinen Geraden mit rein topologischen Mitteln. Unsere Situation reiht sich perfekt in die von [**9**] ein. Mithilfe dieser Mittel berechnen wir die Stokes-Daten der Fourier–Laplace-Transformation des Gauß–Manin-Systems des Landau–Ginzburg-Modells einiger gewichteter projektiver Geraden. Nach einer Vermutung von B. Dubrovin erhält man die Stokes-Matrix des Quantenzusammenhangs unter geeigneten Wahlen als Gram-Matrix der Euler–Poincaré-Paarung χ auf $\mathrm{D^b}\left(\mathrm{Coh}\left(\mathcal{X}\right)\right)$. Wir vergleichen die topologisch berechneten Stokes-Matrizen mit der Gram-Matrix von χ.

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Contents

CHAPTER I

Introduction

This thesis is about $\mathcal{D}$-modules within the field of Algebraic Analysis. $\mathcal{D}$-modules are an algebraic model of systems of linear (partial) differential equations, combining elegantly methods from Algebraic Geometry, Complex Analysis and Category Theory. Let X be a complex manifold (resp. a smooth algebraic variety over the field of complex numbers). The non-commutative sheaf $\mathcal{D}_X$ is the subsheaf of $\mathcal{E}nd_{\mathbb{C}_X}(\mathcal{O}_X)$ generated by $\mathcal{O}_X$ and Θ_X as an algebra over $\mathbb{C}_X$. The non-commutativity is determined by $[\theta, f] = \theta(f)$ for local sections $\theta \in \Theta_X,\ f \in \mathcal{O}_X$. A left-$\mathcal{D}_X$-module $\mathcal{M} \in \mathrm{Mod}(\mathcal{D}_X)$ can be seen as a generalization of systems of linear (partial) differential equations with holomorphic (resp. polynomial) coefficients. In the 1980s, M. Kashiwara and Z. Mebkhout independently of each other proved the equivalence of the bounded derived category of regular holonomic $\mathcal{D}_X$-modules and the bounded derived category of $\mathbb{C}$-constructible sheaves on X:

$$\mathcal{S}\mathrm{ol} : \mathrm{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_X)^{\mathrm{op}} \xrightarrow{\simeq} \mathrm{D}^{\mathrm{b}}_{\mathbb{C}-\mathrm{constr.}}(\mathbb{C}_X)$$

is an equivalence of categories given by the solution functor[1] $\mathcal{S}\mathrm{ol}(\bullet) = \mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}((\bullet), \mathcal{O}_X)$.
It restricts to an equivalence

$$\mathcal{S}\mathrm{ol}[d_X] : \mathrm{Mod}_{\mathrm{rh}}(\mathcal{D}_X)^{\mathrm{op}} \xrightarrow{\simeq} \mathrm{Perv}(\mathbb{C}_X)$$

from the category of regular holonomic $\mathcal{D}_X$-modules to the category of perverse sheaves, where d_X denotes the complex dimension of X. Perverse sheaves being a generalization of representations of the fundamental group of X, this correspondence gives a sophisticated answer to Hilbert's 21st problem in a more general setting and is called *Riemann–Hilbert correspondence*. Within the last years, a lot of effort has been put in generalizing the Riemann–Hilbert equivalence to not necessarily regular singular, i.e., possibly irregular singular holonomic $\mathcal{D}_X$-modules. Using the theory of enhanced ind-sheaves, A. D'Agnolo and M. Kashiwara in [**10**] proved that the enhanced solution functor

$$\mathcal{S}\mathrm{ol}^{\mathrm{E}}_X[d_X] : \mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(\mathcal{D}_X)^{\mathrm{op}} \longrightarrow \mathrm{E}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathrm{I}\mathbb{C}_X)$$

gives a fully faithful embedding of the triangulated category of possibly irregular singular holonomic $\mathcal{D}_X$-modules into a new category of $\mathbb{R}$-constructible enhanced ind-sheaves. Based on that, A. D'Agnolo, M. Hien, G. Morando, and C. Sabbah in [**9**] give a fruitful method for the computation of the Stokes data of the enhanced Fourier–Sato transform of a perverse sheaf F associated to a regular singular holonomic $\mathcal{D}$-module $\mathcal{M}$ on the affine line, the Fourier–Sato transform of F being the enhanced ind-sheaf associated to the Fourier–Laplace transform of $\mathcal{M}$.[2] The work [**9**] recovers results of B. Malgrange [**22**] in a purely topological way.

[1] If X is an algebraic variety, one has to consider the analytified setting, i.e., one has to consider holomorphic solutions $\mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_{X^{\mathrm{an}}}}((\bullet)^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) : \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X)^{\mathrm{op}} \to \mathrm{D}^{\mathrm{b}}_{\mathbb{C}-\mathrm{constr.}}(\mathbb{C}_{X^{\mathrm{an}}})$. In the following, for the sake of notational simplicity, we often omit the superscript $(\bullet)^{\mathrm{an}}$.

[2] It was observed in [**19**] that, by functoriality, the Riemann–Hilbert correspondence of [**10**] interchanges the Fourier–Laplace transform for holonomic $\mathcal{D}$-modules with the Fourier–Sato transform for enhanced ind-sheaves.

In this thesis, we apply the methods of [**9**] to examples arising from mirror symmetry. Mirror symmetry describes in a precise mathematical language the duality of geometric objects which was observed by physicists in the setting of string theory.
From small quantum cohomology of Fano varieties we obtain a class of linear differential equations being regular singular at 0 and irregular singular at ∞. We are interested in the Stokes data at ∞ which describe the change of the asymptotic behavior of its holomorphic solutions when varying the considered direction. Let $\mathcal{X}$ be a Fano variety (or, more generally, a stack) over some field k such that the bounded derived category $\mathrm{D^b}(\mathrm{Coh}(\mathcal{X}))$ of coherent sheaves on X admits a full exceptional collection $\langle E_1, \dots, E_n\rangle$. The small quantum cohomology of $\mathcal{X}$ gives rise to a flat meromorphic connection on the trivial bundle over $\mathbb{P}^1$ with fiber $H^*(\mathcal{X}, \mathbb{C})$ which is regular singular at 0 and irregular singular at ∞. B. Dubrovin in [**14**] conjectured that the Stokes matrix at ∞ of the small quantum connection of $\mathcal{X}$, under appropriate choices, is given by the Gram matrix of the bilinear form

$$\chi(E,F) = \sum_k (-1)^k \dim \mathrm{Ext}^k(E,F), \quad E,F \in \mathrm{D^b}(\mathrm{Coh}(\mathcal{X})),$$

the *Euler–Poincaré pairing*, with respect to some full exceptional collection of $\mathrm{D^b}(\mathrm{Coh}(\mathcal{X}))$. The Gram matrix $S = (s_{ij})_{i,j}$ with respect to a full exceptional collection is upper triangular with ones on the diagonal.
B. Dubrovin himself proved the conjecture for the complex projective line and complex projective plane. D. Guzzetti proved this conjecture in [**17**] for complex projective space $\mathbb{P}^n$ for arbitrary n. It is proven for weighted projective spaces $\mathbb{P}(\omega_0, \dots, \omega_n)$ in [**29**] by S. Tanabé and K. Ueda and in [**8**] by J. A. Cruz Morales and M. van der Put.

By mirror symmetry, the quantum connection of $\mathcal{X}$ is closely related to the Fourier–Laplace transform of the Gauß–Manin system of a Landau–Ginzburg model of $\mathcal{X}$. The zeroth cohomology of the Gauß–Manin system is a regular singular holonomic $\mathcal{D}$-module on the affine line $\mathbb{A}^1$. Its Fourier–Laplace transform is known to be regular singular at 0 and irregular singular at ∞. By work of B. Malgrange it is known that the coefficients of the exponential components at ∞ are of linear type and given by the singularities of the regular singular system. For this kind of $\mathcal{D}$-modules—the Fourier–Laplace transform of some holonomic $\mathcal{D}$-module $\mathcal{M} \in \mathrm{Mod_{rh}}(\mathcal{D}_{\mathbb{A}^1})$, regular everywhere including at infinity—A. D'Agnolo, M. Hien, G. Morando, and C. Sabbah in [**9**] give a purely topological method for the computation of the Stokes data consisting of the formal type and two Stokes matrices as gluing data of the local systems of holomorphic solutions. One considers the perverse sheaf $F \in \mathrm{Perv}_\Sigma(\mathbb{C}_{\mathbb{A}^1})$ associated to the regular singular $\mathcal{D}$-module $\mathcal{M}$ via the Riemann–Hilbert correspondence, where $\Sigma \subset \mathbb{A}^1$ denotes the set of singularities of $\mathcal{M}$. The resulting perverse sheaf can be described by linear algebra data, namely its quiver

$$Q_\Sigma^{(\alpha,\beta)}(F) = (\Psi_\Sigma(F), \Phi_\sigma(F), u_\sigma, v_\sigma)_{\sigma\in\Sigma}.$$

The quiver of F consists of finite dimensional $\mathbb{C}$-vector spaces—the vanishing and (global) nearby cycles of F—and linear maps $u_\sigma \colon \Psi_\Sigma(F) \to \Phi_\sigma(F)$ and $v_\sigma \colon \Phi_\sigma(F) \to \Psi_\Sigma(F)$ such that $1 - u_\sigma v_\sigma$ is invertible for any $\sigma \in \Sigma$ and $\alpha \in \mathbb{A}^1$, $\beta \in (\mathbb{A}^1)^\vee$ determine a total order on Σ and an orientation. The main result in [**9**] is a determination of the Stokes multipliers at ∞ of the Fourier–Sato transform of F—and therefore the Fourier–Laplace transform of $\mathcal{M}$—in terms of the quiver of F.

In this thesis, we compute the Stokes data of the quantum connection of some weighted projective lines $\mathbb{P}(a,b)$, $a,b \in \mathbb{N}$, in a purely topological way following [**9**] and compare them to the Gram matrix of the Euler–Poincaré pairing on $\mathrm{D^b}(\mathrm{Coh}(\mathbb{P}(a,b)))$ with respect to the full exceptional collection $\langle \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(a+b-1)\rangle$ of $\mathrm{D^b}\left(\mathrm{Coh}\left(\mathbb{P}\left(a,b\right)\right)\right)$. For the topological computations we use the Landau–Ginzburg models (on the A-side of mirror symmetry) (X,f) of the weighted projective line. By work of A. Douai, É. Mann, and C. Sabbah, there are explicit formulae for the mirror partner (X,f) of weighted projective spaces. By the regular Riemann–Hilbert correspondence, we associate to the regular singular Gauß–Manin system $H^0\left(\int_f \mathcal{O}\right) \in \mathrm{Mod_{rh}}(\mathcal{D}_{\mathbb{A}^1})$ the perverse sheaf $\mathrm{R}f_*\mathbb{C}[1] \in \mathrm{Perv}_\Sigma(\mathbb{C}_{\mathbb{A}^1})$, where Σ denotes the set of singular values of f. The potential f of the Landau–Ginzburg model of weighted projective lines is proper, cohomologically tame and semismall. Therefore, $\mathrm{R}f_*\mathbb{C}[1]$ is indeed a perverse sheaf and we can apply the topological methods of [**9**] in order to compute the Stokes data of its Fourier–Laplace transform.

Let us give an overview of the structure of this thesis.

In Chapter II, we describe the Stokes phenomenon from a classical point of view by Asymptotic Analysis at the examples of the Airy and Bessel's differential equation.

Chapter III treats Gauß–Manin systems associated to Laurent polynomials and their localized Fourier–Laplace transform. We explicitly compute some examples that will be of interest in the course of the thesis.

In Chapter IV, we compute the quantum connection of some (weighted) projective spaces and describe their relation to the appropriate Gauß–Manin systems. According to Dubrovin's conjecture in the setting of mirror symmetry, the Stokes matrix of the quantum connection of $\mathbb{P}(a,b)$ under appropriate choices equals the Gram matrix of the Euler–Poincaré pairing on $\mathrm{D^b}\left(\mathrm{Coh}\left(\mathbb{P}\left(a,b\right)\right)\right)$ with respect to some full exceptional collection of $\mathrm{D^b}\left(\mathrm{Coh}\left(\mathbb{P}\left(a,b\right)\right)\right)$. We compute the Gram matrix of the Euler–Poincaré pairing with respect to the full exceptional collection $\langle \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(a+b-1)\rangle$ of $\mathrm{D^b}\left(\mathrm{Coh}\left(\mathbb{P}\left(a,b\right)\right)\right)$ for the cases that are of interest for this thesis.

In Chapter V, we carry out the topological computations of Stokes data for the examples $\mathrm{R}f_*\mathbb{C}[1]$ for $f = x^2 + x^{-2}$ and for the Landau–Ginzburg models (X,f) of the weighted projective lines $\mathbb{P}(1,2)$, $\mathbb{P}(1,3)$, and $\mathbb{P}(2,2)$. These are the main results of this thesis and are recorded in

- Theorem V.5 for $\mathrm{R}f_*\mathbb{C}[1]$, where $f = x^2 + x^{-2}$,
- Theorem V.9 for $\mathbb{P}(1,2)$,
- Theorem V.12 for $\mathbb{P}(1,2)$ with an alternate choice of bases,
- Theorem V.14 for $\mathbb{P}(1,3)$,
- Theorem V.18 for $\mathbb{P}(2,2)$.

We emphasize that the computation of the Stokes data uses purely topological methods and, in contrast to present research results, does not require Asymptotic Analysis. Moreover, we give the explicit transformations that deform the Gram matrix of the Euler–Poincaré pairing with respect to the full exceptional collection $\langle \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(a+b-1)\rangle$ into the topologically computed Stokes matrices. The computations for the weighted projective line $\mathbb{P}(1,3)$ have already been published in the article [**28**].

We assume the reader to be familiar with the language of $\mathcal{D}$-modules and refer all the others to the book [**18**] of R. Hotta, K. Takeuchi, and T. Tanisaki as an introduction to this subject.

CHAPTER II

Stokes phenomenon

For regular singular holonomic $\mathcal{D}$-modules, the Riemann–Hilbert correspondence ensures that all the information necessary for the classification is encoded in the monodromy data of its solutions. The irregular singular case behaves fundamentally different, the so called Stokes phenomenon comes into play. The asymptotic behavior of solutions at an irregular singular point changes when crossing certain directions. The gluing data of the local system of its holomorphic solutions is given by Stokes matrices and necessary for the local classification at an irregular singular point. G. Stokes observed this phenomenon when considering the entire solutions of the Airy equation

$$u''(z) - zu(z) = 0,$$

which has an irregular singularity at $z = \infty$.
In this chapter, the Stokes phenomenon is described at the examples of the Airy and Bessel's differential equation in the classical language by means of Asymptotic Analysis.

1. Airy equation

Following [**30**], let us consider the Airy equation in more detail. G. Stokes observed the phenomenon mentioned above when considering the Airy functions—the entire solutions of the Airy equation

$$u''(z) - zu(z) = 0. \tag{1}$$

The differential equation has $z = \infty$ as its only singular point, being irregular singular. The differential equation has two linearly independent entire solutions, called Airy function of first, resp. second kind.
We associate to (1) the differential operator

$$P_{\text{Airy}} = \partial_z^2 - z \in \mathbb{C}[z]\langle\partial_z\rangle. \tag{2}$$

In matrix form, the Airy equation is given by the system

$$\frac{1}{z}\frac{\partial Y}{\partial z} = \begin{pmatrix} 0 & \frac{1}{z} \\ 1 & 0 \end{pmatrix} Y, \tag{3}$$

where we set $Y = (u(z), u'(z))^{\mathrm{t}}$ for $u(z)$ a solution of (1).
The entire solutions of the Airy equation have an integral representation

$$u_j(z) = \int_\Gamma e^{zt} e^{-\frac{t^3}{3}}\, dt \tag{4}$$

for appropriately chosen integration contours Γ. In order to understand how this representation arises, consider the Fourier–Laplace transform induced by the isomorphism of Weyl algebras

$$\widehat{(\bullet)}\colon \mathbb{C}[z]\langle\partial_z\rangle \xrightarrow{\cong} \mathbb{C}[t]\langle\partial_t\rangle, \quad z \mapsto -\partial_t,\ \partial_z \mapsto t.$$

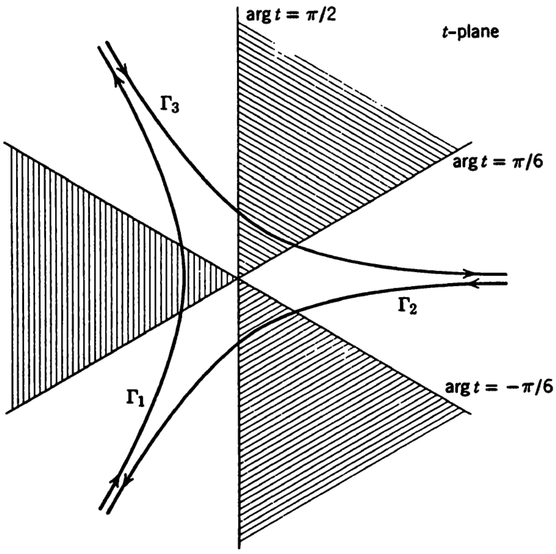


FIGURE 1. [**30**, Figure 22.1]

The Fourier–Laplace transform of the differential operator (2) is given by

$$\widehat{P_{\mathrm{Airy}}} = \partial_t + t^2 \in \mathbb{C}[t]\langle\partial_t\rangle$$

and the associated differential equation by

$$\hat{u}'(t) + t^2\hat{u}(t) = 0. \tag{5}$$

One easily verifies that the function $\hat{u}(t) = e^{-\frac{t^3}{3}}$ is a solution of (5). Then the inverse Fourier–Laplace transform of $\hat{u}(t)$ given by

$$\int e^{zt-\frac{t^3}{3}}\,dt$$

is a solution of equation (1), where we need to find appropriate integration contours such that the integral converges. If we choose loops at finite distance, the integral vanishes according to Cauchy's theorem. Therefore, in order to obtain non-trivial solutions of the equation, the integration contours have to start and end at ∞. The integration contours have to be chosen such that the dominating part of the integrand, i.e., $|e^{\frac{-t^3}{3}}|$, tends to 0 when t tends to ∞. This is the case if t at ∞ is asymptotic to one of the directions within

$$\arg(t) \in \left(-\frac{\pi}{6}, \frac{\pi}{6}\right), \quad \arg(t) \in \left(\frac{\pi}{2}, \frac{5\pi}{6}\right), \quad \arg(t) \in \left(-\frac{5\pi}{6}, -\frac{\pi}{2}\right).$$

By Cauchy's theorem we have a big freedom in choosing the paths—as long as they are asymptotic, at $t = \infty$, to directions lying in the sectors described above. Choose $\Gamma_1, \Gamma_2, \Gamma_3$, as depicted in Figure 1. The integration cycle $\Gamma_1 + \Gamma_2 + \Gamma_3$ on the Riemann sphere is homotopic to a point, since 0 is not a singular point. Therefore,

$$u_1(z) + u_2(z) + u_3(z) \equiv 0,$$

where

$$u_j(z) := \int_{\Gamma_j} e^{zt-\frac{t^3}{3}}\,dt, \quad j = 1, 2, 3 \tag{6}$$

is the inverse Fourier–Laplace transform of $e^{-\frac{t^3}{3}}$. Moreover, by [**30**, (22.8)],

$$\zeta_3^2 u_2(\zeta_3^2 z) = u_1(z) = \zeta_3 u_3(\zeta_3 z), \tag{7}$$

where ζ_3 denotes the primitive third root of unity $e^{\frac{2\pi i}{3}}$.
The function $\frac{u_1(z)}{2\pi i}$ is known as *Airy's integral* and is denoted by

$$\mathrm{Ai}(z) \coloneqq \frac{1}{2\pi i}\int_{\Gamma_1} e^{zt-\frac{t^3}{3}}\,dt \tag{8}$$

Then

$$U(z) = \begin{pmatrix} \mathrm{Ai}(z) & \mathrm{Ai}(\zeta_3 z) \\ \mathrm{Ai}'(z) & \zeta_3\,\mathrm{Ai}'(\zeta_3 z) \end{pmatrix}$$

is a fundamental solution matrix of system (3).
By [**30**, (23.19)], u_1 has the following asymptotics, as z tends to ∞:

$$u_1(z) \sim i\sqrt{\pi} z^{-1/4} e^{-\frac{2}{3}z^{\frac{3}{2}}} \sum_{r=0}^{\infty} a_r z^{-\frac{3r}{2}}, \quad \arg(z) \in (-\pi, \pi).$$

We denote the asymptotic expansion on the right hand side by A. By (7), we obtain

$$u_2(z) \sim -A, \quad \arg(z) \in \left(\frac{\pi}{3}, \frac{7\pi}{3}\right).$$

Again by (7), we obtain

$$u_3(z) \sim \sqrt{\pi} z^{-1/4} e^{\frac{2}{3}z^{\frac{3}{2}}} \sum_{r=0}^{\infty} (-1)^r a_r z^{-\frac{3r}{2}}, \quad \arg(z) \in \left(-\frac{\pi}{3}, \frac{5\pi}{3}\right).$$

We denote the asymptotic expansion on the right hand side by B.
In the asymptotic expansions, the exponential factors $e^{\pm\frac{2}{3}z^{3/2}}$ appear, which determine the asymptotic behavior of the solutions. The Stokes rays are

$$-\pi, -\frac{\pi}{3}, \frac{\pi}{3} \mod 2\pi,$$

i.e., $-\frac{\pi}{3}, -\pi, \frac{\pi}{3} + 2\pi k$ for $k \in \mathbb{Z}$, since there the real part of $\pm z^{3/2}$ vanishes and therefore the dominance relation between the exponential factors changes when varying the argument crossing these rays.
The asymptotic expansions strongly depend on the argument of z. We fix the pair of formal solutions (A, B) and search for holomorphic lifts on sectors. The change on the intersection of these sectors is measured by Stokes matrices.

- On the sector $\left(-\frac{\pi}{3}, \pi\right)$, (u_1, u_3) is the unique holomorphic lift of the pair (A, B).
- On the sector $\left(\frac{\pi}{3}, \frac{5\pi}{3}\right)$, $(-u_2, u_3)$ is the unique holomorphic lift of the pair (A, B).

On the intersection of these two sectors, i.e., on $\arg(z) \in \left(\frac{\pi}{3}, \pi\right)$, the counterclockwise change is described by $(u_1, u_3) \mapsto (-u_2, u_3) = (u_1 + u_3, u_3)$. Therefore, it is described by the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

one of the *Stokes matrices* of the Airy equation.

2. Bessel's differential equation

Another example is Bessel's differential equation of weight n which is given by

$$z^2u''(z) + zu'(z) + (z^2 - n)u(z) = 0.$$

The associated differential operator

$$P_{\mathrm{Bessel},n} = z^2\partial_z^2 + z\partial_z + (z^2 - n) \in \mathbb{C}[z]\langle\partial_z\rangle \tag{9}$$

is called *Bessel's differential operator of weight n*. It is irregular singular at $z = \infty$.
Following [**30**, (15.2)], the multivalued Hankel functions $H_n^j(z), j = 1,2$, are two linearly independent solutions of the differential equation $P_n u = 0$ and have asymptotic representations of the form

$$H_n^1(z) \sim \widehat{H}_n^1(z)z^{-1/2}e^{iz} =: A$$
$$H_n^2(z) \sim \widehat{H}_n^2(z)z^{-1/2}e^{-iz} =: B,$$

where $\widehat{H}_j^n(z)$ are power series in z^{-1}. The expansion for $H_n^1(z)$ is valid on $-\pi < \arg(z) < 2\pi$, the expansion for $H_n^2(z)$ is valid on $-2\pi < \arg(z) < \pi$, $H_n^j(z)$ being interpreted as function on the Riemann surface. [**30**, (15.6)] gives explicit formulae for the analytic continuation of $H_n^1(z)$ and $H_n^2(z)$.
We apply it to Bessel's differential equation of weight 0. On the sector $(-\pi,\pi)$, the pair $\big(H_0^2(z), H_0^1(z)\big)$ is a holomorphic lift of (B,A). On the sector $(0,2\pi)$, $\big(2H_0^1(z) + H_0^2(z), H_0^1(z)\big)$ is a holomorphic lift of (B,A) (cf. [**30**, (15.7)]). Therefore, the Stokes matrix for the Stokes direction $\arg(z) = \pi$ in counterclockwise orientation is given by

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}. \tag{10}$$

CHAPTER III

Gauß–Manin systems

In this chapter, we repeat the definition of Gauß–Manin systems associated to Laurent polynomials, following work of C. Sabbah and A. Douai. In particular, we carry out the computation for several examples arising from Landau–Ginzburg models of weighted projective lines in order to compare them to the quantum connection of the latter. In principle, this relationship is well known from mirror symmetry. In this chapter, we explicitly choose bases which are suitable for the further computations in the course of this thesis.

1. Definitions

Following [**13**, Section 2.c], we define the Gauß-Manin system attached to a Laurent polynomial. Let X be affine, of complex dimension n, and $f\colon X \to \mathbb{A}^1_t$ a regular function on it, where t denotes the coordinate on $\mathbb{A}^1$. An important example is the torus $X = \mathbb{G}_m^n$ and $f \in \mathbb{C}\left[x_1^{\pm 1}, \dots, x_n^{\pm 1}\right]$ a Laurent polynomial in n variables $x_1, \dots, x_n$, which is a regular function $f\colon \mathbb{G}_m^n \to \mathbb{A}^1_t$. The *Gauß–Manin system attached to* f is defined to be the complex $\int_f \mathcal{O}_X$ of $\mathcal{D}_{\mathbb{A}^1_t}$-modules, where $\int_f(\bullet)$ denotes the direct image in the category of $\mathcal{D}$-modules.

Proposition III.1 ([**26**]). The cohomology modules $H^k(\int_f \mathcal{O}_{\mathbb{G}_m^n})$ are naturally equipped with the structure of $\mathbb{C}[t]\langle\partial_t\rangle$-modules which makes them holonomic modules, regular even at ∞. Moreover, $H^k\left(\int_f \mathcal{O}_{\mathbb{G}_m^n}\right) = 0$ for $k \notin [-n+1, 0]$.

Denote by $M \coloneqq H^0(\int_f \mathcal{O}_{\mathbb{G}_m^n})$ the zeroth cohomology of the Gauß–Manin system of f. It is given by

$$M = \Omega^n(X)[\partial_t]/(d - \partial_t df\wedge)\Omega^{n-1}(X)[\partial_t].$$

Denote by $G \coloneqq \widehat{M}\left[\tau^{-1}\right]$ the Fourier–Laplace transform of M localized at $\tau = 0$. G is a free $\mathbb{C}\left[\tau, \tau^{-1}\right]$-module of finite rank (cf. [**11**]). It is given by (cf. [**12**, Section 2.c])

$$G = \Omega^n(X)\left[\tau, \tau^{-1}\right]/(d - \tau df\wedge)\Omega^{n-1}(X)\left[\tau, \tau^{-1}\right].$$

Let us rewrite G in the variable $\theta = \tau^{-1}$. The free $\mathbb{C}\left[\theta, \theta^{-1}\right]$-module

$$G = \Omega^n(X)\left[\theta, \theta^{-1}\right]/(\theta d - df\wedge)\Omega^{n-1}(X)\left[\theta, \theta^{-1}\right]$$

is endowed with a flat connection given as follows. For $\gamma = \left[\sum_{k\in\mathbb{Z}} \omega_k \theta^k\right] \in G$, where $\Omega^n(X) \ni \omega_k = 0$ for almost all k, set (cf. [**16**, Definition 2.3.1])

$$\theta^2 \nabla_{\frac{\partial}{\partial\theta}}(\gamma) = \left[\sum_k f\omega_k\theta^k + \sum_k k\omega_k\theta^{k+1}\right]. \tag{11}$$

It is known that (G, ∇) has a regular singularity at $\theta = \infty$ and possibly an irregular one at $\theta = 0$. Rewriting in $\tau = \theta^{-1}$ yields the irregular singularity at $\tau = \infty$.

