# The spectral parameter of pluriharmonic maps

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## 1. *Introduction*

What is an integrable system? Although this notion seems a bit vague, one of the common features is that the differential equation allows for a one-parameter deformation, depending on the so-called spectral parameter $\lambda$. This is often introduced in a purely formal way. It is the purpose of the present article to discuss the geometric meaning of $\lambda$ in an important case, that of harmonic maps of surfaces and pluriharmonic maps of complex manifolds with values in Riemannian symmetric spaces. We shall link the spectral parameter to the associated family of such maps, which is well known from elementary minimal surface theory; the most prominent example is the deformation of the catenoid into the helicoid. In fact, our geometric theory joins two different approaches to (pluri-) harmonic maps: extended solutions and extended frames.

## 2. *Extended solutions and extended frames*

The equation of a harmonic map $f$ of a Riemann surface $M$ into a compact (not necessarily connected) Lie group $G$ with bi-invariant metric or a totally geodesic submanifold $S \subset G$ (a symmetric space) allows for a spectral parameter $\lambda \in \mathbb{S}^1$. There are two different ways to assign to $f$ a $\lambda$-dependent map, a so-called spectral deformation. The first one goes back to Uhlenbeck [**10**], motivated by earlier work in physics [**9**, **11**, **12**], and the second one was introduced by Burstall and Pedit [**2**]; see also [**4**].

Uhlenbeck introduced the notion of an *extended solution*. This is a family of maps $\Phi_\lambda : M \to G$ depending smoothly on $\lambda \in \mathbb{S}^1$ such that $\Phi_1 = e$ is a constant group element (often one assumes that $\Phi_{-1} = e$ (unit element in $G$) but we would like to make the notion independent under left translations in $G$) and the Maurer–Cartan form (using matrix notation, we write $gx$ for the left translation $L_g x$ (for $x \in G$) as well as for its differential $dL_g x$ (for $x \in \mathfrak{g}$))

$$\beta_\lambda := \Phi_\lambda^{-1} d\Phi_\lambda \in \Omega^1(M, \mathfrak{g}) \tag{1}$$

satisfies

$$\beta_\lambda = (1 - \lambda^{-1})\beta' + (1 - \lambda)\beta'' \tag{2}$$

for some $\beta' \in \Omega^{(1,0)}(M, \mathfrak{g}^c)$ and $\beta'' = \overline{\beta'}$. (By $\Omega^{(1,0)}$, we denote the space of one-forms $\omega$ that are complex linear, $\omega(JX) = i\omega(X)$ for any $X \in TM$, while its complex conjugate $\bar\omega \in \Omega^{(0,1)}$ is anti-linear, $\bar\omega(JX) = -i\bar\omega(X)$.) By [**10**], a map $f : M \to G$ is harmonic if and only if there exists (at least locally) an extended solution $\Phi_\lambda$ with $f = \Phi_{-1}$. Since the inversion $j : G \to G$,

$j(g) = g^{-1}$ is an isometry of $G$ for any bi-invariant metric, $f^{-1} = j \circ f$ is again harmonic, and, up to left translation, the corresponding extended solution is (cf. [**1**])

$$(T\Phi)_\lambda := \Phi_{-\lambda}\Phi_{-1}^{-1}. \tag{3}$$

The map $f$ may take values in a totally geodesic submanifold $S \subset G$ and can then be considered as a harmonic map into $S$ rather than into $G$. In particular, we consider a *Cartan embedded symmetric space* $S$ that is a connected component of the set of order 2 elements,

$$\sqrt{e} = \{s \in G;\ s^2 = e\} \subset G$$

(*standard* Cartan embedding), or a left translate of such a set.

REMARK 1. A standard Cartan embedding $(\sqrt{e})_c$, that is, a connected component of $\sqrt{e}$, may or may not be contained in the identity component $G_o$ of $G$. Using a left translation in $G$, it can be shifted into $G_o$. If one uses the left translation by $s_o$ for some $s_o \in (\sqrt{e})_c$, then $s_o(\sqrt{e})_c = \{s_o s; s \in (\sqrt{e})_c\} \subset G_0$ is called a *pointed Cartan embedding*.

