Pluriharmonic maps into outer symmetric spaces and a subdivision of Weyl chambers

J.-H. Eschenburg, A.-L. Mare and P. Quast

1. *Introduction*

Harmonic maps of surfaces and pluriharmonic maps of Kähler manifolds with values in a symmetric space $S = \mathrm{G}/\mathrm{K}$ come in one-parameter families (*associated families*), parameterized by the circle $\mathbb{S}^1$. Therefore these objects can be considered as maps into loop spaces. There are several ways to do this; one is using the so-called *extended solutions*, which are certain maps from the domain M into the space $\Omega\mathrm{G}$ of (based) loops $\omega : \mathbb{S}^1 \to \mathrm{G}$ with $\omega(1) = e$; cf. [**9**, **13**] (see also [**4**] for the precise relation between associated families and extended solutions and [**3**] for an alternative approach). Of particular importance are extended solutions taking values in the subspace $\Omega_{\mathrm{alg}}(\mathrm{G})$ of *algebraic loops* with finite Fourier expansion; the corresponding harmonic maps are those of *finite uniton number*. Burstall and Guest [**1**] classified (pluri)harmonic maps of finite uniton number into the Lie group G using Morse theory of the energy function on $\Omega_{\mathrm{alg}}(\mathrm{G})$: under the gradient flow of the energy the extended solution is deformed to a particular one, taking values in a critical manifold which is a conjugacy class of a Lie group homomorphism $\gamma : \mathbb{S}^1 \to \mathrm{G}$. In fact, this homomorphism can be chosen as simple as possible (so-called *canonical*): its generator is a simple sum of fundamental dual roots. There are only 2^r such conjugacy classes where $r = \operatorname{rank}\mathrm{G}$. Thus Burstall and Guest obtained 2^r classes of (pluri)harmonic maps with finite uniton number, refining the obvious classification by the size of the uniton number. Replacing G by an *inner* symmetric space $S = \mathrm{G}/\mathrm{K}$, they got an even more restrictive result, but their methods cannot be extended to *outer* symmetric spaces (those where the geodesic symmetry does not belong to the transvection group). It is the purpose of our paper to fill this gap and specialize their result to outer symmetric spaces G/K which are embedded into G via the (pointed) Cartan embedding.

This requires an investigation of (outer) symmetric spaces which might be useful for its own sake. Usually, a principal tool to understand a symmetric space of compact type is the Satake diagram [**7**]. One starts with a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ where $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the eigenspace decomposition of the corresponding Cartan involution, and extends it to a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$. Then the Dynkin diagram of $\mathfrak{g}$ gets an additional structure (colors, arrows) according to the behavior of the Cartan involution σ. Our approach is opposite: we start with a maximal abelian subspace $\mathfrak{t}_{\mathfrak{k}}$ of $\mathfrak{k}$ and extend this to a maximal abelian subspace

The research was partially supported by the Deutsche Forschungsgemeinschaft, DFG-project ES 59/7-1.

$\mathfrak{t}$ of $\mathfrak{g}$. We find a σ-invariant fundamental root system of $\mathfrak{g}$ whose restriction to $\mathfrak{t}_\mathfrak{k}$ is a basis of $\mathfrak{t}_\mathfrak{k}^*$, and we have 2^s simple sums of the dual basis where $s = \operatorname{rank} \mathrm{K} = \dim \mathfrak{t}_\mathfrak{k}$; these classify pluriharmonic maps of finite uniton number into G/K (cf. Theorem 14). Geometrically, we have constructed a subdivision of the Weyl chambers of K by the projections of the Weyl chambers of G, so-called *compartments*; see Theorem 10.

2. *Pluriharmonic maps into compact Lie groups*

A *pluriharmonic map* f is a smooth map from a Kähler manifold M into a Riemannian manifold S whose restriction to every complex curve in M is harmonic, that is, critical for the variation of the energy (see [**5**] for a tensorial definition of pluriharmonicity). We will restrict our attention to target spaces which are totally geodesic submanifolds of a compact connected Lie group G with bi-invariant metric.

Since the work of Uhlenbeck [**13**], harmonic maps of Riemann surfaces (complex curves) into a compact Lie group have been described using extended solutions. Ohnita and Valli [**9**] have applied the same method to pluriharmonic maps $f : M \to \mathrm{G}$ where M is a Kähler manifold of any dimension. An *extended solution* is by definition a smooth map

$$\Phi : M \times \mathbb{S}^1 \longrightarrow \mathrm{G}, \quad (x, \lambda) \longmapsto \Phi(\lambda, x) = \Phi_\lambda(x)$$

with $\Phi_1 = e$ so that the pulled back Maurer–Cartan form satisfies

$$\Phi_\lambda^{-1} \mathrm{d}'\Phi_\lambda = \tfrac{1}{2}(1 - \lambda^{-1})A'; \quad \Phi_\lambda^{-1} \mathrm{d}''\Phi_\lambda = \tfrac{1}{2}(1 - \lambda)A'' \tag{1}$$

for some linear map $A : TM \to \mathfrak{g}$ into the Lie algebra of G, where α' and α'', respectively, denote the complex linear and complex antilinear parts of a one-form α. The map $\Phi_{-1} : M \to \mathrm{G}$ is then pluriharmonic. Conversely, if $f : M \to \mathrm{G}$ is pluriharmonic, then, provided that the domain M is simply connected, there exists an extended solution Φ with $\Phi_{-1} = f$, and this is unique up to left-multiplication with a smooth loop $\omega : \mathbb{S}^1 \to \mathrm{G}$ satisfying $\omega(1) = e$ and $\omega(-1) = e$ (see [**13**, **9**]). We may allow for $\omega(-1) = g \neq e$; then f has to be replaced with its translate gf. Clearly $A = f^{-1}\mathrm{d}f$. It is often convenient to consider Φ as a map into the *loop space*,

$$\Phi : M \longrightarrow \Omega\mathrm{G}, \quad x \longmapsto [\Phi(x) : \lambda \longmapsto \Phi_\lambda(x)]$$

where $\Omega\mathrm{G}$ is the set of all smooth maps $\omega : \mathbb{S}^1 \to \mathrm{G}$ with $\omega(1) = e$.

Now we will assume that G has trivial center. Then it is a matrix group; more precisely, the adjoint representation $\mathrm{Ad} : \mathrm{G} \hookrightarrow \mathrm{GL}(\mathfrak{g}) \subset \mathrm{End}(\mathfrak{g})$ is faithful. Each loop $\omega : \mathbb{S}^1 \to \mathrm{G}$ is represented by a Fourier series, $\omega(\lambda) = \sum_{j \in \mathbb{Z}} a_j \lambda^j$, with coefficients $a_j \in \mathrm{End}(\mathfrak{g})$. A loop ω is called *algebraic* if its Fourier series is finite. More precisely, let $\Omega_k(\mathrm{G})$ be the set of loops in G with Fourier series $\sum_{|j| \leqslant k} a_j \lambda^j$ and

$$\Omega_{\mathrm{alg}}(\mathrm{G}) = \bigcup_k \Omega_k(\mathrm{G})$$

be the set of all algebraic loops. An extended solution Φ is said to have *finite uniton number* if $\Phi(M) \subset \Omega_k(\mathrm{G})$ for some (minimal) k. Replacing Φ by $\omega\Phi$ may change k; the smallest such k is called the *uniton number* of the corresponding pluriharmonic map $f = \Phi_{-1}$. It has been shown by Uhlenbeck and Segal for Riemann surfaces and by Ohnita and Valli for Kähler manifolds that any pluriharmonic map on a compact simply connected Kähler manifold M has finite uniton number.

