Pluriharmonic maps into Kähler symmetric spaces and Sym's formula

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0 Introduction

An important notion for a surface in euclidean 3-space is the *Gauss map* which assigns to each point its normal vector in the sphere $S^2 \subset \mathbb{R}^3$. But can one revert this process and recover the original surface from its Gauss map? In general this is impossible; e.g. for *minimal surfaces* the Gauss map remains the same when we pass to the *associated surfaces*. However, there are surface classes where such a one-to-one correspondence exists. Among them are surfaces of prescribed nonzero constant mean curvature (cmc). By a theorem of Ruh and Vilms [18], an immersed surface $f : M \to \mathbb{R}^3$ is cmc if and only if its Gauss map is harmonic. Vice versa, given a generic harmonic map $h : M \to S$ into the 2-sphere S, there exists precisely

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one cmc surface f with Gauss map h and mean curvature $H = \frac{1}{2}$ (say). It can be constructed from h and its associated family using a famous formula of Sym [20] and Bobenko [1].

The aim of our paper is to generalize this construction to higher dimensions and codimensions. We replace the 2-sphere S by an arbitrary Kähler symmetric space P of compact type and arrive at a new class of Kähler submanifolds of $\mathbb{R}^n$, which could be called "pluri-cmc". To be more precise, we must look a little closer to the original Sym–Bobenko construction: starting with a harmonic map $h : M \to S$, one obtains two weakly conformal maps $f_\pm : M \to \mathbb{R}^3$ with $h = \frac{1}{2}(f_+ - f_-)$. Outside the branch points, f_+ and f_- have Gauss map h and mean curvature $H = -\frac{1}{2}$ and $H = \frac{1}{2}$, respectively. Now let $P = G/K$ be an arbitrary Kähler symmetric space of compact type. It can be viewed as an adjoint orbit in its transvection Lie algebra $\mathfrak{g}$ in the same way as S is an adjoint orbit in $\mathbb{R}^3 = \mathfrak{so}_3$. As before, there is a one-to-one correspondence between *pluriharmonic* maps $h : M \to P$ from a complex manifold M, and pairs of maps $f_+, f_- : M \to \mathfrak{g}$, which are *quasi-holomorphic* (a notion generalizing "weakly conformal") along the common normal vector $h = \frac{1}{2}(f_+ - f_-)$ (Theorem 7.2). At regular points the Riemannian metrics on M induced by $f_\pm$ are Kähler. Moreover, both immersions are 'pluri-cmc', i.e. when restricted to complex one-dimensional submanifolds of M they behave like cmc surfaces in a certain sense (cf. (35)); in particular they allow a very peculiar isometric deformation (associated family).

As it turned out, a modified and less explicit version of the Sym–Bobenko construction was already known to Bonnet [2] (see [12]), and, in fact, the viewpoint of Bonnet is an important tool for our generalization.

1 Parallel surfaces

Let us recall some elementary facts for surfaces in 3-space. Consider an immersion $f : M \to \mathbb{R}^3$ of a two-dimensional manifold M ('surface'). Suppose that M is oriented and that $\nu : M \to S$ is the Gauss map of f, where $S \subset \mathbb{R}^3$ denotes the unit sphere. The surface f gives rise to a family of *parallel surfaces* $f_t = f + t\nu$ for all $t \in \mathbb{R}$ (we always exclude the points where f_t is not regular, i.e. not an immersion). The surfaces f and f_t have the same principal curvature vectors on M, but the principal curvatures κ_1, κ_2 change from $\kappa_j = 1/r_j$ to $\kappa_{j,t} = 1/(r_j - t)$.

Suppose now that f has constant Gaussian curvature 1, i.e. $r_1 r_2 = 1$. Then the parallel surfaces $f_{\mp 1}$ have cmc $H = \pm\frac{1}{2}$ at their regular points:

$$2H = \frac{1}{r_1 \pm 1} + \frac{1}{r_2 \pm 1} = \frac{r_1 + r_2 \pm 2}{r_1 r_2 \pm (r_1 + r_2) + 1} = \pm 1. \tag{1}$$

Further, the metrics on M induced by f_1 and f_{-1} are conformal to each other. In fact, if $v_j \in T_u M$ (for some $u \in M$) is a principal curvature vector for κ_j with $|df.v_j| = |r_j|$, then $|df_t.v_j| = |r_j - t|$. Consequently, the length ratio of the perpendicular vectors $df_t.v_1$ and $df_t.v_2$ is the same for $t = 1$ and $t = -1$ (which proves conformality): using $r_1 r_2 = 1$, we have

$$\frac{r_1 - 1}{r_2 - 1} : \frac{r_1 + 1}{r_2 + 1} = \frac{r_1 r_2 + r_1 - r_2 - 1}{r_1 r_2 - r_1 + r_2 - 1} = -1. \tag{2}$$

Vice versa, starting with a surface $\tilde{f} : M \to \mathbb{R}^3$ of cmc $H = \frac{1}{2}$, its parallel surfaces $\tilde{f}_1$ and $\tilde{f}_2$ have constant Gaussian curvature 1 and cmc $-\frac{1}{2}$, respectively. Moreover, the metrics on M induced by $\tilde{f}$ and $\tilde{f}_2$ are conformal.