REMARK III.2. If f fulfills some tameness condition, this enables one to make statements about the existence and rank of Brieskorn lattices (cf. [**11**] or [**26**]), which might be useful for extending the connection.

REMARK III.3. From now on—for the sake of notational simplicity—we often drop the $[\bullet]$ and work with representatives $\bullet$ of equivalence classes instead.

2. Examples

EXAMPLE. Consider the Laurent polynomial $f = x^2 + x^{-2} \in \mathbb{C}\left[x, x^{-1}\right]$. The free $\mathbb{C}\left[\theta, \theta^{-1}\right]$-module G is given by

$$G = \mathbb{C}\left[x, x^{-1}\right] dx \left[\theta, \theta^{-1}\right] / \left(\theta d - \left(\left(2x - \frac{2}{x^3}\right) dx\right) \wedge\right) \mathbb{C}\left[x, x^{-1}\right]\left[\theta, \theta^{-1}\right].$$

We read that

$$\begin{aligned} &\vdots \\ -2\theta \frac{dx}{x^3} &\equiv 2\frac{dx}{x} - 2\frac{dx}{x^5}, \\ -\theta \frac{dx}{x^2} &\equiv 2dx - 2\frac{dx}{x^4}, \\ 0 &\equiv 2xdx - 2\frac{dx}{x^3}, \\ \theta dx &\equiv 2x^2 dx - 2\frac{dx}{x^2}, \\ 2\theta x dx &\equiv 2x^3 dx - 2\frac{dx}{x}, \\ &\vdots \end{aligned}$$

G is free over $\mathbb{C}\left[\theta, \theta^{-1}\right]$ of rank 4 with basis dx, $\frac{dx}{x}$, $\frac{dx}{x^2}$, $\frac{dx}{x^3}$. Let us compute connection (11) in this basis. We compute that

$$\begin{aligned} \theta^2 \nabla_{\partial_\theta} dx &\equiv x^2 dx + \frac{dx}{x^2} \equiv \frac{\theta}{2} dx + 2\frac{dx}{x^2}, \\ \theta^2 \nabla_{\partial_\theta} \frac{dx}{x} &\equiv x dx + \frac{dx}{x^3} \equiv 2\frac{dx}{x^3}, \\ \theta^2 \nabla_{\partial_\theta} \frac{dx}{x^2} &\equiv dx + \frac{dx}{x^4} \equiv 2dx + \frac{\theta}{2}\frac{dx}{x^2}, \\ \theta^2 \nabla_{\partial_\theta} \frac{dx}{x^3} &\equiv \frac{dx}{x} + \frac{dx}{x^5} \equiv 2\frac{dx}{x} + \theta \frac{dx}{x^3}. \end{aligned}$$

Therefore, the connection in this basis is given by

$$\theta^2 \nabla_{\frac{\partial}{\partial \theta}} = \theta^2 \frac{\partial}{\partial \theta} + \begin{pmatrix} \frac{\theta}{2} & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 2 & 0 & \frac{\theta}{2} & 0 \\ 0 & 2 & 0 & \theta \end{pmatrix}.$$

2.1. Examples arising from complex projective space.

Example (Complex projective line). Let us now consider $X = \mathbb{G}_m$ and the cohomologically tame Laurent polynomial $f(x) = x + x^{-1}$. This is a Landau–Ginzburg model of the complex projective line. The free $\mathbb{C}\left[\theta, \theta^{-1}\right]$-module G is given by

$$G = \mathbb{C}\left[x, x^{-1}\right] dx\left[\theta, \theta^{-1}\right] / \left(\theta d - \left(1 - \frac{1}{x^2}\right) dx \wedge\right) \mathbb{C}\left[x, x^{-1}\right]\left[\theta, \theta^{-1}\right].$$

By considering the equivalence classes of $x^k dx$ for $k \in \mathbb{Z}$, we find that $\frac{dx}{x}, dx$ is a basis of G over the ring of Laurent polynomials $\mathbb{C}\left[\theta, \theta^{-1}\right]$, i.e., G has rank 2. Now let us compute the connection matrix of ∇ in this basis. We compute that

$$\theta^2 \nabla_{\frac{\partial}{\partial\theta}} \frac{dx}{x} \equiv 2dx, \quad \theta^2 \nabla_{\frac{\partial}{\partial\theta}} dx \equiv 2\frac{dx}{x} + \theta dx.$$

Hence connection (11) in the basis $\frac{dx}{x}, dx$ is given by

$$\theta^2 \nabla_{\frac{\partial}{\partial\theta}} = \theta^2 \frac{\partial}{\partial\theta} + \begin{pmatrix} 0 & 2 \\ 2 & \theta \end{pmatrix}.$$

Passing to $t = -\theta^{-1}$ yields

$$t\nabla_{\partial_t} = t\partial_t + \begin{pmatrix} 0 & 2t \\ 2t & -1 \end{pmatrix}, \tag{12}$$

which is, up to the constants on the main diagonal, the quantum connection (20) of $\mathbb{P}^1$. Indeed, the two systems are gauge equivalent after pulling back via the ramification map. For

$$h = \begin{pmatrix} t^{\frac{1}{2}} & 0 \\ 0 & t^{\frac{1}{2}} \end{pmatrix} \in \mathrm{GL}_2\left(\mathbb{C}\left[t^{\frac{1}{2}}, t^{-\frac{1}{2}}\right]\right)$$

we obtain

$$\begin{pmatrix} \frac{1}{2} & 2t \\ 2t & -\frac{1}{2} \end{pmatrix} = h^{-1} \begin{pmatrix} 0 & 2t \\ 2t & -1 \end{pmatrix} h + h^{-1} t \frac{\partial h}{\partial t}.$$

System (12) is gauge equivalent to the connection

$$t\nabla_{\partial_t} = t\partial_t + \begin{pmatrix} 0 & 4t^2 \\ 1 & 0 \end{pmatrix}$$

via the matrix $h = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{4t} & 0 \end{pmatrix} \in \mathrm{GL}_2\left(\mathbb{C}\left[t, t^{-1}\right]\right)$, i.e.,

$$\begin{pmatrix} 0 & 4t^2 \\ 1 & 0 \end{pmatrix} = h^{-1} \begin{pmatrix} 0 & 2t \\ 2t & -1 \end{pmatrix} h + h^{-1} t \frac{\partial h}{\partial t}.$$

Furthermore, via $h = \mathrm{diag}\left(1, \frac{1}{2t}\right)$, we have that

$$\begin{pmatrix} 0 & 2t \\ 2t & -1 \end{pmatrix} = h^{-1} \begin{pmatrix} 0 & 4t^2 \\ 1 & 0 \end{pmatrix} h + h^{-1} t \frac{\partial h}{\partial t}.$$

By the cyclic vector $m = (1, 0)^{\mathrm{t}}$ for the connection

$$\nabla_{t\partial_t} = t\partial_t + \begin{pmatrix} 0 & 4t^2 \\ 1 & 0 \end{pmatrix},$$

we read the relation $(\nabla_{t\partial_t})^2 m - 4t^2 m = 0$ and therefore associate the differential operator $P = (t\partial_t)^2 - 4t^2$. Passing to $s = 2it$ yields Bessel's differential operator (9) of weight 0.

EXAMPLE (Complex projective plane). Consider the Laurent polynomial $f(x_1, x_2) = x_1 + x_2 + \frac{1}{x_1 x_2} \in \mathbb{C}\left[x_1^{\pm 1}, x_2^{\pm 1}\right]$. This is a Landau–Ginzburg model of the complex projective plane. The free $\mathbb{C}\left[\theta, \theta^{-1}\right]$-module G is given by

$$G = \mathbb{C}\left[x_1^{\pm 1}, x_2^{\pm 1}\right] dx_1 \wedge dx_2 \left[\theta, \theta^{-1}\right] / \left(\theta d - \left(\left(1 - \frac{1}{x_1^2 x_2}\right) dx_1 + \left(1 - \frac{1}{x_1 x_2^2}\right) dx_2\right) \wedge\right) \Omega_X^1 \left[\theta, \theta^{-1}\right].$$

By considering the equivalence classes $x_1^k x_2^l dx_1 \wedge dx_2$ for $k, l \in \mathbb{Z}$, one computes that a basis of G over $\mathbb{C}\left[\theta, \theta^{-1}\right]$ is given by

$$\frac{1}{x_1 x_2} dx_1 \wedge dx_2,\ \frac{1}{x_2} dx_1 \wedge dx_2,\ dx_1 \wedge dx_2.$$

Hence G is free of rank 3 over $\mathbb{C}[\theta, \theta^{-1}]$. The action of $\theta^2 \frac{\partial}{\partial \theta}$ on G is given by

$$\theta^2 \nabla_{\frac{\partial}{\partial \theta}} g = \left[\sum_k \left(x_1 + x_2 + \frac{1}{x_1 x_2}\right) \omega_k \theta^k + \sum_k k \omega_k \theta^{k+1}\right]$$

for an element $g = \left[\sum_{k \in \mathbb{Z}, \text{finite}} \omega_k \theta^k\right] \in G$. In the basis $\frac{1}{x_1 x_2} dx_1 \wedge dx_2$, $\frac{1}{x_2} dx_1 \wedge dx_2$, $dx_1 \wedge dx_2$, the connection on G is given by

$$\theta^2 \nabla_{\frac{\partial}{\partial \theta}} = \theta^2 \frac{\partial}{\partial \theta} + \begin{pmatrix} 0 & 0 & 3 \\ 3 & \theta & 0 \\ 0 & 3 & 2\theta \end{pmatrix}. \tag{13}$$

EXAMPLE ($\mathbb{P}^n$). Consider the torus $X = \mathbb{G}_m^n$ and the Laurent polynomial in n variables $f(x_1, \ldots, x_n) = x_1 + \ldots + x_n + \frac{1}{x_1 \cdot x_2 \cdot \ldots \cdot x_n} \in \mathbb{C}\left[x_1^{\pm 1}, \ldots, x_n^{\pm 1}\right]$, which is a regular function on X. This is a Landau–Ginzburg model of the complex projective space $\mathbb{P}^n$. The free $\mathbb{C}\left[\theta, \theta^{-1}\right]$-module G is given by

$$G = \left(\mathbb{C}\left[x_1^{\pm 1}, \ldots, x_n^{\pm 1}\right] dx_1 \wedge \ldots \wedge dx_n\right) \left[\theta, \theta^{-1}\right] / \left(\theta d - df \wedge\right) \Omega^{n-1}(X) \left[\theta, \theta^{-1}\right].$$

It is free of rank $n+1$ with basis

$$\omega_0 \coloneqq \frac{1}{x_1 \cdot \ldots \cdot x_n} dx_1 \wedge \ldots \wedge dx_n,\ x_1 \omega_0,\ x_1 x_2 \omega_0, \ldots,\ x_1 x_2 \cdot \ldots \cdot x_n \omega_0.$$

After passing to $t = -\theta^{-1}$, the connection on G is given by

$$\nabla_{t\partial_t} = t\partial_t + \underbrace{\begin{pmatrix} 0 & 0 & \ldots & & 0 & (n+1)t \\ (n+1)t & -1 & 0 & & \ldots & 0 \\ 0 & (n+1)t & -2 & 0 & & \\ & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & & & & 0 \\ & & & 0 & (n+1)t & -n \end{pmatrix}}_{=: A_{GM, \mathbb{P}^n}}. \tag{14}$$

This system is—up to the constants on the main diagonal of the matrix—system (25), the quantum connection of $\mathbb{P}^n$. Indeed, they are gauge equivalent—to be precise, gauge equivalent after possibly pulling the system back by the ramification map $s \mapsto s^2 = t$—by the matrix $h = \operatorname{diag}\left(t^{\frac{n}{2}}, \ldots, t^{\frac{n}{2}}\right)$, i.e., $A_{Q, \mathbb{P}^n} = h^{-1} A_{GM, \mathbb{P}^n} h + h^{-1} t \frac{\partial h}{\partial t}$. Furthermore, the Gauß–Manin system (14) is gauge equivalent to system (26) via $h = \operatorname{diag}\left(1, \frac{1}{(n+1)t}, \frac{1}{(n+1)^2 t^2}, \ldots, \frac{1}{(n+1)^n t^n}\right)$, i.e., $A_{GM, \mathbb{P}^n} = h^{-1} A'_{\mathbb{P}^n} h + h^{-1} t \frac{\partial h}{\partial t}$.

2.2. Mirror of $\mathbb{P}(1,2)$. A Landau–Ginzburg model of $\mathbb{P}(1,2)$ is given by $\left(\mathbb{G}_m, x^2+x^{-1}\right)$. The localized Fourier–Laplace transform of $H^0\left(\int_f \mathcal{O}\right)$ is the free $\mathbb{C}\left[\theta,\theta^{-1}\right]$-module

$$G = \mathbb{C}\left[x,x^{-1}\right]dx\left[\theta,\theta^{-1}\right]/\left(\theta d - \left(2x - \frac{1}{x^2}\right)dx\wedge\right)\mathbb{C}\left[x,x^{-1}\right]\left[\theta,\theta^{-1}\right].$$

A basis over $\mathbb{C}\left[\theta,\theta^{-1}\right]$ is given by $\frac{dx}{x^2}$, $\frac{dx}{x}$, dx. In this basis, the connection on G is given by

$$\nabla_{\theta\frac{\partial}{\partial\theta}} = \theta\partial_\theta + \begin{pmatrix} 1 & \frac{3}{2\theta} & 0 \\ 0 & 0 & \frac{3}{2\theta} \\ \frac{3}{\theta} & 0 & \frac{1}{2} \end{pmatrix}.$$

Via the cyclic vector $m = (0,1,0)^{\mathrm{t}}$, we read the relation

$$\left(\theta\nabla_{\partial_\theta}\right)^3 m + \frac{3}{2}\left(\theta\nabla_{\partial_\theta}\right)^2 m - \frac{27}{4\theta^3}m = 0$$

and therefore associate the differential operator

$$P = (\theta\partial_\theta)^3 + \frac{3}{2}(\theta\partial_\theta)^2 - \frac{27}{4\theta^3} \in \mathbb{C}\left[\theta,\theta^{-1}\right]\langle\partial_\theta\rangle.$$

Rewriting in $\tau = \theta^{-1}$ yields the operator $P_\tau = -(\tau\partial_\tau)^3 + \frac{3}{2}(\tau\partial_\tau)^2 - \frac{27}{4}\tau^3$. We associate the $\mathbb{C}\left[\tau,\tau^{-1}\right]\langle\partial_\tau\rangle$-module

$$\mathbb{C}\left[\tau,\tau^{-1}\right]\langle\partial_\tau\rangle/\mathbb{C}\left[\tau,\tau^{-1}\right]\langle\partial_\tau\rangle\left(\partial_\tau^3 + \frac{3}{2\tau}\partial_\tau^2 - \frac{1}{2\tau^2} + \frac{27}{4}\right).$$

Proposition III.4. In the basis $\frac{dx}{x}$, dx, xdx, the connection on G is given by

$$\nabla_{\theta\frac{\partial}{\partial\theta}} = \theta\partial_\theta + \begin{pmatrix} 0 & \frac{3}{2\theta} & 0 \\ 0 & \frac{1}{2} & \frac{3}{2\theta} \\ \frac{3}{\theta} & 0 & 1 \end{pmatrix}.$$

By the gauge $h = \operatorname{diag}\left(\theta^{-\frac{1}{2}},\theta^{-\frac{1}{2}},\theta^{-\frac{1}{2}}\right)$ and passing to the variable $-\theta$, this is exactly the quantum connection (28) of $\mathbb{P}(1,2)$.

Remark III.5. This gauge transformation subtracts $\frac{1}{2}$ on the main diagonal, which produces an extra square root in the solutions.

One might also use $\left(\mathbb{G}_m, x + x^{-2}\right)$ as Landau–Ginzburg model. G is then given by

$$G = \mathbb{C}\left[x,x^{-1}\right]dx\left[\theta,\theta^{-1}\right]/\left(\theta d - \left(1 - \frac{2}{x^3}\right)dx\wedge\right)\mathbb{C}\left[x,x^{-1}\right]\left[\theta,\theta^{-1}\right].$$

Proposition III.6. In the basis $\frac{dx}{x}$, $\frac{dx}{x^2}$, $\frac{dx}{x^3}$, the connection is given by

$$\nabla_{\theta\frac{\partial}{\partial\theta}} = \theta\partial_\theta + \begin{pmatrix} 0 & \frac{3}{2\theta} & 0 \\ 0 & \frac{1}{2} & \frac{3}{2\theta} \\ \frac{3}{\theta} & 0 & 1 \end{pmatrix}.$$

Again by the gauge $h = \operatorname{diag}\left(\theta^{-\frac{1}{2}},\theta^{-\frac{1}{2}},\theta^{-\frac{1}{2}}\right)$, which subtracts $\frac{1}{2}$ on the main diagonal entries, and passing to $-\theta$, this is exactly the quantum connection (28) of $\mathbb{P}(1,2)$.

2.3. Mirror of $\mathbb{P}(1,3)$. A Landau–Ginzburg model of $\mathbb{P}(1,3)$ is given by $\left(\mathbb{G}_m, x^3+x^{-1}\right)$.

$$G = \mathbb{C}\left[x,x^{-1}\right]dx\left[\theta,\theta^{-1}\right] / \left(\theta d - \left(3x^2 - \frac{1}{x^2}\right)dx\wedge\right)\mathbb{C}\left[x,x^{-1}\right]\left[\theta,\theta^{-1}\right].$$

G is free of rank 4, basis given by $\frac{dx}{x^3}$, $\frac{dx}{x^2}$, $\frac{dx}{x}$, dx. In this basis, the connection is given by

$$\nabla_{\theta\frac{\partial}{\partial\theta}} = \theta\partial_\theta + \begin{pmatrix} 2 & \frac{4}{3\theta} & 0 & 0 \\ 0 & -\frac{1}{3} & \frac{4}{3\theta} & 0 \\ 0 & 0 & 0 & \frac{4}{3\theta} \\ \frac{4}{\theta} & 0 & 0 & \frac{1}{3} \end{pmatrix}. \tag{15}$$

Via the cyclic vector $m = (0,0,1,0)^{\mathrm{t}}$, we associate the differential operator

$$P_\theta = (\theta\partial_\theta)^4 + 4(\theta\partial_\theta)^3 + \frac{32}{9}(\theta\partial_\theta)^2 - \frac{256}{27\theta^4}.$$

The computation of the cyclic vector was carried out in SAGE. The code can be found in the appendix. Rewriting in $\tau = \theta^{-1}$ yields the differential operator

$$P_\tau = (\tau\partial_\tau)^4 - 4(\tau\partial_\tau)^3 + \frac{32}{9}(\tau\partial_\tau)^2 - \frac{256}{27}\tau^4.$$

Let us consider the basis $\frac{dx}{x}, dx, xdx, x^2dx$ instead. In this basis, the connection is given by

$$\nabla_{\theta\frac{\partial}{\partial\theta}} = \theta\partial_\theta + \begin{pmatrix} 0 & \frac{4}{3\theta} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{4}{3\theta} & 0 \\ 0 & 0 & \frac{2}{3} & \frac{2}{3\theta} \\ \frac{4}{\theta} & 0 & 0 & 1 \end{pmatrix}.$$

By the gauge $h = \mathrm{diag}\left(\theta^{-\frac{1}{2}},\theta^{-\frac{1}{2}},\theta^{-\frac{1}{2}},\theta^{-\frac{1}{2}}\right)$, which subtracts $\frac{1}{2}$ on the main diagonal entries, and passing to $-\theta$, this is exactly the quantum connection (29) of $\mathbb{P}(1,3)$.
Let us consider $\left(\mathbb{G}_m, x+x^{-3}\right)$ as a Landau–Ginzburg model instead. G is given by

$$G = \mathbb{C}\left[x,x^{-1}\right]dx\left[\theta,\theta^{-1}\right] / \left(\theta d - \left(dx - \frac{3}{x^4}dx\right)\wedge\right)\mathbb{C}\left[x,x^{-1}\right]\left[\theta,\theta^{-1}\right]$$

with basis over $\mathbb{C}\left[\theta,\theta^{-1}\right]$ given by $\frac{dx}{x}$, $\frac{dx}{x^2}$, $\frac{dx}{x^3}$, $\frac{dx}{x^4}$.

Proposition III.7. In the basis stated above, the connection is given by

$$\nabla_{\theta\frac{\partial}{\partial\theta}} = \theta\partial_\theta + \begin{pmatrix} 0 & \frac{4}{3\theta} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{4}{3\theta} & 0 \\ 0 & 0 & \frac{2}{3} & \frac{4}{3\theta} \\ \frac{4}{\theta} & 0 & 0 & 1 \end{pmatrix}.$$

By the gauge $h = \mathrm{diag}\left(\theta^{-\frac{1}{2}},\theta^{-\frac{1}{2}},\theta^{-\frac{1}{2}},\theta^{-\frac{1}{2}}\right)$, which subtracts $\frac{1}{2}$ on the main diagonal entries, and passing to $-\theta$, this is exactly the quantum connection (29) of $\mathbb{P}(1,3)$.

2.4. Mirror of $\mathbb{P}(1,n)$. Let $n \in \mathbb{N}_{>0}$. The multiplicative group $\mathbb{G}_m$ together with the Laurent polynomial $f = x + x^{-n}$ is a Landau–Ginzburg model of the weighted projective line $\mathbb{P}(1,n)$. The localized Fourier–Laplace transform of $H^0\left(\int_f \mathcal{O}\right)$ is the free $\mathbb{C}\left[\theta,\theta^{-1}\right]$-module

$$G = \mathbb{C}\left[x,x^{-1}\right]dx\left[\theta,\theta^{-1}\right] / \left(\theta d - \left(1 - \frac{n}{x^{n+1}}\right)dx\wedge\right)\mathbb{C}\left[x,x^{-1}\right]\left[\theta,\theta^{-1}\right].$$

A basis is given by $\frac{dx}{x}, \frac{dx}{x^2}, \ldots, \frac{dx}{x^{n+1}}$. We read the relations

$$\begin{aligned}
&\vdots\\
-n\theta\frac{dx}{x^{n+1}} &\equiv \frac{dx}{x^n} - n\frac{dx}{x^{2n+1}},\\
-(n-1)\theta\frac{dx}{x^n} &\equiv \frac{dx}{x^{n-1}} - n\frac{dx}{x^{2n}},\\
&\vdots\\
-2\theta\frac{dx}{x^3} &\equiv \frac{dx}{x^2} - n\frac{dx}{x^{n+3}},\\
-\theta\frac{dx}{x^2} &\equiv \frac{dx}{x} - n\frac{dx}{x^{n+2}},\\
0 &\equiv dx - n\frac{dx}{x^{n+1}},\\
\theta dx &\equiv x dx - n\frac{dx}{x^n},\\
2\theta dx &\equiv x^2 dx - n\frac{dx}{x^{n-1}},\\
&\vdots
\end{aligned}$$

We compute that $\theta^2\nabla_{\frac{\partial}{\partial\theta}}\frac{dx}{x^k} \equiv \frac{dx}{x^{k-1}} + \frac{dx}{x^{n+k}}$, $k = 1,\ldots,n+1$. Therefore, we obtain the following

PROPOSITION III.8. In the basis $\frac{dx}{x}, \frac{dx}{x^2}, \ldots, \frac{dx}{x^{n+1}}$, the connection on G is given by

$$\theta\nabla_{\frac{\partial}{\partial\theta}} = \theta\partial_\theta + \begin{pmatrix} 0 & \frac{n+1}{n\theta} & 0 & \cdots & \cdots & 0 \\ 0 & \frac{1}{n} & \frac{n+1}{n\theta} & \ddots & & \vdots \\ \vdots & \ddots & \frac{2}{n} & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & & & 0 & \frac{n-1}{n} & \frac{n+1}{n\theta} \\ \frac{n+1}{n\theta} & 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}. \tag{16}$$

By the gauge $h = \mathrm{diag}\left(\theta^{-\frac{1}{2}}, \theta^{-\frac{1}{2}}, \theta^{-\frac{1}{2}}, \theta^{-\frac{1}{2}}\right)$, which subtracts $\frac{1}{2}$ on the main diagonal entries, and passing to $-\theta$, this is exactly the quantum connection (30) of $\mathbb{P}(1,n)$.

CHAPTER IV

Quantum connection and Dubrovin's conjecture

Let X be a Fano variety over the complex numbers. The *(small) quantum cohomology algebra of* X is defined to be the ordinary cohomology $H^*(X,\mathbb{C})$ as a $\mathbb{C}$-vector space—but the multiplication is deformed by the so called *Gromov–Witten invariants*. We will denote this algebra by $(\mathrm{QH}(X),\circ)$, where $\circ$ denotes the product in this algebra. We do not repeat the definitions of quantum cohomology in this thesis, but refer to prevailing work. Fortunately, a lot of computations for the quantum cohomology of weighted projective spaces can be found in the literature, for instance in work of É. Mann.
From quantum cohomology, one deduces a flat meromorphic connection on the trivial bundle over $\mathbb{P}^1$ with fiber $H^*(X,\mathbb{C})$, having 0 and ∞ as singular points, one of them irregular and one of them regular singular. In this chapter, we explicitly compute the quantum connection of some (weighted) projective spaces. According to Dubrovin's conjecture, the Stokes matrix of the quantum connection can be obtained as the Gram matrix of the Euler–Poincaré pairing χ on $\mathrm{D^b}(\mathrm{Coh}(X))$. We compute the Gram matrix of χ for some weighted projective lines that will be of interest in the course of the thesis.

1. Quantum connection of Fano varieties

The quantum connection is a flat meromorphic connection on the trivial bundle over $\mathbb{P}^1$ with fiber $H^*(X,\mathbb{C})$. Let z denote the inhomogeneous coordinate on $\mathbb{A}^1 = \mathbb{P}^1 \smallsetminus \{0\}$. The *quantum connection*[1] is the connection given by [**15**, (2.2.1)]:

$$\nabla_{z\partial_z} = z\frac{\partial}{\partial z} - \frac{1}{z}\left(-K_X\circ \quad\right) + \mu. \tag{17}$$

The first term on the right hand side is ordinary differentiation, the second one is pointwise quantum multiplication by the anticanonical divisor $(-K_X) \in H^2(X,\mathbb{C})$, and the third term is a grading operator defined as follows:

$$\mu(a) \coloneqq \left(\frac{i}{2} - \frac{\dim X}{2}\right)a \quad \text{for } a \in H^i(X,\mathbb{C}). \tag{18}$$

Remark IV.1. In the literature, different variations of the grading operator appear. A common variation, with the notation from above, is the grading operator given by $\mu(a) = \frac{i}{2}a$, which might be more suitable for certain considerations.

The quantum connection is known to have an irregular singularity at $z = 0$ and a regular one at $z = \infty$. In order to obtain the irregular singularity at ∞, let us rewrite (17) in the variable $t = z^{-1}$. Since $z\frac{\partial}{\partial z} = z\frac{\partial t}{\partial z}\frac{\partial}{\partial t} = -t\frac{\partial}{\partial t}$, we obtain

$$\nabla_{t\partial_t} = t\frac{\partial}{\partial t} + t\left(-K_X\circ \quad\right) - \mu, \tag{19}$$

which is now regular singular at $t = 0$ and irregular singular at $t = \infty$.