THEOREM 1 [**9**, **12**, **13**]. *Let M be a compact Kähler manifold and $\Phi : M \to \Omega\mathrm{G}$ be an extended solution. Then there exist some $\omega \in \Omega\mathrm{G}$ and some $k \geqslant 0$ such that $\omega\Phi(M) \subset \Omega_k(\mathrm{G}) \subset \Omega_{\mathrm{alg}}(\mathrm{G})$.*

A special case are those extended solutions Φ where all $\Phi(x)$ are *homomorphic loops*, that is, group homomorphisms from $\mathbb{S}^1$ to G. These are the fixed points of the $\mathbb{S}^1$-action on ΩG given by $(\mu, \omega) \mapsto \mu\omega$ with

$$(\mu\omega)(\lambda) := \omega(\mu\lambda)\omega(\mu)^{-1}.$$

The pluriharmonic maps arising from such Φ are called *isotropic*. They were investigated in [5] (see also [4]). Burstall and Guest [1] have shown that in a certain sense the general situation can be reduced to this particular case: every extended solution of finite uniton number can be deformed to an $\mathbb{S}^1$-invariant one. This is done by applying the gradient flow of the energy functional

$$E : \Omega\mathrm{G} \longrightarrow \mathbb{R}, \quad E(\omega) = \int_0^{2\pi} \left| \frac{d}{dt}\omega(e^{it}) \right|^2 dt, \tag{2}$$

where $|\ |$ is the norm of the canonical inner product on $\mathrm{End}(\mathfrak{g})$,

$$\langle A, B \rangle = \mathrm{trace}\ A^{\mathrm{T}} B. \tag{3}$$

The critical points of E are the closed geodesics γ in G passing through e; these are the homomorphic loops. They come in conjugacy classes $\Omega_\gamma = \{g\gamma g^{-1};\ g \in \mathrm{G}\}$ which form the critical manifolds of the Morse–Bott function E. Clearly, under the flow of $-\nabla E$, each $\omega \in \Omega\mathrm{G}$ is moved eventually into some critical manifold. But if we restrict our attention to $\Omega_{\mathrm{alg}}(\mathrm{G})$, then, also the inverse flow, the flow of $+\nabla E$ has this property. In other words, $\Omega_{\mathrm{alg}}(\mathrm{G})$ is the disjoint union of the so-called *unstable manifolds* U_γ where U_γ is the (finite-dimensional) domain of attraction of the critical manifolds Ω_γ under the flow of ∇E.

Note that the unstable manifolds $U_\gamma \subset \Omega_{\mathrm{alg}}(\mathrm{G})$ have also a group-theoretical description (cf. [10]). Let G^c be the complexification of G, that is, the connected subgroup of $\mathrm{GL}(\mathfrak{g} \otimes \mathbb{C})$ with Lie algebra $\mathfrak{g}^c = \mathfrak{g} \otimes \mathbb{C}$. Let $\Lambda^c := \Lambda_{\mathrm{alg}}(\mathrm{G}^c)$ be the group of all G^c-valued algebraic loops and $\Lambda^+ \subset \Lambda^c$ be the subgroup of all loops whose Fourier series are just polynomials in λ. Then we have a unique decomposition $\Lambda^c = \Omega_{\mathrm{alg}}(\mathrm{G})\,\Lambda^+$, the so-called Iwasawa decomposition, which yields an action of Λ^c on $\Omega_{\mathrm{alg}}(\mathrm{G})$ with stabilizer Λ^+. Since Λ^c, Λ^+ are complex groups, we thus obtain a homogeneous complex structure on $\Omega_{\mathrm{alg}}(\mathrm{G}) = \Lambda^c/\Lambda^+$. Moreover we represent $U_\gamma \subset \Omega_{\mathrm{alg}}(\mathrm{G})$ as the orbit of γ under the subgroup $\Lambda^+ \subset \Lambda^c$. Thus the Morse theoretic decomposition of $\Omega_{\mathrm{alg}}(\mathrm{G})$ into unstable manifolds is nothing else than the orbit decomposition of Λ^+, the *Bruhat decomposition* of $\Omega_{\mathrm{alg}}(\mathrm{G})$.

It turns out that any extended solution $\Phi : M \to \Omega_{\mathrm{alg}}(\mathrm{G})$ takes values *essentially* (that is, up to a subset $D \subset M$ which is the common zero set of finitely many holomorphic functions on M) in a single unstable manifold (Bruhat cell) U_γ. Indeed, $\Omega_k(\mathrm{G})$ is a complex projective variety and the closures of the Bruhat cells are algebraic subvarieties (cf. [11]); moreover, Φ is a holomorphic map and therefore the preimages of the closed Bruhat cells are common zero sets of holomorphic functions (see also [1, Proposition 4.1]). Hence Φ is flowed to a map $\Phi_\infty : M \to \Omega_\gamma$ which happens to be still an extended solution with the same uniton number. More precisely, let $u_\gamma : U_\gamma \to \Omega_\gamma \subset \Omega_{\mathrm{alg}}(\mathrm{G})$ be the map assigning to each $\omega \in \Omega_{\mathrm{alg}}(\mathrm{G})$ the endpoint of the flow line of ∇E. Then we have the following.

THEOREM 2 [1, Propositions 4.1 and 4.2]. *Let M be a Kähler manifold. If $\Phi : M \to \Omega_{\mathrm{alg}}(\mathrm{G})$ is an extended solution such that $\Phi(M) \subset \Omega_k(\mathrm{G})$ for some k, then Φ takes values essentially in an unstable manifold U_γ, and $u_\gamma \circ \Phi : M \setminus D \to \Omega_\gamma$ is another extended solution with the same uniton number.*

Thus the possible uniton numbers can be read off from the $\mathbb{S}^1$-invariant extended solutions taking values in some Ω_γ.

3. *Canonical elements*

Before applying the flow, we may use our freedom of choice for Φ and look for a suitable $\omega \in \Omega_{\mathrm{alg}}(\mathrm{G})$ such that $\tilde{\Phi} = \omega\Phi$ has the least possible uniton number. This will reduce the possible homomorphic loops γ to a finite set, which consists of the so-called canonical loops (see [**1**]).

Let $\gamma : \mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z}) \to \mathrm{G}$ be a homomorphic loop. We fix a maximal torus T of G containing γ. Then the initial vector $\xi = \gamma'(0) \in \mathfrak{t}$ belongs to the integer lattice I of $\mathfrak{t}$, that is, $\exp 2\pi\xi = e$. We may fix a fundamental root system $\alpha_1, \ldots, \alpha_r$ of G such that ξ lies in the corresponding closed Weyl chamber, that is, $\alpha_j(\xi) \geqslant 0$ for $j = 1, \ldots, r$. Let $\xi_1, \ldots, \xi_r$ be the dual basis, that is, $\alpha_j(\xi_k) = \delta_{jk}$. Then we have

$$\xi = \sum_j n_j \xi_j \tag{4}$$

with $n_j \in \mathbb{N}_o = \mathbb{N} \cup \{0\}$ for $j = 1, \ldots, r$. Suppose that $n_j \neq 0$ precisely for the indices $j = j_1, \ldots, j_p$ and put

$$\xi_o := \xi_{j_1} + \ldots + \xi_{j_p} \in I. \tag{5}$$

This is a *canonical element*, that is, a sum of dual fundamental roots that are distinct. It is obtained from ξ by changing all nonzero coefficients n_{j_i} to 1. Let γ_o be the homomorphic loop with $\gamma_o'(0) = \xi_o$. Now Burstall and Guest have shown the following.