2 The Gauss map of cmc surfaces

By a theorem of Ruh and Vilms [18], surfaces of cmc are characterized by the harmonicity of their Gauss maps:

Theorem 2.1 (Ruh–Vilms) *Let M be a Riemann surface and $f : M \to \mathbb{R}^3$ a conformal immersion. Then f has cmc if and only if its Gauss map $\nu : M \to S$ is harmonic.*

Proof Let H be the mean curvature of an immersion $f : M \to \mathbb{R}^3$. For each $u \in M$ and $v \in T_u M$ we have

$$2\partial_v H = -\partial_v \operatorname{trace} d\nu = -\operatorname{trace} \nabla_v d\nu \overset{*}{=} -\operatorname{trace} \langle \nabla d\nu, df.v\rangle = \langle \Delta\nu, df.v\rangle.$$

Here, ∇ denotes the Levi–Civita connection and Δ the Laplacian for the induced metric on M. For "$\overset{*}{=}$", we use the symmetry of $\langle \nabla d\nu, df\rangle$ in all three arguments (Codazzi). Thus $\partial_v H = 0$ for all v if and only if the tangent part of $\Delta\nu$ vanishes (note that $df(T_u M) = T_{\nu(u)} S$), which is the definition of $\nu : M \to S$ being harmonic. □

Now let us consider the inverse problem: given any harmonic map $h : M \to S$ on a Riemann surface M, can we construct a cmc surface $f : M \to \mathbb{R}^3$ with $H = \pm\frac{1}{2}$ and Gauss map $\nu = h$? This question has already been solved by Bonnet in 1853 [2,12] as follows: using the results of the previous section, we know that such surfaces always come in pairs

$$f_\pm = g \pm h, \tag{3}$$

where $g : M \to \mathbb{R}^3$ has constant Gaussian curvature 1. Thus the task is to find g from h. By harmonicity, the vector Δh is normal to S, i.e. it points into the direction of h. This means $h \times \Delta h = 0$ where $\times$ denotes the vector product on $\mathbb{R}^3$. Using conformal coordinates (x, y) on M we have

$$0 = h \times (h_{xx} + h_{yy}) = (h \times h_x)_x + (h \times h_y)_y,$$

where subscripts mean partial derivatives. In other words, the $\mathbb{R}^3$ valued 1-form

$$\gamma = (h \times h_y)dx - (h \times h_x)dy \tag{4}$$

is closed,[1] $d\gamma = 0$. Hence it can be integrated, $\gamma = dg$ for some $g : M \to \mathbb{R}^3$, provided that M is simply connected. In fact, g has the desired properties (cf. [12]) as we will see below (Sect. 5, Remark 2).

Using the almost complex structures j on M and J on S (the vector product with the position vector), we may rewrite (4) as

$$\gamma = h \times dh\, j = J\, dh\, j. \tag{5}$$

Hence from (3) we obtain

$$df_\pm = dh \pm J\, dh\, j. \tag{6}$$

Theorem 2.2 (Bonnet) *Let M be a Riemann surface and $h : M \to S$ a harmonic map, then the 1-form $\gamma = J\, dh\, j$ is closed. Further, if M is simply connected, there is (up to translations) precisely one pair of weakly conformal maps $f_\pm : M \to \mathbb{R}^3$ with cmc $H = \mp\frac{1}{2}$ and Gauss map h at the regular points, and $f_\pm$ is obtained by integrating $df_\pm = dh \pm \gamma$.*

[1] Since harmonic maps are critical for a variational principle (the variation of the energy) which is invariant under the isometry group of S, this formula can also be obtained as a conservation law from the Noether theorem, see [12,17].

Remark Equation (6) looks as if f_- and f_+ were holomorphic and anti-holomorphic, respectively:

$$J\,df_\pm\,j = J\,dh\,j \pm dh = \pm df_\pm. \tag{7}$$

But remember that J is the almost complex structure on S while $f_\pm$ does not take values in S; only the tangent spaces are the same:

$$df_\pm(T_u M) \subset T_{h(u)} S \tag{8}$$

(in fact we have equality). Mappings $f_\pm$ satisfying (7) and (8) will be called *quasi-holomorphic* along h (see Sect. 7). In the present context this simply means weak conformality.

The Bonnet construction involves integrating the 1-form $\gamma = dg$. More recently it was observed by Sym [20] and Bobenko [1][2] that g has a direct geometric meaning in terms of the *associated family* and the *extended solution* of the harmonic map h. We will discuss this construction in a more general setting, using that $(\mathbb{R}^3, \times)$ is a Lie algebra (corresponding to the Lie group SO_3) and S a particular adjoint orbit which is a *Kähler symmetric space* of compact type. In fact, *any* such space allows this kind of embedding (Sect. 3 below). We will also generalize the domain M to a complex manifold of arbitrary dimension (Sect. 4).

3 Kähler symmetric spaces

A Riemannian manifold P is *Kähler* if it carries a parallel isometric almost complex structure J. From $\nabla_X(JZ) = J\nabla_X Z$ we have $R(X,Y)JZ = JR(X,Y)Z$ for all tangent vectors X, Y, Z where R denotes the curvature tensor of P. Consequently $\langle R(X,Y)JZ, JW\rangle = \langle R(X,Y)Z, W\rangle$, and from the block symmetry of R we see

$$R(X,Y) = R(JX, JY). \tag{9}$$

Thus $R(JX,Y) = R(JJX, JY) = -R(X, JY)$, and therefore J is a *derivation* of R at any point p:

$$R(JX,Y)Z + R(X,JY)Z + R(X,Y)JZ = JR(X,Y)Z.$$

Now let $P = G/K$ be *Kähler symmetric* (*hermitian symmetric*) of compact type, i.e. P is Kähler and symmetric of compact type and all the *point symmetries* (geodesic symmetries) s_p are holomorphic. Then at any point $p \in P$ the curvature tensor R is a *Lie triple product* on $T_p P$ and J_p a derivation of R. We may assume $p = eK$. Let