[1] sometimes called *Dubrovin's connection in the z-direction*

2. Quantum connection of complex projective space

2.1. Quantum connection of the complex projective line. Let us make the above explicit for the complex projective line $X = \mathbb{P}^1$. There is an isomorphism of $\mathbb{C}$-algebras

$$H^*\left(\mathbb{P}^1, \mathbb{C}\right) \cong \mathbb{C}\left[h\right]/h^2,$$

where h is the cohomology class of a point in $\mathbb{P}^1$. Moreover, by [**21**, (5.2.4)] for $q = 1$, there is an isomorphism of $\mathbb{C}$-algebras

$$\mathrm{QH}\left(\mathbb{P}^1\right) \cong \mathbb{C}[h]/\left(h^2 - 1\right).$$

We choose $(1, h)$ as a basis of $\mathrm{QH}\left(\mathbb{P}^1\right)$. In this basis, the matrix of quantum multiplication by the anticanonical class $(-K_{\mathbb{P}^1}) = 2h$ is given by

$$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}.$$

Further, the grading μ, given by (18), in this basis is given by the diagonal matrix $\mathrm{diag}\left(-\frac{1}{2}, \frac{1}{2}\right)$. Thus, we obtain the quantum connection (17) of $\mathbb{P}^1$ as

$$\nabla_{z\partial_z} = z\frac{\partial}{\partial z} - \frac{1}{z}\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

2.1.1. *Scalar equation for $\mathbb{P}^1$ and its Fourier–Laplace transform.* Formula (19) for $\mathbb{P}^1$ becomes

$$\nabla_{t\partial_t} = t\frac{\partial}{\partial t} + \underbrace{t\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} - \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}}_{=:A_{\mathbb{P}^1}}. \tag{20}$$

By the cyclic vector $m = (1, 0)^{\mathrm{t}}$ for system (20), we read the relation

$$\left(\nabla_{t\partial_t}\right)^2 m - \nabla_{t\partial_t} m + \left(-4t^2 + \frac{1}{4}\right) m = 0.$$

Therefore, the corresponding differential operator is given by

$$P_{\mathbb{P}^1} = (t\partial_t)^2 - t\partial_t - 4t^2 + \frac{1}{4} \in \mathbb{C}[t]\langle\partial_t\rangle. \tag{21}$$

Hence, by the $\mathcal{D}_{\mathbb{G}_m}$-linear map

$$\mathcal{D}_{\mathbb{G}_m} \longrightarrow \left(\mathcal{O}^2_{\mathbb{G}_m}, \nabla\right), \quad 1 \mapsto m,\ \partial_t \mapsto \nabla_{\partial_t} m,$$

where the connection ∇ is given by (20), we get an induced isomorphism of $\mathcal{D}_{\mathbb{G}_m}$-modules

$$\mathcal{D}_{\mathbb{G}_m}/\mathcal{D}_{\mathbb{G}_m} P_{\mathbb{P}^1} \cong \left(\mathcal{O}^2_{\mathbb{G}_m}, \nabla\right).$$

Computing the Fourier–Laplace transform of (21) via the isomorphism of Weyl algebras

$$\widehat{(\bullet)}\colon \mathbb{C}[t]\langle\partial_t\rangle \longrightarrow \mathbb{C}[\tau]\langle\partial_\tau\rangle, \quad t \mapsto -\partial_\tau,\ \partial_t \mapsto \tau,$$

we obtain the differential operator

$$\widehat{P}_{\mathbb{P}^1} = (\tau^2 - 4)\partial_\tau^2 + 4\tau\partial_\tau + \frac{9}{4} \in \mathbb{C}[\tau]\langle\partial_\tau\rangle,$$

which has two regular singular points at $\tau = \pm 2$.

Observation IV.2. These are exactly the eigenvalues of quantum multiplication by $-K_{\mathbb{P}^1}$.

System (20) is gauge equivalent to the system

$$\nabla_{t\partial_t} = t\partial_t + \underbrace{\begin{pmatrix} 0 & 4t^2 \\ 1 & 0 \end{pmatrix}}_{=:A'_{\mathbb{P}^1}} \tag{22}$$

via the matrix

$$h = \begin{pmatrix} t^{-\frac{1}{2}} & 0 \\ 0 & 2t^{\frac{1}{2}} \end{pmatrix}$$

(cf. [**15**][Comment 2.2.4]), i.e., $A'_{\mathbb{P}^1} = h^{-1}A_{\mathbb{P}^1}h + h^{-1}t\frac{\partial h}{\partial t}$.

Remark IV.3. This gauge transformation identifies *Dubrovin's connection in the anticanonical direction* with *Dubrovin's connection in the z-direction*—both of them being called *quantum connection*, varying from author to author.

Via the cyclic vector $(1,0)^{\mathrm{t}}$, we associate to system (22) the differential operator

$$P = (t\partial_t)^2 - 4t^2 = t^2\partial_t^2 + t\partial_t - 4t^2.$$

To be more precise, there is some ramification behind. We pull back the system via the ramification map $s \mapsto s^2 = t$. By using $t\partial_t = \frac{1}{2}s\partial_s$, system (22) turns into

$$\nabla_{s\partial_s} = s\partial_s + \begin{pmatrix} 0 & 8s^4 \\ 2 & 0 \end{pmatrix}$$

and system (20) turns into

$$\nabla_{s\partial_s} = s\partial_s + \begin{pmatrix} 1 & 4s^2 \\ 4s^2 & -1 \end{pmatrix}.$$

They are gauge equivalent via the matrix

$$h = \begin{pmatrix} s^{-1} & 0 \\ 0 & 2s \end{pmatrix} \in \mathrm{GL}_2\left(\mathbb{C}\left[s, s^{-1}\right]\right).$$

2.2. Quantum connection of the complex projective plane.

2.2.1. *System and scalar equation for $\mathbb{P}^2$.*

$$H^*\left(\mathbb{P}^2, \mathbb{C}\right) \cong \mathbb{C}[h]/h^3,$$

where h denotes the cohomology class of a hyperplane in $\mathbb{P}^2$. Moreover, by [**21**, (5.2.4)][2], there is an isomorphism of $\mathbb{C}$-algebras

$$\mathrm{QH}\left(\mathbb{P}^1\right) \cong \mathbb{C}[h]/\left(h^3 - 1\right).$$

We choose $1, h, h^2$ as a basis and note that $-K_{\mathbb{P}^2} = 3h$. The quantum connection 19 of $\mathbb{P}^2$ is then given by

$$\nabla_{t\partial_t} = t\partial_t + \underbrace{t\begin{pmatrix} 0 & 0 & 3 \\ 3 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} - \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{=:A_{\mathbb{P}^2}}. \tag{23}$$

[2] We always consider the case $q = 1$.

System (23) is gauge equivalent to the system

$$\nabla_{t\partial_t} = t\partial_t + \underbrace{\begin{pmatrix} 0 & 0 & 27t^3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{=:A'_{\mathbb{P}^2}} \tag{24}$$

via

$$h = \begin{pmatrix} t^{-1} & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 9t \end{pmatrix} \in \mathrm{GL}_3\left(\mathbb{C}\left[t, t^{-1}\right]\right),$$

i.e., $A'_{\mathbb{P}^2} = h^{-1} A_{\mathbb{P}^2} h + h^{-1} t \frac{\partial h}{\partial t}$. Via the cyclic vector $(1,0,0)^{\mathrm{t}}$, we compute that the associated differential operator to system (24) is given by $P_{\mathbb{P}^2} \coloneqq (t\partial_t)^3 - 27t^3$. Its Fourier–Laplace transform $(t \mapsto -\partial_\tau,\ \partial_t \mapsto \tau)$ is given by

$$\widehat{P} = \left(27 - \tau^3\right) \partial_\tau^3 - 6\tau^2 \partial_\tau^2 - 7\tau\partial_\tau - 1,$$

with regular singular points $3, 3\zeta_3, 3\zeta_3^2$, where ζ_3 denotes the primitive third root of unity $e^{\frac{2\pi i}{3}}$.

2.2.2. *Landau–Ginzburg model of* $\mathbb{P}^2$. The Landau–Ginzburg model of the complex projective plane is given by the Laurent polynomial in two variables $f(x_1, x_2) = x_1 + x_2 + \frac{1}{x_1 x_2}$, which is a regular function on $X = (\mathbb{G}_m)^2$. The function

$$f\colon (\mathbb{C}^*)^2 \to \mathbb{C}, \quad (x, y) \mapsto x_1 + x_2 + \frac{1}{x_1 x_2},$$

has critical points at $(1,1)$, (ζ_3, ζ_3), and $\left(\zeta_3^2, \zeta_3^2\right)$, hence critical values 3, $3\zeta_3$, and $3\zeta_3^2$.

Observation IV.4. The singularities—all of them being regular singular—of the Fourier–Laplace transform of the quantum connection are exactly the critical values of the Landau–Ginzburg model f.

2.2.3. *Relation to the Gauß–Manin system of* $f(x_1, x_2) = x_1 + x_2 + \frac{1}{x_1 x_2}$. After rewriting the Gauß–Manin connection (13) in $t = -\theta^{-1}$ and hence $\theta\partial_\theta = -t\partial_t$, this system turns into

$$\nabla_{t\partial_t} = t\partial_t + \underbrace{\begin{pmatrix} 0 & 0 & 3t \\ 3t & -1 & 0 \\ 0 & 3t & -2 \end{pmatrix}}_{=:A_{GM,\mathbb{P}^2}}.$$

This is—up to the constants on the main diagonal of the matrix—system (23), the quantum connection of $\mathbb{P}^2$. Indeed, they are gauge equivalent by the matrix $\mathrm{diag}(t,t,t) \in \mathrm{GL}_2\left(\mathbb{C}\left[t, t^{-1}\right]\right)$, i.e.,

$$\begin{pmatrix} 1 & 0 & 3t \\ 3t & 0 & 0 \\ 0 & 3t & -1 \end{pmatrix} = h^{-1} \begin{pmatrix} 0 & 0 & 3t \\ 3t & -1 & 0 \\ 0 & 3t & -2 \end{pmatrix} h + h^{-1} t \frac{\partial h}{\partial t}.$$

Furthermore, the Gauß–Manin system is gauge equivalent to system (24) via $h = \mathrm{diag}\left(1, \frac{1}{3t}, \frac{1}{9t^2}\right)$, i.e.,

$$\begin{pmatrix} 0 & 0 & 3t \\ 3t & -1 & 0 \\ 0 & 3t & -2 \end{pmatrix} = h^{-1} \begin{pmatrix} 0 & 0 & 27t^3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} h + h^{-1} t \frac{\partial h}{\partial t}.$$

2.3. Quantum connection of $\mathbb{P}^n$. For the cohomology ring and the small quantum cohomology of complex projective space, there are isomorphisms (cf. [**21**, (5.2.4)][3])

$$H^*(\mathbb{P}^n, \mathbb{C}) \cong \mathbb{C}[h]/h^{n+1},$$
$$\mathrm{QH}(\mathbb{P}^n) \cong \mathbb{C}[h]/\left(h^{n+1}-1\right),$$

where h denotes the cohomology class of a hyperplane in $\mathbb{P}^n$. We choose $1, h, \ldots, h^n$ as basis and note that $-K_{\mathbb{P}^n} = (n+1)h$.
The quantum connection of $\mathbb{P}^n$ in the basis given above is given by

$$\nabla_{t\partial_t} = t\partial_t + t\underbrace{\begin{pmatrix} 0 & & & n+1 \\ n+1 & & & \\ & \cdots & & \\ & & n+1 & 0 \end{pmatrix} - \begin{pmatrix} -\frac{n}{2} & & & \\ & 1-\frac{n}{2} & & \\ & & \cdots & \\ & & & n-\frac{n}{2} \end{pmatrix}}_{=:A_{Q,\mathbb{P}^n}}, \tag{25}$$

which is regular singular at $t=0$ and irregular singular at $t=\infty$.
Via $h = \mathrm{diag}\left(t^{-\frac{n}{2}}, (n+1)t^{1-\frac{n}{2}}, \ldots, (n+1)^n t^{n-\frac{n}{2}}\right) \in \mathrm{GL}_{n+1}\left(\mathbb{C}\left[t^{\frac{1}{2}}, t^{-\frac{1}{2}}\right]\right)$, this system—to be precise, the pull back of the system via the ramification map $s \mapsto s^2 = t$—is gauge equivalent to system

$$\nabla_{t\partial_t} = t\partial_t + \underbrace{\begin{pmatrix} 0 & 0 & \ldots & 0 & (n+1)^{n+1}t^{n+1} \\ 1 & 0 & & \cdots & 0 \\ & & & \cdots & \\ 0 & \ldots & 0 & 1 & 0 \end{pmatrix}}_{=:A'_{\mathbb{P}^n}}, \tag{26}$$

i.e., $A'_{\mathbb{P}^n} = h^{-1}A_{Q,\mathbb{P}^n}h + h^{-1}t\frac{\partial h}{\partial t}$.[4] Via the cyclic vector $(1,0,\ldots,0)^{\mathrm{t}}$, this system corresponds to the differential operator

$$P_{\mathbb{P}^n} = (t\partial_t)^{n+1} - (n+1)^{n+1}t^{n+1},$$

i.e.,

$$\left(\mathcal{O}^{n+1}_{\mathbb{G}_m}, \nabla\right) \cong \mathbb{C}\left[t,t^{-1}\right]\langle\partial_t\rangle / \mathbb{C}\left[t,t^{-1}\right]\langle\partial_t\rangle\left((t\partial_t)^{n+1} - (n+1)^{n+1}t^{n+1}\right).$$

$P_{\mathbb{P}^n}$ is the pullback of the generalized hypergeometric differential operator $\frac{1}{(n+1)^{2(n+1)}}(z\partial_z)^{n+1} - z$ under $t \mapsto t^{n+1} = z$.

Remark IV.5 (Cf. [**6**]). The hypergeometric series

$$K(t) \coloneqq \sum_{k=0}^{\infty} \frac{1}{(k!)^{n+1}} t^{(n+1)k} \in \mathbb{C}[\![t]\!]$$

is a formal power series solution of $P_{\mathbb{P}^n}u = 0$.

[3]We always consider the case $q = 1$.

[4]These two systems are *Dubrovin's connection in the anticanonical* and *z-direction*, respectively, and are identified via the gauge transformation h.

3. Quantum connection of weighted projective lines

In complete analogy to Section IV.1, the quantum connection is defined for orbifolds—such as weighted projective spaces—using orbifold cohomology. By [**23**, Example 3.20], the orbifold cohomology ring of the weighted projective line $\mathbb{P}(a,b)$ is given by

$$H^*_{\text{orb}}\left(\mathbb{P}(a,b),\mathbb{C}\right) = \mathbb{C}\left[x,y,\xi\right]/\langle xy, ax^{\frac{a}{d}} - by^{\frac{b}{d}}\xi^{n-m}, \xi^d - 1\rangle, \tag{27}$$

where $d = \gcd(a,b)$ and $m,n \in \mathbb{Z}$ such that $am + bn = d$. The grading is given by $\deg x = \frac{1}{A}$, $\deg y = \frac{1}{B}$, $\deg \xi = 0$, where $A = \frac{a}{d}$, $B = \frac{b}{d}$ (cf. [**1**]). Quantum multiplication is computed in

$$\text{QH}_{\text{orb}}\left(\mathbb{P}(a,b),\mathbb{C}\right) = \mathbb{C}\left[x,y,\xi\right]/\langle xy - 1, ax^{\frac{a}{d}} - by^{\frac{b}{d}}\xi^{n-m}, \xi^d - 1\rangle.$$

For $\gcd(a,b) = 1$, $-K_{\mathbb{P}(a,b)}$ is given by the element $\left[x^a + y^b\right] \in H^1_{\text{orb}}(\mathbb{P}(a,b),\mathbb{C})$. Taking into account the scaling of the degree of the cohomology groups by 2, the grading operator is then defined by

$$\mu(a) = \left(i - \frac{\dim X}{2}\right)a \quad \text{for } a \in H^i_{\text{orb}}(\mathbb{P}(a,b),\mathbb{C}).$$

3.1. Quantum connection of $\mathbb{P}(1,2)$. The orbifold cohomology ring of $\mathbb{P}(1,2)$ is given by $H^*_{\text{orb}}(\mathbb{P}(1,2),\mathbb{C}) = \mathbb{C}\left[x,y\right]/\langle xy, x - 2y^2\rangle$ with grading $\deg x = 1$, $\deg y = \frac{1}{2}$. A basis over $\mathbb{C}$ is given by $1, y, y^2$. Quantum multiplication by $-K_{\mathbb{P}(1,2)} = x + y^2 = 3y^2$ in the basis $1, y, y^2$ is given by the matrix

$$\begin{pmatrix} 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{3}{2} \\ 3 & 0 & 0 \end{pmatrix}.$$

The grading operator μ in this basis is given by the diagonal matrix $\operatorname{diag}\left(-\frac{1}{2}, 0, \frac{1}{2}\right)$. Therefore, we obtain the following

Proposition IV.6. The quantum connection of $\mathbb{P}(1,2)$ in the basis stated above is given by

$$\nabla_{z\partial_z} = z\partial_z - \frac{1}{z}\begin{pmatrix} 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{3}{2} \\ 3 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}. \tag{28}$$

3.2. Quantum connection of $\mathbb{P}(1,3)$. The orbifold cohomology ring of $\mathbb{P}(1,3)$ is given by $H^*_{\text{orb}}(\mathbb{P}(1,3),\mathbb{C}) = \mathbb{C}\left[x,y\right]/\langle xy, x - 3y^3\rangle$ with grading $\deg x = 1$, $\deg y = \frac{1}{3}$. A basis over $\mathbb{C}$ is given by $1, y, y^2, y^3$. Quantum multiplication by $-K_{\mathbb{P}(1,3)} = x + y^3 = 4y^3$ in the basis $1, y, y^2, y^3$ is given by the matrix

$$\begin{pmatrix} 0 & \frac{4}{3} & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 0 \\ 0 & 0 & 0 & \frac{4}{3} \\ 4 & 0 & 0 & 0 \end{pmatrix}.$$

The grading operator μ in this basis is given by the diagonal matrix $\operatorname{diag}\left(-\frac{1}{2}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{2}\right)$. Therefore, we obtain the following

PROPOSITION IV.7. The quantum connection of $\mathbb{P}(1,3)$ in the basis stated above is given by

$$\nabla_{z\partial_z} = z\partial_z - \frac{1}{z}\begin{pmatrix} 0 & \frac{4}{3} & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 0 \\ 0 & 0 & 0 & \frac{4}{3} \\ 4 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{6} & 0 & 0 \\ 0 & 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}. \tag{29}$$

3.3. Quantum connection of $\mathbb{P}(1,n)$. Let $n \in \mathbb{N}_{>0}$. The orbifold cohomology ring of $\mathbb{P}(1,n)$ is given by $H^*_{\text{orb}}(\mathbb{P}(1,n),\mathbb{C}) = \mathbb{C}[x,y]/\langle xy, x-ny^n\rangle$ with grading $\deg x = 1$, $\deg y = \frac{1}{n}$. A basis over $\mathbb{C}$ given by $1, y, y^2, \dots, y^n$. Quantum multiplication by $-K_{\mathbb{P}(1,n)} = x + y^n = (n+1)y^n$ in the basis $1, y, y^2, \dots, y^n$ is given by the matrix

$$\begin{pmatrix} 0 & \frac{n+1}{n} & 0 & \cdots & 0 \\ \vdots & \ddots & \frac{n+1}{n} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & \ddots & \frac{n+1}{n} \\ n+1 & 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

The grading operator μ in this basis is given by the diagonal matrix $\operatorname{diag}\left(-\frac{1}{2}, \frac{1}{n}-\frac{1}{2}, \frac{2}{n}-\frac{1}{2}, \dots, \frac{n-1}{n}-\frac{1}{2}, \frac{1}{2}\right)$. Therefore, we obtain the following

PROPOSITION IV.8. The quantum connection of $\mathbb{P}(1,n)$ in the basis $1, y, \dots, y^n$ is given by

$$\nabla_{z\partial_z} = z\partial_z - \frac{1}{z}\begin{pmatrix} 0 & \frac{n+1}{n} & 0 & \cdots & 0 \\ \vdots & \ddots & \frac{n+1}{n} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & \ddots & \frac{n+1}{n} \\ n+1 & 0 & \cdots & \cdots & 0 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} & & & & \\ & \frac{1}{n}-\frac{1}{2} & & & \\ & & \ddots & & \\ & & & \frac{n-1}{n}-\frac{1}{2} & \\ & & & & \frac{1}{2} \end{pmatrix}. \tag{30}$$

3.4. Quantum connection of $\mathbb{P}(2,2)$. For $\gcd(a,b) \neq 1$ formulae get much more complicated and the representation of the orbifold cohomology ring given by (27) does not reflect the geometry of the quantum cohomology. In order to compare the quantum connection to the Gauß–Manin system of its Landau–Ginzburg model, one should use the basis $1, H, H^2, H^3$, as suggested in the work [**23**] of É. Mann. We compute the quantum connection

$$\nabla_{z\partial_z} = z\partial_z - \frac{1}{z}4H \circ \quad + \mu$$

on the trivial bundle with fiber $H^*_{\text{orb}}(\mathbb{P}(2,2),\mathbb{C})$ over $\mathbb{P}^1$ to be given by the following

PROPOSITION IV.9. The quantum connection of $\mathbb{P}(2,2)$ in the basis stated above is given by

$$\nabla_{z\partial_z} = z\partial_z - \frac{1}{z}\begin{pmatrix} 0 & 0 & 0 & \frac{1}{4} \\ 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}. \tag{31}$$

4. Dubrovin's conjecture

Let $\mathcal{X}$ be a Fano variety (resp. a stack) such that the bounded derived category $\mathrm{D^b}(\mathrm{Coh}(\mathcal{X}))$ of coherent sheaves on $\mathcal{X}$ admits a full exceptional collection $\langle E_1, \dots, E_n \rangle$, where the collection $\langle E_1, \dots, E_n \rangle \subset \mathrm{D^b}(\mathrm{Coh}(\mathcal{X}))$ is called

- *exceptional* if $\mathrm{R}\mathcal{H}\mathrm{om}(E_i, E_i) = \mathbb{C}$ for all i and $\mathrm{R}\mathcal{H}\mathrm{om}(E_i, E_j) = 0$ for $i > j$,
- *full* if $\mathrm{D^b}(\mathrm{Coh}(\mathcal{X}))$ is the smallest full triangulated subcategory of $\mathrm{D^b}(\mathrm{Coh}(\mathcal{X}))$ containing $E_1, \dots, E_n$.

REMARK IV.10. B. Dubrovin calls this kind of Fano varieties *good* Fano varieties, but in the following we will not use this terminology.

In [**14**], B. Dubrovin conjectured that, under appropriate choices, the Stokes matrix of the quantum connection of $\mathcal{X}$ equals the Gram matrix of the Euler–Poincaré pairing with respect to some full exceptional collection—modulo some action of the braid group, sign changes and permutations. Then the second Stokes matrix is the transpose of the first one. The Euler–Poincaré pairing is given by the bilinear form

$$\chi(E, F) := \sum_k (-1)^k \dim_{\mathbb{C}} \mathrm{Ext}^k(E, F), \quad E, F \in \mathrm{D^b}(\mathrm{Coh}(\mathcal{X})).$$

The conjecture was proven for weighted projective spaces $\mathbb{P}(\omega_0, \omega_1, \dots, \omega_n)$ by S. Tanabé and K. Ueda in [**29**] and in [**8**] by J. A. Cruz Morales, for Grassmannians by work of S. Galkin, V. Golyshev, and H. Iritani and a refined version by G. Cotti. D. Guzzetti in [**17**] proved this conjecture for complex projective space in any dimension.

For the complex projective space $\mathcal{X} = \mathbb{P}^n$, Beilinson's collection

$$\mathcal{B} := \langle \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n) \rangle$$

is a full exceptional collection of $\mathrm{D^b}(\mathrm{Coh}(\mathcal{X}))$. The entries of the Gram matrix $S_{\mathbb{P}^n,\mathrm{Gram}} = (s_{ij})_{i,j=1,\dots,n+1}$ of χ with respect to $\mathcal{B}$ are given by $s_{ij} = \chi(\mathcal{O}(i-1), \mathcal{O}(j-1))$, hence

$$s_{ii} = 1,\ \ s_{ij} = 0 \text{ for } i > j \text{ and } s_{ij} = \binom{n+j-i}{j-i} \text{ for } i < j,$$

i.e.,

$$S_{\mathbb{P}^n,\mathrm{Gram}} = \begin{pmatrix} 1 & \binom{n+2-1}{2-1} & \binom{n+3-1}{3-1} & \dots & \binom{n+(n+1)-1}{n+1-1} \\ 0 & 1 & \binom{n+3-2}{3-2} & \dots & \binom{n+(n+1)-2}{n+1-2} \\ & & \dots & & \\ & & 0 & 1 & \binom{n+(n+1)-n}{n+1-n} \\ & & & 0 & 1 \end{pmatrix}.$$

For the complex projective line this formula gives the Gram matrix

$$S_{\mathbb{P}^1,\mathrm{Gram}} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \tag{32}$$

4.1. Application to weighted projective lines.

By [**2**, Theorem 2.12],

$$\mathcal{E} := \langle \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(a+b-1) \rangle$$

is a full exceptional collection of $\mathrm{D^b}(\mathrm{Coh}(\mathbb{P}(a,b)))$. Following [**4**, Theorem 4.1], the cohomology of the twisting sheaves for $k \in \mathbb{Z}$ is given by

- $H^0\left(\mathbb{P}(a,b),\mathcal{O}(k)\right) = \bigoplus_{(m,n)\in I_0} \mathbb{C}x^m y^n$, where
$$I_0 = \{(m,n)\in \mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0} \mid am+bn=k\}.$$
- $H^1\left(\mathbb{P}(a,b),\mathcal{O}(k)\right) = \bigoplus_{(m,n)\in I_1} \mathbb{C}x^m y^n$, where
$$I_1 = \{(m,n)\in \mathbb{Z}_{<0}\times\mathbb{Z}_{<0} \mid am+bn=k\}.$$
- $H^i\left(\mathbb{P}(a,b),\mathcal{O}(k)\right) = 0$ for all $i \geq 2$.

Since $\mathcal{E} = \langle \mathcal{O},\mathcal{O}(1),\ldots,\mathcal{O}(a+b-1)\rangle$ is a full exceptional collection of $\mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(\mathbb{P}(a,b)))$, the Gram matrix of χ with respect to $\mathcal{E}$ is upper triangular with ones on the main diagonal. Therefore, we only need to compute $\mathrm{Ext}^k\left(\mathcal{O}(i),\mathcal{O}(j)\right)$ for $i<j$ which is given by $H^k\left(\mathcal{O}\left(j-i\right)\right)$ (cf. [**24**, Lemma 4.5]). Therefore, the zeroth cohomologies of the twisting sheaves $\mathcal{O}\left(j-i\right)$ are the only ones that contribute to the Gram matrix of χ with respect to the full exceptional collection $\mathcal{E}$ of $\mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(\mathbb{P}(a,b)))$.

In the following, we compute the Gram matrix of χ with respect to $\mathcal{E}$ for some weighted projective lines $\mathbb{P}(a,b)$ that are of interest for the course of this thesis.