Theorem 3 [**1**, Theorem 4.5]. *If an extended solution $\Phi : M \to \Omega_{\mathrm{alg}}(\mathrm{G})$ takes values essentially in an unstable manifold U_γ, then there exists $\omega \in \Omega_{\mathrm{alg}}(\mathrm{G})$ such that $\omega\Phi$ takes values essentially in U_{γ_o} where $\gamma_o'(0) = \xi_o$ is given by* (5).

Consequently, there is an estimate of all possible uniton numbers by those of the 2^r canonical homomorphic loops corresponding to the canonical elements. Moreover, one obtains a classification of pluriharmonic maps (up to congruence) which is finer than the one by uniton number: the (suitably chosen) extended solution of such a pluriharmonic map belongs to some U_{γ_o} for a canonical γ_o, and it can be deformed into an $\mathbb{S}^1$-invariant pluriharmonic map whose extended solution takes values in Ω_{γ_o}.

4. *Pluriharmonic maps into compact symmetric spaces*

Let M be a Kähler manifold which is compact and simply connected. We also consider a symmetric space S of compact type which is a *bottom space*, that is, any symmetric space covered by S is isomorphic to S. Let G be the identity component of the isometry group of S. It has trivial center and can be realized as a matrix group via its (faithful) adjoint representation. For any $p \in S$ we denote by s_p the geodesic symmetry at p. The choice of a basepoint $o \in S$ provides an involution σ^o on G, namely the conjugation with s_o:

$$\sigma^o : \mathrm{G} \longrightarrow \mathrm{G}; \quad g \longmapsto s_o g s_o. \tag{6}$$

Then we have $S = \mathrm{G}/\mathrm{K}$ where K $=\mathrm{Fix}(\sigma^o)$, and we obtain an embedding (*Cartan embedding*)

$$\iota^o : S \longrightarrow \mathrm{G}; \quad p = go \ \longmapsto \ s_p s_o = g s_o g^{-1} s_o = g\sigma^o(g^{-1}). \tag{7}$$

The image $\iota^o(S) = P_e$ is the connected component (passing through e) of the fixed-point set of the isometric involution τ^o obtained by composing σ^o with the inversion, that is,

$$\tau^o : \mathrm{G} \longrightarrow \mathrm{G}; \quad g \longmapsto \sigma^o(g^{-1}) = \sigma^o(g)^{-1}. \tag{8}$$

Hence $\iota^o(S) \subset \mathrm{G}$ is totally geodesic.

Consequently, a smooth map $f : M \to S$ is pluriharmonic if and only if so is the composition $\tilde{f} = \iota^o \circ f : M \to \mathrm{G}$. Hence Theorem 3 gives a classification also for the pluriharmonic maps of compact simply connected Kähler manifolds into S, but as we will see, the number of canonical homomorphic loops can now be lessened from 2^r to 2^s where $s = \mathrm{rank}(\mathrm{K})$. Lemma 5 below is the first step to this goal.

The involution τ^o given in equation (8) extends to an involution T^o of $\Omega_{\mathrm{alg}}(\mathrm{G})$,

$$T^o : \Omega_{\mathrm{alg}}(\mathrm{G}) \longrightarrow \Omega_{\mathrm{alg}}(\mathrm{G}); \quad (T^o\omega)(\lambda) = s_o\omega(-\lambda)\omega(-1)^{-1}s_o \tag{9}$$

in the sense that $(T^o\omega)(-1) = \tau^o(\omega(-1))$. Let $\Omega_{\mathrm{alg}}(\mathrm{G})^{T^o} \subset \Omega_{\mathrm{alg}}(\mathrm{G})$ be the fixed-point set of T^o.

Lemma 4 [**6**, p. 285]. *Let $f : M \to S$ be a pluriharmonic map and let x_o be a basepoint in M. We set $o = f(x_o)$. Then there exists a T^o-invariant extended solution $\tilde{\Phi} : M \to \Omega_{\mathrm{alg}}(\mathrm{G})^{T^o}$ with $\tilde{\Phi}_{-1} = \tilde{f} = \iota^o \circ f$.*

Proof. Let $\Phi : M \to \Omega\mathrm{G}$ be an extended solution with the property that $\Phi_{-1} = \tilde{f}$. left-multiplication of Φ with the smooth loop $\omega(\lambda) = \Phi_\lambda(x_o)^{-1}$ satisfying $\omega(1) = e$ and $\omega(-1) = \iota^o(o) = e$ we get the unique extended solution $\tilde{\Phi}$ satisfying $\tilde{\Phi}_{-1} = \tilde{f}$ and $\tilde{\Phi}_\lambda(x_o) = e$ for all $\lambda \in \mathbb{S}^1$. By Theorem 1, $\tilde{\Phi}$ takes values in $\Omega_{\mathrm{alg}}(\mathrm{G})$ after left multiplication by a suitable loop. We note that this loop is itself algebraic, thus $\tilde{\Phi}$ takes values in $\Omega_{\mathrm{alg}}(\mathrm{G})$. Since $\tilde{f}$ is τ^o-invariant, it is easy to check that $T^o\tilde{\Phi}$ satisfies the extended solution equation (1) with $(T^o\tilde{\Phi})_{-1} = \tilde{f}$ and $(T^o\tilde{\Phi})_\lambda(x_o) = e$. Thus $T^o\tilde{\Phi} = \tilde{\Phi}$ by uniqueness. $\square$

Lemma 5. *If $\Phi : M \to \Omega_{\mathrm{alg}}(\mathrm{G})$ is a T^o-invariant extended solution, then Φ takes values essentially in an unstable manifold U_γ, where γ is a homomorphic loop in the subgroup $\mathrm{K} \subset \mathrm{G}$.*

Proof. Note that the energy is a T^o-invariant function on $\Omega_{\mathrm{alg}}(\mathrm{G})$. Thus the flow of ∇E preserves $\Omega_{\mathrm{alg}}(\mathrm{G})^{T^o}$. By Theorem 2, Φ takes values essentially in U_γ for some homomorphic loop $\gamma \in \Omega_{\mathrm{alg}}(\mathrm{G})$, and Φ is flowed onto $u_\gamma \circ \Phi$ with values in Ω_γ. Since Φ and $u_\gamma \circ \Phi$ are T^o-invariant, we may assume $\gamma \in \Omega_{\mathrm{alg}}(\mathrm{G})^{T^o}$. But $T^o\gamma = \sigma^o\gamma$:

$$\begin{aligned}(T^o\gamma)(\lambda) &= s_o\gamma(-\lambda)\gamma(-1)^{-1}s_o \\ &= s_o\gamma(\lambda)\gamma(-1)\gamma(-1)^{-1}s_o \\ &= s_o\gamma(\lambda)s_o \\ &= \sigma^o\gamma(\lambda).\end{aligned}$$

Thus γ is a homomorphic loop in the subgroup $\mathrm{K} \subset \mathrm{G}$. $\square$

5. *Compartments*

As before, let $S = \mathrm{G}/\mathrm{K}$ denote a symmetric space of compact type. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the (± 1)-eigenspace decomposition for the involution $\sigma = \mathrm{Ad}(s_o)$, where s_o is the geodesic reflection at the basepoint $o = e\mathrm{K}$. Fix a maximal abelian subspace $\mathfrak{t}_\mathfrak{k}$ in $\mathfrak{k}$ and enlarge it to a maximal abelian subspace $\mathfrak{t}$ of $\mathfrak{g}$, that is, $\mathfrak{t}_\mathfrak{k} \subset \mathfrak{t}$ (in fact, such an enlargement is unique, cf. Lemma 7 below). The next two lemmas are due to Loos [**8**, p. 125]; for the convenience of the reader we add the (short) proofs.