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p} \tag{10}$$

be the corresponding Cartan decomposition (eigenspace decomposition of $\mathrm{Ad}(s_p)$). Then we may identify $T_p P = \mathfrak{p}$. We extend $J_p : \mathfrak{p} \to \mathfrak{p}$ to a derivation $\hat{J}_p$ of the Lie algebra $\mathfrak{g}$ by putting $\hat{J}_p = 0$ on $\mathfrak{k}$. Since $\mathfrak{g}$ is semisimple, each derivation is inner. Hence we may view $\hat{J}_p \in \mathfrak{g}$ (acting on $\mathfrak{g}$ by $\mathrm{ad}(\hat{J}_p)$). The map

$$\hat{J} : P \to \mathfrak{g}, \quad p \mapsto \hat{J}_p \tag{11}$$

[2] Sym studied surfaces g with Gaussian curvature $K = -1$ which have no parallel cmc surfaces. Bobenko transferred this idea to the case $K = +1$ and to cmc surfaces.

is called the *standard embedding* of P (see [10]). Its image $\tilde{P} = \hat{J}(P) \subset \mathfrak{g}$ is an adjoint orbit: since J is parallel, J_p and J_q are conjugate for an arbitrary $q \in P$ under the transvection g along a geodesic joining $p = eK$ to q. Hence $\hat{J}_q = \mathrm{Ad}(g)\hat{J}_p$. By holomorphicity each $k \in K = G_p$ preserves J_p, thus $\hat{J}_p$ centralizes K and the map $\hat{J} : P \to \mathrm{Ad}(G)\hat{J}_p$ is an equivariant covering (note that the stabilizer Lie algebra of $\hat{J}_p$ is $\mathfrak{k}$). But in fact it is injective. To see this, note that the orbit $\tilde{P} = \mathrm{Ad}(G)\hat{J}_p \subset \mathfrak{g}$ is itself an (extrinsic) hermitian symmetric space with (extrinsic) symmetry $s_p = \mathrm{Ad}(\exp \pi \hat{J}_p)$ and almost complex structure $\mathrm{ad}(\hat{J}_p)|_{T_{\tilde{p}}\tilde{P}}$ where $\tilde{p} = \hat{J}_p$. Since any semisimple hermitian symmetric space is simply connected [13, p. 376], the map $\hat{J}$ is one-to-one. The Riemannian metric on $\tilde{P}$ induced by any $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$ coincides up to a constant with the initial Riemannian metric on each de Rham factor. The tangent and normal spaces of $\tilde{P}$ at $\tilde{p} = \hat{J}_p$ are

$$T_{\tilde{p}}\tilde{P} = \mathrm{ad}(\mathfrak{g})\hat{J}_p = [\mathfrak{p}, \hat{J}_p] = -J_p(\mathfrak{p}) = \mathfrak{p}, \quad N_{\tilde{p}}\tilde{P} = \mathfrak{p}^{\perp} = \mathfrak{k}, \tag{12}$$

thus (10) is also the decomposition into the tangent and normal space of $\tilde{P}$ at $\hat{J}_p$. From now on, we will no longer distinguish between P and $\tilde{P}$. Hence we consider P as a submanifold of $\mathfrak{g}$ where the point $p \in P$ becomes the element $p = \hat{J}_p \in \mathfrak{g}$.

Example 1 Let $P = S \subset \mathbb{R}^3$ be the 2-sphere. For any $p \in S$ we have $T_pS = p^{\perp}$ and $J_p v = p \times v$ for $v \in T_pS$. Let $\mathfrak{so}_3$ be the space of real anti-symmetric 3×3-matrices (the Lie algebra of SO_3). The mapping $\mathbb{R}^3 \to \mathfrak{so}_3 : w \mapsto A_w$ with $A_w x := w \times x$ is a linear isomorphism which transforms the vector product into the Lie product and the usual SO_3-action on $\mathbb{R}^3$ into the adjoint action on $\mathfrak{so}_3$. Thus the sphere $S \subset \mathbb{R}^3$, which is the SO_3-orbit of e_3, is mapped onto the adjoint orbit of $A_{e_3} = \hat{J}_{e_3}$.

Example 2 Let $P = G_k(\mathbb{C}^n) = U_n/(U_k \times U_{n-k})$ be the complex Grassmannian of k-dimensional linear subspaces of $\mathbb{C}^n$. Identifying each complex subspace with its orthogonal projection, we embed P as a U_n-conjugacy class into the space of hermitian or (after multiplying with $i = \sqrt{-1}$) anti-hermitian $n \times n$-matrices which form the Lie algebra $\mathfrak{u}_n$ of the unitary group U_n; this is the standard embedding.

4 Pluriharmonic maps

Let $P = G/K$ be a semisimple symmetric space and M a simply connected complex manifold with almost complex structure j. We will also use the corresponding rotations

$$r_\theta = (\cos\theta)I + (\sin\theta)j : TM \to TM \tag{13}$$

for any $\theta \in [0, 2\pi]$. A smooth map $h : M \to P$ is called *pluriharmonic* if $h|_C$ is harmonic for any complex one-dimensional submanifold (complex curve) $C \subset M$, or, in other terms, if the (1,1) part of the Hessian $\nabla dh^{(1,1)}$, the so called *Levi form*, vanishes:

$$\nabla dh(v, w) + \nabla dh(jv, jw) = 0 \tag{14}$$

for any two tangent vectors v, w on M.[3]

Pluriharmonic maps always come in one-parameter families, called *associated families*, defined as follows (cf. [4,9]): the differential of a smooth map $f : M \to P$ is a vector bundle

[3] In order to define the Hessian one has to choose locally a Kähler metric on M. However, the definition of pluriharmonicity is independent of the choice of this metric.

homomorphism $\varphi = df : TM \to E = f^*TP$. Vice versa, given any vector bundle E (over M) endowed with a connection and a bundle homomorphism $\varphi : TM \to E$, we may ask if φ is the differential of a smooth map f; such a homomorphism (or E valued 1-form) φ will be called *integrable*. If this holds, E can be identified with f^*TP and, in particular, E carries a parallel Lie triple product on its fibres. Assuming that E is already equipped with such a structure, one obtains the following precise integrability condition for φ (see [8]): there exists a map $f : M \to P$ and a parallel vector bundle isometry $\Phi : f^*TP \to E$ preserving the Lie triple structure such that

$$\varphi = \Phi\, df. \tag{15}$$

Both f and Φ are unique up to translation with some $g \in G$.