Proposition IV.11. For $\mathbb{P}(1,2)$ we get the cohomology groups
$$H^0(\mathcal{O}(1)) \cong \mathbb{C},\ H^0(\mathcal{O}(2)) \cong \mathbb{C}^2$$
and therefore the Gram matrix of χ with respect to $\mathcal{E} = \langle\mathcal{O},\mathcal{O}(1),\mathcal{O}(2)\rangle$ is given by
$$S_{\mathbb{P}(1,2),\mathrm{Gram}} = \begin{pmatrix}1&1&2\\0&1&1\\0&0&1\end{pmatrix}. \tag{33}$$

Proposition IV.12. For $\mathbb{P}(1,3)$ we get the cohomology groups
$$H^0(\mathcal{O}(1)) \cong H^0(\mathcal{O}(2)) \cong \mathbb{C},\ H^0(\mathcal{O}(3)) \cong \mathbb{C}^2$$
and therefore the Gram matrix of χ with respect to $\mathcal{E} = \langle\mathcal{O},\mathcal{O}(1),\mathcal{O}(2),\mathcal{O}(3)\rangle$ is given by
$$S_{\mathbb{P}(1,3),\mathrm{Gram}} = \begin{pmatrix}1&1&1&2\\0&1&1&1\\0&0&1&1\\0&0&0&1\end{pmatrix}. \tag{34}$$

Proposition IV.13. For $\mathbb{P}(1,n)$ we get the cohomology groups
$$H^0(\mathcal{O}(1)) \cong H^0(\mathcal{O}(2)) \cong \ldots \cong H^0(\mathcal{O}(n-1)) \cong \mathbb{C},\ H^0(\mathcal{O}(n)) \cong \mathbb{C}^2$$
and therefore the Gram matrix of χ with respect to $\mathcal{E} = \langle\mathcal{O},\mathcal{O}(1),\ldots,\mathcal{O}(n)\rangle$ is given by
$$S_{\mathbb{P}(1,n),\mathrm{Gram}} = \begin{pmatrix}1&1&\cdots&1&2\\0&1&\ddots&&1\\\vdots&\ddots&\ddots&\ddots&\vdots\\\vdots&&\ddots&\ddots&1\\0&\cdots&\cdots&0&1\end{pmatrix}. \tag{35}$$

Proposition IV.14. For $\mathbb{P}(2,2)$ we get the cohomology groups

$$H^0(\mathcal{O}(1)) \cong H^0(\mathcal{O}(3)) \cong 0,\ H^0(\mathcal{O}(2)) \cong \mathbb{C}^2$$

and therefore the Gram matrix of χ with respect to $\mathcal{E} = \langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \mathcal{O}(3) \rangle$ is given by

$$S_{\mathbb{P}(2,2),\mathrm{Gram}} = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{36}$$

CHAPTER V

Topological computation of Stokes data

In this chapter, we compute the Stokes multipliers of the enhanced Fourier–Sato transform of some perverse sheaves on the affine line $\mathbb{A}^1$ in a purely topological way, following the recent work [**9**] of A. D'Agnolo, M. Hien, G. Morando, and C. Sabbah. We focus on examples arising from weighted projective lines via mirror symmetry.

Let $\mathcal{M} \in \mathrm{Mod}_{\mathrm{rh}}(\mathcal{D}_{\mathbb{A}^1})$ be a holonomic $\mathcal{D}$-module on the affine line with singularities $\Sigma \subset \mathbb{A}^1$, all of them being regular singular. The singularities of the Fourier–Laplace transform $\widehat{\mathcal{M}}$ are known to be 0, being regular singular, and ∞, being irregular singular. It is known by work of B. Malgrange (cf. [**22**]) that the exponential factors of $\widehat{\mathcal{M}}$ at ∞ are of linear type with coefficients given by the singularities of $\mathcal{M}$. By the regular Riemann–Hilbert correspondence, one associates to $\mathcal{M}$ a perverse sheaf $F \in \mathrm{Perv}_{\Sigma}(\mathbb{C}_{\mathbb{A}^1})$.

The main result in [**9**] is a determination of the Stokes data of the enhanced Fourier–Sato transform of F in terms of the quiver associated to F. The quiver associated to the perverse sheaf $F \in \mathrm{Perv}_{\Sigma}(\mathbb{C}_{\mathbb{A}^1})$ is defined to be

$$Q_{\Sigma}^{(\alpha,\beta)}(F) = \left(\Psi(F), \Phi_{\sigma}(F), u_{\sigma}, v_{\sigma}\right)_{\sigma\in\Sigma},$$

consisting of the finite dimensional $\mathbb{C}$-vector spaces of global nearby cycles and vanishing cycles of F at σ and linear maps $u_{\sigma}\colon \Psi(F) \to \Phi_{\sigma}(F)$ and $v_{\sigma}\colon \Phi_{\sigma}(F) \to \Psi(F)$ such that $1 - u_{\sigma}v_{\sigma}$ is invertible for every $\sigma \in \Sigma$.

The enhanced Fourier–Sato transform of F being the enhanced ind-sheaf associated to $\widehat{\mathcal{M}}$ by the enhanced solution functor, this procedure yields the Stokes data of $\widehat{\mathcal{M}}$ at ∞.

We apply this procedure to examples arising from mirror symmetry. It is known by results of A. Douai, É. Mann, and C. Sabbah that the quantum connection of the (weighted) projective line $\mathbb{P}(a,b)$ is given by the Fourier–Laplace transform $\widehat{H^0\left(\int_f \mathcal{O}\right)}$ of the Gauß–Manin system associated to its mirror partner $\left(\left\{x^a y^b = 1\right\} \subset (\mathbb{C}^*)^2, f = x + y\right)$. For $\gcd(a,b) = 1$, the mirror partner can be described as $(\mathbb{G}_m, f)$ with the Laurent polynomial $f = x^a + x^{-b} \in \mathbb{C}\left[x, x^{-1}\right]$.

One important example is the perverse sheaf $\mathrm{R}f_*^{\mathrm{an}}\mathbb{C}_{\mathbb{G}_m^{\mathrm{an}}}[1] \in \mathrm{Perv}_{\Sigma}(\mathbb{C}_{\mathbb{A}^1})$, where $\Sigma = \{\pm 2\}$ denotes the set of critical values of $f = x + x^{-1}$. This turns out to be the perverse sheaf associated to the regular singular Gauß–Manin system $H^0\left(\int_f \mathcal{O}\right)$. Hence the application of the method developed in [**9**] yields the Stokes multipliers of the Fourier–Laplace transform of $\int_f \mathcal{O}$ at ∞ and therefore the Stokes multipliers of the quantum connection of $\mathbb{P}^1$ at ∞.

In this chapter, we repeat the computation for the $\mathbb{P}^1$ case carried out in [**9**, Section 7] and, among others, apply it to the mirror partners of the weighted projective lines $\mathbb{P}(1,2)$, $\mathbb{P}(1,3)$, and $\mathbb{P}(2,2)$.

1. Topological computations

1.1. General procedure. Denote by $\Re(\bullet)$ (resp. $\Im(\bullet)$) the real (resp. imaginary) part of a complex number. Moreover, we canonically identify $\mathbb{A}^1$ with its dual space $\left(\mathbb{A}^1\right)^\vee$ and denote by $\langle\alpha,\beta\rangle = \alpha\cdot\beta$ the perfect pairing of $\mathbb{A}^1$ and $\left(\mathbb{A}^1\right)^\vee$. We fix $\alpha\in\mathbb{A}^1$ and $\beta\in\left(\mathbb{A}^1\right)^\vee\smallsetminus\{0\}$ such that

$$\Re\left(\langle\alpha,\beta\rangle\right) = 0,$$
$$\Re\left(\langle\sigma_i-\sigma_j,\beta\rangle\right)\neq 0\quad\forall\sigma_i,\sigma_j\in\Sigma,\ \sigma_i\neq\sigma_j.$$

For an embedding $i_A\colon A\hookrightarrow\mathbb{A}^1$, denote by $\mathbb{C}_A \coloneqq (i_A)_! i_A^{-1}\mathbb{C}_{\mathbb{A}^1}$.

Definition V.1. Let $\Sigma\subset\mathbb{A}^1$ be finite. Let $\sigma\in\Sigma$ and $F\in\mathrm{D}^{\mathrm{b}}_{\mathbb{C}-\mathrm{constr.}}(\mathbb{C}_{\mathbb{A}^1},\Sigma)$, i.e., F is $\mathbb{C}$-constructible with respect to the stratification $(\mathbb{A}^1\smallsetminus\Sigma,\Sigma)$. The complex of

- *nearby cycles at* $\sigma\in\Sigma$ is defined by
$$\Psi_\sigma^{(\alpha,\beta)}(F)\coloneqq\Psi_\sigma(F)\coloneqq\mathrm{R}\Gamma_c\left(\mathbb{A}^1;\mathbb{C}_{\ell_\sigma^\times}\otimes F\right),$$
- *vanishing cycles at* $\sigma\in\Sigma$ is defined by
$$\Phi_\sigma^{(\alpha,\beta)}(F)\coloneqq\Phi_\sigma(F)\coloneqq\mathrm{R}\Gamma_c\left(\mathbb{A}^1;\mathbb{C}_{\ell_\sigma}\otimes F\right),$$
- *global nearby cycles* is defined by
$$\Psi^{(\alpha,\beta)}(F)\coloneqq\Phi(F)\coloneqq\mathrm{R}\Gamma_c\left(\mathbb{A}^1;\mathbb{C}_{\mathbb{A}^1\smallsetminus\ell_\Sigma}\otimes F\right)[1],$$

where $\ell_\sigma\coloneqq\ell_\sigma(\alpha)\coloneqq\sigma+\mathbb{R}_{\geq 0}\alpha$, $\ell_\sigma^\times\coloneqq\ell_\sigma^\times(\alpha)\coloneqq\sigma+\mathbb{R}_{>0}\alpha$ and $\ell_\Sigma\coloneqq\bigcup_{\sigma\in\Sigma}\ell_\sigma$.

Denote by
$$p_\beta\colon\mathbb{A}^1\to\mathbb{R},\quad z\mapsto\Re\left(\langle z,\beta\rangle\right)$$
the $\mathbb{R}$-linear projection. It defines a total order on Σ by
$$\sigma<_\beta\sigma' :\Leftrightarrow p_\beta(\sigma)<p_\beta(\sigma').$$
We enumerate the elements of Σ as
$$\sigma_1<_\beta\sigma_2<_\beta\ldots<_\beta\sigma_n.$$
Set $-\infty \eqqcolon r_0,\ r_i\coloneqq p_\beta\left(\sigma_i\right),\ r_{n+1}\coloneqq+\infty\quad(i=1,\ldots,n)$. The open bands
$$B_{\sigma_i}\coloneqq B_{\sigma_i}\left(\beta\right)\coloneqq p_\beta^{-1}\left(\left(r_{i-1},r_{i+1}\right)\right)$$
cover $\mathbb{A}^1$ and satisfy $B_{\sigma_i}\left(\beta\right)\cap\Sigma=\{\sigma_i\}$. Denote by
$$B_{\sigma_i}^{>}\coloneqq B_{\sigma_i}^{>}\coloneqq p_\beta^{-1}\left(\left(r_i,r_{i+1}\right)\right),\quad B_{\sigma_i}^{\leq}\coloneqq B_{\sigma_i}^{\leq}\left(\beta\right)\coloneqq p_\beta^{-1}\left(\left(r_{i-1},r_{i+1}\right]\right).$$
The map $u_\sigma\colon\Psi(F)\to\Phi_\sigma(F)$ arises from the short exact sequence $0\to\mathbb{C}_{\ell_\sigma^\times}\to\mathbb{C}_{\ell_\sigma}\to\mathbb{C}_\sigma\to 0$, $v_\sigma\colon\Phi_\sigma(F)\to\Psi(F)$ is induced by the short exact sequence $0\to\mathbb{C}_{B_\sigma\smallsetminus\ell_\sigma}\to\mathbb{C}_{B_\sigma}\to\mathbb{C}_{\ell_\sigma}\to 0$ as described in [**9**, Section 4.2].
By [**9**, Theorem 5.2.2], the Stokes multipliers $S_{\pm\beta}$ of the enhanced Fourier–Sato transform of F at ∞ are then given by
$$S_\beta=\begin{pmatrix}1 & u_1v_2 & u_1v_3 & \ldots & u_1v_n\\ & 1 & u_2v_3 & \ldots & u_2v_n\\ & & \ddots & & \vdots\\ & & & & 1\end{pmatrix}\in\mathrm{End}^+\left(\Phi_\Sigma\right),$$

$$S_{-\beta} = \begin{pmatrix} \mathbb{T}_1 & & & & \\ -u_2 v_1 & \mathbb{T}_2 & & & \\ -u_3 v_1 & -u_3 v_2 & \ddots & & \\ \vdots & \vdots & & \ddots & \\ -u_n v_1 & -u_n v_2 & \dots & -u_n v_{n-1} & \mathbb{T}_n \end{pmatrix} \in \operatorname{End}^-(\Phi_\Sigma),$$

where $\mathbb{T}_i := \mathbb{T}_{\sigma_i} := 1 - u_{\sigma_i} v_{\sigma_i} \in \operatorname{End}(\Phi_{\sigma_i})$, $\Phi_\Sigma := \bigoplus_{\sigma \in \Sigma} \Phi_\sigma(F)$ and $\operatorname{End}^\pm(\bullet)$ denotes upper and lower triangular block matrices, respectively. The matrices $S_{\pm\beta}$ describe passing from $H_\alpha = \left\{ z \in \left(\mathbb{A}^1\right)^\vee \smallsetminus \{0\} \mid \Re(\langle \alpha, z \rangle) \geq 0 \right\}$ to $H_{-\alpha} = \left\{ z \in \left(\mathbb{A}^1\right)^\vee \smallsetminus \{0\} \mid \Re(\langle \alpha, z \rangle) \leq 0 \right\}$ crossing $h_{\pm\beta} := \pm\beta\mathbb{R}_{>0}$, $H_{\pm\alpha}$ being considered as closed sectors at ∞.

1.2. Application to two examples.

Example 1. The pair $\left(\mathbb{G}_m, f = x + x^{-1}\right)$ is a Landau–Ginzburg model of the complex projective line. Since $f = x + x^{-1}$ is a proper map, we know that

$$\left(\int_f \mathcal{O}\right)^{\mathrm{an}} \simeq \int_{f^{\mathrm{an}}} \mathcal{O}^{\mathrm{an}} \quad \text{and} \quad \int_f \mathcal{O} \simeq \int_{f_!} \mathcal{O}.$$

By the adjunction formula [**18**, Corollary 2.7.3] for proper morphisms we compute that

$$\begin{aligned} \mathcal{S}\mathrm{ol}\left(\int_f \mathcal{O}_{\mathbb{G}_m}\right) &= \mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{D}^{\mathrm{an}}_{\mathbb{A}^1}}\left(\left(\int_f \mathcal{O}_{\mathbb{G}_m}\right)^{\mathrm{an}}, \mathcal{O}^{\mathrm{an}}_{\mathbb{A}^1}\right) \simeq \mathrm{R}f^{\mathrm{an}}_* \, \mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_{\mathbb{G}_m^{\mathrm{an}}}}\left(\mathcal{O}^{\mathrm{an}}_{\mathbb{G}_m}, f^{\mathrm{an},\dagger}\mathcal{O}^{\mathrm{an}}_{\mathbb{A}^1}\right) \\ &\simeq \mathrm{R}f^{\mathrm{an}}_* \, \mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{D}^{\mathrm{an}}_{\mathbb{G}_m}}\left(\mathcal{O}^{\mathrm{an}}_{\mathbb{G}_m}, \mathrm{L}f^{\mathrm{an},*}\mathcal{O}^{\mathrm{an}}_{\mathbb{A}^1}[1-1]\right) \simeq \mathrm{R}f^{\mathrm{an}}_* \, \mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_{\mathbb{G}_m^{\mathrm{an}}}}\left(\mathcal{O}^{\mathrm{an}}_{\mathbb{G}_m}, \mathcal{O}^{\mathrm{an}}_{\mathbb{G}_m}\right) \\ &\simeq \mathrm{R}f^{\mathrm{an}}_* \mathbb{C}_{\mathbb{G}_m^{\mathrm{an}}} \in \mathrm{D}^{\mathrm{b}}_{\mathbb{C}-\mathrm{constr.}}(\mathbb{C}_{\mathbb{A}^1}, \Sigma), \end{aligned}$$

where $\Sigma = \{\pm 2\}$ denotes the set of critical values of f. In the following, for the sake of notational simplicity, we often omit the superscript $(\bullet)^{\mathrm{an}}$ as well as the subscripts $(\bullet)_{\mathbb{G}_m}$ and $(\bullet)_{\mathbb{A}^1}$.
Outside of $\Sigma = \{\pm 2\} \subset \mathbb{A}^1$, f is a two-sheeted covering. We now consider the perverse sheaf

$$F := \mathrm{R}f_*\mathbb{C}[1] \in \mathrm{Perv}_\Sigma(\mathbb{C}_{\mathbb{A}^1})$$

which is concentrated in degree -1.
Following [**9**, Section 7], where this example is carried out explicitly, we compute the quiver $Q^{(\alpha,\beta)}_\Sigma(F) = (\Psi, \Phi_\sigma, u_\sigma, v_\sigma)_{\sigma\in\Sigma}$ of F and by means of it the Stokes multipliers at ∞ of the Fourier–Sato transform of F. The exponential components at ∞ of the Fourier–Sato transform of F are known to be of linear type, with coefficients given by ± 2. Therefore, the Stokes directions are given by $\pm\frac{\pi}{2}$.
We have to choose α such that $\ell_{\pm 2}(\alpha)$[1] does not intersect any other critical values of f, i.e., we have to choose $\alpha \notin \mathbb{R}$. We fix $\alpha = i \in \mathbb{A}^1$, $\beta = 1 \in \left(\mathbb{A}^1\right)^\vee \smallsetminus \{0\}$, which determines the closed sectors centered at infinity

$$H_i = \left\{ z \in \left(\mathbb{A}^1\right)^\vee \smallsetminus \{0\} \mid \Im(z) \leq 0 \right\}, \; H_{-i} = \left\{ z \in \left(\mathbb{A}^1\right)^\vee \smallsetminus \{0\} \mid \Im(z) \geq 0 \right\}.$$

Then

$$H_i \cap H_{-i} = h_1 \cup h_{-1},$$

where $h_{\pm\beta} = \pm\mathbb{R}_{>0}\beta \subset \left(\mathbb{A}^1\right)^\vee$. Moreover, the choice of α and β induces the following order of the singularities:

$$\sigma_1 := -2 <_\beta 2 =: \sigma_2$$

[1] In [**9**, Section 7], ℓ_2 is denoted by a^+, ℓ_{-2} by a^-.

Choose a base point $e \in \mathbb{A}^1$ with $\Re(e) > 2$ and denote its preimages under f by e_1, e_2, as depicted in Figure 2.
The nearby cycles of F at ± 2 are given by

$$\Psi_{\pm 2}(F) = \mathrm{R}\Gamma_c\left(\mathbb{A}^1; \mathbb{C}_{\ell^\times_{\pm 2}} \otimes F\right) \cong \bigoplus_{j=1,2} \mathbb{C}_{e_j} \cong \mathbb{C}^2.$$

The global nearby cycles of F are

$$\Psi(F) = \mathrm{R}\Gamma_c\left(\mathbb{A}^1; \mathbb{C}_{\mathbb{A}^1 \smallsetminus \ell_\Sigma} \otimes F\right) \cong \Psi_{\pm 2}(F) \cong \mathbb{C}^2.$$

Furthermore, we fix isomorphisms

$$i^{-1}_{\pm 2} F[-1] \cong \mathbb{C}_{f^{-1}(\pm 2)} \cong \mathbb{C},$$

where i_σ denotes the embedding $i_\sigma \colon \{\sigma\} \hookrightarrow \mathbb{A}^1$.
The distinguished triangle

$$F \otimes \mathbb{C}_{\mathbb{A}^1 \smallsetminus \Sigma} \to F \to F \otimes \mathbb{C}_\Sigma \xrightarrow{+1}$$

induces the short exact sequence of perverse sheaves

$$0 \to F \otimes \mathbb{C}_\Sigma[-1] \to F \otimes \mathbb{C}_{\mathbb{A}^1 \smallsetminus \Sigma} \to F \to 0.$$

Applying the quiver functor $Q^{(\alpha,\beta)}$ yields the exact diagram of quivers

$$\begin{tikzcd}
 & {i_2}^{-1}F[-1] \arrow[r, "b_2"] \arrow[d, shift left] & \Psi(F) \arrow[r, "u_2"] \arrow[d, shift left, "1-T_2"] & \Phi_2(F) \arrow[d, shift left, "v_2"] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[u, shift left] \arrow[d, shift left] & \Psi(F) \arrow[r, "1"] \arrow[u, shift left, "1"] \arrow[d, shift left, "1"'] & \Psi(F) \arrow[r] \arrow[u, shift left, "u_2"] \arrow[d, shift left, "u_{-2}"'] & 0. \\
 & {i_{-2}}^{-1}F[-1] \arrow[r, "b_{-2}"] \arrow[u, shift left] & \Psi(F) \arrow[r, "u_{-2}"] \arrow[u, shift left, "1-T_{-2}"'] & \Phi_{-2}(F) \arrow[u, shift left, "v_{-2}"'] &
\end{tikzcd}$$

We obtain the quiver of F as the cokernel of the left part of the diagram.
Let us start with the computation of the monodromy operators $T_{\pm 2}$. Consider loops γ^+ and γ^- starting at e and running around 2 resp. -2 in counterclockwise orientation as depicted in Figure 2. Consider the two lifts γ^+_j of γ^+ (resp. lifts γ^-_j of γ^-) under f starting at e_j. The way in which e_1, e_2 are interchanged yields the monodromy T_2 at 2 (resp. T_{-2} at -2). We read from Figure 2 the monodromy operators in the basis e_1, e_2:

$$T_{-2} = T_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The mappings $b_{\pm 2}$ are induced by the boundary map, starting from $\ell^\times_{\pm 2} \hookrightarrow \ell_{\pm 2} \hookleftarrow \{\pm 2\}$, assigning the starting point to the lifts of $\ell^\times_{\pm 2}$. Denote by $\ell^j_{\pm 2}$ the preimage of $\ell_{\pm 2}$ under f that intersects $\gamma^\pm_j$. We read that $\ell^j_{\pm 2} \mapsto f^{-1}(\pm 2)$, $j = 1, 2$, and therefore obtain

$$b_{-2} = b_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

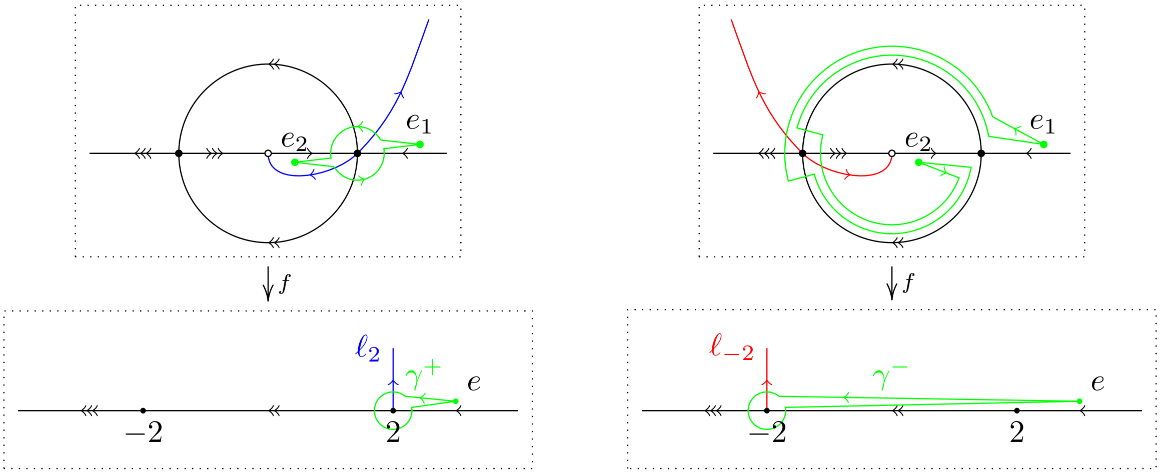


FIGURE 2. Cf. [**9**, Figure 17]

Then the quiver of F is computed as

$$\begin{tikzcd} \Phi_2(F) \arrow[d, shift left, "v_2"] \\ \Psi(F) \arrow[u, shift left, "u_2"] \arrow[d, shift right, "u_{-2}"'] \\ \Phi_{-2}(F) \arrow[u, shift right, "v_{-2}"'] \end{tikzcd} \simeq \operatorname{coker}\left(\begin{tikzcd} \mathbb{C} \arrow[r, "\binom{1}{1}"] \arrow[d, shift left] & \mathbb{C}^2 \arrow[d, shift left, "\begin{pmatrix}1&-1\\-1&1\end{pmatrix}"] \\ 0 \arrow[r] \arrow[u, shift left] \arrow[d, shift right] & \mathbb{C}^2 \arrow[u, shift left, "\mathrm{id}"] \arrow[d, shift right, "\mathrm{id}"'] \\ \mathbb{C} \arrow[r, "\binom{1}{1}"'] \arrow[u, shift right] & \mathbb{C}^2 \arrow[u, shift right, "\begin{pmatrix}1&-1\\-1&1\end{pmatrix}"'] \end{tikzcd}\right) \simeq \begin{tikzcd} \mathbb{C} \arrow[d, shift left, "\binom{1}{-1}"] \\ \mathbb{C}^2 \arrow[u, shift left, "(1\ -1)"] \arrow[d, shift right, "(1\ -1)"'] \\ \mathbb{C} \arrow[u, shift right, "\binom{1}{-1}"'] \end{tikzcd},$$

where we identified the cokernel of $b_{\pm 2}$ with $\mathbb{C}$ via $[(v_1, v_2)^{\mathrm{t}}] = [(v_1 - v_2, 0)^{\mathrm{t}}]$. Therefore,

$$\mathbb{T}_{-2} = 1 - u_{-2}v_{-2} = -1, \quad \mathbb{T}_2 = 1 - u_2 v_2 = -1.$$

Following [**9**, Theorem 5.2.2], the Stokes multipliers at ∞ of the Fourier–Laplace transform of $H^0\left(\int_f \mathcal{O}\right)$ are then given by

$$S_\beta = \begin{pmatrix} 1 & u_{-2}v_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

and

$$S_{-\beta} = \begin{pmatrix} \mathbb{T}_{-2} & 0 \\ -u_2 v_{-2} & \mathbb{T}_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix}.$$

$S_{\pm\beta}$ describes crossing $h_{\pm\beta}$ from H_α to $H_{-\alpha}$.

OBSERVATION V.2. S_β coincides with the Stokes matrix (10) of Bessel's differential equation computed by means of Asymptotic Analysis in Section II.2.

OBSERVATION V.3. S_β coincides with the Gram matrix (32) of the Euler–Poincaré pairing on $\mathrm{D^b}\left(\mathrm{Coh}\left(\mathbb{P}^1\right)\right)$ with respect to the full exceptional collection $\langle \mathcal{O}, \mathcal{O}(1) \rangle$.