The approach by extended frames in turn uses the projection $\pi : G \to S = G/K$ rather than a Cartan embedding $\iota : S \to G$. Starting with a map $f : M \to S$, one first chooses a lift ('frame') $F : M \to G$ with $f = \pi \circ F$; in fact, $F$ may be defined only locally on some open subset $M_o \subset M$. If $S \subset G$ is standard Cartan embedded, then $G$ acts by conjugation on $S$, and the relation between $F$ and $f$ is given by

$$f = F s_o F^{-1} \tag{4}$$

for some $s_o \in \sqrt{S}$ (which is the point reflection at $eK \in G/K$).

Let $\alpha = F^{-1}dF \in \Omega^1(M_o, \mathfrak{g})$ be the corresponding Maurer–Cartan form. We decompose $\alpha = \alpha_{\mathfrak{k}} + \alpha_{\mathfrak{p}}$ according to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ corresponding to $\mathrm{Ad}(s_o)$, that is, $\mathrm{Ad}(s_o) = I$ on $\mathfrak{k}$ and $\mathrm{Ad}(s_o) = -I$ on $\mathfrak{p}$. Then $f$ is harmonic if and only if the modified one-form

$$\alpha_\lambda = \alpha_{\mathfrak{k}} + \lambda^{-1}\alpha'_{\mathfrak{p}} + \lambda\alpha''_p \tag{5}$$

is integrable, that is,

$$\alpha_\lambda = F_\lambda^{-1} dF_\lambda \tag{6}$$

for some smooth map $F_\lambda : M_o \to G$ depending smoothly on $\lambda \in \mathbb{S}^1$ with $F_1 = F$; this is called an *extended frame*. Moreover, all maps $f_\lambda := \pi \circ F_\lambda : M_o \to S$ are harmonic.

Both approaches have been extended from harmonic maps of surfaces to pluriharmonic maps of Kähler manifolds; cf. [**8**] for Uhlenbeck's theory and [**3**, **5**] for the extended frame method. A map $f$ on a Kähler manifold $M$ (in fact, it suffices that $M$ is a complex manifold; locally, we may always choose a Kähler metric, and the notion of pluriharmonicity is independent of the choice of that metric) is called *pluriharmonic* if its restriction to any complex curve in $M$ is harmonic, or, in more technical terms, if its Levi form $Ddf^{(1,1)}$ (the (1,1) part of its hessian) vanishes. Everything we have said remains unchanged after replacing the word 'harmonic' everywhere by 'pluriharmonic'.

The two approaches are related to each other by the following theorem.

THEOREM 1. *Let $f : M \to S \subset G$ be a pluriharmonic map, where $S = G/K$ is (standard) Cartan embedded into $G$. Let $F : M_o \to G$ be a local frame for $f$, that is, $f = \pi \circ F = F s_o F^{-1}$. Then extended solutions and extended frames for $f$ are related by*

$$F_\lambda = \Phi_\lambda F. \tag{7}$$

*More precisely, suppose that two families of maps $F_\lambda, \Phi_\lambda : M_o \to G$, with $\lambda \in \mathbb{S}^1$, satisfying (7), are given. Then $F_\lambda$ is an extended frame for $f$ if and only if $\Phi_\lambda$ is an extended solution for $f$.*

*Proof.* Differentiation of (7) yields

$$\alpha_\lambda = F^{-1}\beta_\lambda F + \alpha, \tag{8}$$

where $\alpha_\lambda = F_\lambda^{-1} dF_\lambda$, $\alpha = \alpha_1 = F^{-1}dF$ and $\beta_\lambda = \Phi_\lambda^{-1} d\Phi_\lambda$. Using (4), we relate $f$ to $\alpha$ as follows:

$$f^{-1}df = F(s_o \alpha s_o - \alpha)F^{-1} = -2\tilde{\alpha}, \tag{9}$$

where

$$\tilde{\alpha} := F\alpha_{\mathfrak{p}}F^{-1}; \tag{10}$$

we remember that $s_o(\alpha_{\mathfrak{k}} + \alpha_{\mathfrak{p}})s_o = \alpha_{\mathfrak{k}} - \alpha_{\mathfrak{p}}$.

