Spectral invariants in Rabinowitz-Floer homology and global Hamiltonian perturbations

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ABSTRACT. Spectral invariant were introduced in Hamiltonian Floer homology by Viterbo, Oh, and Schwarz. We extend this concept to Rabinowitz Floer homology. As an application we derive new quantitative existence results for leaf-wise intersections. The importance of spectral invariants for the presented application is that spectral invariants allow us to derive existence of critical points of the Rabinowitz action functional even in degenerate situations where the functional is not Morse.

1. INTRODUCTION

We consider an autonomous Hamiltonian system (M, ω, F) where (M, ω) is a symplectic manifold and $F : M \longrightarrow \mathbb{R}$ is a smooth time-independent function. The dynamics is given by the flow ϕ_F^t of the Hamiltonian vector field X_F which is defined implicitly by $\omega(X_F, \cdot) = dF(\cdot)$. Since F is autonomous the energy hypersurface $S = F^{-1}(0)$ is preserved under ϕ_F^t. Therefore, S is foliated by leaves $L_x := \{\phi_F^t(x) \mid t \in \mathbb{R}\}$, $x \in S$.

It is a challenging problem to compare the system F before and after a global perturbation occurring in the time interval $[0, 1]$. Such a perturbation is described by a function $H : M \times [0, 1] \longrightarrow \mathbb{R}$. J. Moser observed in [Mos78] that it is not possible to destroy all trajectories of the unperturbed system if the perturbation is sufficiently small, that is, there exists $x \in S$

$$\phi_H^1(x) \in L_x \, . \tag{1.1}$$

Such a point x is referred to as a leaf-wise intersection. Equivalently, there exists $(x, \eta) \in S \times \mathbb{R}$ such that

$$\phi_F^\eta(x) = \phi_H^1(x) \, . \tag{1.2}$$

We point out that the time shift η is uniquely defined by the above equation unless the leaf L_x is closed. If the time shift is negative then the perturbation moves the system back into its own past. Likewise, if the time shift is positive the perturbation moves the system forward into its own future.

Already the existence problem for leaf-wise intersections is highly non-trivial. The search for leaf-wise intersections was initiated by Moser in [Mos78] and pursued further in [Ban80, Hof90, EH89, Gin07, Dra08, AF08b, Zil08, AF08a, Gur09, Kan09, Mer10]. We refer to [AF08a] for a brief history.

To our knowledge the size of possible time shifts η has not been studied so far.

Theorem 1. *Let B be a closed manifold with $\dim \mathrm{H}_*(\mathscr{L}_B) = \infty$ where $\mathscr{L}_B = C^\infty(S^1, B)$. Let $(M := T^*B, \omega)$ be its cotangent bundle and $F : M \longrightarrow \mathbb{R}$ be a smooth function such*

that $S := F^{-1}(0)$ is a regular level set which is fiber-wise star-shaped. We assume that $H : M \times [0,1] \longrightarrow \mathbb{R}$ has compact support. Then there exist $(x, \eta) \in S \times \mathbb{R}$ such that

$$\phi_F^\eta(x) = \phi_H^1(x) \tag{1.3}$$

with arbitrarily large positive and negative time shifts η.

Remark 1.1. Thus, in classical Hamiltonian dynamical systems perturbations can move the system arbitrarily far into the past and future.

Remark 1.2. Theorem 1 cannot be true for arbitrary energy surfaces S. Indeed if S is Hamiltonianly displaceable there are no leaf-wise intersections at all for a displacing Hamiltonian H.

Corollary 1.3. *Under the assumptions of Theorem 1 there exists infinitely many leaf-wise intersections or a leaf-wise intersection x where L_x is closed. The latter we refer to as periodic leaf-wise intersections.*

We recall that if $\dim B \geq 2$ generically there are no periodic leaf-wise intersection, therefore, generically there exist infinitely many leaf-wise intersections, see [AF08a].

We use our variational approach to leaf-wise intersections by interpreting them as critical points of a perturbed Rabinowitz action functional, see [AF08b]. Rabinowitz Floer homology for unit cotangent bundle can be expressed with help of the homology $H_*(\mathscr{L}_B)$ of the free loop space $\mathscr{L}_B$ of B, see [CFO09, AS09]. Hence, if the perturbed Rabinowitz action functional is Morse it has to have infinitely many critical points. The main difficulty in proving Theorem 1 is to extend this result to degenerate situations in which Rabinowitz Floer homology cannot be directly defined. To overcome this problem we define spectral invariants for Rabinowitz Floer homology. Spectral invariants were introduced by Viterbo [Vit92], Oh [Oh97, Oh99], and Schwarz [Sch00] in the context of Hamiltonian Floer homology. An interesting and useful feature in Hamiltonian Floer theory is the relation between spectral invariants and the pair-of-pants product. This direction is not needed for the applications in the present article and therefore not pursued. It is an interesting problem for the future to study product structures in Rabinowitz Floer homology and their relations to spectral invariants.

If the Rabinowitz functional is Morse the spectral invariants are defined by a standard minimax procedure. In order to extend them to arbitrary Rabinowitz action functionals one has to proof a local Lipschitz property. This is the main technical issue and occupies most of this article. Spectral invariants are useful since even in the degenerate case they assign critical values to a Rabinowitz Floer homology class.

2. A VARIATIONAL APPROACH TO LEAF-WISE INTERSECTIONS

We recall from [AF08a] the notion of Moser pair.

Definition 2.1. A pair $\mathfrak{M} = (F, H)$ of Hamiltonian functions $F, H : M \times S^1 \longrightarrow R$ is called a Moser pair if it satisfies

$$F(\cdot, t) = 0 \quad \forall t \in [\tfrac{1}{2}, 1] \qquad \text{and} \qquad H(\cdot, t) = 0 \quad \forall t \in [0, \tfrac{1}{2}] , \qquad (2.1)$$

and F is of the form $F(x, t) = \rho(t)f(x)$ for some smooth map $\rho : S^1 \to [0, 1]$ with $\int_0^1 \rho(t)dt = 1$ and $f : M \longrightarrow \mathbb{R}$. We denote the set of Moser pairs by $\mathcal{MP}(M)$.

