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SQUARE ROOTS OF HAMILTONIAN DIFFEOMORPHISMS

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Abstract. In this article we prove that on any closed symplectic manifold there exists an arbitrarily C^∞-small Hamiltonian diffeomorphism not admitting a square root.

1. Introduction

Let (M,ω) be a closed symplectic manifold, i.e. $\omega \in \Omega^2(M)$ is a non-degenerate, closed 2-form. To a function $L : S^1 \times M \to \mathbb{R}$ we associate the Hamiltonian vector field X_L by setting

$$\omega(X_{L_t}, \cdot) = -dL_t(\cdot) \tag{1}$$

where $L_t(x) := L(t,x)$. The flow $\phi_L^t : M \to M$ of the vector field X_{L_t} is called a Hamiltonian flow. For simplicity we abbreviate

$$\phi_L = \phi_L^1\ . \tag{2}$$

The Hamiltonian diffeomorphisms form the Lie group $\mathrm{Ham}(M,\omega)$ with Lie algebra being the smooth functions modulo constants. We refer the reader to the book [MS98] for the basics in symplectic geometry.

In this article we prove the following Theorem.

Theorem 1. *In any C^∞-neighborhood of the identity in $\mathrm{Ham}(M,\omega)$ there exists a Hamiltonian diffeomorphism ϕ which has no square root, i.e. for all Hamiltonian diffeomorphism ψ (not necessarily close to the identity)*

$$\psi^2 \neq \phi \tag{3}$$

holds.

An immediate corollary of Theorem 1 is the following.

Corollary 2. *The exponential map*

$$\begin{aligned} \mathrm{Exp} : C^\infty(M,\mathbb{R})/\mathbb{R} &\to \mathrm{Ham}(M,\omega) \\ [L] &\mapsto \phi_L \end{aligned} \tag{4}$$

is not a local diffeomorphism.

In the proof of the Theorem we use the following beautiful observation by Milnor [Mil84, Warning 1.6]. Milnor observed that an obstruction to the existence of a square root is an odd number of $2k$-cycles, see next section for details. The main work in this article is to construct an example in the symplectic category.

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2. Milnor's observation

We define

$$CM^k := M^k/(\mathbb{Z}/k) \tag{5}$$

where $\mathbb{Z}/k$ acts by cyclic shifts on M^k. We write elements of CM^k as

$$[x_1,\dots,x_k] \in CM^k\ . \tag{6}$$

The space of k-cycles of a diffeomorphism $\phi : M \to M$ is

$$\mathscr{C}^k(\phi) := \big\{[x_1,\dots,x_k] \in CM^k \mid \phi^j(x_i) \neq x_i\, \forall j = 1,\dots,k-1,\ \phi(x_i) = x_{i+1}\big\}\ . \tag{7}$$

We point out that if $[x_1,\dots,x_k] \in \mathscr{C}^k(\phi)$ then $\phi^k(x_i) = x_i$ for $i = 1,\dots,k$.

Proposition 3 (Milnor [Mil84]). *If $\phi = \psi^2$ then $\mathscr{C}^{2k}(\phi)$ admits a free $\mathbb{Z}/2$-action. In particular, $\#\mathscr{C}^{2k}(\phi)$ is even if $\mathscr{C}^{2k}(\phi)$ is a finite set.*

For the convenience of the reader we include a proof of Milnor's ingenious observation.

Proof. We define

$$\begin{aligned} I : \mathscr{C}^{2k}(\phi) &\to \mathscr{C}^{2k}(\phi) \\ [x_1,\dots,x_{2k}] &\mapsto [\psi(x_1),\dots,\psi(x_{2k})]\ . \end{aligned} \tag{8}$$

Since $\psi\circ\phi = \phi\circ\psi$ and $\psi^2 = \phi$ the map I is well-defined and an involution. We assume by contradiction that $[x_1,\dots,x_{2k}]$ is a fixed point of I, i.e. there exists $0 \leq r \leq 2k-1$

$$\psi(x_i) = x_{i+r} \tag{9}$$

where we read indices $\mathbb{Z}/2k$-cyclically. Using $x_{i+r} = \phi^r(x_i)$ we get

$$\psi(x_i) = \phi^r(x_i) = \psi^{2r}(x_i) \tag{10}$$

and thus

$$\psi^{2r-1}(x_i) = x_i\ . \tag{11}$$

In particular,

$$x_i = \psi^{2r-1}(x_i) = \psi^{2r-1}(\psi^{2r-1}(x_i)) = \psi^{4r-2}(x_i) = \phi^{2r-1}(x_i)\ . \tag{12}$$

In summary we have

$$x_i = \phi^{2r-1}(x_i) \quad \text{and} \quad x_i = \phi^{2k}(x_i)\ . \tag{13}$$