Now suppose that $F_\lambda$ is an extended frame for $f$. Then from (5) and (8) we obtain

$$\beta_\lambda = F(\alpha_\lambda - \alpha)F^{-1} = (\lambda^{-1} - 1)\tilde{\alpha}' + (\lambda - 1)\tilde{\alpha}''. \tag{11}$$

Thus $\beta_\lambda = \Phi_\lambda^{-1} d\Phi_\lambda$ satisfies (2) with $\beta = -\tilde{\alpha}$. In particular, $\beta_1 = 0$, and hence $\Phi_1 =$ constant, and we have

$$\Phi_{-1}^{-1} d\Phi_{-1} = \beta_{-1} = -2\tilde{\alpha} \overset{(9)}{=} f^{-1}df, \tag{12}$$

whence $\Phi_{-1} = f$ up to a left translation. Thus $\Phi_\lambda$ is an extended solution for $f$.

Conversely, suppose that $\Phi_\lambda$ is an extended solution for $f = \Phi_{-1}$. Let $F_\lambda = \Phi_\lambda F$ and $\alpha_\lambda = F_\lambda^{-1} dF_\lambda$. From (2) and (8) we obtain

$$\alpha_\lambda - \alpha = F^{-1}\beta_\lambda F = (1 - \lambda^{-1})\tilde{\beta}' + (1 - \lambda)\tilde{\beta}'', \tag{13}$$

where

$$\tilde{\beta}_\lambda = F^{-1}\beta_\lambda F. \tag{14}$$

We show first that the right-hand side of (13) takes values in (the complexification of) $\mathfrak{p}$. In fact, from (9) we have, on the one hand, that

$$F^{-1}(f^{-1}df)F = -2\alpha_{\mathfrak{p}}, \tag{15}$$

and, on the other hand, by (2), for $\lambda = -1$ and (14), we have

$$F^{-1}(f^{-1}df)F = F^{-1}(\Phi_{-1}^{-1} d\Phi_{-1})F = F^{-1}\beta_{-1}F = 2\tilde{\beta}. \tag{16}$$

Thus

$$\tilde{\beta} = -\alpha_{\mathfrak{p}} \in \mathfrak{p}, \tag{17}$$

and hence $\alpha_\lambda - \alpha$ takes values in $\mathfrak{p}$; cf. (13). This implies that the $\mathfrak{k}$-components of $\alpha_\lambda$ and $\alpha$ are equal. Further, (13) shows that

$$\begin{aligned}
(\alpha_\lambda)_{\mathfrak{p}} &= -\tilde{\beta} + (1 - \lambda^{-1})\tilde{\beta}' + (1 - \lambda)\tilde{\beta}'' \\
&= -\lambda^{-1}\tilde{\beta}' - \lambda\tilde{\beta}'' \\
&= \lambda^{-1}\alpha_{\mathfrak{p}}' + \lambda\alpha_{\mathfrak{p}}'',
\end{aligned}$$

and hence we have proved (5). □

## 3. *Associated families*

We want to outline a third approach [7] that gives a geometric interpretation of both extended solutions and extended frames in a single theory. The starting point is the observation that a pluriharmonic map $f$ allows a one-parameter deformation of pluriharmonic maps $f_\lambda$, the so-called *associated family*.