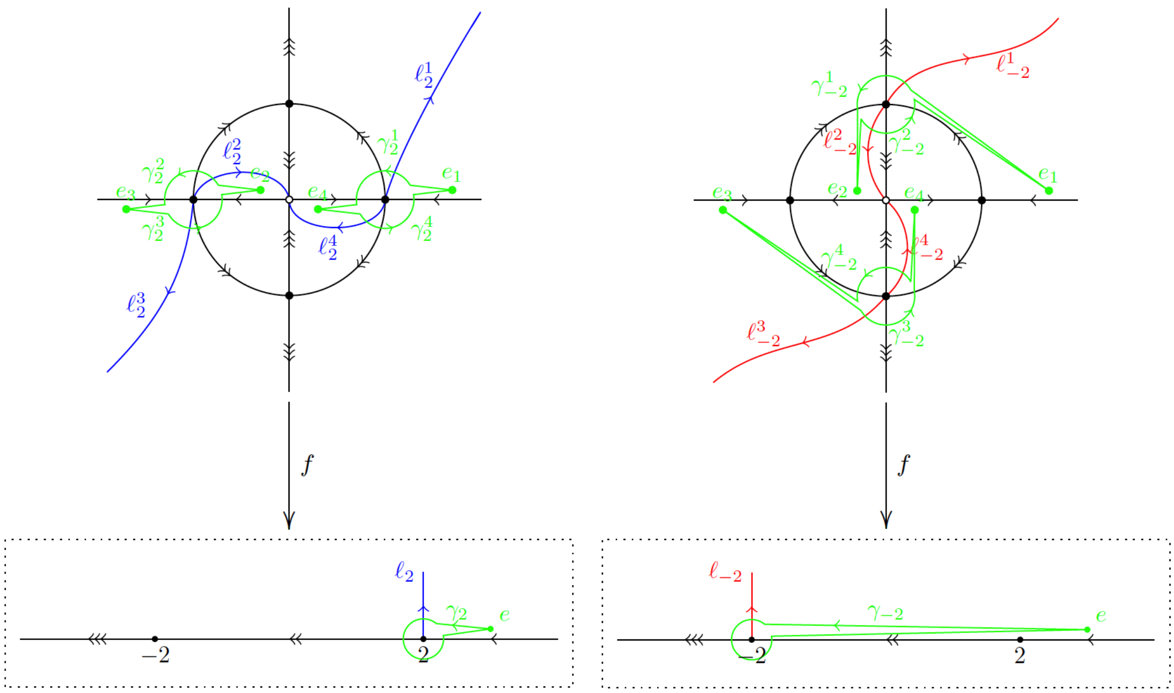


FIGURE 3. $f = x^2 + x^{-2}$, plotted with Ti*k*Z

Example 2. We consider the cohomologically tame Laurent polynomial $f = x^2 + x^{-2} \colon \mathbb{G}_m \to \mathbb{A}^1$.

REMARK V.4. One should be aware that this is not a mirror partner of $\mathbb{P}(2,2)$.

Its critical points are given by $\{\pm 1, \pm i\}$. The critical values of f are $\Sigma = \{\pm 2\}$ with $f^{-1}(2) = \{\pm 1\}$, $f^{-1}(-2) = \{\pm i\}$. Outside of Σ, f is a covering of degree 4. In the following, we investigate the perverse sheaf

$$F \coloneqq \mathrm{R}f_*\mathbb{C}[1] \in \mathrm{Perv}_{\{\pm 2\}}\left(\mathbb{C}_{\mathbb{A}^1}\right).$$

F is indeed a perverse sheaf, since f is semismall[2] Outside of Σ, F is the local system $\mathbb{C}^4_{\mathbb{A}^1 \smallsetminus \Sigma}$, hence $F \in \mathrm{Perv}_\Sigma\left(\mathbb{C}_{\mathbb{A}^1}\right)$.
The exponential components at ∞ of the Fourier–Sato transform of F are of linear type, with coefficients given by ± 2. Hence the Stokes directions are given by $\pm\frac{\pi}{2}$. For our computations we have to choose α such that $\ell_{\sigma_i}(\alpha)$ does not intersect any other singular value of f, i.e., $\alpha \notin \mathbb{R}$. We choose $\alpha = i$, $\beta = 1$. This yields the following ordering of the singularities: $\sigma_1 \coloneqq -2 <_\beta 2 \eqqcolon \sigma_2$. Moreover, this choice leads to the closed sectors centered at infinity

$$H_i = \left\{z \in \left(\mathbb{A}^1\right)^\vee \smallsetminus \{0\} \mid \Im(z) \leq 0\right\}, \; H_{-i} = \left\{z \in \left(\mathbb{A}^1\right)^\vee \smallsetminus \{0\} \mid \Im(z) \geq 0\right\}.$$

Then

$$H_i \cap H_{-i} = h_1 \cup h_{-1},$$

where $h_{\pm\beta} = \pm\mathbb{R}_{>0}\beta \subset \left(\mathbb{A}^1\right)^\vee$.

[2]By [**5**, Definition 2.1.1], a proper holomorphic map $f \colon X \to Y$ of irreducible varieties is *semismall* if $\dim\left\{y \in Y \mid \dim f^{-1}(y) = k\right\} + 2k \leq \dim X$ for every k.

Choose a base point $e \in \mathbb{A}^1$ and label its preimages under f by e_1, e_2, e_3, e_4, as depicted in Figure 3. The nearby and global nearby cycles of $F = \mathrm{R}f_*\mathbb{C}[1]$ are computed to be

$$\Psi_{\pm 2}(F) \cong \bigoplus_{e_j \in f^{-1}(e)} \mathbb{C}_{e_j} \cong \mathbb{C}^4,$$

$$\Psi(F) \cong \Psi_{\pm 2}(F) \cong \mathbb{C}^4.$$

Furthermore, we fix isomorphisms

$$i_{\sigma_i}^{-1} F[-1] \cong \bigoplus_{\sigma_i^j \in f^{-1}(\sigma_i)} \mathbb{C}_{\sigma_i^j} \cong \mathbb{C}^2.$$

By considering Figure 3, in the basis e_1, e_2, e_3, e_4 we obtain the monodromies at ± 2

$$T_{-2} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The maps $b_{\pm 2}$ are induced by the boundary map in homology, which assigns the starting point to the lifts $\ell^1_{\pm 2}, \ldots, \ell^4_{\pm 2}$ of $\ell_{\pm 2}$. We label the preimages of $\sigma_1 = -2$ by $\sigma_1^1 = i, \sigma_1^2 = -i$ and the preimages of $\sigma_2 = 2$ by $\sigma_2^1 = 1, \sigma_2^2 = -1$.
From Figure 3 we read the following:

σ_1: $\ell^1_{-2} \mapsto \sigma_1^1$, $\ell^2_{-2} \mapsto \sigma_1^1$, $\ell^3_{-2} \mapsto \sigma_1^2$, $\ell^4_{-2} \mapsto \sigma_1^2$,

σ_2: $\ell^1_{2} \mapsto \sigma_2^1$, $\ell^2_{2} \mapsto \sigma_2^2$, $\ell^3_{2} \mapsto \sigma_2^2$, $\ell^4_{2} \mapsto \sigma_2^1$.

Therefore,

$$b_{-2} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We obtain the quiver of F as the cokernel

$$\begin{tikzcd} \Phi_2(F) \arrow[d, shift left, "v_2"] \\ \Psi(F) \arrow[u, shift left, "u_2"] \arrow[d, shift right, "u_{-2}"'] \\ \Phi_{-2}(F) \arrow[u, shift right, "v_{-2}"'] \end{tikzcd} \simeq \operatorname{coker} \left(\begin{tikzcd} \mathbb{C}^2 \arrow[r, "b_2"] \arrow[d, shift left] & \mathbb{C}^4 \arrow[d, shift left, "1-T_2"] \\ 0 \arrow[r] \arrow[u, shift left] \arrow[d, shift right] & \mathbb{C}^4 \arrow[u, shift left, "1"] \arrow[d, shift right, "1"'] \\ \mathbb{C}^2 \arrow[r, "b_{-2}"'] \arrow[u, shift right] & \mathbb{C}^4 \arrow[u, shift right, "1-T_{-2}"'] \end{tikzcd} \right).$$

We identify the cokernel of b_2 with $\mathbb{C}^2$ via

$$\left[\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} \right] = \left[\begin{pmatrix} w_1 - w_4 \\ 0 \\ w_3 - w_2 \\ 0 \end{pmatrix} \right]$$

and the cokernel of b_{-2} with $\mathbb{C}^2$ via

$$\left[\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} \right] = \left[\begin{pmatrix} w_1 - w_2 \\ 0 \\ w_3 - w_4 \\ 0 \end{pmatrix} \right].$$

Under these identifications, we obtain

$$u_{-2} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix},$$

$$v_{-2} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

In summary, we obtain $\left(\Phi_2(F) \underset{u_2}{\overset{v_2}{\rightleftarrows}} \Psi(F) \underset{u_{-2}}{\overset{v_{-2}}{\leftrightarrows}} \Phi_{-2}(F)\right) \simeq \left(\mathbb{C}^2 \underset{u_2}{\overset{v_2}{\rightleftarrows}} \mathbb{C}^4 \underset{u_{-2}}{\overset{v_{-2}}{\leftrightarrows}} \mathbb{C}^2\right)$. Therefore, we obtain the following

Theorem V.5. The Stokes multipliers $S_{\pm\beta}$ of the enhanced Fourier–Sato transform of F at ∞ are, in the chosen bases, given by

$$S_\beta = \begin{pmatrix} \mathbb{1} & u_{-2}v_2 \\ 0 & \mathbb{1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$S_{-\beta} = \begin{pmatrix} \mathbb{T}_{-2} & 0 \\ -u_2v_{-2} & \mathbb{T}_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & 0 & -1 \end{pmatrix} = -S_\beta^{\mathrm{t}}.$$

$S_{\pm\beta}$ describes crossing $h_{\pm\beta}$ from H_α to $H_{-\alpha}$.

2. Mirror of $\mathbb{P}(a,b)$

In the following sections, we compute the Stokes data at ∞ of the Fourier–Laplace transform of the Gauß–Manin system attached to the mirror partners of some weighted projective lines $\mathbb{P}(a,b)$. Remember that the complex projective line is given by $\left(\mathbb{C}^2\right)^*/\mathbb{C}^*$, where the action of $\lambda \in \mathbb{C}^*$ is given by $(x,y) \mapsto (\lambda x, \lambda y)$. The *weighted* projective line $\mathbb{P}(a,b)$, for $a,b \in \mathbb{N}_{>0}$, is given by the very same construction under the following weighted action of $\lambda \in \mathbb{C}^*$:

$$(x,y) \mapsto \left(\lambda^a x, \lambda^b y\right). \tag{37}$$

Remark V.6. The weighted projective line $\mathbb{P}(a,b)$ can be obtained by the Proj construction for the graded ring $\mathbb{C}[x,y]$ with $\deg x = a$, $\deg y = b$. In general, $\mathbb{P}(a,b)$ is a singular projective variety. Furthermore, it is Fano and toric.

Example. The weighted projective line $\mathbb{P}(1,n)$, $n \in \mathbb{N}$, is the toric variety of the fan associated to the Newton polytope $\Delta_\infty(f) = \mathrm{conv}_{\mathbb{R}}(\{1\},\{-n\})$ of $f = x + x^{-n} \in \mathbb{C}\left[x, x^{-1}\right]$ at ∞.

Since the weighted projective line is not smooth and therefore does not fit into the setting of mirror symmetry, it is more natural to define the weighted projective line as the quotient stack instead as follows (cf. [**2**]):

$$\mathbb{P}(a,b) \coloneqq \left[\left(\mathbb{A}^2_{\mathbb{C}} \smallsetminus \{0\}\right)/\mathbb{G}_m\right]$$

in the category of stacks, where the action of $\mathbb{G}_m$ is twisted as described in (37).

A Landau–Ginzburg model of the weighted projective line $\mathbb{P}(a,b)$ is given by the curve $\{x^a y^b = 1\} \subset (\mathbb{C}^*)^2$ together with the superpotential $f = x + y$ (cf., e.g., [**23**]). The case $a = b = 1$ is the usual projective line, as treated in Section V.1.2, with Landau–Ginzburg model $f\colon \mathbb{G}_m \to \mathbb{A}^1$, $x \mapsto x + x^{-1}$. If $a = 1$, one recovers the Landau–Ginzburg model as $(\mathbb{G}_m, f = x + x^{-b})$. Denote by

$$f^{(a,b)}(z) \coloneqq z^b + \frac{1}{z^a}, \quad a,b \in \mathbb{N} \smallsetminus \{0\}.$$

For the case that $\gcd(a,b) = 1$, $(\mathbb{G}_m, f^{(a,b)})$ is a Landau–Ginzburg model of $\mathbb{P}(a,b)$. Let us give a reasoning for that. We describe the mirror of $\mathbb{P}(a,b)$ as the affine scheme $\operatorname{Spec}\left(\mathbb{C}[x,y]/(x^a y^b - 1)\right)$ endowed with the regular function $x + y$. Since a and b are coprime, there exist $m, n \in \mathbb{Z}$ such that $am + bn = 1$. Without loss of generality, assume that $m \leq 0$ and therefore $n \geq 0$. The ring homomorphisms defined by

$$\begin{aligned}\phi\colon \mathbb{C}[x,y]/(x^a y^b - 1) &\rightleftarrows \mathbb{C}[z,z^{-1}]\colon \psi,\\ \phi(x) = z^b,\ \phi(y) &= z^{-a},\\ \psi(z) &= x^n y^{-m},\end{aligned}$$

are inverse to each other. The regular function $x + y$ under ϕ corresponds to $z^b + z^{-a}$.

Remark V.7. One may also choose $m \geq 0$ and therefore $n \leq 0$. Then the ring homomorphisms defined by

$$\begin{aligned}\phi\colon \mathbb{C}[x,y]/(x^a y^b - 1) &\rightleftarrows \mathbb{C}[z,z^{-1}]\colon \psi,\\ \phi(x) = z^{-b},\ \phi(y) &= z^{a},\\ \psi(z) &= x^{-n} y^{m},\end{aligned}$$

are inverse to each other, the regular function $x+y$ under ϕ corresponds to $z^a + z^{-b}$. Therefore, one may work either with $(\mathbb{G}_m, z^a + z^{-b})$ or $(\mathbb{G}_m, z^b + z^{-a})$ as a Landau–Ginzburg model of $\mathbb{P}(a,b)$.

The maps $f^{(a,b)}$ are proper. Moreover, $f^{(a,b)}$ is semismall and therefore we know that $\mathrm{R}f_*^{(a,b)}\mathbb{C}[1] \in \operatorname{Perv}(\mathbb{C}_{\mathbb{A}^1})$ (cf. [**5**]). Denote by Σ the critical values of $f^{(a,b)}$. On $\mathbb{A}^1 \smallsetminus \Sigma$, $\mathrm{R}f_*^{(a,b)}\mathbb{C}[1]$ is the local system $\mathbb{C}^{a+b}_{\mathbb{A}^1\smallsetminus\Sigma}$, hence $\mathrm{R}f^{(a,b)}\mathbb{C}[1] \in \operatorname{Perv}_\Sigma(\mathbb{C}_{\mathbb{A}^1})$.

Remark V.8. The Laurent polynomials $z^a + z^{-b}$ are convenient and non-degenerate with respect to their Newton polytope at ∞. Therefore, they are cohomologically tame.

By the regular Riemann–Hilbert correspondence, to the perverse sheaf $\mathrm{R}f_*^{(a,b)}\mathbb{C}[1] \in \operatorname{Perv}_\Sigma(\mathbb{C}_{\mathbb{A}^1})$ there is an associated regular singular holonomic $\mathcal{D}_{\mathbb{A}^1}$-module on the affine line with regular singularities Σ. The latter is given by the Gauß–Manin module $\int_{f^{(a,b)}} \mathcal{O}_{\mathbb{G}_m}$, since

$$\begin{aligned}\mathcal{S}\mathrm{ol}\left(\int_{f^{(a,b)}} \mathcal{O}_{\mathbb{G}_m}\right) &= \mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{D}^{\mathrm{an}}_{\mathbb{A}^1}}\left(\left(\int_{f^{(a,b)}} \mathcal{O}_{\mathbb{G}_m}\right)^{\mathrm{an}}, \mathcal{O}^{\mathrm{an}}_{\mathbb{A}^1}\right) \simeq \mathrm{R}f^{\mathrm{an}}_* \mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{D}^{\mathrm{an}}_{\mathbb{G}_m}}\left(\mathcal{O}^{\mathrm{an}}_{\mathbb{G}_m}, f^{\mathrm{an},\dagger}\left(\mathcal{O}^{\mathrm{an}}_{\mathbb{A}^1}\right)\right)\\ &\simeq \mathrm{R}f^{\mathrm{an}}_* \mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{D}^{\mathrm{an}}_{\mathbb{G}_m}}\left(\mathcal{O}^{\mathrm{an}}_{\mathbb{G}_m}, \mathrm{L}f^{\mathrm{an},*}\left(\mathcal{O}^{\mathrm{an}}_{\mathbb{A}^1}\right)\right) \simeq \mathrm{R}f^{\mathrm{an}}_* \mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{D}^{\mathrm{an}}_{\mathbb{G}_m}}\left(\mathcal{O}^{\mathrm{an}}_{\mathbb{G}_m}, \mathcal{O}^{\mathrm{an}}_{\mathbb{G}_m}\right)\\ &\simeq \mathrm{R}f^{\mathrm{an}}_* \mathbb{C}_{\mathbb{G}^{\mathrm{an}}_m},\end{aligned}$$

where we used the adjunction formula for proper morphism and, from the second line on, abbreviated $f^{(a,b)}$ to f. It follows that $\int_{f^{(a,b)}} \mathcal{O}_{\mathbb{G}_m}$ is concentrated in degree 0, i.e.,

$$\int_{f^{(a,b)}} \mathcal{O}_{\mathbb{G}_m} \simeq H^0\left(\int_{f^{(a,b)}} \mathcal{O}_{\mathbb{G}_m}\right).$$

Hence, by the methods of [**9**], we may compute the Stokes multipliers of the Fourier–Laplace transform of the Gauß–Manin system associated to $f^{(a,b)}$ by carrying out the topological computations for the perverse sheaf $\mathrm{R}f_*^{(a,b)}\mathbb{C}[1]$. In the following sections, we explicitly compute some examples. The plots in the following sections were produced in the open source computer algebra system SAGE and can be found in the appendix of this thesis.

3. Mirror of $\mathbb{P}(1,2)$

We consider the Laurent polynomial $f = x^2 + x^{-1} \colon \mathbb{G}_m \to \mathbb{A}^1$. Its critical points are given by $\left\{\frac{1}{\sqrt[3]{2}}\zeta_3^k\right\}_{k=0,1,2}$. The critical values of f are

$$\Sigma = \left\{\frac{3}{\sqrt[3]{4}}, \frac{3}{\sqrt[3]{4}}\zeta_3, \frac{3}{\sqrt[3]{4}}\zeta_3^2\right\}.$$

The blue area in Figure 4 shows where the real (resp. imaginary) part of f is greater than or equal to 0. In Figure 5, the preimage of the imaginary (resp. real) axis is depicted in blue (resp. red) color.

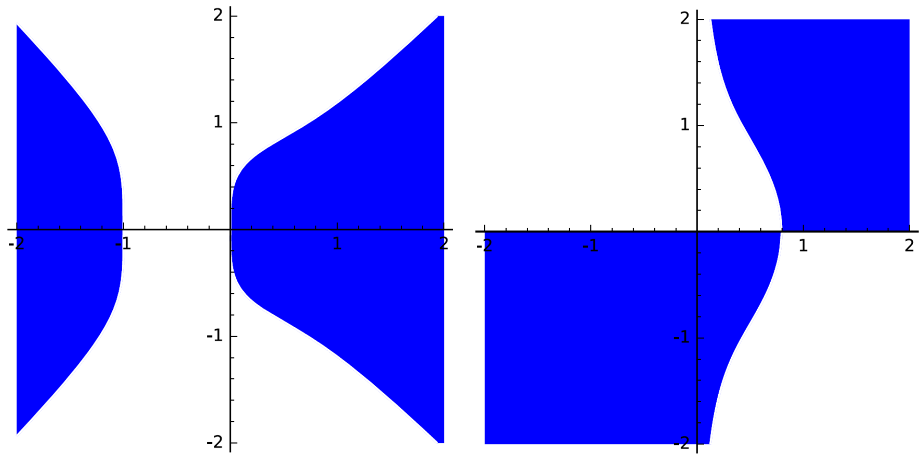


FIGURE 4. LHS: $\{x \mid \Re(f(x)) \geq 0\}$, RHS: $\{x \mid \Im(f(x)) \geq 0\}$

For our computations, we choose $\alpha = 1$, $\beta = -i$. Then $\Re(\langle\alpha,\beta\rangle) = 0$, $\Im(\langle\alpha,\beta\rangle) = -1$. This induces the following ordering on Σ:

$$\sigma_1 := \frac{3}{\sqrt[3]{4}}\zeta_3^2 <_\beta \sigma_2 := \frac{3}{\sqrt[3]{4}} <_\beta \sigma_3 := \frac{3}{\sqrt[3]{4}}\zeta_3.$$

The critical points are double inverse images of the critical values. Outside of Σ, f is a covering of degree 3. In the figures, we assign the following colors: σ_1 red, σ_2 purple, σ_3 blue.
Since f is semismall, $\mathrm{R}f_*\mathbb{C}[1] \in \mathrm{Perv}(\mathbb{C}_{\mathbb{A}^1})$. Outside of Σ, $\mathrm{R}f_*\mathbb{C}$ is the local system $\mathbb{C}^3_{\mathbb{A}^1\setminus\Sigma}$ and therefore

$$F := \mathrm{R}f_*\mathbb{C}[1] \in \mathrm{Perv}_\Sigma(\mathbb{C}_{\mathbb{A}^1}).$$

The exponential factors at ∞ of the Fourier–Laplace transform of F are of linear type with coefficients given by the σ_i.
In Figure 6, the preimages of horizontal lines passing through σ_i are plotted.

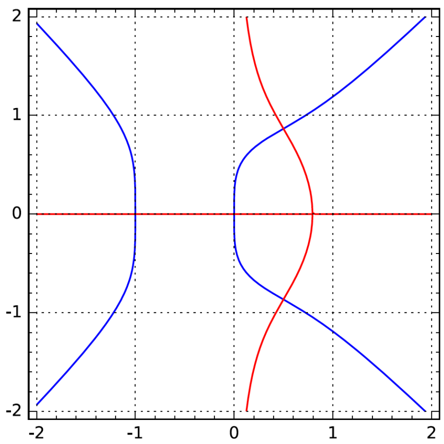


FIGURE 5. Preimage of the imaginary (resp. real) axis in blue (resp. red) color

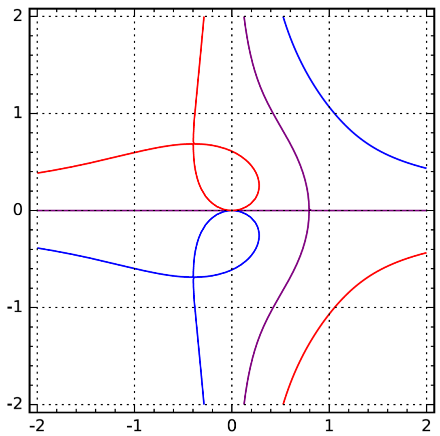


FIGURE 6. Preimages of horizontal lines through σ_i

We choose a base point $e \in \mathbb{A}^1$ with $\Re(e) > \Re(\sigma_i)$ for all i and consider loops γ_{σ_i} starting at e and running around the σ_i in clockwise[3] orientation as depicted in Figure 8. We denote by $\gamma^j_{\sigma_i}$ the preimage of γ_{σ_i} starting at e_j as depicted in Figure 8. We label the preimages of e (resp. σ_i) by e_1, e_2, e_3 (resp. σ_i^1, σ_i^2) as in Figure 7. Furthermore, we consider the half-lines $\ell_{\sigma_i} = \sigma_i + \mathbb{R}_{\geq 0}\alpha$. We denote by $\ell^j_{\sigma_i}$ the preimage of ℓ_{σ_i} which first intersects $\gamma^j_{\sigma_i}$.
The nearby cycles of the perverse sheaf F are given by

$$\Psi_{\sigma_i}(F) \cong \bigoplus_{e_j \in f^{-1}(e)} \mathbb{C}_{e_j} \cong \mathbb{C}^3,$$

$$\Psi(F) \cong \Psi_{\sigma_i}(F) \cong \mathbb{C}^3.$$

Furthermore, we fix isomorphisms

$$i^{-1}_{\sigma_i}(F)[-1] \cong \bigoplus_{\sigma_i^j \in f^{-1}(\sigma_i)} \mathbb{C}_{\sigma_i^j} \cong \mathbb{C}^2.$$

[3]Clockwise orientation since $\Im(\langle \alpha, \beta \rangle) = -1 < 0$.

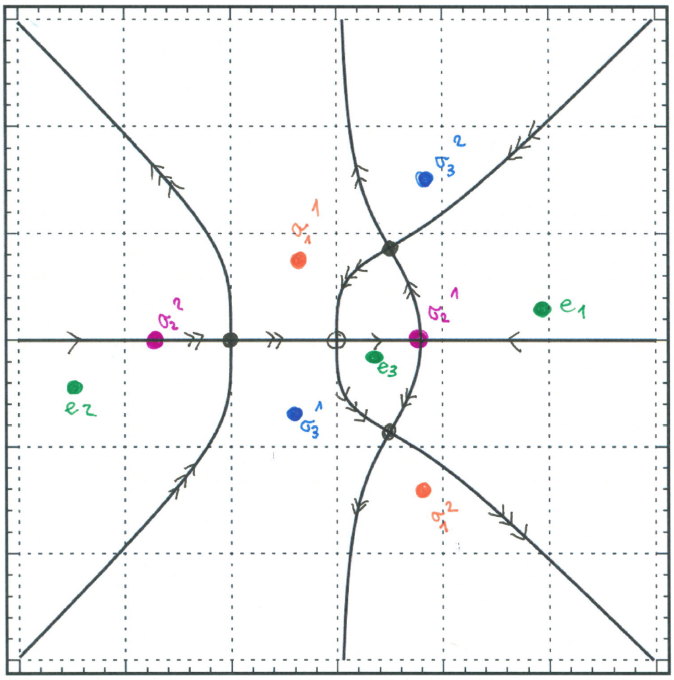


FIGURE 7. Labeling of the preimages e_j of e and σ_i^j of σ_i

From Figure 8 we read the monodromies in the basis e_1, e_2, e_3 to be given by

$$T_{\sigma_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \; T_{\sigma_2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \; T_{\sigma_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The mappings b_{σ_i} in the bases $\ell^1_{\sigma_i}, \ell^2_{\sigma_i}, \ell^3_{\sigma_i}$ and σ_i^1, σ_i^2 are induced by the boundary value map, assigning to ℓ_{σ_i} its origin σ_i. From Figure 6 we read

σ_1: $\ell^1_{\sigma_1} \mapsto \sigma_1^2$, $\ell^2_{\sigma_1} \mapsto \sigma_1^1$, $\ell^3_{\sigma_1} \mapsto \sigma_1^1$,
σ_2: $\ell^1_{\sigma_2} \mapsto \sigma_2^1$, $\ell^2_{\sigma_2} \mapsto \sigma_2^2$, $\ell^3_{\sigma_2} \mapsto \sigma_2^1$,
σ_3: $\ell^1_{\sigma_3} \mapsto \sigma_3^2$, $\ell^2_{\sigma_3} \mapsto \sigma_3^1$, $\ell^3_{\sigma_3} \mapsto \sigma_3^1$.