LEMMA 6. *The space* $\mathfrak{t}$ *is invariant under* σ. *Therefore* $\mathfrak{t} = \mathfrak{t}_{\mathfrak{k}} \oplus \mathfrak{t}_{\mathfrak{p}}$, *where* $\mathfrak{t}_{\mathfrak{p}}$ *is an abelian subalgebra of* $\mathfrak{p}$ *which need not to be maximal.*

Proof. Let $X \in \mathfrak{t}$; then $X + \sigma(X) \in \mathfrak{k}$ and for all $A \in \mathfrak{t}_{\mathfrak{k}}$ we have $[A, X + \sigma(X)] = [A, X] + [\sigma(A), \sigma(X)] = [A, X] + \sigma([A, X]) = 0$. Since $\mathfrak{t}_{\mathfrak{k}}$ is maximal abelian in $\mathfrak{k}$, we get $X + \sigma(X) \in \mathfrak{t}_{\mathfrak{k}}$. Thus $\sigma(X) \in \mathfrak{t}$. □

Now we consider the root system Δ of $\mathfrak{g}$ corresponding to the maximal abelian subalgebra $\mathfrak{t}$. Using an invariant inner product, we consider $\Delta \subset \mathfrak{t}^* = \mathfrak{t}$. For any $\alpha \in \Delta$, the corresponding root space in $\mathfrak{g}^c = \mathfrak{g} \otimes \mathbb{C}$ is denoted by $\mathfrak{g}_\alpha$. Recall that an element of $\mathfrak{t}$ is $\mathfrak{g}$-*regular* if it does not lie in the kernel of any root.

LEMMA 7. *No* $\mathfrak{g}$-*root vanishes on* $\mathfrak{t}_{\mathfrak{k}}$. *Hence* $\mathfrak{t}_{\mathfrak{k}}$ *contains* $\mathfrak{g}$-*regular elements. In particular* $\mathfrak{t}$ *is the unique maximal abelian extension of* $\mathfrak{t}_{\mathfrak{k}}$.

Proof. Assume that there are some $\alpha \in \Delta$ with $\alpha(A) = 0$ for all $A \in \mathfrak{t}_{\mathfrak{k}}$. Then $\alpha \in \mathfrak{t}_{\mathfrak{p}}$ and $\sigma\alpha = -\alpha$. Since σ is an automorphism of $\mathfrak{g}$ preserving $\mathfrak{t}$, we have $\sigma(\mathfrak{g}_\alpha) = \mathfrak{g}_{\sigma\alpha} = \mathfrak{g}_{-\alpha}$. Choose any nonzero $X_\alpha \in \mathfrak{g}_\alpha$. Then $X_\alpha + \sigma X_\alpha \in \mathfrak{k}^c = \mathfrak{k} \otimes \mathbb{C}$. On the other hand, $X_\alpha + \sigma X_\alpha$ commutes with $\mathfrak{t}_{\mathfrak{k}}$ since

$$[X_\alpha + \sigma X_\alpha, A] = i\alpha(A)(X_\alpha - \sigma X_\alpha) = 0$$

for all $A \in \mathfrak{t}_{\mathfrak{k}}$. The real and imaginary parts of $X_\alpha + \sigma X_\alpha$ are contained in $\mathfrak{k}$ and still commute with $\mathfrak{t}_{\mathfrak{k}}$. Thus they belong to $\mathfrak{t}_{\mathfrak{k}}$ which is maximal abelian in $\mathfrak{k}$. On the other hand, $X_\alpha + \sigma X_\alpha$ is a nonzero vector in $\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}$, but this subspace has zero intersection with $\mathfrak{t}^c \supset \mathfrak{t}_{\mathfrak{k}}^c$, which is a contradiction. □

The connected components of the set of $\mathfrak{g}$-regular elements are the *Weyl chambers* in $\mathfrak{t}$. Each nonempty intersection of $\mathfrak{t}_{\mathfrak{k}}$ with a Weyl chamber will be called a *compartment*. An automorphism of $\mathfrak{g}$ preserving $\mathfrak{t}$ permutes Δ as well as the set of Weyl chambers (being the connected components of $\mathfrak{t} \setminus \bigcup_{\alpha \in \Delta} \ker \alpha$). But each Weyl chamber C intersecting $\mathfrak{t}_{\mathfrak{k}}$ contains a nonzero element which is fixed under σ, thus $\sigma(C) = C$. To C corresponds a unique root basis (fundamental root system)

$$\Delta_o = \{\alpha_1, \ldots, \alpha_r\} \tag{10}$$

such that $C \subset \mathfrak{t}$ is precisely the subset where $\alpha_1, \ldots, \alpha_r$ take positive values. This is also σ-invariant, that is, σ acts on Δ_o as a permutation: $\sigma\alpha_j = \alpha_{\sigma j}$.

LEMMA 8. *Let* $\pi_{\mathfrak{k}} : \mathfrak{t} \to \mathfrak{t}_{\mathfrak{k}}$, $H \mapsto \frac{1}{2}(H + \sigma H)$ *be the orthogonal projection of* $\mathfrak{t}$ *onto* $\mathfrak{t}_{\mathfrak{k}}$. *Then* $\pi_{\mathfrak{k}}(\Delta_o)$ *is a basis of* $\mathfrak{t}_{\mathfrak{k}}$, *and the compartment* $C \cap \mathfrak{t}_{\mathfrak{k}}$ *is a simplicial cone: It is precisely the subset of* $\mathfrak{t}_{\mathfrak{k}}$ *where all elements of the basis* $\pi_{\mathfrak{k}}(\Delta_o)$ *are positive.*

Proof. The set $\pi_{\mathfrak{k}}(\Delta_o)$ clearly generates $\mathfrak{t}_{\mathfrak{k}} = \pi_{\mathfrak{k}}(\mathfrak{t})$ since Δ_o generates $\mathfrak{t}$. Moreover, any linear relation among the elements $\pi_{\mathfrak{k}}(\alpha_j) = \frac{1}{2}(\alpha_j + \alpha_{\sigma j})$ is also a linear relation between the α_j with essentially the same coefficients, and this is trivial since Δ_o is linearly independent. Further, for any $H \in \mathfrak{t}_{\mathfrak{k}}$ we have $\langle \pi_{\mathfrak{k}}(\alpha_j), H \rangle = \langle \alpha_j, \pi_{\mathfrak{k}}(H) \rangle = \langle \alpha_j, H \rangle$, hence $\langle \pi_{\mathfrak{k}}(\alpha_j), H \rangle$ is positive for all j if and only if $H \in C \cap \mathfrak{t}_{\mathfrak{k}}$. Thus $C \cap \mathfrak{t}_{\mathfrak{k}}$ is a simplicial cone in $\mathfrak{t}_{\mathfrak{k}}$, namely the convex set bounded by the s hyperplanes $\ker \pi_{\mathfrak{k}}(\alpha_j)$ (where $s = \dim \mathfrak{t}_{\mathfrak{k}}$). □

The basis $\pi_{\mathfrak{k}}(\Delta_o)$ of $\mathfrak{t}_{\mathfrak{k}}$ will be called *semi-fundamental*; it will replace the fundamental root basis used in Theorem 3. In the next section we will see that all compartments are conjugate and subdivide the Weyl chambers of $\mathfrak{k}$.