Now assume that a smooth map $h : M \to P$ is given, thus $\varphi_0 = dh$ is integrable. We may ask if the rotated differential $\varphi_\theta = dh\, r_\theta$ is integrable for all $\theta \in [0, 2\pi]$ as well. This question was answered in [9]: the integrability condition holds for all φ_θ if and only if h is pluriharmonic. In this case we have a family of pluriharmonic maps $h_\theta : M \to P$ (the *associated family* of h) and parallel bundle isometries $\Phi_\theta : f^*TP \to f_\theta^*TP$ preserving the curvature tensor (Lie triple product) of P such that

$$dh_\theta = \Phi_\theta\, dh\, r_\theta \tag{16}$$

holds for all $\theta \in [0, 2\pi]$. We can always assume $\Phi_0 = I$, and, if P is an *inner* symmetric space (which means that $-I$ lies in the identity component of K acting on $\mathfrak{p}$), we may choose additionally $\Phi_\pi = -I$, due to $r_\pi = -I$ (see [4]). Since $\Phi_\theta(u)$ maps $T_{f(u)}P$ onto $T_{f_\theta(u)}P$ preserving the metric and the curvature tensor, it is the differential of a unique element of G mapping $f(u)$ to $f_\theta(u)$. This will be called $\Phi_\theta(u)$ again and it defines a family of mappings $\Phi_\theta : M \to G$ with $\Phi_0 = e$ and, if P is inner, $\Phi_\pi(u) = s_{h(u)}$, where $s_q \in G$ denotes the point symmetry at q for any $q \in P$.

Remark Pluriharmonic maps have often been described in terms of moving frames. If we choose (locally) a frame F for h (i.e. a smooth map $F : M_o \to G$ with $F(u)p = h(u)$ for any $u \in M_o \subset M$, where $p = eK \in P = G/K$), we obtain also a frame for each h_θ, namely

$$F_\theta = \Phi_\theta F. \tag{17}$$

Then the corresponding Maurer–Cartan form[4] $\omega_\theta = F_\theta^{-1} dF_\theta \in \Omega^1(M, \mathfrak{g})$ satisfies

$$\omega_\theta = \omega_\mathfrak{k} + \omega_\mathfrak{p}\, r_\theta = \omega_\mathfrak{k} + \lambda^{-1}\omega'_\mathfrak{p} + \lambda\omega''_p \tag{18}$$

due to (16) and the parallelism of Φ_θ (see [4]). Here we put $\lambda = e^{-i\theta}$, and $\omega_\mathfrak{k}$, $\omega_\mathfrak{p}$ are the components of $\omega = \omega_0 = F^{-1}dF$ in the Cartan decomposition (10), while $\omega'_\mathfrak{p}$, $\omega''_\mathfrak{p}$ are the restrictions of the (complexified) 1-form $\omega_\mathfrak{p} : TM \otimes \mathbb{C} \to \mathfrak{p} \otimes \mathbb{C}$ to

$$T'M = \{v - ijv;\ v \in TM\}, \quad T''M = \{v + ijv;\ v \in TM\}, \tag{19}$$

the $(\pm i)$-eigenbundles of j. As a consequence of (17) and (18) we obtain

$$\begin{aligned} \Phi_\theta^{-1} d\Phi_\theta &= \mathrm{Ad}(F)(\omega - \omega\, r_\theta) \\ &= (1 - \lambda^{-1})\, \mathrm{Ad}(F)\omega'_\mathfrak{p} + (1 - \lambda)\, \mathrm{Ad}(F)\omega''_\mathfrak{p}. \end{aligned} \tag{20}$$

This shows that Φ_θ is an *extended solution* in the sense of Uhlenbeck [22], generalized to the pluriharmonic case by Ohnita and Valli [15].

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[4] To keep the notation simple we assume that G is a matrix group.

One may show that $\mathrm{Ad}(F)\omega_{\mathfrak{p}} = \frac{1}{2} s_h ds_h$ where $s : P \to G$, $p \mapsto s_p$ is the Cartan embedding and $s_h = s \circ h$.

5 The Kähler symmetric case

Let us restrict our attention to a *Kähler* symmetric space $P = G/K$ of compact type. Using the standard embedding we consider P as an adjoint orbit in $\mathfrak{g}$. Then the almost complex structure J_p at any $p \in P \subset \mathfrak{g}$ is just $\mathrm{ad}(p)$, restricted to the tangent space $T_pP = \mathrm{ad}(\mathfrak{g})p \subset \mathfrak{g}$.