For a Moser pair $\mathfrak{M} = (F, H)$ the perturbed Rabinowitz action functional is defined by

$$\mathcal{A}^{\mathfrak{M}} : \mathscr{L}_M \times \mathbb{R} \longrightarrow \mathbb{R}$$

$$(v, \eta) \mapsto - \int_0^1 v^* \lambda - \int_0^1 H(v, t)dt - \eta \int_0^1 F(v, t)dt \qquad (2.2)$$

where $\mathscr{L}_M := C^\infty(S^1, M)$. A critical point (v, η) of $\mathcal{A}^{\mathfrak{M}}$ is a solution of

$$\left. \begin{aligned} \partial_t v &= \eta X_F(v, t) + X_H(v, t) \\ \int_0^1 F(v, t)dt &= 0 \end{aligned} \right\} \qquad (2.3)$$

In his pioneering work [Rab78] Rabinowitz studied the case of the unperturbed functional, that is, the case $H = 0$. In this situation critical points correspond to closed characteristics on the energy hypersurface $F^{-1}(0)$.

In [AF08b] we observed that critical points of the perturbed Rabinowitz action functional $\mathcal{A}^{\mathfrak{M}}$ give rise to leaf-wise intersections.

Proposition 2.2 ([AF08b]). *Let (v, η) be a critical point of $\mathcal{A}^{\mathfrak{M}}$ then $x := v(\tfrac{1}{2}) \in F^{-1}(0)$ and*

$$\phi_H^1(x) \in L_x \qquad (2.4)$$

thus, x is a leaf-wise intersection.

3. Rabinowitz Floer homology

Rabinowitz Floer homology is the semi-infinite Morse homology associated to the Rabinowitz action functional. In the unperturbed case it has been constructed in [CF09] under the assumption that the energy hypersurface $F^{-1}(0)$ is a smooth restricted contact-type hypersurface. This construction in the unperturbed case has been extended to stable hypersurfaces in [CFP09]. In [AF08b] we extended the construction in the case of restricted contact-type hypersurface to the perturbed Rabinowitz action functionals. In this article we continue our study of the perturbed Rabinowitz action functional for restricted contact-type hypersurfaces.

Let $(W, \omega = d\lambda)$ be a compact, exact symplectic manifold with contact type boundary $\Sigma = \partial W$, that is, the Liouville vector field L defined by $i_L \omega = \lambda$ points outward along Σ. In particular, $(\Sigma, \alpha := \lambda|_\Sigma)$ is contact. We denote by M the completion of W obtained by attaching the positive half of the symplectization of Σ, that is, $(M = W \cup_\Sigma (\Sigma \times \mathbb{R}_+), \omega = d\lambda)$ where λ is extended by $e^r \alpha$, $r \in \mathbb{R}_+$, over $\Sigma \times \mathbb{R}_+$. Since W is compact and exact the negative half $\Sigma \times \mathbb{R}_-$ of the symplectization embeds into W. In the following we will identify $\Sigma \times \mathbb{R}$ with its embedding into M.

We choose a smooth function $\rho : S^1 = \mathbb{R}/\mathbb{Z} \to [0, 1]$ with $\int_0^1 \rho(t)dt = 1$ and $\rho(t) = 0$ for $t \in [\tfrac{1}{2}, 1]$. We fix $0 < \delta < 1$ once and for all and choose a smooth monotone function

$\beta : \mathbb{R} \longrightarrow \mathbb{R}$ with

$$\beta(r) = \begin{cases} r & \text{for } |r| \leq \delta/2 \\ \delta & \text{for } r \geq \delta \\ -\delta & \text{for } r \leq -\delta \end{cases} \tag{3.1}$$

For later convenience we require in addition that

$$0 \leq \beta'(s) \leq 2 . \tag{3.2}$$

For any smooth function $f : \Sigma \longrightarrow \mathbb{R}$ we define

$$F_f(y, t) := \begin{cases} \beta(r - f(x)) \, \rho(t) & \text{for } y = (x, r) \in \Sigma \times \mathbb{R} \\ -\delta \, \rho(t) & \text{for } y \in M \setminus (\Sigma \times \mathbb{R}) \end{cases} \tag{3.3}$$

We denote by $\Sigma_f := \{(x, f(x)) \mid x \in \Sigma\} \subset M$ the graph of f over Σ and abbreviate $F := F_0$.

Lemma 3.1. The 1-form $\alpha_f := \lambda|_{\Sigma_f} = e^f \alpha$ is a contact form on Σ_f with Reeb vector field R_f given by $X_{G_f}|_{\Sigma_f}$ where X_{G_f} is the Hamiltonian vector field of the function $G_f(x, r) := r - f(x) : \Sigma \times \mathbb{R} \longrightarrow \mathbb{R}$. In particular,

$$\lambda(X_{G_f}) = 1 . \tag{3.4}$$

PROOF. That α_f is a contact form is straight forward to check. In order to prove $R_f = X_{G_f}|_{\Sigma_f}$ we first note that $\Sigma_f = G_f^{-1}(0)$ and thus $X_{G_f}|_{\Sigma_f}$ is indeed tangent to Σ_f. It remains to check the following two equations on Σ_f

$$i_{X_{G_f}} d\alpha_f = 0, \tag{3.5}$$

$$\alpha_f(X_{G_f}) = 1 . \tag{3.6}$$

The defining equation of X_{G_f} is

$$i_{X_{G_f}} \left(e^r (dr \wedge \alpha + d\alpha) \right) = dG_f . \tag{3.7}$$

On $\Sigma_f = \{r = f(x)\}$ this reads

$$i_{X_{G_f}} \left(\underbrace{e^f (df \wedge \alpha + d\alpha)}_{=d\alpha_f} \right) = dG_f|_{\Sigma_f} = 0 . \tag{3.8}$$