It has already been observed by Weierstraß that minimal surfaces in euclidean space come in one-parameter families, so-called *associated families*. The best-known example is the deformation of the catenoid into the helicoid by cutting the catenoid along a vertical meridian and then move the two ends of the cut upwards and downwards apart from each other (http://page.mi.fu-berlin.de/polthier/Calendar/Kalender86/Kalender86.htm). Starting with a surface $f = f_o$, the associated family is an isometric deformation $f_\theta$ with the following three properties.

(i) Up to parallel translation, each tangent plane remains unchanged during the deformation.
(ii) Principal curvatures are preserved, while principal curvature lines rotate.
(iii) The deformation is periodic, that is, $f_{\theta+2\pi} = f_\theta$, and after a half period $\pi$ we see the same object in opposite orientation.

In fact, denoting by $\mathsf{R}_\theta$ the rotation by the angle $\theta$ in the tangent plane of the surface, the above properties are expressed by

$$df \circ \mathsf{R}_\theta = df_\theta. \tag{18}$$

One may ask which (other) surfaces $f : M \to \mathbb{R}^n$ allow an associated family (18). It is enough to consider the 90° rotation $J = \mathsf{R}_{\pi/2}$ since

$$\mathsf{R}_\theta = (\cos\theta)I + (\sin\theta)J. \tag{19}$$

We need to find a map $g : M \to \mathbb{R}^n$ with

$$df \circ J = dg.$$

If $M$ is simply connected (which we will always assume), then this is equivalent to

$$d(df \circ J) = 0.$$

From $df = f_x dx + f_y dy$ we see that $df \circ J = f_y dx - f_x dy$ and hence

$$d(df \circ J) = f_{yy} dx \wedge dy - f_{xx} dy \wedge dx = \Delta f \, dx \wedge dy,$$

where $\Delta f = \text{trace } Ddf = f_{xx} + f_{yy}$ is the Laplacian. Hence an associated family exists if and only if $f$ is *harmonic*, that is, $\Delta f = 0$. In particular, this applies to minimal surfaces that are just conformal harmonic maps.

Now we replace the euclidean space $\mathbb{R}^n$ by an arbitrary symmetric space $S = G/K$. Furthermore, we replace the surface by a Kähler manifold $M$, that is, a Riemannian manifold with a parallel, almost complex structure $J$. We can still define the parallel tensors $\mathsf{R}_\theta$ by (19) and ask the following question: Given a smooth map $f : M \to S$, under what condition is the one-form $df \circ \mathsf{R}_\theta$ integrable, that is, the differential of a map $f_\theta$? To make this question more precise, recall that $df_x$ is a linear map from $T_x M$ to $T_{f(x)}S$ and hence it defines a bundle map $df : TM \to f^*TS$. Thus (18) has to be modified since $f_\theta^*TS$ and $f^*TS$ are different vector bundles. What we need is an isomorphism $\Phi_\theta$ between these bundles such that

$$df_\theta = \Phi_\theta \circ df \circ \mathsf{R}_\theta, \tag{20}$$

where the isomorphism $\Phi_\theta : f^*TS \to f_\theta^*TS$, like parallel translation in euclidean space $\mathbb{R}^n$, preserves the metric and the Lie triple structure (curvature tensor) on $TS$ and is parallel with respect to the induced connections on $f^*TS$ and $f_\theta^*TS$. A family of pairs $(f_\theta, \Phi_\theta)$ with $f_0 = f$ and $\Phi_0 = I$ satisfying (20) will be called an *associated family* for $f : M \to S$.

In [**6**], the question has been discussed in a more general setting. Let there be a vector bundle $E \to M$ and an $E$-valued one-form (bundle homomorphism) $\varphi : TM \to E$. Suppose that the fibres of $E$ carry a connection $\nabla$ and a parallel Lie triple structure $R^S$ on the fibres that is isomorphic to the Lie triple structure of a Riemannian symmetric space $S$. When does there exist a smooth map $f : M \to S$ with $df = \Phi \circ d\varphi$, using a suitable isomorphism $\Phi : E \to f^*TS$? The answer was given in [**6**]: $f$ exists and is unique up to isometries of $S$ if and only if $d^\nabla \varphi = 0$ and $R^\nabla = \varphi^* R^S$. The proof is an application of the Frobenius integrability theorem. Applying this theory to $\varphi_\theta = df \circ \mathsf{R}_\theta$, one obtains the following.