In general, if

$$z = \phi^a(z) \quad \text{and} \quad z = \phi^b(z) \tag{14}$$

for $a, b \in \mathbb{Z}$ then

$$z = \phi^{\mathrm{lcd}(a,b)}(z) \tag{15}$$

since by the Euclidean algorithm there exists $n_1, n_2 \in \mathbb{Z}$ with

$$\mathrm{lcd}(a,b) = n_1 a + n_2 b\ . \tag{16}$$

In our specific situation $2r-1$ is odd and $2k$ is even and thus

$$1 \leq \mathrm{lcd}(2r-1, 2k) < 2k \tag{17}$$

contradicting the assumption $\phi^j(x_i) \neq x_i\, \forall j = 1,\dots,2k-1$. This proves the Proposition. □

3. Proof of Theorem 1

Let (M,ω) be a closed symplectic manifold. We fix a Darboux chart $B^{2N}(R) \cong B \subset M$ where $B^{2N}(R)$ is the open ball of radius R in $\mathbb{R}^{2N}$. For an integer $k \geq 1$ and a positive number $\delta > 0$ we choose a smooth function $\rho : [0,R^2] \to \mathbb{R}$ satisfying the following

$$\begin{cases} \dfrac{\pi}{2k} \geq \rho'(r) > 0 \ , \\ \rho'(r) = \dfrac{\pi}{2k} \quad \Longleftrightarrow \quad r = \frac{1}{2}R^2 \ , \\ \rho'|_{\left[\frac{8}{9}R^2, R^2\right]} = \delta > 0 \ . \end{cases} \tag{18}$$

We set for $1 \leq \nu \leq N$

$$\zeta(\nu) := \begin{cases} 1 & \nu = N \\ \frac{9}{10} & \text{else} \end{cases} \tag{19}$$

and define

$$\begin{aligned} H : B^{2N}(R) &\to \mathbb{R} \\ z &\mapsto \rho\left(\sum_{\nu=1}^{N} \zeta(\nu)|z_\nu|^2\right) \ . \end{aligned} \tag{20}$$

We denote by $\phi_H^t : B^{2N}(R) \to B^{2N}(R)$ the induced Hamiltonian flow. We recall that the Hamiltonian flow of $z \mapsto |z|^2$ is given by $z \mapsto \exp(2it)z$ thus

$$\left(\phi_H^t(z)\right)_\nu = \exp\left[\rho'\left(\sum_{\nu=1}^{N} \zeta(\nu)|z_\nu|^2\right) 2i\zeta(\nu)t\right] z_\nu \ . \tag{21}$$

We point out that ϕ_H^t preserves the quantities $|z_\nu|$, $\nu = 1, \dots, N$.

Lemma 4. *The fixed points of ϕ_H^{2k} are precisely $z = 0$ and the circle*

$$C := \left\{(z_1, \dots, z_N) \in B^{2N}(R) \mid |z_N|^2 = \tfrac{1}{2}R^2 \text{ and } z_1 = \dots = z_{N-1} = 0\right\} \ . \tag{22}$$

Moreover, ϕ_H acts on C by rotation of the last coordinate by an angle of $\frac{\pi}{k}$.

Proof. Assume $\phi_H^{2k}(z) = z$ which is equivalent to

$$\exp\left[\rho'\left(\sum_{\nu=1}^{N} \zeta(\nu)|z_\nu|^2\right) 2i\zeta(\nu)2k\right] z_\nu = z_\nu, \quad \nu = 1, \dots, N \ , \tag{23}$$

thus, either $z_\nu = 0$ or

$$\rho'\left(\sum_{\nu=1}^{N} \zeta(\nu)|z_\nu|^2\right) 4k\zeta(\nu) \in 2\pi\mathbb{Z} \ . \tag{24}$$

From $\rho'(r) \leq \frac{\pi}{2k}$ we conclude that $z_1 = \dots = z_{N-1} = 0$. Moreover, $z_N = 0$ or

$$\rho'\left(\sum_{\nu=1}^{N} \zeta(\nu)|z_\nu|^2\right) = \rho'(|z_N|^2) = \frac{\pi}{2k} \tag{25}$$

holds. In summary, either $z = 0$ or $z \in C$. This together with (21) proves the Lemma. $\square$

We now perturb H. For this we fix a smooth cut-off function $\beta : [0, R^2] \to [0,1]$ satisfying

$$\beta\big|_{\left[\frac{1}{3}R^2, \frac{2}{3}R^2\right]} = 1 \quad \text{and} \quad \beta\big|_{\left[0, \frac{1}{9}R^2\right] \cup \left[\frac{8}{9}R^2, R^2\right]} = 0 \tag{26}$$

and set

$$F(z) := \beta(|z_N|^2) \cdot \mathrm{Re}\left(\frac{z_N^k}{|z_N|^k}\right) : B^{2N}(R) \to \mathbb{R} \tag{27}$$

where Re is the real part. If we introduce new coordinates $(z_1, \ldots, z_{N-1}, r, \vartheta)$, where $z_N = r\exp(i\vartheta)$, the function F equals

$$F(z) = \beta(r^2)\cos\big(k\vartheta\big)\ . \tag{28}$$

We point out that the Hamiltonian diffeomorphism $\phi_H \circ \phi_{\epsilon F}$ maps $B^{2N}(R)$ into itself.