This proves the equation (3.5). To prove (3.6) we observe

$$1 = dG_f\left(\frac{\partial}{\partial r}\right) = i_{\frac{\partial}{\partial r}} i_{X_{G_f}} \left(e^r (dr \wedge \alpha + d\alpha) \right) = e^r dr \left(\frac{\partial}{\partial r}\right) \alpha(X_{G_f}) = e^r \alpha(X_{G_f}) . \tag{3.9}$$

On $\Sigma_f = \{r = f(x)\}$ this becomes

$$1 = e^f \alpha(X_{G_f}) = \alpha_f(X_{G_f}) . \tag{3.10}$$

$\square$

Definition 3.2. We set

$$\mathcal{H} := \{H \in C^\infty(M \times S^1) \mid H \text{ has compact support and } H(t, \cdot) = 0 \quad \forall t \in [0, \tfrac{1}{2}]\} \tag{3.11}$$

Remark 3.3. It's easy to see that the $\text{Ham}(M, \omega) \equiv \{\phi_H^1 \mid H \in \mathcal{H}\}$, e.g. [AF08b], where ϕ_H^1 is the time-1-map of the Hamiltonian flow of H.

Definition 3.4. We define the subset $\mathcal{MP}(\Sigma)$ of Moser pairs

$$\mathcal{MP}(\Sigma) := \{\mathfrak{M} = (F_f, H) \mid f \in C^\infty(\Sigma), \ H \in \mathcal{H}\} \tag{3.12}$$

where F_f is defined in equation (3.3). We call $\mathfrak{M} \in \mathcal{MP}(\Sigma)$ a Moser pair adapted to Σ.

Proposition 2.2 implies that for $\mathfrak{M} \in \mathcal{MP}(\Sigma)$ critical points of the Rabinowitz action functional $\mathcal{A}^{\mathfrak{M}}$ are leaf-wise intersections on Σ_f. We choose a compatible almost complex structure $\widetilde{J}$ on M such that on a δ-neighborhood of Σ_f the almost complex structure is SFT-like with respect to the contact form α_f, see [BEH+03]. That is, $\widetilde{J}$ interchanges the Reeb vector field R_f and Liouville vector field L, preserves the contact distribution, and is translationally invariant. Here δ is the universally chosen constant, for instance as in the definition of F_f, see (3.3). Now we change $\widetilde{J}$ to J by requiring

$$JR_f = e^{r-f(x)}L, \quad JL = e^{-r+f(x)}R_f \tag{3.13}$$

and that $J = \widetilde{J}$ on the contact distribution. Then J still is a compatible almost complex structure. Such a J is called twisted SFT-like.

Remark 3.5. Since J is twisted SFT-like we have on a δ-neighborhood

$$||X_{G_f}|| = ||L|| = 1 \tag{3.14}$$

and since $\lambda(X_{G_f}) = 1$

$$||\lambda|| = 1 . \tag{3.15}$$

Let $\mathfrak{M} \in \mathcal{MP}(\Sigma)$ be an adapted Moser pair. The norm of the gradient of $\mathcal{A}^{\mathfrak{M}}$ equals

$$||\nabla \mathcal{A}_H^F(u,\eta)||^2 = ||\partial_t u - X_H(u,t) - \eta X_F(u,t)||_{L^2}^2 + \left| \int_0^1 F(u(t),t)dt \right|^2 \tag{3.16}$$

where the L^2-norm is taken with respect to the metric $g(\cdot,\cdot) := \omega(\cdot,J\cdot)$. We denote by $\mathscr{L}$ the component of the contractible loops in M.

Definition 3.6. A gradient flow line of $\mathcal{A}^{\mathfrak{M}}$ is (formally) a map $w = (u,\eta) \in C^\infty(\mathbb{R}, \mathscr{L} \times \mathbb{R})$ solving the ODE

$$\partial_s w(s) + \nabla \mathcal{A}^{\mathfrak{M}}(w(s)) = 0 , \tag{3.17}$$

where the gradient is taken with respect to metric $\mathfrak{m}$ defined as follows. Let $(\hat{u}_1, \hat{\eta}_1)$ and $(\hat{u}_2, \hat{\eta}_2)$ be two tangent vectors in $T_{(u,\eta)}(\mathscr{L} \times \mathbb{R})$. We set

$$\mathfrak{m}\big((\hat{u}_1, \hat{\eta}_1), (\hat{u}_2, \hat{\eta}_2)\big) := \int_0^1 g(\hat{u}_1, \hat{u}_2)dt + \hat{\eta}_1 \hat{\eta}_2 . \tag{3.18}$$

According to Floer's interpretation, [Flo88], this means that u and η are smooth maps $u : \mathbb{R} \times S^1 \longrightarrow M$ and $\eta : \mathbb{R} \longrightarrow \mathbb{R}$ solving

$$\left. \begin{aligned} &\partial_s u + J(u)\big(\partial_t u - X_H(u,t) - \eta X_F(u,t)\big) = 0 \\[2mm] &\partial_s \eta - \int_0^1 F_f(u,t)dt = 0. \end{aligned} \right\} \tag{3.19}$$

Definition 3.7. A Moser pair $\mathfrak{M}$ is called regular if $\mathcal{A}^{\mathfrak{M}}$ is Morse.

We recall the following

Proposition 3.8 ([AF08b]). *A generic Moser pair is regular.*

We need the following slightly stronger version here.