6. *A subdivision of Weyl chambers of* K

In order to study the compartments more closely, we split the $\mathfrak{g}$-root system Δ into two complementary subsets, $\Delta = \Delta' \sqcup \Delta''$, with

$$\Delta' = \{\alpha \in \Delta;\ \mathfrak{g}_\alpha \not\subset \mathfrak{p}^c\}, \quad \Delta'' = \{\alpha \in \Delta;\ \mathfrak{g}_\alpha \subset \mathfrak{p}^c\}. \tag{11}$$

The following result is related to [2, Lemma 3.11].

Lemma 9. *The restriction of every $\mathfrak{g}$-root in Δ' to $\mathfrak{t}_{\mathfrak{k}}$ is a $\mathfrak{k}$-root. Moreover, every $\mathfrak{k}$-root arises in this way, that is, $\Delta_{\mathfrak{k}} = \{\alpha|_{\mathfrak{t}_{\mathfrak{k}}};\ \alpha \in \Delta'\}$.*

Proof. Let $\alpha \in \Delta'$ and $0 \neq X_\alpha \in \mathfrak{g}_\alpha$. We decompose X_α into its $\mathfrak{k}^c$- and $\mathfrak{p}^c$-components:

$$X_\alpha = X_\alpha^{\mathfrak{k}} + X_\alpha^{\mathfrak{p}}. \tag{12}$$

Since $\alpha \in \Delta'$, we have $X_\alpha^{\mathfrak{k}} \neq 0$. Let $A \in \mathfrak{t}_{\mathfrak{k}}$. On the one hand, from $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$ and $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ we get $[A, X_\alpha^{\mathfrak{k}}] \in \mathfrak{k}^c$ and $[A, X_\alpha^{\mathfrak{p}}] \in \mathfrak{p}^c$, and on the other hand we have $[A, X_\alpha] = i\alpha(A)X_\alpha$ which yields

$$[A, X_\alpha^{\mathfrak{k}}] = i\alpha(A)X_\alpha^{\mathfrak{k}}, \quad [A, X_\alpha^{\mathfrak{p}}] = i\alpha(A)X_\alpha^{\mathfrak{p}}.$$

The first of these equations shows that $\alpha|_{\mathfrak{t}_{\mathfrak{k}}}$ is a root of $\mathfrak{k}$, and the corresponding root space is $\pi_{\mathfrak{k}}(\mathfrak{g}_\alpha)$ where $\pi_{\mathfrak{k}} : \mathfrak{g}^c \to \mathfrak{k}^c$ is the (complexified) orthogonal projection. From $\mathfrak{g}^c = \mathfrak{t}^c + \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha$ we get $\mathfrak{k}^c = \pi_{\mathfrak{k}}(\mathfrak{g}^c) = \mathfrak{t}_{\mathfrak{k}}^c + \sum_{\alpha\in\Delta} \pi_{\mathfrak{k}}(\mathfrak{g}_\alpha)$. Hence $\mathfrak{k}^c$ is spanned by $\mathfrak{t}_{\mathfrak{k}}^c$ and the vectors $X_\alpha^{\mathfrak{k}}$, $\alpha \in \Delta$, and there is no room for another $\mathfrak{k}$-root. □

Theorem 10. *Let $S = \mathrm{G}/\mathrm{K}$ be an outer symmetric space. Then*:

(i) *each compartment lies in a $\mathfrak{k}$-Weyl chamber*;
(ii) *every $\mathfrak{k}$-Weyl chamber is decomposed into the same number of compartments*;
(iii) *any two compartments are conjugate under* G.

Proof. By Lemma 9, $\mathfrak{g}$-regular elements in $\mathfrak{t}_{\mathfrak{k}}$ are also $\mathfrak{k}$-regular. Thus a compartment is contained in a connected component of $\mathfrak{k}$-regular elements, that is, in a $\mathfrak{k}$-Weyl chamber.

Now let $C_1^{\mathfrak{k}}$ and $C_2^{\mathfrak{k}}$ be any two $\mathfrak{k}$-Weyl chambers. Then there exists an element $k \in \mathrm{K}$ such that $C_2^{\mathfrak{k}} = \mathrm{Ad}_{\mathrm{K}}(k)C_1^{\mathfrak{k}}$. But $\mathrm{Ad}_{\mathrm{K}}(k)$ is the restriction of $\mathrm{Ad}_{\mathrm{G}}(k)$ to $\mathfrak{k}$, and since $\mathrm{Ad}_{\mathrm{G}}(k)$ preserves maximal abelian subalgebras of $\mathfrak{g}$, it maps the unique maximal abelian subalgebra containing $C_1^{\mathfrak{k}}$ (see Lemma 7) onto the (unique) one containing $C_2^{\mathfrak{k}}$. Moreover $\mathrm{Ad}_{\mathrm{G}}(k)$ preserves $\mathfrak{g}$-regular vectors and therefore $\mathfrak{g}$-Weyl chambers. Thus it maps the $\mathfrak{g}$-Weyl chambers intersecting $C_1^{\mathfrak{k}}$ onto the $\mathfrak{g}$-Weyl chambers intersecting $C_2^{\mathfrak{k}}$ and $\mathrm{Ad}_{\mathrm{G}}(k^{-1})$ does the converse.

Each compartment has s walls, being the kernels of the s elements of $\pi_{\mathfrak{k}}(\Delta_o)$. By Lemma 9 we see that the kernels of the elements of $\pi_{\mathfrak{k}}(\Delta')$ are root hyperplanes of $\mathfrak{k}$, that is, walls of the $\mathfrak{k}$-Weyl chamber. The only 'new' walls are the kernels of elements in $\pi_{\mathfrak{k}}(\Delta'') = \Delta''$ (note that $\sigma\alpha = \alpha$ for $\alpha \in \Delta''$). The reflection at such a wall $\ker\alpha$ with $\alpha \in \Delta''$ acts on $\mathfrak{t}$ as a Weyl group element of G which preserves $\mathfrak{t}_{\mathfrak{k}}$. Thus it sends the compartment to an adjacent one. Hence any two adjacent compartments within a $\mathfrak{g}$-Weyl chamber are conjugate, and by iteration, any two compartments are conjugate. □

Since this decomposition of Weyl chambers may be of independent interest, we are giving here a more detailed description which is not needed in our main result. There are three different cases.

Case A: inner symmetric spaces. A symmetric space S of compact type is called *inner* if its geodesic symmetries are contained in its transvection group, or equivalently, if the involution σ is inner. It is well known that a symmetric space S of compact type is inner if and only if the rank of $\mathfrak{g}$ and $\mathfrak{k}$ coincide, that is, if every maximal torus of $\mathfrak{k}$ is also a maximal torus of $\mathfrak{g}$ (see [**7**, p. 424]). Thus we have $\mathfrak{t}_{\mathfrak{k}} = \mathfrak{t}$. Hence the $\mathfrak{k}$-roots are precisely those $\mathfrak{g}$-roots whose root spaces are contained in $\mathfrak{k}^c$, so that $\Delta_{\mathfrak{k}} \subset \Delta$. Therefore every $\mathfrak{g}$-Weyl chamber in $\mathfrak{t}$ is contained in a $\mathfrak{k}$-Weyl chamber as a compartment. Since $\mathfrak{p}$ is not abelian (as S is of compact type) there must be a $\mathfrak{g}$-root whose root space is not contained in $\mathfrak{k}^c$, so that $\Delta \neq \Delta_{\mathfrak{k}}$. Thus the $\mathfrak{g}$-Weyl chambers in $\mathfrak{t}$ are smaller than the $\mathfrak{k}$-Weyl chambers and every $\mathfrak{k}$-Weyl chamber splits into several compartments.