Therefore, the mappings b_{σ_i} are given by

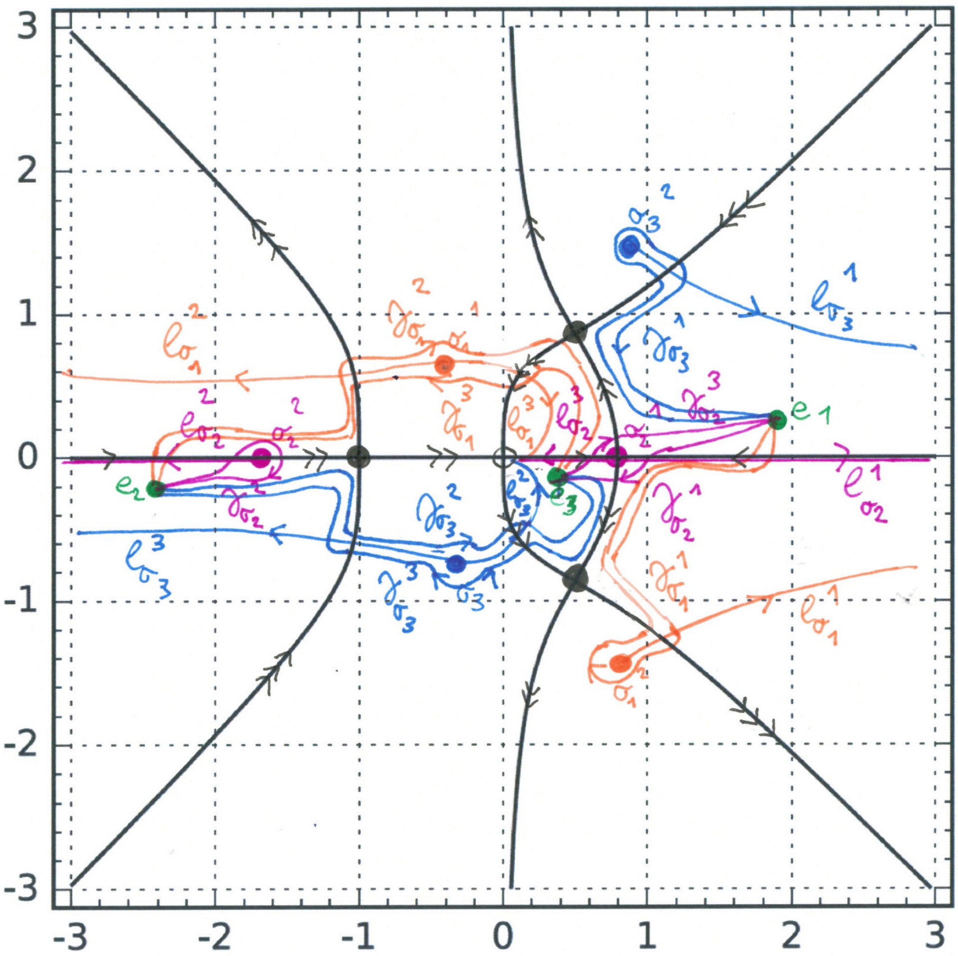
$$b_{\sigma_1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}, \; b_{\sigma_2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \; b_{\sigma_3} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

We identify the cokernel of b_{σ_1} with $\mathbb{C}$ via

$$\left[\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right] = \left[\begin{pmatrix} 0 \\ v_2 - v_3 \\ 0 \end{pmatrix}\right]$$

and therefore

$$u_{\sigma_1} = \begin{pmatrix} 0 & 1 & -1 \end{pmatrix}, \; v_{\sigma_1} = u_{\sigma_1}^{\mathrm{t}}.$$



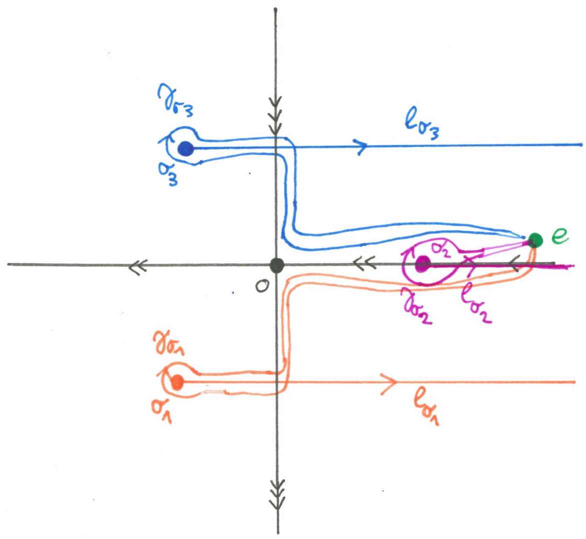


FIGURE 8. ℓ_{σ_i}, γ_{σ_i} and their preimages under f

We identify the cokernel of b_{σ_2} with $\mathbb{C}$ via

$$\left[\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right] = \left[\begin{pmatrix} v_1 - v_3 \\ 0 \\ 0 \end{pmatrix}\right]$$

and therefore

$$u_{\sigma_2} = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}, \ v_{\sigma_2} = u_{\sigma_2}^{\mathrm{t}}.$$

We identify the cokernel of b_{σ_3} with $\mathbb{C}$ via

$$\left[\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right] = \left[\begin{pmatrix} 0 \\ v_2 - v_3 \\ 0 \end{pmatrix}\right]$$

and therefore

$$u_{\sigma_3} = \begin{pmatrix} 0 & 1 & -1 \end{pmatrix}, \ v_{\sigma_3} = u_{\sigma_3}^{\mathrm{t}}.$$

In summary, we obtain the following

THEOREM V.9. The Stokes matrices in the chosen bases are given by

$$S_\beta = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad S_{-\beta} = \begin{pmatrix} -1 & 0 & 0 \\ -1 & -1 & 0 \\ -2 & -1 & -1 \end{pmatrix}.$$

$S_{\pm\beta}$ describes passing $\pm\beta$ from $H_\alpha = \left\{w \in \left(\mathbb{A}^1\right)^\vee \smallsetminus \{0\} \mid \arg(w) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right\}$ to $H_{-\alpha} = \left\{w \in \left(\mathbb{A}^1\right)^\vee \smallsetminus \{0\} \mid \arg(w) \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]\right\}$.

OBSERVATION V.10. S_β coincides with the Gram matrix (33) of the Euler–Poincaré pairing on $\mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(\mathbb{P}(1,2)))$ with respect to the full exceptional collection $\langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)\rangle$. Following Dubrovin's conjecture, the second Stokes matrix of the quantum connection is the transpose of the first one. From our topological computations for the Fourier–Laplace transform of the Gauß–Manin system of $x^2 + x^{-1}$, we obtain the two Stokes matrices S_β and $S_{-\beta} = -S_\beta^{\mathrm{t}}$. The solutions of the quantum connection differ from the Fourier–Laplace transform of the Gauß–Manin system by an additional root—the connection matrices differ by the matrix $\mathrm{diag}\left(\frac{1}{2}, \frac{1}{2}\right)$. This yields a minus sign both in the topological and formal monodromy. In our topological computations, the formal monodromy is encoded in the Stokes matrices. This explains the appearing minus sign.

REMARK V.11. The very same sign issue for the Stokes matrices appears in the following sections.

4. Mirror of $\mathbb{P}(1,2)$, variant

We consider the Laurent polynomial $f = x + x^{-2}$. For our topological computations, we choose $\alpha = 1$, $\beta = i$. This induces the following total order on the set of singular values of f:

$$\sigma_1 := \frac{3}{\sqrt[3]{4}}\zeta_3 <_\beta \sigma_2 := \frac{3}{\sqrt[3]{4}} <_\beta \sigma_3 := \frac{3}{\sqrt[3]{4}}\zeta_3^2$$

The closed sectors at ∞ are given by

$$H_\alpha = \left\{z \in \left(\mathbb{A}^1\right)^\vee \smallsetminus \{0\} \mid \Re(z) \geq 0\right\},\ H_{-\alpha} = \left\{z \in \left(\mathbb{A}^1\right)^\vee \smallsetminus \{0\} \mid \Re(z) \leq 0\right\}.$$

Figure 9 shows where f has real (resp. imaginary) part greater than or equal to 0. In Figure 10, the preimage of the imaginary (resp. real) axis is depicted in blue (resp. red) color. In Figure 11, the preimages of horizontal lines passing through the singular values σ_i are depicted in the following colors: σ_1: red, σ_2: green, σ_3: blue.

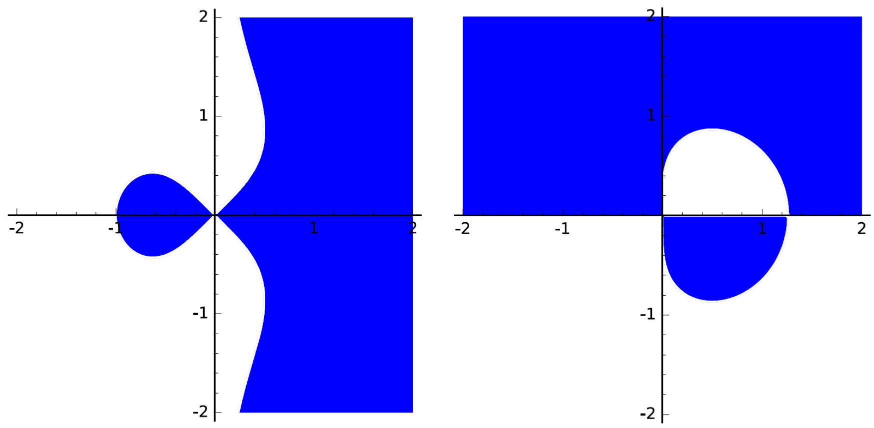


FIGURE 9. LHS: $\{x \mid \Re(f(x)) \geq 0\}$, RHS: $\{x \mid \Im(f(x)) \geq 0\}$

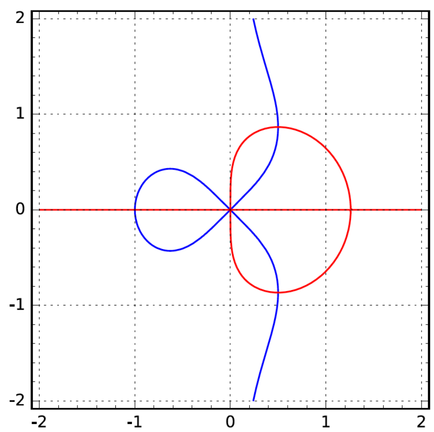


FIGURE 10. Preimage of the imaginary (resp. real) axis in blue (resp. red) color

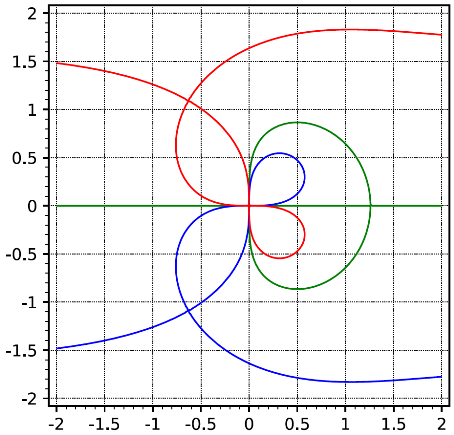


FIGURE 11. Preimages of horizontal lines passing through σ_i

We choose a base point $e \in \mathbb{A}^1$ as depicted in yellow color in Figure 12 and label its preimages under f by e_1, e_2, e_3, as depicted in Figure 12. We label the preimages of σ_i under f by σ_i^1, σ_i^2, as depicted in Figure 12. We choose paths γ_{σ_i} starting at e and running around the singular value σ_i in counterclockwise orientation (since $\Im(\langle \alpha, \beta \rangle) > 0$) as depicted in Figure 13. The preimage of γ_{σ_i} under f starting at e_j is denoted by $\gamma_{\sigma_i}^j$. From Figure 13 we read the monodromies

$$T_{\sigma_1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; T_{\sigma_2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \; T_{\sigma_3} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In Figure 14, the half-lines ℓ_{σ_i} and their preimages under f are depicted. We label by $\ell_{\sigma_i}^j$ the preimage of ℓ_{σ_i} which intersects $\gamma_{\sigma_i}^j$. From the figure we read

σ_1: $\ell_{\sigma_1}^1 \mapsto \sigma_1^1$, $\ell_{\sigma_1}^2 \mapsto \sigma_1^1$, $\ell_{\sigma_1}^3 \mapsto \sigma_1^2$, therefore $b_{\sigma_1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{\mathrm{t}}$,

σ_2: $\ell_{\sigma_2}^1 \mapsto \sigma_2^1$, $\ell_{\sigma_2}^2 \mapsto \sigma_2^2$, $\ell_{\sigma_2}^3 \mapsto \sigma_2^1$, therefore $b_{\sigma_2} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^{\mathrm{t}}$,

σ_3: $\ell_{\sigma_3}^1 \mapsto \sigma_3^1$, $\ell_{\sigma_3}^2 \mapsto \sigma_3^1$, $\ell_{\sigma_3}^3 \mapsto \sigma_3^2$, therefore $b_{\sigma_3} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{\mathrm{t}}$.

We identify the cokernel of b_{σ_1} with $\mathbb{C}$ via

$$\left[\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right] = \left[\begin{pmatrix} v_1 - v_2 \\ 0 \\ 0 \end{pmatrix}\right]$$

and therefore $u_{\sigma_1} = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}$, $v_{\sigma_1} = u_{\sigma_1}^{\mathrm{t}}$. We identify the cokernel of b_{σ_2} with $\mathbb{C}$ via

$$\left[\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right] = \left[\begin{pmatrix} v_1 - v_3 \\ 0 \\ 0 \end{pmatrix}\right]$$

and therefore $u_{\sigma_2} = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}$, $v_{\sigma_2} = u_{\sigma_2}^{\mathrm{t}}$.

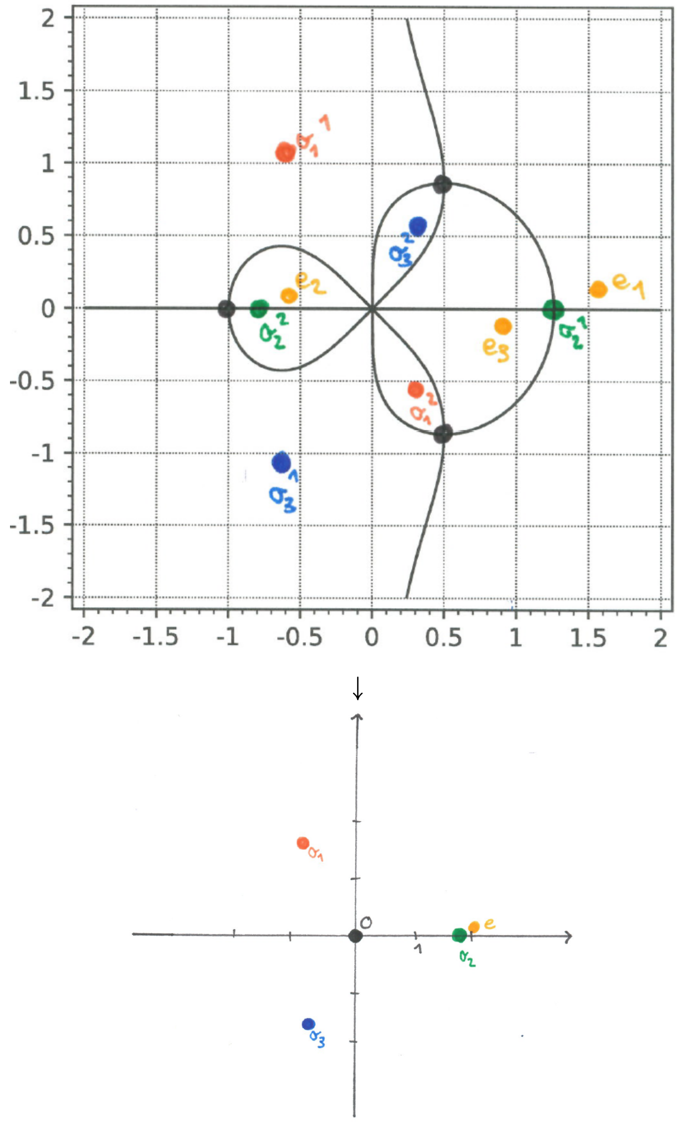


FIGURE 12. e, σ_i and their preimages under f

We identify the cokernel of b_{σ_3} with $\mathbb{C}$ via

$$\left[\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right] = \left[\begin{pmatrix} v_1 - v_2 \\ 0 \\ 0 \end{pmatrix}\right]$$

and therefore $u_{\sigma_3} = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}$, $v_{\sigma_3} = u_{\sigma_3}^{\mathrm{t}}$.

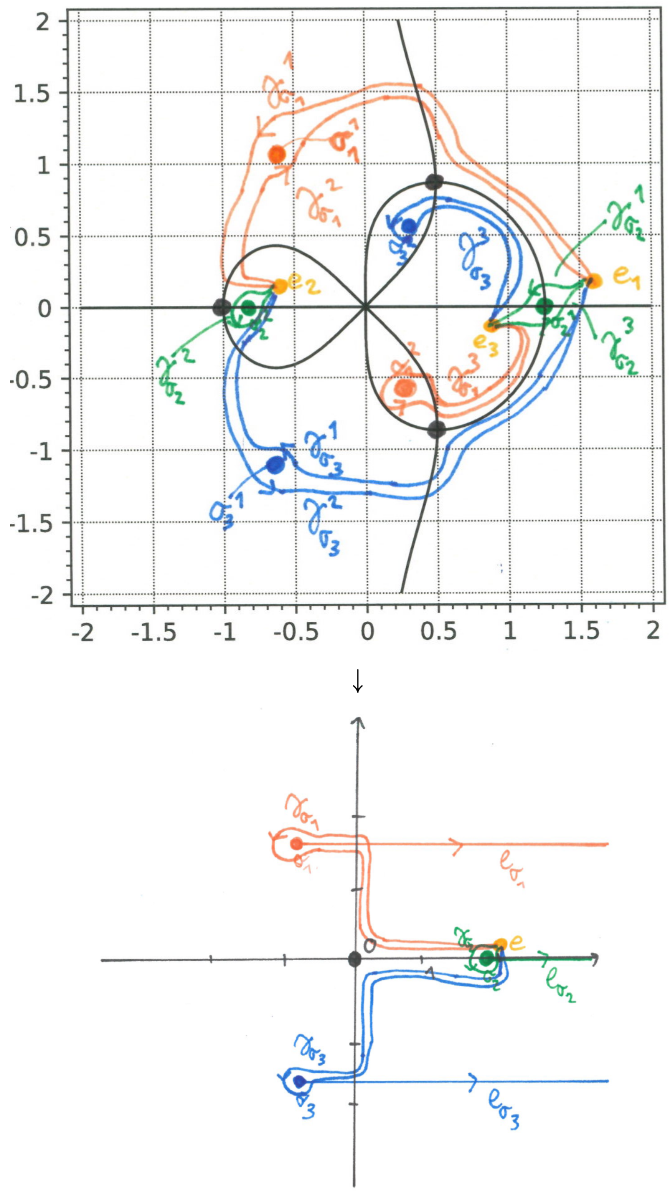


FIGURE 13. γ_{σ_i} and their preimages under f

Therefore, we obtain the following

THEOREM V.12. The Stokes multipliers in the chosen bases are given by

$$S_\beta = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad S_{-\beta} = \begin{pmatrix} -1 & 0 & 0 \\ -1 & -1 & 0 \\ -2 & -1 & -1 \end{pmatrix}.$$

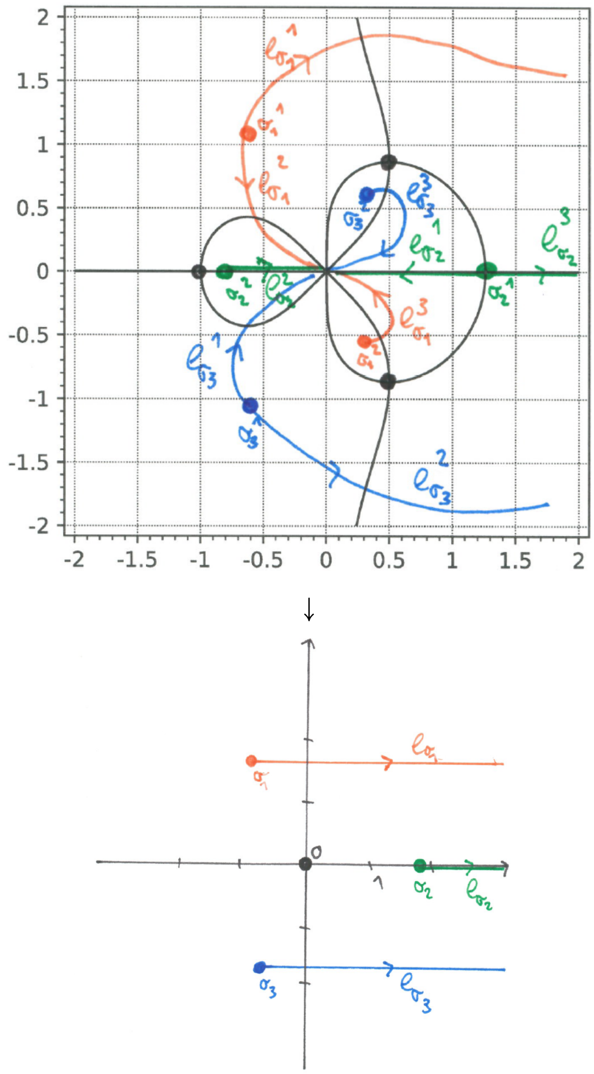


FIGURE 14. ℓ_{σ_i} and their preimages under f

OBSERVATION V.13. We observe that S_β coincides with the Gram matrix (33) of the Euler–Poincaré pairing on $\mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(\mathbb{P}(1,2)))$ with respect to the full exceptional collection $\langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2) \rangle$. Up to a sign (cf. Observation V.10), $S_{-\beta}$ is the transpose of the Gram matrix (33).

5. Mirror of $\mathbb{P}(1,3)$

We consider the Laurent polynomial $f = x + x^{-3}\colon \mathbb{G}_m \to \mathbb{A}^1$. Its critical points are given by $\left\{\pm\sqrt[4]{3}, \pm i\sqrt[4]{3}\right\}$, each being a double inverse image of the corresponding critical value. The critical values of f are

$$\Sigma = \left\{\pm\frac{4}{\sqrt[4]{27}}, \pm\frac{4i}{\sqrt[4]{27}}\right\}.$$

We fix $\alpha = e^{\frac{\pi i}{8}} \in \mathbb{A}^1$, $\beta = e^{\frac{3\pi i}{8}} \in \left(\mathbb{A}^1\right)^\vee$, such that $\Re(\langle\alpha,\beta\rangle) = 0$, $\Im(\langle\alpha,\beta\rangle) = 1$. This induces the following ordering on Σ:

$$\sigma_1 \coloneqq \frac{4i}{\sqrt[4]{27}} <_\beta \sigma_2 \coloneqq -\frac{4}{\sqrt[4]{27}} <_\beta \sigma_3 \coloneqq \frac{4}{\sqrt[4]{27}} <_\beta \sigma_4 \coloneqq -\frac{4i}{\sqrt[4]{27}}.$$

In Figure 16, the σ_i are depicted in the following colors:

- σ_1: green,
- σ_2: red,
- σ_3: purple,
- σ_4: orange.

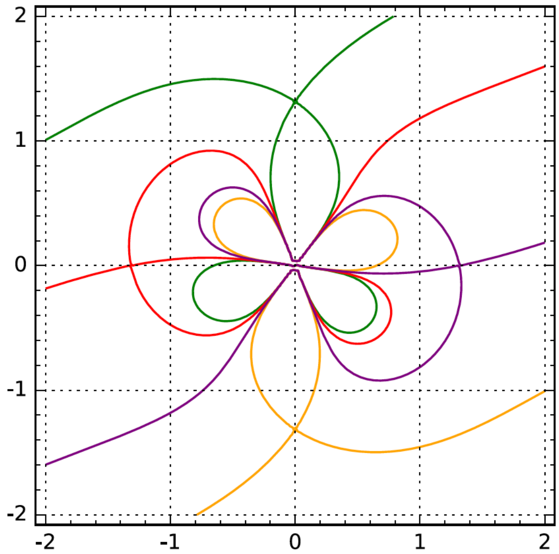


FIGURE 15. Preimages under $f = x + \frac{1}{x^3}$

We consider lines passing through the singular values with phase $\frac{\pi}{8}$, as depicted in Figure 16. The preimages of these lines under f are plotted in Figure 15. The blue area in Figure 17 shows where f has real (resp. imaginary) part greater than or equal to 0. In Figure 18, the preimage of the imaginary (resp. real) axis is plotted in blue (resp. red) color.

Outside of Σ, f is a covering of degree 4. In the following, we consider the perverse sheaf

$$F \coloneqq \mathrm{R}f_*\mathbb{C}[1] \in \mathrm{Perv}_\Sigma\left(\mathbb{C}_{\mathbb{A}^1}\right).$$

Its nearby and global nearby cycles are given by

$$\Psi_{\sigma_i}(F) \cong \bigoplus_{e_j \in f^{-1}(e)} \mathbb{C}_{e_j} \cong \mathbb{C}^4,$$

$$\Psi(F) \cong \Psi_{\sigma_i}(F) \cong \mathbb{C}^4.$$

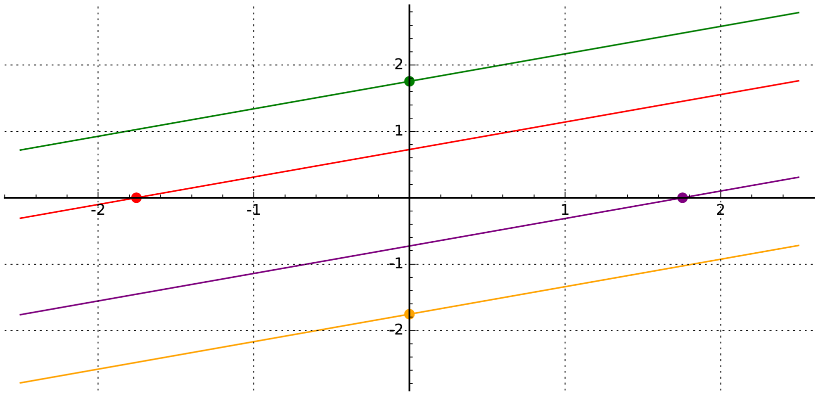


FIGURE 16. Lines passing through σ_i with phase $\frac{\pi}{8}$

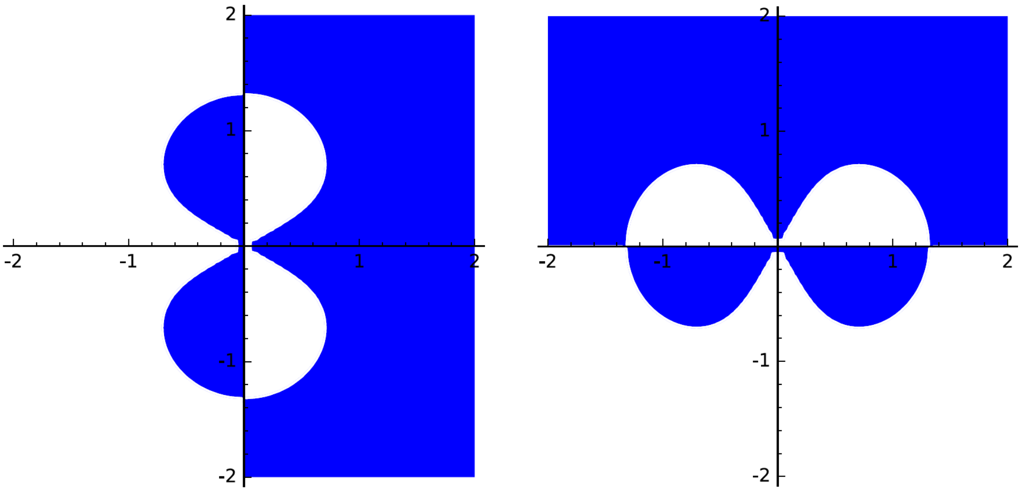


FIGURE 17. LHS: $\{x \mid \Re(f(x)) \geq 0\}$, RHS: $\{x \mid \Im(f(x)) \geq 0\}$

Furthermore, we fix isomorphisms

$$i_{\sigma_i}^{-1}(F)[-1] \cong \bigoplus_{\sigma_i^j \in f^{-1}(\sigma_i)} \mathbb{C}_{\sigma_i^j} \cong \mathbb{C}^3.$$

The exponential components at ∞ of the Fourier–Sato transform of F are of linear type, with coefficients given by the $\sigma_i \in \Sigma$. Its Stokes rays are therefore given by $\left\{0, \pm\frac{\pi}{4}, \pm\frac{\pi}{2}, \pm\frac{3\pi}{4}, \pi\right\}$. Consider the closed sectors centered at infinity

$$H_\alpha = \left\{w \in \left(\mathbb{A}^1\right)^\vee \smallsetminus \{0\} \mid \arg(w) \in \left[-\frac{5\pi}{8}, \frac{3\pi}{8}\right]\right\}, \; H_{-\alpha} = \left\{w \in \left(\mathbb{A}^1\right)^\vee \smallsetminus \{0\} \mid \arg(w) \in \left[\frac{3\pi}{8}, \frac{11\pi}{8}\right]\right\}.$$

Then

$$H_\alpha \cap H_{-\alpha} = h_\beta \cup h_{-\beta},$$

where $h_{\pm\beta} = \pm\mathbb{R}_{>0}\beta \subset \left(\mathbb{A}^1\right)^\vee$.