Proposition 3.9 ([AF08b]). *A generic adapted Moser pair is regular (see Definition 3.4).*

PROOF. We note that the property of $\mathcal{A}^{(F,H)}$ being Morse is in fact a property of the hypersurface $\Sigma = F^{-1}(0)$ as long as the defining function F has 0 as a regular value as is apparent from the proof of Proposition A.2 in [AF08b]. Moreover, the property of Σ_f of being a graph is a C^1-open condition. Thus, the assertion follows from Proposition 3.8. $\qquad\square$

For a regular contact-type Moser pair $\mathfrak{M}$ the Rabinowitz Floer homology $\mathrm{RFH}_*(\mathfrak{M})$ is defined from the following chain complex

$$\mathrm{RFC}_k(\mathfrak{M}) := \Big\{ \xi = \sum_{c \colon \mu_{\mathrm{CZ}}(c)=k} \xi_c\, c \mid \#\{c \in \mathrm{Crit}\mathcal{A}^{\mathfrak{M}} \mid \xi_c \neq 0 \in \mathbb{Z}/2 \text{ and } \mathcal{A}^{\mathfrak{M}}(c) \geq \kappa\} < \infty\ \forall \kappa \in \mathbb{R} \Big\}$$

(3.20)

where the boundary operator is defined by counting gradient flow lines of $\mathcal{A}^{\mathfrak{M}}$ in the sense of Floer homology, see [CF09] for details.

If the Moser pair is of the form $\mathfrak{M} = (F_f, 0)$ then $\mathcal{A}^{\mathfrak{M}}$ is never Morse. But for a generic F_f the action functional $\mathcal{A}^{\mathfrak{M}}$ is Morse-Bott with critical manifold being the disjoint union of constant solutions of the form $(p, 0)$, $p \in \Sigma_f$, and a family of circles corresponding to closed characteristics of ω on Σ_f.

Definition 3.10. An adapted Moser pair is called weakly regular if it is of the form just described or if it is regular. The set of adapted weakly regular Moser pairs is denoted by $\mathcal{MP}^{reg}(\Sigma)$.

Remark 3.11. For adapted weakly regular Moser pairs $\mathfrak{M}$ Rabinowitz Floer homology $\mathrm{RFH}_*(\mathfrak{M})$ can still be defined by taking the critical points of a Morse function on the critical manifolds as generators, see [CF09] for details.

For $\mathfrak{M}_0, \mathfrak{M}_1 \in \mathcal{MP}^{reg}(\Sigma)$ there exist canonical isomorphisms

$$\zeta_{\mathfrak{M}_0}^{\mathfrak{M}_1} : \mathrm{RFH}_*(\mathfrak{M}_0) \longrightarrow \mathrm{RFH}_*(\mathfrak{M}_1) \tag{3.21}$$

called continuation homomorphisms. They satisfy

$$\zeta_{\mathfrak{M}_1}^{\mathfrak{M}_2} \circ \zeta_{\mathfrak{M}_0}^{\mathfrak{M}_1} = \zeta_{\mathfrak{M}_0}^{\mathfrak{M}_2}, \qquad \zeta_{\mathfrak{M}}^{\mathfrak{M}} = \mathrm{id}_{\mathrm{RFH}(\mathfrak{M})} . \tag{3.22}$$

We refer the reader to [CF09] for details.

Definition 3.12. The inverse limit defined with respect to the continuation homomorphism is denoted by

$$\mathrm{RFH}_* \equiv \mathrm{RFH}_*(\Sigma, M) := \varprojlim \mathrm{RFH}_*(\mathfrak{M}). \tag{3.23}$$

Moreover, we refer by

$$\zeta^{\mathfrak{M}} : \mathrm{RFH}_* \longrightarrow \mathrm{RFH}_*(\mathfrak{M}) \tag{3.24}$$

to the canonical map which in our case is an isomorphism.

Remark 3.13. The main difficulty in defining Floer homology is compactness up to breaking of gradient flow lines. The new obstacle in Rabinowitz Floer homology is to establish uniform L^∞ bounds for the Lagrange multiplier $\eta(s)$ along gradient flow lines with fixed asymptotics. The crucial ingredient is a period-action inequality for almost critical points. This has been established in the current set-up in [AF08b, Lemma 2.11]. In this article we present an enhanced version of this lemma, see Lemma 3.15. This enhancement is needed to study continuity properties of spectral invariants in Rabinowitz Floer homology.

We recall the definition of the cut-off function $\beta : \mathbb{R} \longrightarrow \mathbb{R}$

$$\beta(r) = \begin{cases} r & \text{for } |r| \leq \delta/2 \\ \delta & \text{for } r \geq \delta \\ -\delta & \text{for } r \leq -\delta \end{cases} \tag{3.25}$$

and

$$F_f(y,t) := \begin{cases} \beta(r - f(x))\,\rho(t) & \text{for } y = (x,r) \in \Sigma \times \mathbb{R} \\ -\delta\,\rho(t) & \text{for } y \in M \setminus (\Sigma \times \mathbb{R}) \end{cases} \tag{3.26}$$

Definition 3.14. We introduce a semi-norm on the set $\mathcal{H}$, see Definition 3.2, by

$$\kappa(H) := \int_0^1 \max |\lambda(x)[X_H(x,t)] - H(x,t)|\,dt \quad \forall H \in \mathcal{H}\ . \tag{3.27}$$

Lemma 3.15. For all $(u, \eta) \in C^\infty(S^1, M) \times \mathbb{R}$ with

$$\|\nabla \mathcal{A}^{\mathfrak{M}}(u, \eta)\| < \frac{\delta}{4} \tag{3.28}$$

we have the estimate

$$|\eta| \leq \frac{2}{2 - \delta}\Big(|\mathcal{A}^{\mathfrak{M}}(u, \eta)| + \delta/4 + \kappa(H)\Big) \tag{3.29}$$

where the norm of the gradient is given in equation (3.16).

Remark 3.16. We point out the constants appearing in Lemma 3.15 are independent of the function $f \in C^\infty(\Sigma)$ appearing in the Moser pair $\mathfrak{M} = (F_f, H)$.

PROOF. We define

$$U_{\frac{\delta}{2}}(f) := \{(x, r) \mid x \in \Sigma,\ r \in (f(x) - \delta/2, f(x) + \delta/2)\} \tag{3.30}$$

<u>Claim 1</u>: Assume that $u(t) \in U_{\frac{\delta}{2}}(f)$ for all $t \in [0, \frac{1}{2}]$, then

$$|\eta| \leq \frac{2}{2 - \delta}\Big(|\mathcal{A}^{\mathfrak{M}}(u, \eta)| + \|\nabla \mathcal{A}^{\mathfrak{M}}(u, \eta)\| + \kappa(H)\Big)\ , \tag{3.31}$$

where $\kappa(H)$ has been defined in Definition 3.14.