THEOREM 2 [**7**]. *Let $M$ be a Kähler manifold, let $S$ be a compact symmetric space and let $f : M \to S$ be a smooth map. There exists an associated family $(f_\theta, \Phi_\theta)$ for $f$ (unique up to isometries $g_\theta$ of $S$) if and only if $f$ is pluriharmonic.*

In this case, $\Phi_\theta$ is an isometric bundle isomorphism between $f^*TS$ and $f_\theta^*TS$ (that is, it maps $T_{f(x)}S$ isometrically onto $T_{f_\theta(x)}S$ for any $x \in M$) that is parallel and preserves the curvature tensor $R^S$ of $S$. Thus $\Phi_\theta(x)$ is the differential of an isometry of $S$. (If $S$ is not simply connected, then this is not true in general, but it is true for the identity component of the isometry group; note that $\Phi_\theta(x)$ can be connected to $\Phi_0(x) = I$.) of $S$. Since an isometry is uniquely determined by its differential at a single point, $\Phi_\theta$ may be viewed as an element of the isometry group $G$ of $S$.

In [**3**], the connection to the extended frames was given as the following.

THEOREM 3 [**3**]. *Let $f : M \to S = G/K$ be pluriharmonic with the (local) frame $F : M_o \to G$ and the associated family $(f_\lambda, \Phi_\lambda)$ with $\lambda = e^{-i\theta}$. Then*

$$F_\lambda = \Phi_\lambda F \tag{21}$$

*is an extended frame in the sense of* [**2**, **4**].

*Proof.* The idea of the proof is as follows. We have to show that $\alpha_\lambda = F_\lambda^{-1} dF_\lambda$ satisfies (5). We split $\alpha_\lambda = \alpha_\mathfrak{k}^\lambda + \alpha_\mathfrak{p}^\lambda$. The property $\alpha_\mathfrak{k}^\lambda = \alpha_\mathfrak{k}$ is obtained as follows. From (8) we have

$$\alpha_\lambda - \alpha = F^{-1} \beta_\lambda F \tag{22}$$

and the right-hand side of (22) takes values in $\mathfrak{p}$ due to the parallelism of $\Phi_\lambda$ (see Lemma 1 below). Moreover, $d\pi(F_\lambda \alpha_\mathfrak{p}^\lambda) = df_\lambda$, where $\pi : G \to G/K$ is the projection and $df_\lambda = \Phi_\lambda \circ df \circ \mathsf{R}_{-\theta}$. Since $T'M$ and $T''M$ are the $\pm i$ eigenspaces of $J$ and the $e^{\mp i\theta}$ eigenspaces of $\mathsf{R}_{-\theta}$, respectively, we have $df \circ \mathsf{R}_{-\theta} = \lambda^{-1} d'f + \lambda d''f$, which shows (5). □

LEMMA 1. *Let $f, \tilde{f} : M \to S = G/K$ be smooth maps and let $\Phi : f^*TS \to \tilde{f}^*TS$ be an isometric bundle isomorphism preserving $R^S$. Then $\Phi$ is parallel if and only if $F^{-1}(\Phi^{-1} d\Phi)F$ takes values in $\mathfrak{p}$ for any frame $F$ of $f$.*