Now we deal with two almost complex structures: j on M and J on P. Recall that the definition of a pluriharmonic map $h : M \to P$ involves only j, not J (which is not present in the general case). However, for Kähler symmetric spaces we have another characterization of pluriharmonic maps in terms of both j and J which generalizes the first part of Bonnet's Theorem 2.2:

Theorem 5.1 *Let $P \subset \mathfrak{g}$ be a Kähler symmetric space of compact type, M a complex manifold and $h : M \to P$ a smooth map. Then h is pluriharmonic if and only if the $\mathfrak{g}$ valued 1-form $\gamma = J\,dh\,j = [h, dh\,j]$ is closed.*

Proof We have $d\gamma(v, w) = \partial_v\gamma(w) - \partial_w\gamma(v) - \gamma(\nabla_v w - \nabla_w v)$ and

$$\begin{aligned}\partial_v\gamma(w) &= \partial_v[h, \partial_{jw}h] \\ &= [\partial_v h, \partial_{jw}h] + [h, \partial_v\partial_{jw}h].\end{aligned} \tag{21}$$

Thus we obtain

$$\begin{aligned}d\gamma(v, w) = {} & [dh.v, dh.jw] - [dh.w, dh.jv] \\ & + [h, \nabla dh(v, jw) - \nabla dh(w, jv)],\end{aligned} \tag{22}$$

where h is considered as a map into the ambient space $\mathfrak{g}$ rather than into P. The normal and tangent spaces of P at the point $h = h(u) \in P$ (which we may consider as the base point $p = eK$) form the Cartan decomposition (10). Since the kernel of $\mathrm{ad}(h)$ is $\mathfrak{k}$, the term in the second line of (22) is in $\mathfrak{p}$ while the two terms in the first line belong to $\mathfrak{k}$, due to $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. Thus we have $d\gamma = 0$ if and only if

$$[dh.v, dh.jw] - [dh.w, dh.jv] = 0, \tag{23}$$

$$(\nabla dh(v, jw) - \nabla dh(w, jv))^T = 0, \tag{24}$$

where $(\,)^T$ denotes the component in T_hP. The second Eq. (24) says precisely that $h : M \to P$ is pluriharmonic. The first one, (23), is a consequence of the pluriharmonicity whenever P is a compact symmetric space: if $h : M \to P$ is pluriharmonic, we have $R(dh.a, dh.b) = 0$ for all $a, b \in T'M$ (see [9,15]). For $a = v - ijv$ and $b = w - ijw$ this gives (23); recall that the Lie bracket on $\mathfrak{p}$ is the curvature operator of P (up to sign). □

Remark 1 All arguments can be generalized to metrics of arbitrary signature (see [14,19]). However, in the indefinite case we can no more conclude $R(dh(T'M), dh(T'M)) = 0$ from the pluriharmonicity of $h : M \to P$. However, this extra condition is extremely useful; e.g. it is necessary for an associated family to exist. It was an additional assumption in [19] (called S^1*-pluriharmonicity*). Maybe the closedness of the form $J\,dh\,j$ would be the better definition.

Remark 2 If M is simply connected, we can integrate γ and find a smooth mapping $g : M \to \mathfrak{g}$ with $dg = \gamma = J\,dh\,j$. Using (21) we compute its Hessian

$$\nabla dg(v, w) = [dh.v, dh.jw] + [h, \nabla dh(v, jw)]. \tag{25}$$

In the Bonnet case (dim $M = 2$, $P = S$), the map g at regular points is the surface with Gaussian curvature $K = 1$, see Sect. 1 and [12]. This is not completely obvious since g is not isometric, not even conformal. The second fundamental form α^g of g (assuming that g is an immersion) is the normal part of its Hessian (25). In the surface case, there is no normal part inside TS, thus we get (omitting the symbol 'dh')

$$\alpha^g(v, w) = [v, jw] = v \times jw. \tag{26}$$

Hence $\alpha^g(v, jv) = 0$ and $\alpha^g(v, v) = [v, jv] = \alpha^g(jv, jv)$ and further

$$\begin{aligned} \langle \alpha^g(v, v), \alpha^g(jv, jv)\rangle - |\alpha^g(v, jw)|^2 &= \langle [v, jv], [v, jv]\rangle \\ &= \langle [[v, jv]v], jv\rangle \\ &= -\langle R(v, jv)v, jv\rangle \\ &= |v|^2|jv|^2 - \langle v, jv\rangle^2. \end{aligned} \tag{27}$$

Comparing with the Gauss equations for the surface g in $\mathbb{R}^3$ we see that g has Gaussian curvature $K = 1$.

Remark 3 The case where M is a surface and $P = \mathbb{C}P^n = G_1(\mathbb{C}^{n+1}) \subset \mathfrak{su}_{n+1}$ was recently considered in [11].

6 Extending Sym's construction

For any pluriharmonic map $h : M \to P = G/K$ and its associated family (h_θ, Φ_θ) with framing $F_\theta = \Phi_\theta F$ we define the *Sym map* (putting $\delta = \frac{\partial}{\partial\theta}|_{\theta=0}$ and using $\Phi_0 = I$)

$$k := (\delta F)F^{-1} = (\delta\Phi)\Phi_0^{-1} = \delta\Phi : M \to \mathfrak{g}. \tag{28}$$

This was introduced by Sym [20] in the case $P = S$. It is of particular importance in the Kähler symmetric case where P is an adjoint orbit in the Lie algebra $\mathfrak{g}$. Thus the group G acts on $P \subset \mathfrak{g}$ by the adjoint representation, and the defining Eq. (16) for the associated family now becomes

$$dh_\theta = \mathrm{Ad}(\Phi_\theta)\,dh\,r_\theta. \tag{29}$$

On the other hand, the isometry $\Phi_\theta(u)$ also maps $h(u)$ onto $h_\theta(u)$:

$$h_\theta = \mathrm{Ad}(\Phi_\theta)h. \tag{30}$$

Differentiating this last equation,

$$dh_\theta = \mathrm{ad}(d\Phi_\theta)h + \mathrm{Ad}(\Phi_\theta)dh,$$

and comparing with (29) we obtain

$$\mathrm{Ad}(\Phi_\theta)\,dh(r_\theta - I) = [d\Phi_\theta, h]. \tag{31}$$

Now we differentiate once more, this time with respect to θ at $\theta = 0$, using $\Phi_0 = e$, $\delta\Phi_\theta = k$ and $r_0 = I$, $\delta r_\theta = j$:

$$dh\, j = [\delta d\Phi_\theta, h] = -J_h\, dk,$$

where $J_h = \mathrm{ad}(h)$ is the complex structure on $T_h P$. Summing up we get:

Theorem 6.1 *The Sym map $k = \delta\Phi$ integrates the Bonnet form γ:*

$$dk = J\, dh\, j = \gamma. \tag{32}$$

Thus we have seen that the Sym map k is (up to a translation) nothing else than the Bonnet map g (we will call it *Bonnet–Sym–Bobenko map*).

7 Generalizing cmc surfaces

As we saw in the first section, cmc surfaces in 3-space always come in pairs $f_\pm$ where $\nu = \frac{1}{2}(f_+ - f_-)$ is the Gauss map. More precisely, cmc surfaces with $|H| = \frac{1}{2}$ can be characterized as pairs of immersions $f_\pm : M \to \mathbb{R}^3$, defined on a Riemann surface M, being conformal ('quasi-holomorphic') and having common harmonic Gauss map $h = \frac{1}{2}(f_+ - f_-)$. If M is simply connected, there is an explicit one-to-one correspondence between harmonic maps $h : M \to S$ and cmc surfaces (f_+, f_-); the reverse correspondence $h \rightsquigarrow (f_+, f_-)$ is given by the Bonnet–Sym–Bobenko construction (see Theorem 2.2). In this form, cmc surfaces can be generalized to higher dimension and codimension.

First we have to give a precise definition of quasi-holomorphicity. Let $P \subset \mathbb{R}^n$ be a submanifold whose induced metric is Kähler. Further, let M be any complex manifold and $h : M \to P$ a smooth map. Let j and J denote the almost complex structures on M and P. Then J induces a complex structure J_h on the fibres of h^*TP, i.e. $J_{h(u)}$ acts on $T_{h(u)}P$ for any $u \in M$. A smooth map $f : M \to \mathbb{R}^n$ is called $(\mp)$*quasi-holomorphic along h* if

(1) $df(T_u M) \subset dh(T_u M)$ for any $u \in M$,
(2) $J_h\, df\, j = \pm df$.

Lemma 7.1 *If $f : M \to P$ is quasi-holomorphic along h, then f is a Kähler immersion on its regular set $M_{\mathrm{reg}} = \{u \in M;\ df_u$ injective$\}$, i.e. j is an isometric parallel almost complex structure for the induced metric on M_{reg}.*

Proof J_h is isometric and parallel in the bundle h^*TP which contains $df(TM)$, and df intertwines j and $\mp J_h$. □

Theorem 7.2 *Let $P = G/K$ be a Kähler symmetric space of compact type with its standard embedding $P \subset \mathfrak{g}$ and let M be a simply connected complex manifold. Then there is a one-to-one correspondence (up to translations) between pluriharmonic maps $h : M \to P$ with its associated family (h_θ, Φ_θ) on the one side and on the other side pairs of maps $f_\pm : M \to \mathfrak{g}$ with common pluriharmonic normal $h = \frac{1}{2}(f_+ - f_-) : M \to P$ such that $f_\pm$ is $\mp$-quasi-holomorphic along h. The reverse correspondence $h \rightsquigarrow (f_+, f_-)$ is given by*

$$f_\pm = g \pm h, \tag{33}$$

using the Bonnet–Sym–Bobenko map $g = \delta\Phi : M \to \mathfrak{g}$.

Proof Starting with a pluriharmonic map $h : M \to P$, we only have to show that the mappings $f_\pm$ defined by (33) are quasi-holomorphic and $df_\pm(TM) \perp h$. But note that

$$df_\pm = dg \pm dh = J\, dh\, j \pm dh,$$

and hence $J\, df_\pm\, j = -dh \pm J\, dh\, j = \pm df_\pm$. Further, $\partial_v h \perp h$ (any adjoint orbit lies in a sphere and is therefore perpendicular to the position vector) and $J_h \partial_{jv} h = [h, \partial_{jv} h] \perp h$, thus $\partial_v f_\pm \perp h$.

Vice versa, starting with a quasi-holomorphic pair of maps (f_+, f_-) such that $h = \frac{1}{2}(f_+ - f_-)$ is pluriharmonic and normal to both f_+, f_-, we have to show that $g = \frac{1}{2}(f_+ + f_-)$ is the Bonnet–Sym–Bobenko map. This follows from the quasi-holomorphicity:

$$J\, dg\, j = \frac{1}{2}(J\, df_+\, j + J\, df_-\, j) = \frac{1}{2}(df_+ - df_-) = dh,$$

and therefore $dg = J\, dh\, j = \gamma$. □

Our last theorem summarizes the properties of these mappings.

Theorem 7.3 *Let $P \subset \mathfrak{g}$ be Kähler symmetric, M a simply connected complex manifold and $h : M \to P$ a pluriharmonic map. Let (f_+, f_-) be the quasi-holomorphic pair along h defined in Theorem (7.2). Suppose that $f = f_+$ is an immersion. Then we have:*

(1) *f is a Kähler immersion with second fundamental form*

$$\alpha(v, w) = [dh.v, df.jw] + J_h(\nabla^P_v dh).jw + (\nabla^P_v dh).w, \tag{34}$$

where $J_h = \mathrm{ad}(h)$ and $\nabla^P dh$ is the Hessian of $h : M \to P$.