Proof of Claim 1. We compute using Lemma 3.1

$$|\mathcal{A}^{\mathfrak{M}}(u,\eta)| = \left| -\int_0^1 u^*\lambda - \int_0^1 H(t,u(t))dt - \eta\int_0^1 F_f(t,u(t))dt \right|$$

$$= \left| -\int_0^1 \lambda(u(t))\big[\partial_t u - X_H(t,u) - \eta X_{F_f}(t,u)\big] + \int_{\frac{1}{2}}^1 \lambda(u(t))\big[X_H(t,u)\big]dt \right.$$

$$\left. + \int_0^{\frac{1}{2}} \lambda(u(t))\big[\eta \underbrace{X_{F_f}(t,u)}_{=\rho(t)X_{G_f}(u(t))} \big]dt - \int_0^1 H(t,u(t))dt - \eta\int_0^1 F_f(t,u(t))dt \right|$$

$$\underbrace{\hphantom{+ \int_0^{\frac{1}{2}} \lambda(u(t))\big[\eta X_{F_f}(t,u) \big]dt}}_{=\eta}$$

$$= \left| \eta\Big(1 - \int_0^1 \underbrace{F_f(t,u(t))}_{\in(-\delta/2,\delta/2)} dt\Big) - \int_0^1 \lambda(u(t))\big[\partial_t u - X_H(t,u) - \eta X_{F_f}(t,u)\big] \right.$$

$$\left. + \int_0^1 \Big[\lambda(u(t))\big[X_H(t,u)\big] - H(t,u(t))\Big]dt \right|$$

$$\geq \frac{|\eta|(2-\delta)}{2} - \underbrace{||\lambda_{|U_\delta(f)}||_{C^0}}_{=1} ||\partial_t u - X_H(t,u) - \eta X_{F_f}(t,u)||_{L^1} - \kappa(H)$$

$$\geq \frac{|\eta|(2-\delta)}{2} - ||\partial_t u - X_H(t,u) - \eta X_{F_f}(t,u)||_{L^2} - \kappa(H)$$

$$\geq \frac{|\eta|(2-\delta)}{2} - ||\nabla\mathcal{A}^{\mathfrak{M}}(u,\eta)||_{L^2} - \kappa(H)$$

where $||\lambda_{|U_\delta}||_{C^0} = 1$ since J is twisted SFT-like on $U_\delta(f)$. This inequality implies Claim 1. $\qquad\square$

<u>Claim 2</u>: If for (u,η) there exists $t \in [0,\frac{1}{2}]$ with $u(t) \notin U_{\frac{\delta}{2}}(f)$ then $||\nabla_s\mathcal{A}^{\mathfrak{M}}(u,\eta)|| \geq \frac{\delta}{4}$.

Proof of Claim 2. If in addition $u(t) \notin U_{\frac{\delta}{4}}(f)$ holds for all $t \in [0,\frac{1}{2}]$ then using (3.16)

$$||\nabla\mathcal{A}^{\mathfrak{M}}(u,\eta)|| \geq \left| \int_0^1 F_f(u(t),t)dt \right| \geq \frac{\delta}{4}\int_0^1 \rho(t)dt = \frac{\delta}{4} . \qquad (3.32)$$

Otherwise there exists $t' \in [0,\frac{1}{2}]$ with $u(t') \in U_{\frac{\delta}{4}}(f)$. Thus, we can find $0 \leq a < b \leq \frac{1}{2}$ such that either

$$u(a) \in \partial U_{\frac{\delta}{4}}(f), \quad u(b) \in \partial U_{\frac{\delta}{2}}(f) \quad \text{and} \quad u(t) \in U_{\frac{\delta}{2}}(f)\setminus U_{\frac{\delta}{4}}(f) \,\forall t \in [a,b] \qquad (3.33)$$

or

$$u(a) \in \partial U_{\frac{\delta}{2}}(f), \quad u(b) \in \partial U_{\frac{\delta}{4}}(f) \quad \text{and} \quad u(t) \in U_{\frac{\delta}{2}}(f)\setminus U_{\frac{\delta}{4}}(f) \,\forall t \in [a,b] . \qquad (3.34)$$

We only treat the first case here. The second is completely analogous. We recall from Lemma 3.1 the definition $G_f(x, r) = r - f(x)$.

$$||\nabla \mathcal{A}^{\mathfrak{M}}(u, \eta)|| \geq ||\partial_t u - X_H(u, t) - \eta X_{F_f}(u, t)||_{L^2}$$

$$\geq \left(\int_a^b ||\partial_t u - \underbrace{X_H(u, t)}_{=0} - \eta X_{F_f}(u, t)||^2 dt \right)^{\frac{1}{2}}$$

$$\geq \left(\int_a^b \frac{1}{||\nabla G_f||^2} \left| g_t(\partial_t u, \nabla G_f) - \eta \underbrace{g(X_{F_f}(u, t), \nabla G_f)}_{=0} \right|^2 dt \right)^{\frac{1}{2}}$$

$$\geq \frac{1}{||\nabla G_f|_{U_{\frac{\delta}{2}}(f)}||_{C^0}} \left(\int_a^b \left| \frac{d}{dt} G_f(u(t)) \right|^2 dt \right)^{\frac{1}{2}} \tag{3.35}$$

$$\geq \frac{1}{||\nabla G_f|_{U_{\frac{\delta}{2}}(f)}||_{C^0}} \int_a^b \left| \frac{d}{dt} G_f(u(t)) \right| dt$$

$$\geq \frac{1}{||\nabla G_f|_{U_{\frac{\delta}{2}}(f)}||_{C^0}} \int_a^b \frac{d}{dt} G_f(u(t)) dt$$

$$\geq \frac{\delta}{4 ||\nabla G_f|_{U_{\frac{\delta}{2}}(f)}||_{C^0}}$$

$$= \frac{\delta}{4}$$

where we used $g(X_{F_f}, \nabla G_f) = dG_f(X_{F_f}) = dG_f(\rho(t) X_{G_f}) = 0$ since on $U_{\frac{\delta}{2}}(f)$ it holds $F_f = \rho(t) G_f$. Moreover, according to Remark 3.5 we have $||\nabla G_f|| = ||X_{G_f}|| = 1$ on $U_{\frac{\delta}{2}}(f)$. This proves Claim 2. $\square$