*Proof.* Let $x(t)$ be any smooth curve in $M$ with $x(t_o) = x_o$ fixed. Consider the curves $c(t) = f(x(t))$ and $\tilde{c}(t) = \tilde{f}(c(t))$ in $S$. Then $\Phi$ is parallel if and only if $\Phi(t) := \Phi(x(t))$ maps parallel frames along $c$ onto parallel frames along $\tilde{c}$. Parallel frames along a curve $c$ in $S = G/K$ are the horizontal lifts $C$ of $c$, where horizontal subspaces in $TG$ are left translates of $\mathfrak{p}$. In other words, $C(t) \in G$ with $\pi \circ C = c$, where $\pi : G \to G/K$, and $C(t)^{-1}C'(t) \in \mathfrak{p}$. Likewise, $\tilde{C}(t) = \Phi(t)C(t)$ is horizontal if and only if $\tilde{C}(t)^{-1}\tilde{C}'(t) \in \mathfrak{p}$. One the other hand we have

$$\tilde{C}' = (\Phi C)' = \Phi' C + \Phi C'$$

and therefore

$$\tilde{C}^{-1}\tilde{C}' = (\Phi C)^{-1}(\Phi C)' = C^{-1}\Phi^{-1}\Phi' C + C^{-1}C'.$$

The second term on the right-hand side lies in $\mathfrak{p}$ due to the horizontality of $C$. Thus $\tilde{C}$ is horizontal if and only if $C^{-1}\Phi^{-1}\Phi' C \in \mathfrak{p}$. Choosing $C(t_o) = F(x_o)$, we have proved our claim. $\square$

REMARK 2. Recall that we are considering $\Phi(t)$ as an element of $G$. A similar identification takes place for frames: Strictly speaking, a frame $C(t)$ at $p = f(x(t))$ is a basis of $T_pS$ which arises by applying some $g \in G$ (more precisely: its differential $dg_o$) to a fixed basis $e_1, \dots, e_n$ of $T_oS$ where $o = eK \in G/K$. Usually we identify $C(t)$ with $g$. The equality $\tilde{C}(t) = \Phi(t)C(t)$ can be understood in two ways, linked by the chain rule: either both $\tilde{C}(t)$ and $C(t)$ are considered as $n$-tuples of tangent vectors, mapped onto each other by the homomorphism $\Phi(t)$, or $\tilde{C}(t), \Phi(t), C(t) \in G$ and the right hand side is a product of group elements. We are adopting the second view point.

From Theorems 1 and 3 we obtain the following theorem.

THEOREM 4. *Let $S \subset G$ be a Cartan embedded symmetric space and let $f : M \to S$ be pluriharmonic with the associated family $(f_\lambda, \Phi_\lambda)$. Then $\Phi_\lambda$ is an extended solution of $f$.*

Thus the theory of associated families $(f_\lambda, \Phi_\lambda)$ combines aspects of both theories as follows: $\Phi_\lambda$ is the extended solution and $F_\lambda = \Phi_\lambda F$ is the extended frame with $\pi \circ F_\lambda = f_\lambda$. Moreover, we have achieved a geometric interpretation of $\Phi_\lambda$ that persists even if no embedding of $S$ into $G$ is given: $\Phi_\lambda$ is the isomorphism between the bundles $f^*TS$ and $f_\lambda^*TS$ which is needed to define the associated family; cf. (20).

Note that a solution $(f_\lambda, \Phi_\lambda)$ of (20) for any single $\lambda$ is unique up to left translation with some $g_\lambda \in G$. In particular, for $\lambda = -1$ or $\theta = \pi$, we have $\mathsf{R}_\pi = -I$ and thus $(f_{-1} = f,\ \Phi_{-1} = -I)$ is a special solution with $\Phi_{-1}(x) = s_{f(x)} \in G$ (geodesic reflection at the point $f(x)$). Thus a general solution will be a left translate of this map, and hence we see immediately that $\Phi_{-1}$ is the composition of $f$ with a Cartan embedding (which follows also from Theorem 3).