Lemma 5. *There exists $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$*

$$\#\mathscr{C}^{2k}(\phi_H \circ \phi_{\epsilon F}) = 1\ . \tag{29}$$

Proof. We set

$$D := \left\{(z_1, \ldots, z_{N-1}, r, \vartheta) \in C \mid \vartheta = \frac{j\pi}{k},\ j = 0, \ldots, 2k-1\right\} \tag{30}$$

where C is defined in Lemma 4. The same lemma implies that ϕ_H acts on D as a cyclic permutation sending $\frac{j\pi}{k}$ to $\frac{(j+1)\pi}{k}$. Moreover, we have

$$\phi_{\epsilon F} z = z \tag{31}$$

for $z \in D$ since $D \subset \mathrm{Crit}F$. In particular, D corresponds precisely to a single element in $\mathscr{C}^{2k}(\phi_H \circ \phi_{\epsilon F})$. It remains to show that there are no other $2k$-cycles. We prove something stronger, namely that for sufficiently small $\epsilon > 0$ the only other fixed point of $\big(\phi_H \circ \phi_{\epsilon F}\big)^{2k}$ is $z = 0$.

For $0 < a < b$ we set

$$A(a,b) := \big\{(z_1, \ldots, z_{N-1}, r, \vartheta) \in B^{2N}(R) \mid r \in [aR^2, bR^2]\big\}\ . \tag{32}$$

We observe that on $A(\frac{1}{3}, \frac{2}{3})$ we have $\beta = 1$ and thus the flow of ϵF is given by

$$(z_1, \ldots, z_{N-1}, r, \vartheta) \mapsto (z_1, \ldots, z_{N-1}, \sqrt{-2\epsilon k\sin(k\vartheta)t + r^2}, \vartheta)\ . \tag{33}$$

In particular, if we set

$$\bar{\epsilon} := \frac{7R^4}{324k^2} \tag{34}$$

then for $0 < \epsilon < \bar{\epsilon}$ we conclude that

$$\big(\phi_H \circ \phi_{\epsilon F}\big)^{2k}\big(A(\tfrac{4}{9}, \tfrac{5}{9})\big) \subset A(\tfrac{1}{3}, \tfrac{2}{3})\ , \tag{35}$$

since ϕ_H^t preserves the r coordinate. Fix $w \in A(\frac{4}{9}, \frac{5}{9})$ with $\big(\phi_H \circ \phi_{\epsilon F}\big)^{2k}(w) = w$ and set for $j = 0, \ldots, 2k$

$$\begin{aligned} z_\nu^j &:= P_{z_\nu}\Big(\big(\phi_H \circ \phi_{\epsilon F}\big)^j(w)\Big)\ , \quad \nu = 1, \ldots, N-1, \\ r^j &:= P_r\Big(\big(\phi_H \circ \phi_{\epsilon F}\big)^j(w)\Big)\ , \\ \vartheta^j &:= P_\vartheta\Big(\big(\phi_H \circ \phi_{\epsilon F}\big)^j(w)\Big)\ , \end{aligned} \tag{36}$$

where P_{z_ν}, P_r, and P_ϑ are the projections on the respective coordinates. It follows from equation (33) that

$$P_{z_\nu}\Big(\big(\phi_H\circ\phi_{\epsilon F}\big)^j(w)\Big)=P_{z_\nu}\big(\phi_H^j(w)\big)\quad \nu=1,\ldots,N-1\;. \tag{37}$$

By the same argument as in the proof of Lemma 4 we conclude

$$z_\nu^j=0\quad\forall\nu=1,\ldots,N-1\text{ and }\forall j=0,\ldots,2k\;. \tag{38}$$