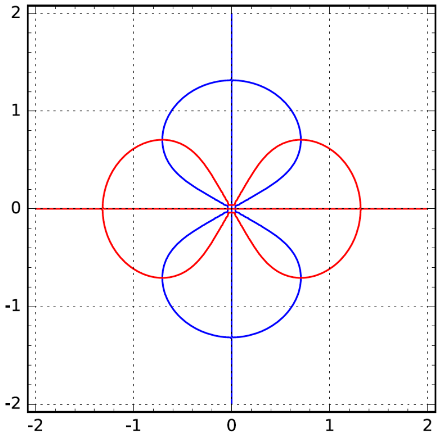


FIGURE 18. Preimage of the imaginary (resp. real) axis in blue (resp. red) color

The preimages of

- $\frac{4}{\sqrt[4]{27}}$ are $\sqrt[4]{3}$ (double), $\frac{-1-\sqrt{2}i}{\sqrt[4]{27}}$ and $\frac{-1+\sqrt{2}i}{\sqrt[4]{27}}$,
- $-\frac{4}{\sqrt[4]{27}}$ are $-\sqrt[4]{3}$ (double), $\frac{1-\sqrt{2}i}{\sqrt[4]{27}}$ and $\frac{1+\sqrt{2}i}{\sqrt[4]{27}}$,
- $i\frac{4}{\sqrt[4]{27}}$ are $i\sqrt[4]{3}$ (double), $\frac{-i-\sqrt{2}}{\sqrt[4]{27}}$ and $\frac{-i+\sqrt{2}}{\sqrt[4]{27}}$,
- $-i\frac{4}{\sqrt[4]{27}}$ are $-i\sqrt[4]{3}$ (double), $\frac{i+\sqrt{2}}{\sqrt[4]{27}}$ and $\frac{i-\sqrt{2}}{\sqrt[4]{27}}$.

We fix a base point e with $\Re(e) > \Re(\sigma_i)$ for all i and denote its preimages by e_1, e_2, e_3, e_4, as depicted in Figure 19. We consider loops γ_{σ_i}, starting at e and running around the singular value γ_i in counterclockwise orientation[4] as depicted in Figure 19. We denote by $\gamma_{\sigma_i}^j$ the preimage of γ_{σ_i} starting at e_j, $j = 1, 2, 3, 4$.
By considering Figure 19, we obtain, in the basis e_1, e_2, e_3, e_4, the monodromies

$$T_{\sigma_1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad T_{\sigma_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$T_{\sigma_3} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad T_{\sigma_4} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In order to obtain the maps b_{σ_i}, we consider the half-lines $\ell_{\sigma_i} := \sigma_i + \alpha\mathbb{R}_{\geq 0}$. We denote their preimages under f by $\{\ell_{\sigma_i}^j\}_{j=1,2,3,4}$, depending on which $\gamma_{\sigma_i}^j$ they intersect first. We label the preimages of σ_i, $i = 1, 2, 3, 4$, by $\sigma_i^1, \sigma_i^2, \sigma_i^3$ as in Figure 20. The maps b_{σ_i} encode which lift ℓ_{σ_i} starts at which preimage of σ_i, induced by the corresponding boundary map in homology. More explicitly, from Figure 20, we read the following:

[4] counterclockwise orientation since the imaginary part of $\langle \alpha, \beta \rangle$ is positive

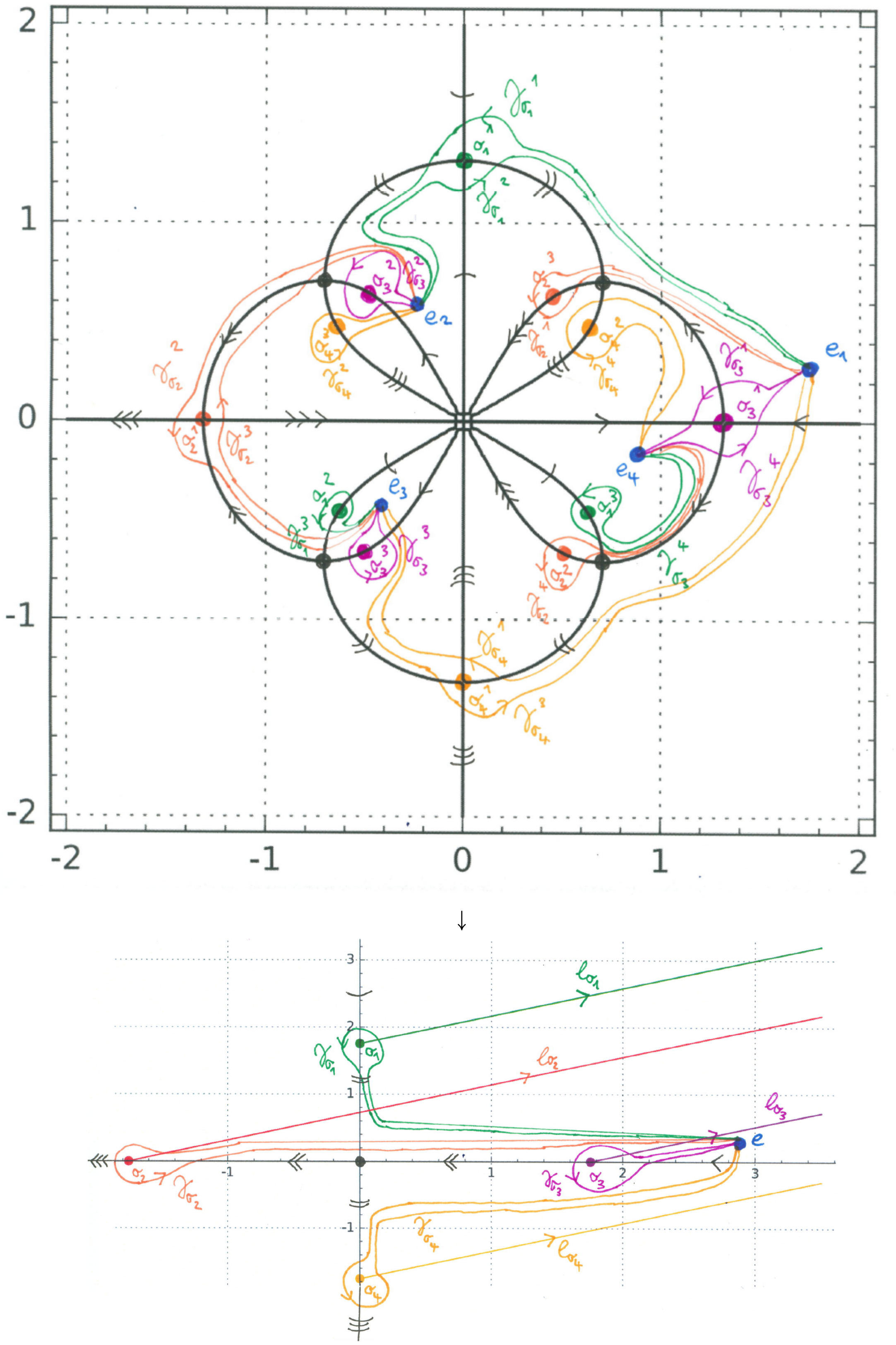


FIGURE 19. Monodromy of $f = x + \frac{1}{x^3}$

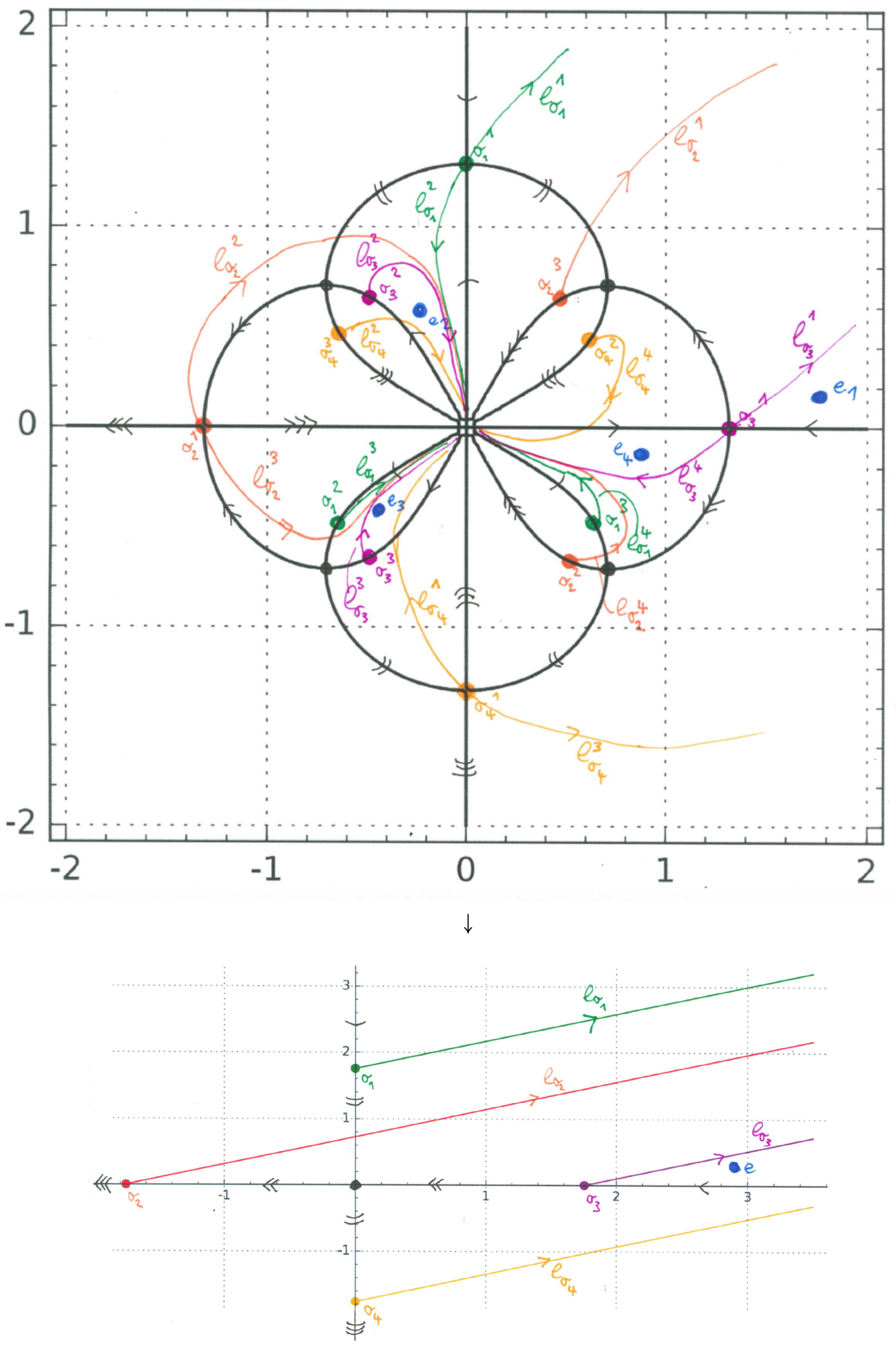


FIGURE 20. Preimages of ℓ_{σ_i} under f

σ_1: $\ell^1_{\sigma_1} \mapsto \sigma^1_1$, $\ell^2_{\sigma_1} \mapsto \sigma^1_1$, $\ell^3_{\sigma_1} \mapsto \sigma^2_1$, $\ell^4_{\sigma_1} \mapsto \sigma^3_1$. Therefore, b_{σ_1} is given by $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{t}$.

σ_2: $\ell^1_{\sigma_2} \mapsto \sigma^3_2$, $\ell^2_{\sigma_2} \mapsto \sigma^1_2$, $\ell^3_{\sigma_2} \mapsto \sigma^1_2$, $\ell^4_{\sigma_2} \mapsto \sigma^2_2$. Therefore, b_{σ_2} is given by $\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}^{t}$.

σ_3: $\ell^1_{\sigma_3} \mapsto \sigma^1_3$, $\ell^2_{\sigma_3} \mapsto \sigma^2_3$, $\ell^3_{\sigma_3} \mapsto \sigma^3_3$, $\ell^4_{\sigma_3} \mapsto \sigma^1_3$. Therefore, b_{σ_3} is given by $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{t}$.

σ_4: $\ell^1_{\sigma_4} \mapsto \sigma^1_4$, $\ell^2_{\sigma_4} \mapsto \sigma^3_4$, $\ell^3_{\sigma_4} \mapsto \sigma^1_4$, $\ell^4_{\sigma_4} \mapsto \sigma^2_4$. Therefore, b_{σ_4} is given by $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}^{t}$.

We obtain, in the ordered bases $\sigma^1_i, \sigma^2_i, \sigma^3_i$ and $\ell^1_{\sigma_i}, \ell^2_{\sigma_i}, \ell^3_{\sigma_i}, \ell^4_{\sigma_i}$ each:

$$b_{\sigma_1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; b_{\sigma_2} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$
$$b_{\sigma_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \; b_{\sigma_4} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Denote by $u_i \coloneqq u_{\sigma_i}$, $v_i \coloneqq v_{\sigma_i}$ and $\mathbb{T}_i \coloneqq \mathbb{T}_{\sigma_i}$. We obtain u_i, v_i, for $i = 1,2,3,4$, by computing the cokernels of the diagrams

$$\begin{tikzcd} i^{-1}_{\sigma_i} F[-1] \arrow[r, "b_{\sigma_i}"] \arrow[d, shift left] & \Psi(F) \arrow[d, shift left, "1-T_i"] \\ 0 \arrow[r] \arrow[u, shift left] & \Psi(F) \arrow[u, shift left, "1"] \end{tikzcd}$$

We identify the cokernels of b_{σ_i} in the following way:

- $\operatorname{coker} b_{\sigma_1} \simeq \mathbb{C}$ via $\left[\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}\right] = \left[\begin{pmatrix} v_1 - v_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right]$,
- $\operatorname{coker} b_{\sigma_2} \simeq \mathbb{C}$ via $\left[\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}\right] = \left[\begin{pmatrix} 0 \\ v_2 - v_3 \\ 0 \\ 0 \end{pmatrix}\right]$,
- $\operatorname{coker} b_{\sigma_3} \simeq \mathbb{C}$ via $\left[\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}\right] = \left[\begin{pmatrix} v_1 - v_4 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right]$,
- $\operatorname{coker} b_{\sigma_4} \simeq \mathbb{C}$ via $\left[\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}\right] = \left[\begin{pmatrix} v_1 - v_3 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right]$.

We obtain that $(\Phi_i(F) \underset{u_i}{\overset{v_i}{\rightleftarrows}} \Psi(F)) \simeq (\mathbb{C} \underset{u_i}{\overset{v_i}{\rightleftarrows}} \mathbb{C}^4)$, where

$$u_1 = \begin{pmatrix}1 & -1 & 0 & 0\end{pmatrix},\ u_2 = \begin{pmatrix}0 & 1 & -1 & 0\end{pmatrix},\ u_3 = \begin{pmatrix}1 & 0 & 0 & -1\end{pmatrix},\ u_4 = \begin{pmatrix}1 & 0 & -1 & 0\end{pmatrix}$$

and $v_i = u_i^{\mathrm{t}}$. Remembering carefully all the choices, we obtain the following

THEOREM V.14. The Stokes multipliers in the chosen bases are given by

$$S_\beta = \begin{pmatrix}1 & u_1v_2 & u_1v_3 & u_1v_4\\ 0 & 1 & u_2v_3 & u_2v_4\\ 0 & 0 & 1 & u_3v_4\\ 0 & 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & -1 & 1 & 1\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1\end{pmatrix},$$

$$S_{-\beta} = \begin{pmatrix}\mathbb{T}_1 & 0 & 0 & 0\\ -u_2v_1 & \mathbb{T}_2 & 0 & 0\\ -u_3v_1 & -u_3v_2 & \mathbb{T}_3 & 0\\ -u_4v_1 & -u_4v_2 & -u_4v_3 & -\mathbb{T}_4\end{pmatrix} = \begin{pmatrix}-1 & 0 & 0 & 0\\ 1 & -1 & 0 & 0\\ -1 & 0 & -1 & 0\\ -1 & -1 & -1 & -1\end{pmatrix} = -S_\beta^{\mathrm{t}}.$$

$S_{\pm\beta}$ describes crossing $h_{\pm\beta}$ from H_α to $H_{-\alpha}$.

OBSERVATION V.15. The pair $\left(\mathbb{G}_m, f = x + x^{-3}\right)$ is mirror to $\mathbb{P}(1,3)$. According to Dubrovin's conjecture, the Stokes matrix—under appropriate choices—is given by the Gram matrix of the Euler–Poincaré pairing χ on $\mathrm{D^b}(\mathrm{Coh}(\mathbb{P}(1,3)))$ with respect to some full exceptional collection. The Gram matrix with respect to the full exceptional collection $\mathcal{E} := \langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \mathcal{O}(3)\rangle$ is given by

$$S_{\mathbb{P}(1,3),\mathrm{Gram}} = \begin{pmatrix}1 & 1 & 1 & 2\\ 0 & 1 & 1 & 1\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

Via the action of the braid β_1 of the braid group B_4 on the Gram matrix, we find that it is equivalent to the Stokes matrix, that we computed topologically. The braid β_1 acts on the Gram matrix as follows (cf. [**17**]):

$$S_{\mathbb{P}(1,3),\mathrm{Gram}} \mapsto S^{\beta_1}_{\mathbb{P}(1,3),\mathrm{Gram}} := A^{\beta_1}\left(S_{\mathbb{P}(1,3),\mathrm{Gram}}\right) \cdot S_{\mathbb{P}(1,3),\mathrm{Gram}} \cdot \left(A^{\beta_1}\left(S_{\mathbb{P}(1,3),\mathrm{Gram}}\right)\right)^{\mathrm{t}},$$

where $A^{\beta_1}\left(S_{\mathbb{P}(1,3),\mathrm{Gram}}\right)$ is given by

$$A^{\beta_1}\left(S_{\mathbb{P}(1,3),\mathrm{Gram}}\right) = \begin{pmatrix}0 & 1 & 0 & 0\\ 1 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

We obtain that

$$A^{\beta_1}\left(S_{\mathbb{P}(1,3),\mathrm{Gram}}\right) S_{\mathbb{P}(1,3),\mathrm{Gram}} \left(A^{\beta_1}\left(S_{\mathbb{P}(1,3),\mathrm{Gram}}\right)\right)^{\mathrm{t}} = \begin{pmatrix}1 & -1 & 1 & 1\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1\end{pmatrix} = S_\beta.$$

REMARK V.16. $S^{\beta_1}_{\mathrm{Gram}} = S_\beta$ is the Gram matrix (34) of the Euler–Poincaré pairing with respect to the right mutation $\mathbb{R}_1\mathcal{E}$ of the full exceptional collection $\mathcal{E}$ (cf. [**7**, Prop. 13.1]). The action of the braid $\beta_1 \in B_4$ should correspond to a counterclockwise rotation of β. Therefore, we could expect to have the braid β_1 acting on our Stokes matrix.

6. Action of S_n, B_n and sign changes—an interpretation

There are natural actions of the symmetric group S_n, the braid group B_n, and sign changes on the Stokes data, reflecting variations in the choices involved to determine the Stokes matrices. In this section, we set into relation these actions—on the topologically computed Stoked data on the one side, on the Gram matrix of the Euler–Poincaré pairing on the other side. The mentioned actions on the Gram matrix are described in a very nice way in [**7**], for instance. We want to stress that the described correspondences of the actions reflect our intuition after having considered various examples, but are not proven in a rigorous way.

Permutations. In our topological computations, we freely choose a numbering $e_1, e_2, \ldots, e_n$ of the preimages of e under the map f. The action of a permutation $\sigma \in S_n$ should correspond to renumbering the preimages of e as $e_{\sigma(1)}, \ldots, e_{\sigma(n)}$.

Sign changes. In our topological computations, we compute the quiver of the perverse sheaf F as a cokernel. Choosing another isomorphism yields a minus sign in the corresponding entry.

Braids. We suspect the action of an elementary braid to correspond to a counterclockwise rotation of $\beta \in \left(\mathbb{A}^1\right)^\vee$, all the other choices being adopted coherently.

7. Outlook: Mirror of $\mathbb{P}(1,n)$

In this section, we give an outlook on the topological computation of the Stokes data of the quantum connection of $\mathbb{P}(1,n)$, $n \in \mathbb{N}_{>0}$. For this purpose, we consider $\mathbb{G}_m$ with the Laurent polynomial $f = x + x^{-n} \in \mathbb{C}\left[x, x^{-1}\right]$, being a Landau–Ginzburg model of $\mathbb{P}(1,n)$. The critical points of

$$f\colon \mathbb{C} \smallsetminus \{0\} \to \mathbb{C}, \quad x \mapsto x + x^{-n},$$

are given by $\left\{ \sqrt[n+1]{n}\zeta_{n+1}^k \right\}_{k=0,\dots,n}$, where ζ_{n+1} denotes the $(n+1)$st primitive root of unity $e^{\frac{2\pi i}{n+1}}$. We compute that

$$f\left(\sqrt[n+1]{n}\zeta_{n+1}^k \right) = \frac{n+1}{\sqrt[n+1]{n^n}}\zeta_{n+1}^k, \quad k = 0, 1, \dots, n,$$

$\sqrt[n+1]{n}\zeta_{n+1}^k$ being a double inverse image of $\frac{n+1}{\sqrt[n+1]{n^n}}\zeta_{n+1}^k$. Therefore, the critical values of f are given by

$$\Sigma = \left\{ \frac{n+1}{\sqrt[n+1]{n^n}}\zeta_{n+1}^k \right\}_{k=0,1,\dots,n}.$$

Outside of Σ, f is a covering of degree $n+1$. Furthermore, f is semismall and therefore $F \coloneqq \mathrm{R}f_*\mathbb{C}[1] \in \mathrm{Perv}_\Sigma(\mathbb{C}_{\mathbb{A}^1})$. F is the perverse sheaf associated to $H^0\left(\int_f \mathcal{O}\right)$ by the regular Riemann–Hilbert correspondence. We distinguish two cases.

- If n is even, -1 is not a $(n+1)$st root of unity and we can choose $\alpha = 1$, $\beta = i$, which induces a counterclockwise orientation.
- If n is odd, -1 is a $(n+1)$st root of unity. In this case we choose β with argument between the real axis and the first Stokes ray, i.e., $\beta = e^{i\varepsilon}$ and $\alpha = e^{i\left(\frac{\pi i}{2} - \varepsilon\right)}$ for suitable small $\varepsilon \in \mathbb{R}_{>0}$. This induces a counterclockwise orientation again.

As a base point for our monodromy computations we choose e with $\mathfrak{R}(e)$ slightly bigger than $\mathfrak{R}\left(\frac{n+1}{\sqrt[n+1]{n^n}}\right)$ and $\mathfrak{I}(e) > 0$ very small. We label the preimages of e by $e_1, e_2, \dots, e_{n+1}$, ascending with respect to their phase in counterclockwise orientation, where e_1 denotes the preimages with smallest phase ($\in [0, 2\pi)$) and by e_{n+1} the preimage with biggest phase. We label by σ_i^1 the double inverse image of $\sigma_i \in \Sigma$, the remaining preimages σ_i^j ascending in counterclockwise orientation.

The nearby and global nearby cycles of the perverse sheaf F are given by

$$\Psi_{\sigma_i}(F) \cong \bigoplus_{e_j \in f^{-1}(e)} \mathbb{C}_{e_j} \cong \mathbb{C}^{n+1},$$

$$\Psi(F) \cong \Psi_{\sigma_i}(F) \cong \mathbb{C}^{n+1}.$$

Furthermore, we fix isomorphisms

$$i_{\sigma_i}^{-1}F[-1] \cong \bigoplus_{\sigma_i^j \in f^{-1}(\sigma_i)} \mathbb{C}_{\sigma_i^j} \cong \mathbb{C}^n.$$

Remark V.17. We have not carried out the combinatorics yet, but in principle it should be possible to compute the Stokes multipliers of $\widehat{H^0\left(\int_f \mathcal{O}\right)}$ for each n individually by means of the procedure used in this chapter.

8. Mirror of $\mathbb{P}(2,2)$

A Landau–Ginzburg model of $\mathbb{P}(2,2)$ is given by the curve $\{x^2y^2 = 1\} \subset (\mathbb{C} \setminus \{0\})^2$ together with the potential $f = x + y$. This splits into two disjoint components $U_1 := \{xy + 1 = 0\}$ and $U_2 := \{xy - 1 = 0\}$. f restricts to $f_1 = x - x^{-1}$ on U_1 and to $f_2 = x + x^{-1}$ on U_2, where we identified $y = -x^{-1}$ and $y = x^{-1}$, respectively. The blue area in

- Figure 21 shows where f_1 has real (resp. imaginary) part greater than or equal to 0,
- Figure 22 shows where f_2 has real (resp. imaginary) part greater than or equal to 0.

In Figure 23, the preimages of the real (resp. imaginary) axis under f_1 and f_2 are plotted.
f has singular fibers at $\Sigma := \{\pm 2i, \pm 2\}$. For our topological computations, we consider the perverse sheaf $F = \mathrm{R}f_*\mathbb{C}[1] \in \mathrm{Perv}_\Sigma(\mathbb{C}_{\mathbb{A}^1})$. The exponential components at ∞ of the Fourier–Sato transform of F are of linear type, with coefficients given by the $\sigma_i \in \Sigma$. The Stokes rays are therefore given by $\{0, \pm\frac{\pi}{4}, \pm\frac{\pi}{2}, \pm\frac{3\pi}{4}, \pi\}$.