## 4. *Totally geodesic submanifolds in Lie groups*

Thus far, we have linked extended solutions, extended frames and associated families only when the symmetric space $S$ is Cartan embedded in a Lie group $G$. Nevertheless, all three theories extend beyond this case. Extended solutions $\Phi_\lambda$ take values in a compact Lie group $G$ and $f = \Phi_{-1}$ may lie in any closed totally geodesic submanifold $S \subset G$ (not only Cartan embeddings), while extended frames and associated families do not make use of embeddings of $S$ at all. Thus we assume that $S \subset G$ is a general closed totally geodesic submanifold and $f : M \to S \subset G$ a pluriharmonic map. Is there still a relation between extended solutions $\Phi_\lambda$ and extended frames $F_\lambda$ of $f$? This seems unclear because $\Phi_\lambda$ and $F_\lambda$ take values in different groups as follows. While $\Phi_\lambda$ is $G$-valued, $F_\lambda$ maps into the transvection group of $S$, which will now be called $H$ (rather than $G$).

Nevertheless, there is a link between the two groups: $H$ is finitely covered by the group of the transvections of $G$ keeping $S \subset G$ invariant. The group $G \times G$ acts on $G$ by left and right translation, $(g_1, g_2)g = g_1 g g_2^{-1}$, and this action (after dividing out the ineffective diagonal of the centre) is the transvection group of $G$. In fact, by definition, a transvection on $G$ is the composition of any two point reflections $s_g, s_{\tilde{g}}$ for $g, \tilde{g} \in G$. We have $s_g(p) = gp^{-1}g$ and hence

$$s_g(s_{\tilde{g}}(p)) = g(\tilde{g}p^{-1}\tilde{g})^{-1}g = g\tilde{g}^{-1}p\tilde{g}^{-1}g \tag{23}$$

for any $p \in G$. Thus $s_g s_{\tilde{g}}$ is the action on $G$ of $(g\tilde{g}^{-1}, g^{-1}\tilde{g}) \in G \times G$.

If $S \subset G$ is a closed totally geodesic submanifold, then the transvections along $S$ are just restrictions to $S$ of the transvections $s_g s_{\tilde{g}}$ with $g, \tilde{g} \in S$. Thus the transvection group $H$ of $S$ is finitely covered by the group $\tilde{H} \subset G \times G$ generated by the set

$$\Gamma = \{(g\tilde{g}^{-1}, g^{-1}\tilde{g}); g, \tilde{g} \in S\} \subset G \times G. \tag{24}$$

The extended frames $F_\lambda$ of a pluriharmonic map $f : M \to S$ take values in $H$. They will be lifted to $\tilde{H}$ and then called $\tilde{F}_\lambda$. Thus $\tilde{F}_\lambda$ and $\tilde{\Phi}_\lambda = \tilde{F}_\lambda \tilde{F}_1^{-1}$ take values in $\tilde{H}$. Since $\tilde{\Phi}_{-1}(x)$ acts on $S$ as $s_o s_{f(x)}$ for a fixed $o \in S$, we may assume that $\tilde{\Phi}_{-1}(x) = (of(x)^{-1}, o^{-1}f(x))$ for all $x \in M$.

On the other hand, we have the Uhlenbeck extended solution $\Phi_\lambda : M \to G$ with $\Phi_{-1} = f : M \to S \subset G$. Embedding $S$ totally geodesically into $\tilde{H} \subset G \times G$ via

$$i_o : S \ni g \mapsto (og^{-1}, o^{-1}g) \in \Gamma \subset \tilde{H} \subset G \times G, \tag{25}$$

($i_o$ is a lift to $\tilde{H}$ of the pointed Cartan embedding $\iota_o : S \to H$, $p \mapsto s_o s_p$) we obtain a pluriharmonic map $i_o \circ f = (of^{-1}, o^{-1}f) : M \to G \times G$. This is a left translate in $G \times G$ of the pluriharmonic map $(f^{-1}, f)$ with the extended solution

$$\hat{\Phi}_\lambda = ((T\Phi)_\lambda, \Phi_\lambda); \tag{26}$$

cf. (3). Thus we have two extended solutions $\tilde{\Phi}_\lambda$ and $\hat{\Phi}_\lambda$ for (left translates of) $i_o \circ f$ that, by unicity, must agree up to left translations in $G \times G$. We have proved the following theorem.