Proposition 11. *If S is an inner symmetric space of compact type, then every compartment in a $\mathfrak{k}$-Weyl chamber is a full $\mathfrak{g}$-Weyl chamber. Every $\mathfrak{k}$-Weyl chamber contains at least two compartments.*

Case B: symmetric spaces of rank-split type. A symmetric space S of compact type is called of *rank-split type*, if the rank of S, that is the maximal dimension of a totally geodesic submanifold in S, or equivalently the dimension of a maximal abelian subset in $\mathfrak{p}$, is just the difference between the ranks of $\mathfrak{g}$ and of $\mathfrak{k}$. Using our splitting $\mathfrak{t} = \mathfrak{t}_{\mathfrak{k}} \oplus \mathfrak{t}_{\mathfrak{p}}$ this implies that $\mathfrak{t}_{\mathfrak{p}}$ is a maximal abelian subspace of $\mathfrak{p}$. A symmetric space S is of rank-split type if all irreducible factors of its universal cover are also of rank-split type. Note that in this case $\mathfrak{k}$ is semisimple. If not, the universal cover of S would have at least one hermitian symmetric de Rham factor, which is not outer and therefore not of rank-split type.

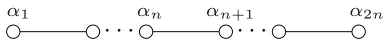
Lemma 12. *If S is of rank-split type, then $\Delta' := \{\alpha \in \Delta;\ \mathfrak{g}_\alpha \not\subset \mathfrak{p}^c\} = \Delta$.*

Proof. Since S is of rank-split type, $\mathfrak{t} = \mathfrak{t}_{\mathfrak{k}} \oplus \mathfrak{t}_{\mathfrak{p}}$, where $\mathfrak{t}_{\mathfrak{p}}$ is a *maximal* abelian subspace of $\mathfrak{p}$. Let $\alpha \in \Delta$ and assume that $\mathfrak{g}_\alpha \subset \mathfrak{p}^c$. Let $0 \neq X_\alpha \subset \mathfrak{g}_\alpha$; then $X_\alpha \notin \mathfrak{t}^c$. The Cartan relation $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ implies that for all $A \in \mathfrak{t}_{\mathfrak{p}}$ we have $[A, X_\alpha] \subset \mathfrak{k}^c$. But on the other hand $[A, X_\alpha] = i\alpha(A)X_\alpha \in \mathfrak{p}^c$. Similarly $[A, \overline{X_\alpha}] = -i\alpha(A)\overline{X_\alpha} \in \mathfrak{p}^c$ for all $A \in \mathfrak{t}_{\mathfrak{p}}$. Since $[\mathfrak{p}^c, \mathfrak{p}^c] \subset \mathfrak{k}^c$ and $\mathfrak{p}^c \cap \mathfrak{k}^c = \{0\}$ we get $[\mathfrak{t}_{\mathfrak{p}}, X_\alpha] = [\mathfrak{t}_{\mathfrak{p}}, \overline{X_\alpha}] = 0$. Thus the real part $\mathrm{Re}\,(X_\alpha)$ of X_α, given by $\mathrm{Re}\,(X_\alpha) = \frac{1}{2}(X_\alpha + \overline{X_\alpha}) \in \mathfrak{p}$, which is not contained in $\mathfrak{t}_{\mathfrak{p}}$ satisfies $[\mathfrak{t}_{\mathfrak{p}}, \mathrm{Re}\,(X_\alpha)] = 0$, a contradiction, since $\mathfrak{t}_{\mathfrak{p}}$ is maximal abelian in $\mathfrak{p}$. □

Let $C_{\mathfrak{k}}$ be a $\mathfrak{k}$-Weyl chamber in $\mathfrak{t}_{\mathfrak{k}}$. Since by Lemmas 9 and 12 the restriction of every $\mathfrak{g}$-root in $\mathfrak{t}$ to $\mathfrak{t}_{\mathfrak{k}}$ is a $\mathfrak{k}$-root and vice versa, the kernels of the $\mathfrak{k}$-roots in $\mathfrak{t}_{\mathfrak{k}}$ are precisely the intersections of the kernels of $\mathfrak{g}$-roots in $\mathfrak{t}$ with $\mathfrak{t}_{\mathfrak{k}}$. Thus every $\mathfrak{k}$-Weyl chamber in $\mathfrak{t}_{\mathfrak{k}}$ is an intersection of a $\mathfrak{g}$-Weyl chamber in $\mathfrak{t}$ with $\mathfrak{t}_{\mathfrak{k}}$.

Lemma 13. *If S is of rank-split type, then a $\mathfrak{k}$-Weyl chamber is itself a compartment and all compartments are conjugate by inner automorphisms of $\mathfrak{k}$.*

Case C: non rank-split type outer symmetric spaces. Finally consider an irreducible outer symmetric space S of compact type, which is not of rank-split type. In most cases, every $\mathfrak{k}$-Weyl chamber contains two or more compartments, but an exception occurs for the outer symmetric space $S = \mathrm{SU}_{2n+1}/\mathrm{SO}_{2n+1}$ which is not of rank-split type. In this case the Dynkin diagram of $\mathfrak{su}_{2n+1}$ (type A_{2n}) shows that no simple root of a $\mathfrak{g}$-Weyl chamber C intersecting $\mathfrak{t}_{\mathfrak{k}}$ is σ-invariant.



Thus all simple roots of C restrict to $\mathfrak{k}$-roots, that is, $C \cap \mathfrak{t}_{\mathfrak{k}}$ is a $\mathfrak{k}$-Weyl chamber.

We would like to end this section with a list of all (local isometry classes of) irreducible outer symmetric spaces of compact type. We start our list with the compact connected simple Lie groups equipped with a bi-invariant metric (they are all symmetric spaces of rank-split type). Any of the remaining spaces is locally isometric to one of the G/K in Table 1 (cf., for example, [**2**, p. 38], and [**7**, Chapter X, Section 6]).

7. *Classification of pluriharmonic maps*

Let G/K be a bottom symmetric space with involution σ such that $\mathrm{K} = \mathrm{Fix}(\sigma)$. Let also $\mathrm{T}_{\mathrm{K}} \subset \mathrm{K}$ be a maximal torus in the connected component of the identity of K, and $\mathrm{T} \subset \mathrm{G}$ the maximal torus of G such that $\mathrm{T}_{\mathrm{K}} \subset \mathrm{T}$ (cf. Lemma 7). The Lie algebras of T and T_{K} are, respectively, $\mathfrak{t}$ and $\mathfrak{t}_{\mathfrak{k}}$. Let $\Delta_o \subset \mathfrak{t}^*$ be a σ-invariant fundamental root system of $\mathfrak{g}$ with dual basis $\{\xi_1, \ldots, \xi_r\}$. A canonical element

$$\xi_o = \xi_{j_1} + \ldots + \xi_{j_p} \tag{13}$$

and its corresponding homomorphic loops γ_o will be called *S-canonical* if $\sigma\xi_o = \xi_o$.