To prove the Lemma we observe that the assumption $||\nabla \mathcal{A}^{\mathfrak{M}}(u, \eta)|| < \frac{\delta}{4}$ excludes the case treated in Claim 2. $\square$

4. Warmup – Spectral Invariants in Morse homology

In this section we explain spectral invariants in the finite dimensional case. The main construction scheme is already visible in the finite dimensional, nevertheless, the proof of local Lipschitz continuity is much easier.

Let M be a closed manifold and $f : M \longrightarrow \mathbb{R}$ a Morse function. We recall that the Morse chain complex $\mathrm{CM}_*(f)$ is the graded $\mathbb{Z}/2$ vector space generated by the set $\mathrm{Crit}(f)$ of critical points of f. The grading is given by the Morse index μ_{Morse} of f. The boundary operator $\partial : \mathrm{CM}_*(f) \longrightarrow \mathrm{CM}_{*-1}(f)$ is defined on generators by counting gradient flow lines. Indeed, we choose a Riemannian metric g on M such that stable and unstable manifold with respect to the negative gradient flow of $\nabla f = \nabla^g f$ intersect transversely, that is, $W^s(x) \pitchfork W^u(y)$ for

all $x, y \in \mathrm{Crit}(f)$. Then the moduli space

$$\widehat{\mathcal{M}}(x_-, x_+) := \left\{ \gamma : \mathbb{R} \longrightarrow M \mid \dot{\gamma} + \nabla f(\gamma) = 0, \ \lim_{s \to \pm\infty} \gamma(s) = x_\pm \right\} \tag{4.1}$$

is a smooth manifold of dimension $\dim \widehat{\mathcal{M}}(x_-, x_+) = \mu_{\mathrm{Morse}}(x_-) - \mu_{\mathrm{Morse}}(x_+)$. Moreover, $\mathbb{R}$ acts by shifting the s-coordinate and we denote the quotient by

$$\mathcal{M}(x_-, x_+) := \widehat{\mathcal{M}}(x_-, x_+)/\mathbb{R} \ . \tag{4.2}$$

Moreover, if $\mu_{\mathrm{Morse}}(x_-) - \mu_{\mathrm{Morse}}(x_+) = 1$ then $\mathcal{M}(x_-, x_+)$ is a finite set. We set

$$m(x_-, x_+) := \#_2 \mathcal{M}(x_-, x_+) \tag{4.3}$$

the mod 2 number of elements in $\mathcal{M}(x_-, x_+)$. Then we can define the differential $\partial = \partial(f, g)$ as a linear map which is given on generators by

$$\partial x_- := \sum_{\substack{x_+ \in \mathrm{Crit}(f) \\ \mu_{\mathrm{Morse}}(x_-) - \mu_{\mathrm{Morse}}(x_+) = 1}} m(x_-, x_+)\, x_+ \ . \tag{4.4}$$

It is a deep theorem in Morse homology that the identity

$$\partial \circ \partial = 0 \tag{4.5}$$

holds, see [Sch93] for details. Then

$$\mathrm{HM}_*(f, g) := \mathrm{H}_*(\mathrm{CM}_\bullet(f), \partial(f, g)) \tag{4.6}$$

is the Morse homology of the pair (f, g).

Up to canonical isomorphisms Morse homology does not depend on the Morse-Smale pair (f, g). These canonical isomorphisms are called continuation homomorphisms and are constructed in the following way. For two Morse-Smale pairs $(f_\pm, g_\pm)$ we choose a $T > 0$ and a smooth family $\{(f_s, g_s)\}_{s \in \mathbb{R}}$ of functions $f_s : M \longrightarrow \mathbb{R}$ and Riemannian metrics g_s such that

$$f_s = \begin{cases} f_- \text{ for } s \leq -T \\ f_+ \text{ for } s \geq T \end{cases} \qquad g_s = \begin{cases} g_- \text{ for } s \leq -T \\ g_+ \text{ for } s \geq T \end{cases} \tag{4.7}$$

For critical points $x_\pm \in \mathrm{Crit}(f_\pm)$ we consider the moduli spaces

$$\mathcal{N}(x_-, x_+) = \mathcal{N}(x_-, x_+; f_s, g_s) := \left\{ \gamma : \mathbb{R} \longrightarrow M \mid \dot{\gamma}(s) + \nabla^{g_s} f_s(\gamma(s)) = 0, \ \lim_{s \to \pm\infty} \gamma(s) = x_\pm \right\} \ . \tag{4.8}$$

A homotopy (f_s, g_s) is called regular if the moduli space $\mathcal{N}(x_-, x_+)$ is a smooth manifold of dimension $\dim \mathcal{N}(x_-, x_+) = \mu_{\mathrm{Morse}}(x_-) - \mu_{\mathrm{Morse}}(x_+)$. A generic homotopy is regular. Moreover, in the special case $f_s = f_- = f_+$ and $g_s = g_- = g_+$ we have the identity

$$\mathcal{N}(x_-, x_+) = \widehat{\mathcal{M}}(x_-, x_+) \ . \tag{4.9}$$

If $\mu_{\mathrm{Morse}}(x_-) - \mu_{\mathrm{Morse}}(x_+) = 0$ the space $\mathcal{N}(x_-, x_+)$ is compact and we set

$$n(x_-, x_+) := \#_2 \mathcal{N}(x_-, x_+) \ . \tag{4.10}$$

Then we can define a linear map

$$Z = Z(f_s, g_s) : \mathrm{CM}_*(f_-) \longrightarrow \mathrm{CM}_*(f_+)$$

$$x_- \mapsto \sum_{\substack{x_+ \in \mathrm{Crit}(f_+) \\ \mu_{\mathrm{Morse}}(x_-) - \mu_{\mathrm{Morse}}(x_+) = 0}} n(x_-, x_+)\, x_+ \ . \tag{4.11}$$