Next, it follows from the flow equations (21) and (33)

$$0<\vartheta_{j+1}-\vartheta_j\le\frac{\pi}{k}\mod 2\pi\;. \tag{39}$$

By (18) equality holds if and only if $r_{j+1}=\frac{1}{2}R^2$. Using again $\big(\phi_H\circ\phi_{\epsilon F}\big)^{2k}(w)=w$ we deduce

$$\vartheta_{2k}-\vartheta_0=0\mod 2\pi \tag{40}$$

and therefore

$$r_0=r_1=\ldots=r_{2k}=\tfrac{1}{2}R^2\;. \tag{41}$$

In summary

$$w=(0,\ldots,0,\tfrac{1}{2}R^2,\vartheta_0) \tag{42}$$

with $\vartheta_0\in\frac{\pi}{k}\mathbb{Z}$, i.e. $w\in D$. Thus, we proved that the only $2k$-cycle of $\phi_H\circ\phi_{\epsilon F}$ in the region $A(\frac{4}{9},\frac{5}{9})$ is the one corresponding to the set D. Therefore it remains to prove that after possibly shrinking $\bar\epsilon$ there are no other fixed points of $\big(\phi_H\circ\phi_{\epsilon F}\big)^{2k}$ outside $A(\frac{4}{9},\frac{5}{9})$ except for $z=0$. We argue by contradiction.

We assume that there exists a sequence $\epsilon_m\to 0$ and a sequence $(z^m)_{m\in\mathbb{N}}$ of points in $B^{2N}(R)\setminus A(\frac{4}{9},\frac{5}{9})$ with

$$\big(\phi_H\circ\phi_{\epsilon_m F}\big)^{2k}(z^m)=z^m\quad\forall m\in\mathbb{N}\;. \tag{43}$$

By compactness we may assume that $z^m\to z^*\in B^{2N}(R)\setminus \mathrm{int}A(\frac{4}{9},\frac{5}{9})$ with

$$\phi_H^{2k}(z^*)=z^*\;. \tag{44}$$

It follows from Lemma 4 that $z^*=0$ and thus for M sufficiently large

$$z^m\in B^{2N}(\tfrac{1}{3}R)\quad\forall m\ge M\;. \tag{45}$$

Then by definition of β the restriction of $\phi_{\epsilon_m F}$ to the ball $B^{2N}(\frac{1}{3}R)$ equals the identity. Moreover, since ϕ_H fixes all balls centered at zero we have

$$z^m=\big(\phi_H\circ\phi_{\epsilon_m F}\big)^{2k}(z^m)=\phi_H^{2k}(z^m)\quad\forall m\ge M\;. \tag{46}$$

Applying again Lemma 4 we conclude that $z^m=0$ for all $m\ge M$. This proves the Lemma. □

Remark 6. Proposition 3 together with Lemma 5 implies that for all $0<\epsilon<\epsilon_0$ the Hamiltonian diffeomorphism $\phi_H\circ\phi_{\epsilon F}:B^{2N}(R)\to B^{2N}(R)$ has no square root.

We are now in the position to prove Theorem 1.

Proof of Theorem 1. We choose $k\in\mathbb{Z}$, $\delta>0$ and $0<\epsilon<\epsilon_0$ (cp. Lemma 5) so that the Hamiltonian diffeomorphism

$$\phi_H\circ\phi_{\epsilon F}:B^{2N}(R)\to B^{2N}(R) \tag{47}$$

has precisely one $2k$-cycle. By construction $\phi_H\circ\phi_{\epsilon F}$ equals the map

$$(z_1,\ldots,z_N)\mapsto\Big(e^{\frac{9i\delta}{5}}z_1,\ldots,e^{\frac{9i\delta}{5}}z_{N-1},e^{2i\delta}z_N\Big) \tag{48}$$

near the boundary of $B^{2N}(R)$. Indeed, if $z \in \partial B^{2N}(R)$ then we conclude

$$\sum_{\nu=1}^{N} \zeta(\nu)|z_\nu|^2 \geq \frac{9}{10}\sum_{\nu=1}^{N}|z_\nu|^2 = \frac{9}{10}R^2 > \frac{8}{9}R^2 \tag{49}$$

and therefore $\rho'(\sum_{\nu=1}^{N}\zeta(\nu)|z_\nu|^2) = \delta$. Next, we extend the Hamiltonian function of $\phi_H \circ \phi_{\epsilon F}$ to $\widetilde{H} : S^1 \times M \to \mathbb{R}$ which we can choose to be autonomous outside the Darboux ball B. If we choose $\delta > 0$ sufficiently small we can guarantee that outside B the only periodic orbits of $\widetilde{H}$ of period less or equal to $2k$ are critical points of $\widetilde{H}$, see [HZ94], in particular line 4 & 5 on page 185. In particular, $\phi_{\widetilde{H}}$ has still precisely one $2k$-cycle. Finally, by choosing k sufficiently large and δ and ϵ sufficiently small, $\phi_H \circ \phi_{\epsilon F}$ and thus $\phi_{\widetilde{H}}$ can be chosen to lie in an arbitrary C^∞-neighborhood of the identity on $B^{2N}(R)$ resp. M. Therefore, with Proposition 3 the Theorem follows. □

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