(2) *For each $v \in TM$ we have*

$$\alpha(v, v) + \alpha(jv, jv) = [J_h df.v, df.v] = \alpha^P_h(df.v, df.v), \tag{35}$$

where α^P_h denotes the second fundamental form of $P \subset \mathfrak{g}$ at $h \in P$.

(3) *Fixing a point $u \in M$ we denote by $\mathfrak{p} = T_{h(u)}P$ and $\mathfrak{k} = N_{h(u)}P$ the tangent and normal spaces of $P \subset \mathfrak{g}$ at $h(u)$. Then the corresponding components of α at u satisfy*

$$\alpha^{(1,1)}_{\mathfrak{p}} = 0, \tag{36}$$

$$\alpha^{(2,0)}_{\mathfrak{k}} = (h^*\alpha^P)^{(2,0)} = [J_h\, dh, dh]^{(2,0)}, \tag{37}$$

where $\alpha^{(1,1)}$ and $\alpha^{(2,0)}$ are the restrictions of α (after complexification) to $T'M \otimes T''M$ and $T'M \otimes T'M$, respectively.

(4) *The associated family h_θ of h leads to a one-parameter family $f_\theta : M \to \mathfrak{g}$ of isometric immersions with*

$$df_\theta = \mathrm{Ad}(\Phi_\theta) df\, r_\theta, \tag{38}$$

and the second fundamental form α_θ of f_θ satisfies

$$\alpha_{\theta,\mathfrak{p}}(v, w) = \mathrm{Ad}(\Phi_\theta)\alpha_{\mathfrak{p}}(v, r_\theta w) \tag{39}$$

$$\alpha_{\theta,\mathfrak{k}}(v, w) = \mathrm{Ad}(\Phi_\theta)\alpha_{\mathfrak{k}}(r_\theta v, r_\theta w). \tag{40}$$

Proof (1) By Lemma 7.1 f is a Kähler immersion. We equip M with the induced (Kähler) metric. Then f is an isometric immersion and α is just its Hessian, $\alpha = \nabla df = \nabla dg + \nabla dh$. Form (25) we obtain

$$\alpha(v, w) = [dh.v, dh.jw] + [h, (\nabla_v dh).jw] + (\nabla_v dh).w. \tag{41}$$

The middle term $[h, \nabla dh(v, jw)]$ of (41) can be replaced by $J_h \nabla^P dh(v, jw)$ where $\nabla^P dh$ is the $\mathfrak{p}$-projection of ∇dh (i.e. the Hessian of $h : M \to P$) since $\mathrm{ad}(h) = \mathrm{ad}(\hat{J}_h)$ vanishes on $\mathfrak{k}$ and acts as $J = J_h$ on $\mathfrak{p}$ (see Sect. 3). The last term $\nabla dh(v, w)$ splits into its $\mathfrak{p}$ and $\mathfrak{k}$ components where the $\mathfrak{k}$-component is given by the second fundamental form α^P of $P \subset \mathfrak{g}$ which is $\alpha^P(X, Y) = [JX, Y]$ for all $X, Y \in \mathfrak{p}$.[5] Thus we obtain

$$\alpha(v, w) = [dh.v, dh.jw] + [Jdh.v, dh.w] + [h, (\nabla^P_v dh).jw] + (\nabla^P_v dh).w.$$

For the second term on the right hand side we have

$$[Jdh.v, dh.w] = -[dh.v, Jdh.w] = [dh.v, Jdh.jjw] = [dh.v, dg.jw],$$

and combining this with the first term we obtain (34).

(2) The right hand side of (34) is already decomposed into its components with respect to $\mathfrak{k}$ and $\mathfrak{p}$ (note that $df(T_u M) \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$), and (36) follows from (23). To prove (37) note that $\alpha = \nabla dh + \nabla dg$, and

$$\nabla dg = \nabla[h, dh\, j] = [dh, dh\, j] + [h, \nabla dh\, j].$$

The $\mathfrak{k}$-component of the second term $[h, \nabla dh\, j]$ vanishes since $\mathrm{ad}(h) = \mathrm{ad}(\hat{J}_h)$ takes values in $\mathfrak{p}$. The first term $[dh, dh\, j]$ is anti-symmetric on $T'M \otimes T'M$ (where j is just a scalar factor i), but $\nabla dg_{\mathfrak{k}}^{(2,0)}$ is symmetric, so it must be zero. We are left with the $(2, 0)$ component of $(\nabla dh)_{\mathfrak{k}} = \alpha^P(dh, dh)$ (mind that $\mathfrak{k}$ is the normal space of $P \subset \mathfrak{g}$ at $p = h(u)$).

(3) In order to prove (35), we only have to consider the $\mathfrak{k}$-part of (34) since the expression $\alpha(v, v) + \alpha(jv, jv)$ belongs to the $(1, 1)$-part of α whose $\mathfrak{p}$-component vanishes by (36). We have

$$\alpha(v, v) + \alpha(jv, jv) = [dh.v, df.jv] + [dh.jv, df.jjv],$$

and since $df\, j = -J\, df$ (due to the quasi-holomorphicity of f), the second term is

$$[dh.jv, df.jjv] = -[dh.jv, J\, df.jv] = [J\, dh.jv, df.jv] = [dg.v, df.jv].$$

Thus the two terms add up to $[df.v, df.jv] = -[df.v, J\, df.v] = [J\, df.v, df.v]$ which proves (35).