We denote $\partial_{\pm} := \partial(f_{\pm}, g_{\pm})$. In the same manner as $\partial \circ \partial = 0$ one proves in Morse homology

$$Z \circ \partial_- = \partial_+ \circ Z , \qquad (4.12)$$

see [Sch93]. In particular, on homology we obtain the map

$$\zeta : \mathrm{HM}_*(f_-, g_-) \longrightarrow \mathrm{HM}_*(f_+, g_+) \qquad (4.13)$$

which is the continuation homomorphism. By a homotopy-of-homotopies argument it is proved that ζ is independent of the chosen homotopy (f_s, g_s), see [Sch93]. Moreover, the continuation homomorphism is functorial in the following sense. If we fix three Morse-Smale pairs (f_a, g_a), (f_b, g_b), and (f_c, g_c) we denote the corresponding continuation homomorphisms by $\zeta_a^b : \mathrm{HM}_*(f_a, g_a) \longrightarrow \mathrm{HM}_*(f_b, g_b)$ and similarly ζ_a^c and ζ_b^c. Then we have the following identities

$$\zeta_a^c = \zeta_b^c \circ \zeta_a^b \quad \text{and} \quad \zeta_a^a = \mathrm{id}_{\mathrm{HM}_*(f_a, g_a)} . \qquad (4.14)$$

In particular, we conclude that ζ_a^b is an isomorphism with inverse ζ_b^a.

Definition 4.1. Let (f, g) be a Morse-Smale pair. For $\xi = \sum_x \xi_x x \neq 0 \in \mathrm{CM}_*(f)$ we set

$$f(\xi) := \max\{f(x) \mid \xi_x \neq 0\} \qquad (4.15)$$

and for $X \neq 0 \in \mathrm{HM}_*(f, g)$ we set

$$\sigma(X) := \min\{f(\xi) \mid X = [\xi]\} . \qquad (4.16)$$

We call $\sigma(X)$ the spectral value of X. Thus, σ is a map

$$\sigma : \bigcup_{(f,g) \text{ Morse-Smale}} \mathrm{HM}_*(f, g) \longrightarrow \mathbb{R} . \qquad (4.17)$$

Theorem 4.2. *Let $(f_{\pm}, g_{\pm})$ be two Morse-Smale pairs. Let $X \neq 0 \in \mathrm{HM}_*(f_-, g_-)$ then*

$$\min(f_+ - f_-) \leq \sigma(\zeta(X)) - \sigma(X) \leq \max(f_+ - f_-) . \qquad (4.18)$$

Remark 4.3. The estimate in Theorem 4.2 is sharp as can be seen for example by choosing $f_+ = f_-+\text{const}$.

An immediate corollary of Theorem 4.2 is the following.

Corollary 4.4. *The spectral invariant $\sigma(X)$ does not depend on the Riemannian metric g.*

As preparation of the proof of Theorem 4.2 we consider the following special homotopy. We fix a smooth monotone function $\beta : \mathbb{R} \longrightarrow [0, 1]$ satisfying $\beta(s) = 0$ for $s \leq -T$ and $\beta(s) = 1$ for $s \geq T$. Then we set

$$f_s := \beta(s)f_+ + (1 - \beta(s))f_- = \beta(s)(f_+ - f_-) + f_- \qquad (4.19)$$

and choose any homotopy g_s from g_- to g_+.

Lemma 4.5. *Let (f_s, g_s) as above. If $\mathcal{N}(x_-, x_+; f_s, g_s) \neq \emptyset$ we have*

$$f_+(x_+) - f_-(x_-) \leq \max(f_+ - f_-) \qquad (4.20)$$

PROOF. We choose an element $\gamma \in \mathcal{N}(x_-, x_+; f_s, g_s)$ and estimate

$$
\begin{aligned}
f_+(x_+) - f_-(x_-) &= \int_{-\infty}^{\infty} \frac{d}{ds} f_s(\gamma(s)) ds \\
&= \int_{-\infty}^{\infty} \left\{ df_s(\gamma(s))[\dot{\gamma}(s)] + \frac{\partial f}{\partial s}(\gamma(s)) \right\} ds \\
&= \int_{-\infty}^{\infty} \left\{ -df_s(\gamma(s)) \Big[\nabla^{g_s} f_s(\gamma(s)) \Big] + \beta'(s)(f_+ - f_-)(\gamma(s)) \right\} ds \\
&= \int_{-\infty}^{\infty} \left\{ \underbrace{-g_s(\gamma(s)) \Big[\nabla^{g_s} f_s(\gamma(s)), \nabla^{g_s} f_s(\gamma(s)) \Big]}_{\leq 0} + \underbrace{\beta'(s)}_{\geq 0} \underbrace{(f_+ - f_-)(\gamma(s))}_{\leq \max(f_+ - f_-)} \right\} ds \\
&\leq \max(f_+ - f_-) \int_{-\infty}^{\infty} \beta'(s) ds \\
&= \max(f_+ - f_-) .
\end{aligned}
$$

$$(4.21)$$

$\square$

Corollary 4.6. *Let $X \neq 0 \in \mathrm{HM}_*(f_-, g_-)$, then*

$$
\sigma(\zeta(X)) - \sigma(X) \leq \max(f_+ - f_-) . \tag{4.22}
$$

PROOF. We first assume that the homotopy $f_s = \beta(s)f_+ + (1 - \beta(s))f_-$ is regular. Let $\xi = \sum_x \xi_x x \in \mathrm{CM}_*(f_-)$ be a representative of X. Then