- $f^{-1}(2) = \{(1,1) \in U_2, (1-\sqrt{2}, 1+\sqrt{2}) \in U_1, (1+\sqrt{2}, 1-\sqrt{2}) \in U_1\}$, $(1,1)$ being the double inverse image,
- $f^{-1}(-2) = \{(-1,-1) \in U_2, (-1-\sqrt{2}, -1+\sqrt{2}) \in U_1, (-1+\sqrt{2}, -1-\sqrt{2}) \in U_1\}$, $(-1,-1)$ being the double inverse image,
- $f^{-1}(2i) = \{(i,i) \in U_1, (i+\sqrt{2}i, i-\sqrt{2}i) \in U_2, (i-\sqrt{2}i, i+\sqrt{2}i) \in U_2\}$, (i,i) being the double inverse image,
- $f^{-1}(-2) = \{(-i,-i) \in U_1, (-i+\sqrt{2}i, -i-\sqrt{2}i) \in U_2, (-i-\sqrt{2}i, -i+\sqrt{2}i) \in U_2\}$, $(-i,-i)$ being the double inverse image.

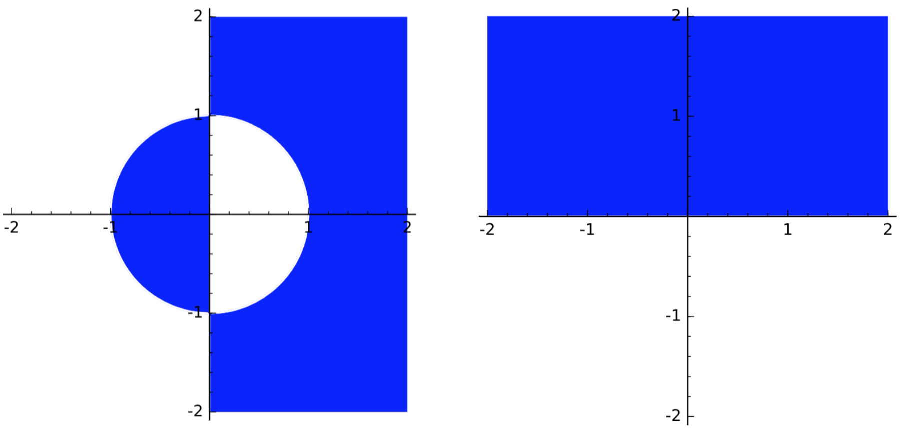


FIGURE 21. LHS: $\{x \mid \Re(f_1(x)) \geq 0\}$, RHS: $\{x \mid \Im(f_1(x)) \geq 0\}$

We choose $\alpha = e^{3\pi i/8}$, $\beta = e^{9\pi i/8}$. This induces the following order on Σ:

$$\sigma_1 := 2 <_\beta \sigma_2 := -2i <_\beta \sigma_3 := 2i <_\beta \sigma_4 := -2.$$

Denote by $\ell_{\sigma_i} = \sigma_i + \mathbb{R}_{\geq 0}\alpha$.
As in the previous examples, only the lifts of γ_{σ_i} and ℓ_{σ_i} around the double preimages of σ_i, which we denote by σ_i^1, contribute to the monodromy and the cokernel of b_{σ_i}. Therefore, in our figures, we restricted to this information.

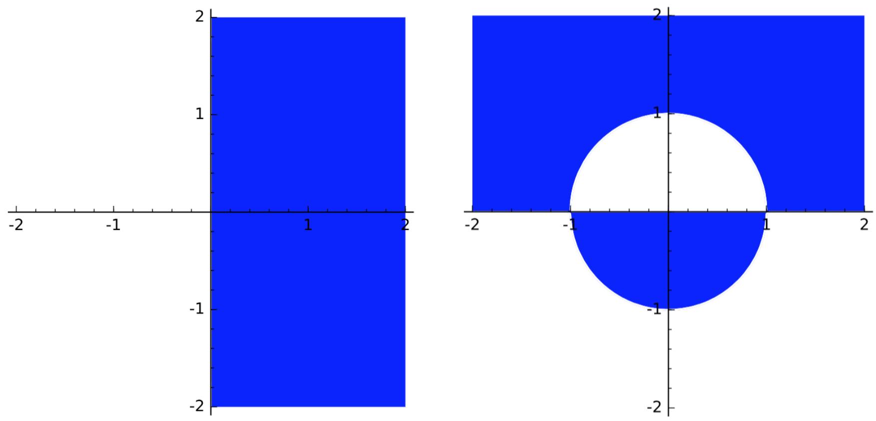


FIGURE 22. LHS: $\{x \mid \Re(f_2(x)) \geq 0\}$, RHS: $\{x \mid \Im(f_2(x)) \geq 0\}$

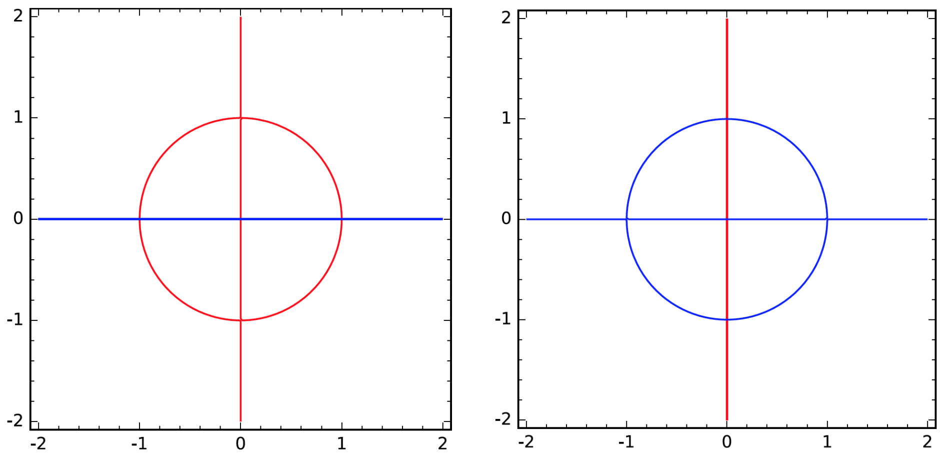


FIGURE 23. Preimage of the real (resp. imaginary) axis in blue (resp. red) color under f_1 (LHS) and f_2 (RHS)

From Figure 26 we read the monodromies in the basis e_1, e_2, e_3, e_4 to be

$$T_{\sigma_1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; T_{\sigma_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$T_{\sigma_3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \; T_{\sigma_4} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

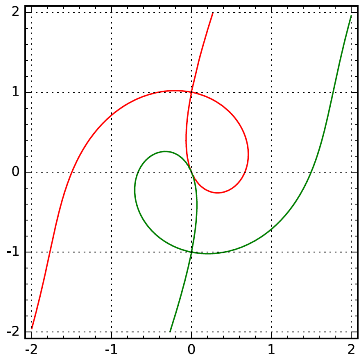


FIGURE 24. Preimages under f_1 of lines passing through σ_2 and σ_3 with phase $3\pi/8$

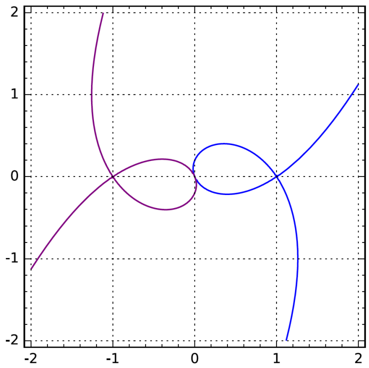


FIGURE 25. Preimages under f_2 of lines passing through σ_1 and σ_4 with phase $3\pi/8$

Taking into account Figure 27, we identify the cokernel of

- b_{σ_1} with $\mathbb{C}$ via $\left[\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}\right] = \left[\begin{pmatrix} v_1 - v_3 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right]$,
- b_{σ_2} with $\mathbb{C}$ via $\left[\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}\right] = \left[\begin{pmatrix} 0 \\ v_2 - v_4 \\ 0 \\ 0 \end{pmatrix}\right]$,
- b_{σ_3} with $\mathbb{C}$ via $\left[\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}\right] = \left[\begin{pmatrix} 0 \\ v_2 - v_4 \\ 0 \\ 0 \end{pmatrix}\right]$,
- b_{σ_4} with $\mathbb{C}$ via $\left[\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}\right] = \left[\begin{pmatrix} v_1 - v_3 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right]$.

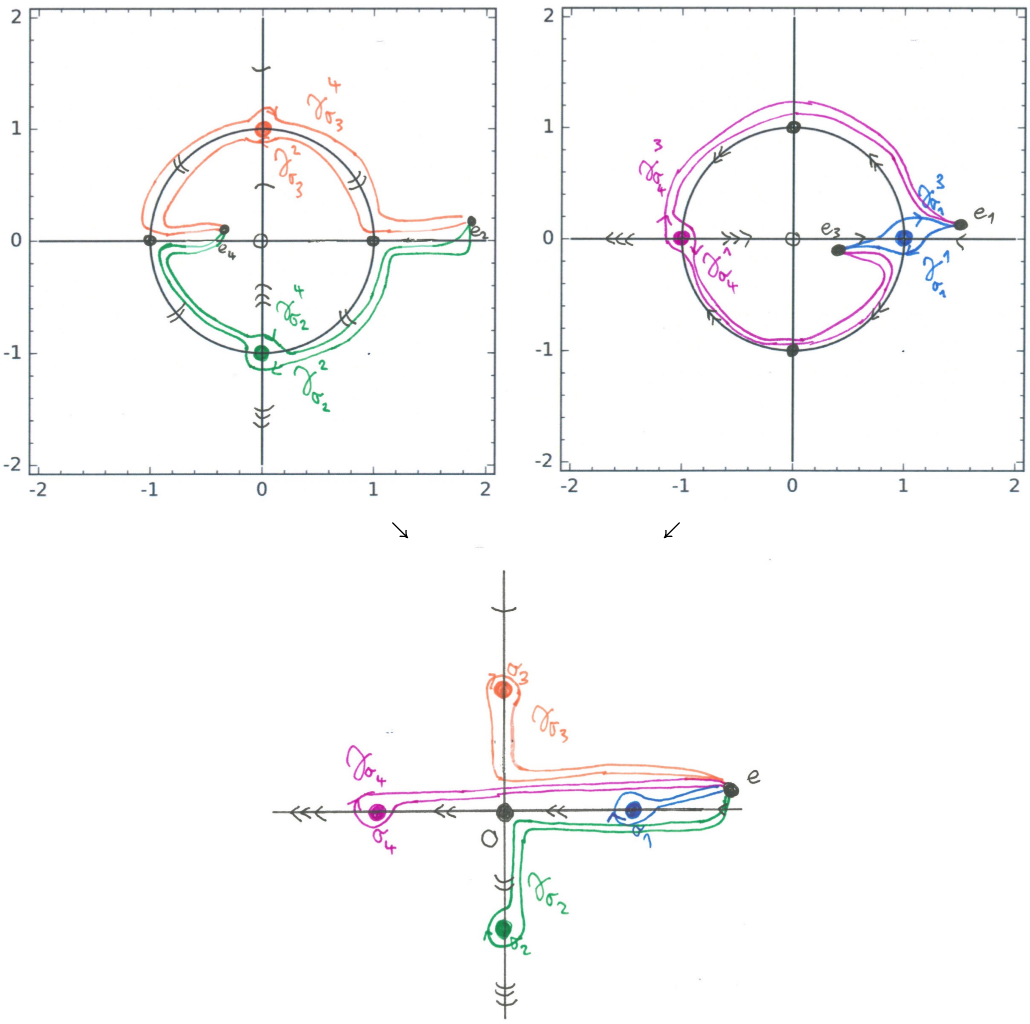


FIGURE 26. Preimages of γ_{σ_i} under f_1 (LHS) and f_2 (RHS)

We therefore obtain $u_{\sigma_1} = \begin{pmatrix}1 & 0 & -1 & 0\end{pmatrix} = u_{\sigma_4}$, $u_{\sigma_2} = \begin{pmatrix}0 & 1 & 0 & -1\end{pmatrix} = u_{\sigma_3}$ and $v_i = u_i^{\mathrm{t}}$. In summary, we obtain the following

THEOREM V.18. The Stokes multipliers in the chosen bases are given by

$$S_\beta = \begin{pmatrix}1 & 0 & 0 & 2\\ 0 & 1 & 2 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}, \quad S_{-\beta} = -S_\beta^{\mathrm{t}}.$$

$S_{\pm\beta}$ describes passing $\pm\beta\mathbb{R}_{>0} \subset \left(\mathbb{A}^1\right)^\vee \setminus \{0\}$ from $H_\alpha = \left\{w \in \left(\mathbb{A}^1\right)^\vee \setminus \{0\} \mid \arg(w) \in \left[-\frac{7\pi}{8}, \frac{\pi}{8}\right]\right\}$ to $H_{-\alpha} = \left\{w \in \left(\mathbb{A}^1\right)^\vee \setminus \{0\} \mid \arg(w) \in \left[\frac{\pi}{8}, \frac{9\pi}{8}\right]\right\}$.

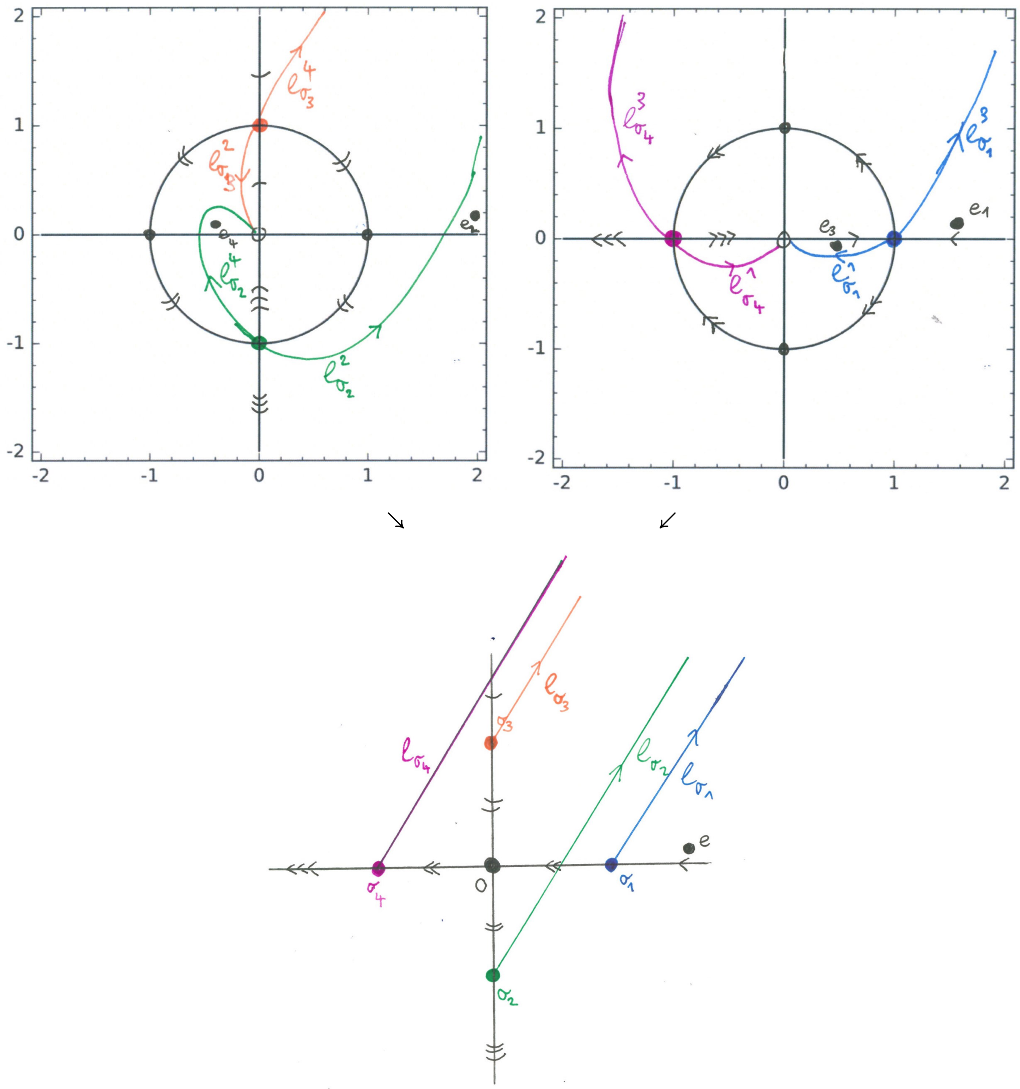


FIGURE 27. Preimages of ℓ_{σ_i} under f_1 (LHS) and f_2 (RHS)

OBSERVATION V.19. By the permutation

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in S_4,$$

acting on the Gram matrix $S_{\mathbb{P}(2,2),\text{Gram}}$ as $P \cdot S_{\mathbb{P}(2,2),\text{Gram}} \cdot P^{-1}$ (cf. [**17**, Section 6.c]), we find that the Gram matrix (36) is transformed to S_β.

Appendix: Implementations in SAGE

Some function plots, computations of cyclic vectors and computations in the Weyl algebra were implemented in the open source computer algebra system SAGE. At this point we give the codes.

1. Weyl algebra

For computations in the Weyl algebra, we used the following implementation, where x denotes the coordinate and d the derivative with respect to x:

```
A.<x,d> = FreeAlgebra (QQ, 2)
D = A.g_algebra({ d*x : x*d+1 })
(x,d) = D.gens()
```

2. Cyclic vectors

We give the code of the computation of the associated differential operator attached to the Gauß–Manin system (15) attached to the Laurent polynomial $x^3 + x^{-1}$ via the cyclic vector $m = (0,0,1,0)^{\mathrm{t}}$.

```
R = QQ['t']; (t,) = R._first_ngens(1)
FF = FractionField(R)
M = matrix(FF, [ [2,4/(3*t),0,0], [0,-1/3,4/(3*t),0], [0,0,0,4/(3*t)], [4/t,0,0,1/3] ])
m0 = vector(FF, [0,0,1,0])
m1 = t*diff(m0,t) + M*m0
m2 = t*diff(m1,t) + M*m1
m3 = t*diff(m2,t) + M*m2
m4 = t*diff(m3,t) + M*m3
W = []
W.append(m0)
W.append(m1)
W.append(m2)
W.append(m3)
W.append(m4)
N = matrix(FF,5,4) # k-th row is (td_t)^k*m
for i in range(5):
N[i] = W[i]
R = N.transpose()
ker = R.right_kernel() # k-th entry is coefficient of (td_t)^k (k=0,1,2,3,4)
ker
```

3. Function plots

3.1. $f = z^2 + z^{-2}$**.** The figures for $f = z^2 + z^{-2}$ in Section V.1.2, were plotted with Ti*k*Z. At this point, we give the SAGE code for reconstructing the function plot.

```
x,y = var('x,y')
# f(z) = z^2 + 1/z^2, z = x + i*y
f(x,y) = (x+i*y)^2 + 1/(x+i*y)^2
re(x,y) = f.real()
im(x,y) = f.imag()
alpha = pi/2 # alpha=i, beta=1
sigma1 = -2
sigma2 = 2
print 'region, where imaginary part of f >=0:'
region_plot(im>=0, (-2,2), (-2,2))
print 'region, where real part of f >=0:'
region_plot(re>=0, (-2,2), (-2,2))
print 'preimages of axes (imaginary blue, real red):'
a = implicit_plot(re==0, (-2,2), (-2,2), color='blue', gridlines=true)
b = implicit_plot(im==0, (-2,2), (-2,2), color='red', gridlines=true)
a+b
print 'lifts of l_sigma1: green'
c = implicit_plot(re==-2, (-2,2), (-2,2), color='green', gridlines=true)
c
print 'lifts of l_sigma2: orange'
d = implicit_plot(re==2, (-2,2), (-2,2), color='orange', gridlines=true)
d
```

3.2. Mirror of $\mathbb{P}(1,2)$**.** We consider the Laurent polynomial $f = z^2 + z^{-1}$ as a Landau–Ginzburg model of $\mathbb{P}(1,2)$, as used for the topological computations in Section V.3.

```
x,y = var('x,y')
# f(z) = z^2 + 1/z, z = x + i*y
f(x,y) = (x+i*y)^2 + 1/(x+i*y)
re(x,y) = f.real()
im(x,y) = f.imag()
alpha = 1
# beta = -i
sigma1 = 3/(4^(1/3))*e^(4*pi*i/3) # ≈ -0.94-1.64*i
sigma2 = 3/(4^(1/3))
sigma3 = 3/(4^(1/3))*e^(2*pi*i/3) # ≈ -0.94+1.64*i
print 'region, where imaginary part of f >=0:'
region_plot(im>=0, (-2,2), (-2,2))
print 'region, where real part of f >=0:'
region_plot(re>=0, (-2,2), (-2,2))
print 'preimages of axes (imaginary blue, real red):'
a = implicit_plot(re==0, (-2,2), (-2,2), color='blue', gridlines=true)
```

```
b = implicit_plot(im==0, (-2,2), (-2,2), color='red', gridlines=true)
a+b
print 'lifts of l_sigma2: purple'
c = implicit_plot(im==0, (-2,2), (-2,2), color='purple', gridlines=true)
print 'lifts of l_sigma3: blue'
d = implicit_plot(im==sigma3.imag(), (-2,2), (-2,2), color='blue', gridlines=true)
print 'lifts of l_sigma1: red:'
e = implicit_plot(im==sigma1.imag(), (-2,2), (-2,2), color='red', gridlines=true)
c+d+e
```

3.3. Mirror of $\mathbb{P}(1,2)$, variant. We consider the Laurent polynomial $f = z + z^{-2}$ as a Landau–Ginzburg model of $\mathbb{P}(1,2)$, as used for the topological computations in Section V.4.

```
x,y = var('x,y')
f(x,y) = (x+i*y) + 1/(x+i*y)^2
re(x,y) = f.real()
im(x,y) = f.imag()
alpha = 1
# beta = i
sigma1 = 3/(4^(1/3))*e^(2*pi*i/3)
sigma2 = 3/(4^(1/3))
sigma3 = 3/(4^(1/3))*e^(4*pi*i/3)
print 'region, where imaginary part of f >=0:'
region_plot(im>=0, (-2,2), (-2,2))
print 'region, where real part of f >=0:'
region_plot(re>=0, (-2,2), (-2,2))
print 'preimages of axes (imaginary blue, real red):'
a = implicit_plot(re==0, (-2,2), (-2,2), color='blue', gridlines=true)
b = implicit_plot(im==0, (-2,2), (-2,2), color='red', gridlines=true)
a+b
print 'lifts of l_sigma2: green'
c = implicit_plot(im==0, (-2,2), (-2,2) color='green', linewidth=1.0, gridlines=true)
print 'lifts of l_sigma3: blue'
d = implicit_plot(im==sigma3.imag(), (-2,2), (-2,2), color='blue', linewidth=1.0, grid-
lines=true)
print 'lifts of l_sigma1: red'
e = implicit_plot(im==sigma1.imag(), (-2,2), (-2,2), color='red', linewidth=1.0, grid-
lines=true)
c+d+e
```

3.4. Mirror of $\mathbb{P}(1,3)$. We consider the Laurent polynomial $f = z + z^{-3}$ as a Landau–Ginzburg model of $\mathbb{P}(1,3)$, as used for the topological computations in Section V.5.

```
x, y = var('x,y')
f(x,y) = (x+i*y) + (1/(x+i*y)^3)
# f(z) = z + 1/z^3, z=x+i*y
re(x,y) = f.real()
im(x,y) = f.imag()
alpha = pi/8
# beta = 3*pi/8
sigma3 = 4/(27^(1/4))
# sigma1 = sigma3*i, sigma2 = -sigma3, sigma3 = -sigma3*i
l1 = plot(tan(alpha)*x+sigma3, (x,0,3.5), gridlines=true, color='green')
l2 = plot(tan(alpha)*(x+sigma3), (x,-sigma3,3.5), gridlines=true, color='red')
l3 = plot(tan(alpha)*(x-sigma3), (x,sigma3,3.5), gridlines=true, color='purple')
l4 = plot(tan(alpha)*x-sigma3, (x,0,3.5), gridlines=true, color='orange')
pt1 = point((0,sigma3), color='green', size=50)
pt2 = point((-sigma3,0), color='red', size=50)
pt3 = point((sigma3,0), color='purple', size=50)
pt4 = point((0,-sigma3), color='orange', size=50)
l1+l2+l3+l4+pt1+pt2+pt3+pt4
print 'region, where imaginary part of f >= 0:'
k = region_plot(im>=0, (-2,2),(-2,2))
k
print 'region, where real part of f >= 0:'
l = region_plot(re>=0, (-2,2),(-2,2))
l
print 'preimages of axes (imaginary blue, real red):'
a = implicit_plot(re==0, (-2,2), -(2,2), color='blue', gridlines=true)
b = implicit_plot(im==0, (-2,2),(-2,2), color='red'), gridlines=true)
a+b
print 'lifts of l_sigma1: green'
c = implicit_plot(im-tan(alpha)*re-sigma3, (-2,2), (-2,2), color='green', gridlines=true)
print 'lifts of l_sigma4: orange'
d = implicit_plot(im-tan(alpha)*re+sigma3, (-2,2), (-2,2), color='orange' gridlines=true)
print 'lifts of l_sigma2: red'
e = implicit_plot(im-tan(alpha)*(re+sigma3), (-2,2), (-2,2), color='red', gridlines=true)
print 'lifts of l_sigma3: purple'
f = implicit_plot(im-tan(alpha)*(re-sigma3), (-2,2), (-2,2), color='purple', gridlines=true)
c+d+e+f
```

3.5. Mirror of $\mathbb{P}(2,2)$. We consider the Landau–Ginzburg model of $\mathbb{P}(2,2)$, as described in Section V.8.

```
x,y = var('x,y')
f(x,y) = (x+i*y) + 1/(x+i*y)
g(x,y) = (x)+i*y) - 1/(x+i*y)
print 'f=z+1/z, g=z-1/z'
alpha = 3*pi/8
# beta = 9*pi/8
sigma1 = 2
ref(x,y) = f.real()
imf(x,y) = f.imag()
reg(x,y) = g.real()
img(x,y) = g.imag()
print 'region, where imaginary part of f >=0:'
region_plot(imf>=0, (-2,2), (-2,2))
print 'region, where real part of f >=0:'
region_plot(ref>=0, (-2,2), (-2,2))
print 'preimages of real (blue) and imaginary (red) axis under f:'
a = implicit_plot(ref==0, (-2,2), (-2,2), color='red')
b = implicit_plot(imf==0, (-2,2), (-2,2), color='blue')
a+b
print 'region, where imaginary part of g >=0:'
region_plot(img>=0, (-2,2), (-2,2))
print 'region, where real part of g >=0:'
region_plot(reg>=0, (-2,2), (-2,2))
print 'preimages of real (blue) and imaginary (red) axis under g:'
c = implicit_plot(reg==0, (-2,2), (-2,2), color='red')
d = implicit_plot(img==0, (-2,2), (-2,2), color='blue')
c+d
print 'preimages of l_sigma1 (blue) and l_sigma4 (purple) under f=z+1/z:'
j = implicit_plot(imf-tan(alpha)*(ref-2)==0, (-2,2), (-2,2) color='blue', gridlines=true)
k = implicit_plot(imf-tan(alpha)*(ref+2)==0, (-2,2), (-2,2), color='purple', gridlines=true)
j+k
print 'preimages of l_sigma3 (red) and l_sigma2 (green) under g=z-1/z:'
l = implicit_plot(img-tan(alpha)*reg-2==0, (-2,2), (-2,2), color='red', gridlines=true)
m = implicit_plot(img-tan(alpha)*reg+2==0, (-2,2), (-2,2), color='green', gridlines=true)
l+m
```

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