Theorem 5. *Let $S \subset G$ be a totally geodesic submanifold and let $f : M \to S$ be a pluriharmonic map. Then $G$-valued extended solutions $\Phi_\lambda$ and $H$-valued extended frames $F_\lambda$ for $f$ correspond in the sense*

$$((T\Phi)_\lambda, \Phi_\lambda) = \tilde{F}_\lambda \tilde{F}_1^{-1} \tag{27}$$

*up to left translations in $G \times G$, where $\tilde{F}_\lambda$ is a lift of $F_\lambda$ to $\tilde{H} \subset G \times G$.*

Remark 3. If $S \subset G$ is standard Cartan embedded, that is, $S = (\sqrt{e})_o$, then $\Gamma = \{(s\tilde{s}, s\tilde{s});\ s, \tilde{s} \in (\sqrt{e})_o\} \subset \Delta G \subset G \times G$; see (24). On the other hand, an extended solution $\Phi_\lambda$ with $\Phi_{-1} \in \sqrt{e}$ can be chosen to be invariant under the twist $T$ defined in (3) (cf [**1**]) and hence $\hat{\Phi}_\lambda = (\Phi_\lambda, \Phi_\lambda)$. Thus we are back to Theorem 4 in this case.

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## *References*

**1.** F. E. Burstall and M. A. Guest, 'Harmonic two-spheres in compact symmetric spaces, revisited', *Math. Ann.* 309 (1997) 541–572.

**2.** F. E. Burstall and F. Pedit, 'Harmonic maps via Adler–Kostant–Symes theory', *Harmonic maps and integrable systems* (eds A. P. Fordy and J. C. Wood; Vieweg, Braunschweig, 1994).

**3.** J. Dorfmeister and J.-H. Eschenburg, 'Pluriharmonic maps, loop groups and twistor theory', *Ann. Global Anal. Geom.* 24 (2003) 301–321.

**4.** J. Dorfmeister, F. Pedit and H. Wu, 'Weierstrass type representations of harmonic maps into symmetric spaces', *Comm. Anal. Geom.* 6 (1998) 633–668.

**5.** J.-H. Eschenburg 'Pluriharmonic maps, twisted loops and twistors', *DMV-Jahresberichte* 106 (2004) 39–48.

**6.** J.-H. Eschenburg and R. Tribuzy, 'Existence and uniqueness of maps into affine homogeneous spaces', *Rend. Sem. Mat. Univ. Padova* 69 (1993) 11–18.

**7.** J.-H. Eschenburg and R. Tribuzy, 'Associated families of pluriharmonic maps and isotropy', *Manuscripta Math.* 95 (1998) 295–310.

**8.** Y. Ohnita and G. Valli, ‘Pluriharmonic maps into compact Lie groups and factorization into unitons’, *Proc. London Math. Soc.* 61 (1990) 546–570.

**9.** K. Pohlmeyer, ‘Integrable Hamiltonian systems and interactions through quadratic constraints’, *Comm. Math. Phys.* 46 (1976) 207–221.

**10.** K. Uhlenbeck, ‘Harmonic maps into Lie groups (classical solutions of the chiral model)’, *J. Differential Geom.* 30 (1989) 1–50.

**11.** V. E. Zakharov and A. V. Mikhailov, ‘Relativistically invariant two-dimensional models of field theory which are integrable by means of the inverse scattering problem method’, *Soviet Phys. JETP* 47 (1978) 1017–1027.

**12.** V. E. Zakharov and A. B. Shabat, ‘Integration of non-linear equations of mathematical physics by the method of inverse scattering II’, *Funct. Anal. Appl.* 13 (1979) 13–22.