Theorem 14. *Let M be a compact simply connected Kähler manifold and $f : M \to S$ be a pluriharmonic map where $S = \mathrm{G}/\mathrm{K}$ is a bottom symmetric space. Choose a basepoint $o \in S$ which lies in the image of f. Let $\iota^o : S \to \mathrm{G}$ be the Cartan embedding. Then there is an S-canonical homomorphic loop γ_o in* K *and an extended solution Φ of the harmonic map $\tilde{f} = \iota^o \circ f : M \to \mathrm{G}$ (up to left translation in* G*) which takes values essentially in the S-canonical Bruhat cell U_{γ_o}. Thus there are at most 2^s classes of pluriharmonic maps, where $s = \mathrm{rank}(\mathrm{K})$.*

Proof. By Lemmas 4 and 5, the pluriharmonic map $\tilde{f} = \iota^o \circ f : M \to \mathrm{G}$ admits a T^o-invariant extended solution Φ which takes values essentially in U_γ for some homomorphic loop $\gamma : \mathbb{R}/(2\pi\mathbb{Z}) \to \mathrm{K}$. Then γ takes values in a maximal torus in K. We may assume that this is actually T_{K}, since conjugating γ with any group element does not change the Bruhat cell U_γ. The vector $\xi = \gamma'(0) \in \mathfrak{t}_{\mathfrak{k}}$ lies in the closure of some compartment. By Theorem 10, all compartments are conjugate. Thus, for the same reason as above, we may further assume that

Table 1. *Irreducible compact outer symmetric spaces of type I.*

G/K	rank G	rank K	rank G/K
$\mathrm{SU}_{2n}/\mathrm{SO}_{2n}$	$2n-1$	n	$2n-1$
$\mathrm{SU}_{2n+1}/\mathrm{SO}_{2n+1}$	$2n$	n	$2n$
$\mathrm{SU}_{2n}/\mathrm{Sp}_n$	$2n-1$	n	$n-1$
$G_{2k-1}(\mathbb{R}^{2n})$, $1 \leqslant k \leqslant n$	n	$n-1$	$\min\{2k-1, 2n-2k+1\}$
E_6/Sp_4	6	4	6
E_6/F_4	6	4	2

ξ is in the closure of the compartment determined by Δ_o. This means that we can write

$$\xi = n_1\xi_{j_1} + \ldots + n_p\xi_{j_p}$$

with $n_1, \ldots, n_p \in \mathbb{N} = \{1, 2, \ldots\}$. By Theorem 3, we can change ξ to the canonical element

$$\xi_o = \xi_{j_1} + \ldots + \xi_{j_p}. \tag{14}$$

More precisely, (a translate of) $\tilde{f}$ has another extended solution $\tilde{\Phi}$ taking values essentially in U_{γ_o} where γ_o is the homomorphic loop with $\gamma_o'(0) = \xi_o$.

We claim that $\sigma\xi_o = \xi_o$, in other words, ξ_o is S-canonical. Indeed, since $\xi \in \mathfrak{t}_\mathfrak{k}$, we have $\sigma\xi = \xi$. Thus, if $j \in \{j_1, \ldots, j_p\}$ then $\sigma j \in \{j_1, \ldots, j_p\}$ as well. This implies our claim.

Let $s = \operatorname{rank} \mathrm{K} = \dim \mathfrak{t}_\mathfrak{k}$ and put $k = r - s$. The fundamental basis $\Delta_o = \{\alpha_1, \ldots, \alpha_r\}$ can be labeled such that $\sigma\alpha_j = \alpha_{s+j}$ for $j \leqslant k$ and $\sigma\alpha_j = \alpha_j$ for $k+1 \leqslant j \leqslant s$. The corresponding semi-fundamental basis is $\pi_\mathfrak{k}(\Delta_o) = \{\rho_1, \ldots, \rho_s\}$ with

$$\rho_j = \begin{cases} \frac{1}{2}(\alpha_j + \alpha_{s+j}) & \text{for } j \leqslant k, \\ \alpha_j & \text{for } k+1 \leqslant j \leqslant s. \end{cases}$$

The dual basis for $\pi_\mathfrak{k}(\Delta_o)$ is $\{\eta_1, \ldots, \eta_s\}$ with

$$\eta_j = \begin{cases} \xi_j + \xi_{s+j} & \text{for } j \leqslant k, \\ \xi_j & \text{for } k+1 \leqslant j \leqslant s. \end{cases} \tag{15}$$

Combining pairs of dual roots joined by σ in the sum (14) we obtain

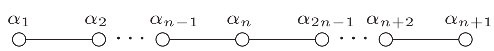
$$\xi_o = \eta_{j_1} + \ldots + \eta_{j_q} \tag{16}$$

where $q \leqslant p$ and the η_j are defined in (15). Since $\{j_1, \ldots, j_q\}$ may be an arbitrary subset of $\{1, \ldots, s\}$, there are 2^s such elements. The empty set corresponds to $\xi_o = 0$ with $\gamma_o = e$ and $U_{\gamma_o} = \{e\}$; the corresponding pluriharmonic maps are constant. □

Remarks. (1) For inner symmetric spaces S, Theorem 14 is not an improvement of Theorem 3: in this case we have $\mathrm{T_K} = \mathrm{T}$, thus all ξ_j are σ-invariant and the notions 'S-canonical' and 'canonical' are the same. However, for this case Burstall and Guest [**1**, 5.4] have improved Theorem 3 in a different way.

(2) Pluriharmonic maps with values in G can also be classified according to their minimal uniton number. This was done by Burstall and Guest [**1**]. For a given G, they have also obtained an explicit upper bound for the minimal uniton number, which they denoted by $r(\mathrm{G})$. As it turns out, Theorem 14 does not improve that bound.

Example. Let us consider the case when S is the adjoint space of $\mathrm{SU}_{2n}/\mathrm{SO}_{2n}$. Then $r = 2n-1$, $s = n$ and $k = r - s = n - 1$. As in the proof of Theorem 14, we can label the elements of $\Delta_o = \{\alpha_1, \ldots, \alpha_{2n-1}\}$ in such a way that $\sigma\alpha_j = \alpha_{n+j}$, for $1 \leqslant j \leqslant n-1$ and $\sigma\alpha_n = \alpha_n$. In the Dynkin diagram below, σ should be thought of as the reflection through the knot α_n.



To be more specific, we choose $\mathfrak{t}_\mathfrak{k}$ to be the subspace of $\mathfrak{so}_{2n}$ which consists of all matrices of the form

$$\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix},$$

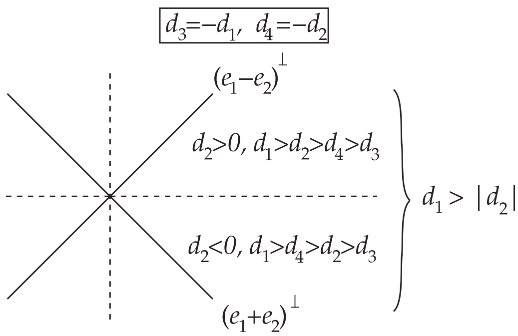


FIGURE 1. *Compartments for* $\mathrm{SU}_4/\mathrm{SO}_4$.

where D is an $n \times n$ diagonal matrix with real entries. The maximal abelian extension of $\mathfrak{t}_\mathfrak{k}$ in $\mathfrak{su}_{2n}$ is $\mathfrak{t} := A\mathfrak{t}'A^{-1}$ where $\mathfrak{t}'$ consists of all diagonal matrices $\mathrm{Diag}(id_1, \dots, id_{2n})$ with $d_1, \dots, d_{2n} \in \mathbb{R}$, $\sum_{j=1}^{2n} d_j = 0$ and

$$A := \frac{1}{\sqrt{2}} \begin{pmatrix} I_n & iI_n \\ iI_n & I_n \end{pmatrix}.$$

In fact, $\mathfrak{t}$ consists of all matrices

$$\begin{pmatrix} iE & D \\ -D & iE \end{pmatrix},$$

where D and E are real diagonal matrices and trace $E = 0$. The map σ is given by complex conjugation of matrices. It turns out that the restriction of σ to $\mathfrak{t}$ is given by

$$A\mathrm{Diag}(id_1, \dots, id_{2n})A^{-1} \longmapsto -A\mathrm{Diag}(id_{n+1}, \dots, id_{2n}, id_1, \dots, id_n)A^{-1}.$$