(4) Each pluriharmonic map h_θ associated with h gives a Bonnet–Sym–Bobenko map g_θ with

$$\begin{aligned} dg_\theta &= J_{h_\theta} dh_\theta\, j \\ &= J_{h_\theta}\, \mathrm{Ad}(\Phi_\theta) dh\, r_\theta\, j \\ &= \mathrm{Ad}(\Phi_\theta) J_h dh\, j\, r_\theta \\ &= \mathrm{Ad}(\Phi_\theta) dg\, r_\theta. \end{aligned} \tag{42}$$

But we also have

$$dh_\theta = \mathrm{Ad}(\Phi_\theta) dh\, r_\theta, \tag{43}$$

(see (29)), and therefore we obtain (38) from $df_\theta = dg_\theta + dh_\theta$. Since $\mathrm{Ad}(\Phi_\theta)$ is an isometry of $\mathfrak{g}$ and r_θ is an isometry for the Kähler metric on M induced by f, the immersions f_θ are

[5] We have $\langle \alpha^P(X, Y), \xi \rangle = \langle \partial_X Y, \xi \rangle = -\langle Y, \partial_X \xi \rangle$ for any $\xi \in \mathfrak{k}$. The vector $X \in T_p P$ can be expressed by the action of a one-parameter group $g_t = \exp t\hat{X}$ for some $\hat{X} \in \mathfrak{p}$, more precisely, $X = \frac{d}{dt}|_{t=0} \mathrm{Ad}(g_t) p = [\hat{X}, p] = -J\hat{X}$. Hence $\hat{X} = JX$. Now $\partial_X \xi = \frac{d}{dt}|_{t=0} \mathrm{Ad}(g_t)\xi = [\hat{X}, \xi] = [JX, \xi]$, and $\langle \alpha^P(X, Y), \xi \rangle = -\langle Y, [JX, \xi] \rangle = -\langle [Y, JX], \xi \rangle$.

isometric. From the $\mathfrak{k}$-part of (34) we get (replacing dh with dh_θ and using (43),

$$\begin{aligned}\alpha_{\theta,\mathfrak{k}}(v,w) &= [\mathrm{Ad}(\Phi_\theta)dh.r_\theta v, Ad(\Phi_\theta)df.r_\theta jw]\\ &= \mathrm{Ad}(\Phi_\theta)[dh.r_\theta v, df.jr_\theta w]\\ &= \mathrm{Ad}(\Phi_\theta)\alpha(r_\theta v, r_\theta w)\end{aligned}$$

which proves (40). Finally, (39) can be concluded from the $\mathfrak{p}$-part of (34) observing

$$\nabla^P_v dh_\theta = \nabla^P_v(\Phi_\theta\, dh\, r_\theta) = \Phi_\theta(\nabla^P_v dh)r_\theta,$$

which holds because r_θ and Φ_θ (viewed as a homomorphism $h^*TP \to h^*_\theta TP$) are parallel. □

Concluding remarks

(1) Equation (35) is the generalization of the cmc property $H = -\frac{1}{2}$: it says that for any complex one-dimensional submanifold (complex curve) $C \subset M$, the mean curvature vector of the surface $f|_C$ in $\mathfrak{g}$ is given by the second fundamental form of P along $h|_C$. If M is itself a surface and $P = S^2$ with the position vector as unit normal, then $\langle\alpha(v,v) + \alpha(jv,jv), h\rangle = -\langle df.v, df.v\rangle$ and hence f has cmc $H = -\frac{1}{2}$. Due to (35), we would like to call the immersion f 'pluri-cmc' although in general the mean curvature vector is not constant (not even of constant length) along $f|_C$.

(2) If h is *isotropic pluriharmonic* (see [9]), i.e. h admits a trivial associated family $h_\theta = h$, the maps $f_\pm$ are twistor lifts of other isotropic pluriharmonic maps, see [16]. If h is even holomorphic (which is stronger), then $f_+ = 0$ and $f_- = 2h$.

(3) All three maps $e = f, g, h$ have *associated families* e_θ formed in the same way:

$$de_\theta = \mathrm{Ad}(\Phi_\theta)de\, r_\theta \tag{44}$$

Geometrically this means that the tangent space $de_u(T_uM)$ which is a subspace of the J-closure of $dh_u(T_uM)$ (i.e. the smallest complex subspace of $T_{h(u)}P$ containing $dh_u(T_uM)$) is moved in a parallel way for all three cases, using the same automorphism $\mathrm{Ad}(\Phi_\theta(u))$.

(4) There is an important difference between the case of cmc surfaces in 3-space and the higher dimensional analogues: if $f : M \to P$ is pluriharmonic but not (anti)-holomorphic, the dimension of M is strictly smaller than the one of P, with the only exception $P = S^2$. In fact, the flatness of $dh(T'M) \subset h^*TP \otimes \mathbb{C}$ determines a dimension bound, see [7,21]. This difference is reflected in the appearance of $\alpha_\mathfrak{p}$ which does not occur in the cmc case.

(5) There is yet another notion generalizing cmc surfaces, the so called *ppmc* submanifolds, see [3]. These are Kähler submanifolds $M \subset \mathbb{R}^n$ with parallel $\alpha^{(1,1)}$, and they are characterized by the pluriharmonicity of their Gauss map. Our present generalization is different: note that the pluriharmonic map $h : M \to P$ is not the (Grassmann-valued) Gauss map of $f_\pm$ but just one distinguished unit normal vector of $f_\pm$. This is the usual Gauss map only for surfaces in 3-space ($P = S^2$). A flaw of the ppmc notion is the difficulty of finding interesting examples, see also [5,6]. In contrast, the Bonnet–Sym–Bobenko construction gives many nontrivial examples of 'pluri-cmc' submanifolds.

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