We make the identification

$$\mathfrak{t} = \left\{ (d_1, \dots, d_{2n}) \in \mathbb{R}^{2n} \ : \ \sum_{j=1}^{2n} d_j = 0 \right\}.$$

We neglect the conjugation by A and turn $\mathfrak{t}_\mathfrak{k}$ into a subspace of $\mathfrak{t}$. It consists of all vectors $(d_1, \dots, d_{2n})$ which satisfy $d_{n+1} = -d_1, \dots, d_{2n} = -d_n$. A σ-invariant Weyl chamber C in $\mathfrak{t}$ is the one determined by

$$d_1 \geqslant d_2 \geqslant \dots \geqslant d_n \geqslant d_{2n} \geqslant d_{2n-1} \geqslant \dots \geqslant d_{n+1}.$$

Figure 1 describes the compartment $C \cap \mathfrak{t}_\mathfrak{k}$ in the special case $n = 2$. One can see there that any $\mathfrak{k}$-Weyl chamber is divided into two compartments. This turns out to be true for any $n \geqslant 2$. Denote by ϵ_j the jth coordinate function on $\mathbb{R}^{2n}$, which is given by $(d_1, \dots, d_{2n}) \mapsto d_j$, where $1 \leqslant j \leqslant 2n$. The roots whose kernels bound C are the following functions $\mathfrak{t} \to \mathbb{R}$:

$$\begin{aligned} &\alpha_1 = \epsilon_1 - \epsilon_2,\ \alpha_2 = \epsilon_2 - \epsilon_3,\ \dots,\ \alpha_{n-1} = \epsilon_{n-1} - \epsilon_n, \\ &\alpha_n = \epsilon_n - \epsilon_{2n}, \\ \alpha_{n+1} = \epsilon_{n+2} &- \epsilon_{n+1},\ \alpha_{n+2} = \epsilon_{n+3} - \epsilon_{n+2},\ \dots,\ \alpha_{2n-1} = \epsilon_{2n} - \epsilon_{2n-1}. \end{aligned}$$

They form the fundamental root system Δ_o. They fit into the Dynkin diagram above. The elements of the basis of $\mathfrak{t}$ which is dual to Δ_o are $\xi_1, \dots, \xi_{2n-1}$, where $\xi_j \in \mathfrak{t}$ is determined by $\alpha_k(\xi_j) = \delta_{kj}$, $1 \leqslant k \leqslant 2n - 1$. Concretely, we have $\xi_j = (s, \dots, s, t, \dots, t)$, where the entry t starts at index $j + 1$ and $s, t \in \mathbb{R}$ are such that $\xi_j \perp (1, \dots, 1)$ and $\alpha_j(\xi_j) = 1$. We write a vector $(x_1,\ldots,x_{2n})$ in $\mathbb{R}^{2n}$ as

$$(x_1,\ldots,x_{2n}) = ((x_1,\ldots,x_n),(x_{n+1},\ldots,x_{2n})).$$

With this convention, we have

$$\begin{aligned}
\xi_1 &= \left(\left(\frac{2n-1}{2n},-\frac{1}{2n},\ldots,-\frac{1}{2n}\right),\left(-\frac{1}{2n},\ldots,-\frac{1}{2n}\right)\right),\\
\xi_n &= \left(\left(\frac{1}{2},\ldots,\frac{1}{2}\right),\left(-\frac{1}{2},\ldots,-\frac{1}{2}\right)\right),\\
\xi_{n+1} &= \left(\left(\frac{1}{2n},\frac{1}{2n},\ldots,\frac{1}{2n}\right),\left(-\frac{2n-1}{2n},\frac{1}{2n},\ldots,\frac{1}{2n}\right)\right).
\end{aligned}$$

Similar formulas give the remaining elements ξ_j. Like in the proof of Theorem 14, we set

$$\begin{aligned}
\eta_1 &:= \xi_1+\xi_{n+1} = ((1,0,\ldots,0),(-1,0,\ldots,0)),\\
\eta_2 &:= \xi_2+\xi_{n+2} = ((1,1,0,\ldots,0),(-1,-1,0,\ldots,0)),\\
&\vdots\\
\eta_{n-1} &:= \xi_{n-1}+\xi_{2n-1} = ((1,\ldots,1,0),(-1,\ldots,-1,0)),\\
\eta_n &:= \xi_n = \left(\left(\frac{1}{2},\ldots,\frac{1}{2}\right),\left(-\frac{1}{2},\ldots,-\frac{1}{2}\right)\right).
\end{aligned}$$

The S-canonical elements are 0 and the vectors of the form $\eta_{j_1}+\ldots+\eta_{j_q}$, where $1\leqslant j_1<\ldots<j_q\leqslant n$.

Acknowledgements. It is a pleasure for us to thank F. Burstall and M. Guest for several useful hints and discussions. We are also thankful to the referee for several valuable suggestions.

References

1. F. E. Burstall and M. A. Guest, 'Harmonic two-spheres in compact symmetric spaces, revisited' *Math. Ann.* 309 (1997) 541–572.

2. F. E. Burstall and J. H. Rawnsley, *Twistor theory for symmetric spaces,* Lecture Notes in Mathematics 1424 (Springer, Berlin, 1990).

3. J. Dorfmeister and J.-H. Eschenburg, 'Pluriharmonic maps, loop groups and twistor theory' *Ann. Global Anal. Geom.* 24 (2003) 301–321.

4. J.-H. Eschenburg and P. Quast, 'The spectral parameter of pluriharmonic maps', *Bull. London Math. Soc.* 42 (2010) 229–236.

5. J.-H. Eschenburg and R. Tribuzy, 'Associated families of pluriharmonic maps and isotropy', *Manuscripta Math.* 95 (1998) 295–310.

6. M. A. Guest and Y. Ohnita, 'Loop group actions on harmonic maps and their applications', *Harmonic maps and integrable systems*, Aspects of Mathematics E23 (Vieweg, Braunschweig, 1994) 273–292.

7. S. Helgason, *Differential geometry, Lie groups, and symmetric spaces* (Academic Press, New York, 1978).

8. O. Loos, *Symmetric spaces II: compact spaces and classification* (Benjamin, New York, 1968).

9. Y. Ohnita and G. Valli, 'Pluriharmonic maps into compact Lie groups and factorization into unitons', *Proc. London Math. Soc.* (3) 61 (1990) 546–570.

10. A. N. Pressley, 'The energy flow on the loop space of a compact Lie group', *J. London Math. Soc.* (2) 26 (1982) 557–566.

11. A. N. Pressley and G. B. Segal, *Loop groups* (Oxford University Press, Oxford, 1986).

12. G. B. Segal, 'Loop groups and harmonic maps', *Advances in homotopy theory*, London Mathematical Society Lecture Note Series 139 (Cambridge University Press, Cambridge, 1989) 153–164.

13. K. Uhlenbeck, 'Harmonic maps into Lie groups (classical solutions of the chiral model)', *J. Differential Geom.* 30 (1989) 1–50.

J.-H. Eschenburg
Institut für Mathematik
Universität Augsburg
D-86135 Augsburg
Germany

eschenburg@math.uni-augsburg.de

A.-L. Mare
Department of Mathematics and Statistics
University of Regina
Regina SK
Canada S4S 0A2

mareal@math.uregina.ca

P. Quast
Institut für Mathematik
Universität Augsburg
D-86135 Augsburg
Germany

peter.quast@math.uni-augsburg.de