$$
Z(\xi) = \sum_y \underbrace{\left(\sum_x \xi_x n(x, y) \right)}_{\eta_y} y \tag{4.23}
$$

and thus

$$
f_+(Z(\xi)) = \max\{ f_+(y) \mid \eta_y \neq 0 \} . \tag{4.24}
$$

Now we choose $y \in \mathrm{Crit}(f_+)$ s.t. $f_+(Z(\xi)) = f_+(y)$. Since $\eta_y \neq 0$ there exists $x \in \mathrm{Crit}(f_-)$ such that $\xi_x n(x, y) \neq 0$, i.e. $\xi_x \neq 0$ and $n(x, y) \neq 0$. In particular, $\mathcal{N}(x, y) \neq \emptyset$ and by Lemma 4.5 we conlude

$$
f_+(y) - f_-(x) \leq \max(f_+ - f_-) . \tag{4.25}
$$

Then using $\xi_x \neq 0$ we estimate

$$
f_+(Z(\xi)) - f_-(\xi) \leq f_+(y) - f_-(x) \leq \max(f_+ - f_-) \tag{4.26}
$$

and

$$
\begin{aligned}
\sigma(\zeta(X)) - \sigma(X) &= \min\{ f_+(\eta) \mid [\eta] = \zeta(X) \} - \min\{ f_-(\xi) \mid [\xi] = X \} \\
&\leq \min\{ f_+(\eta) \mid \eta = Z(\xi),\ [\xi] = X \} - \min\{ f_-(\xi) \mid [\xi] = X \} \\
&\leq \min\{ f_+(Z(\xi)) \mid [\xi] = X \} - \min\{ f_-(\xi) \mid [\xi] = X \} \\
&\leq \min\{ f_-(\xi) + \max(f_+ - f_-) \mid [\xi] = X \} - \min\{ f_-(\xi) \mid [\xi] = X \} \\
&= \max(f_+ - f_-) .
\end{aligned}
\tag{4.27}
$$

If the homotopy f_s from above is not regular then we can approximate it by regular homotopies. The Corollary follows by noting that the estimate of Lemma 4.5 is correct up to an arbitrarily small error. $\square$

Proof of Theorem 4.2. According to Corollary 4.6 we have

$$\sigma(\zeta(X)) - \sigma(X) \leq \max(f_+ - f_-) . \tag{4.28}$$

and thus by symmetry

$$\sigma(X) - \sigma(\zeta(X)) = \sigma(\zeta^{-1}(\zeta(X))) - \sigma(\zeta(X)) \leq \max(f_- - f_+) = -\min(f_+ - f_-) . \tag{4.29}$$

$\square$

With help of the continuation homomorphism we define the inverse limit

$$\mathrm{HM}_* := \varprojlim \mathrm{HM}_*(f,g) . \tag{4.30}$$

Thus, for any Morse-Smale pair (f,g) we have an isomorphism

$$\zeta^{(f,g)} : \mathrm{HM}_* \longrightarrow \mathrm{HM}_*(f,g). \tag{4.31}$$

Definition 4.7. For a Morse function f and $Y \neq 0 \in \mathrm{HM}_*$ we set

$$\sigma_f(Y) := \sigma\Big(\zeta^{(f,g)}(Y)\Big) \tag{4.32}$$

where g is any Riemannian metric so that (f,g) is Morse-Smale. Moreover, for fixed $Y \neq 0 \in \mathrm{HM}_*$ we define

$$\rho_Y : \{f \in C^\infty \mid f \text{ is Morse}\} \longrightarrow \mathbb{R}$$
$$f \mapsto \sigma_f(Y) . \tag{4.33}$$

Remark 4.8. σ_f is well-defined, see Corollary 4.4.

Corollary 4.9. *Let $Y \neq 0 \in \mathrm{HM}_*$ and $f_\pm$ be Morse functions then*

$$|\rho_Y(f_+) - \rho_Y(f_-)| \leq ||f_+ - f_-||_{C^0(\Sigma)} := \max_M |f_+ - f_-| . \tag{4.34}$$

That is, ρ_Y is 1-Lipschitz continuous with respect to the C^0 norm.

PROOF. We choose $g_\pm$ such that $(f_\pm, g_\pm)$ are Morse-Smale and set $X_\pm := \zeta^{(f_\pm, g_\pm)}(Y)$. Then

$$\begin{aligned} |\rho_Y(f_+) - \rho_Y(f_-)| &= |\sigma(X_+) - \sigma(X_-)| \\ &\leq \max\{\max(f_- - f_+), -\min(f_+ - f_-)\} \\ &= ||f_+ - f_-||_{C^0(\Sigma)} \end{aligned} \tag{4.35}$$

where in the inequality we use Theorem 4.2 and $X_+ = \zeta(X_-)$. $\square$

Definition 4.10. For $f \in C^1(M)$ we define the spectrum of f

$$\mathfrak{S}(f) := f\big(\mathrm{Crit}(f)\big) \tag{4.36}$$

to be the set of critical values of f.

Corollary 4.11. *Let $Y \neq 0 \in \mathrm{HM}_*$. The map ρ_Y has a unique extension to a 1-Lipschitz continuous function $\rho_Y : C^0(M) \longrightarrow \mathbb{R}$. Moreover, if $f \in C^1(M)$ then $\rho_Y(f)$ is in the spectrum of f:*

$$\rho_Y(f) \in \mathfrak{S}(f) . \tag{4.37}$$

PROOF. We recall that $\{f \in C^\infty \mid f \text{ is Morse}\}$ is dense in C^0. Therefore, for $f \in C^0(M)$ there exist Morse functions f_n with $||f_n - f||_{C^0} \to 0$. By Corollary 4.9

$$|\rho_Y(f_n) - \rho_Y(f_m)| \leq ||f_n - f_m||_{C^0} \tag{4.38}$$

the sequence $(\rho_Y(f_n))$ is a Cauchy sequence in $\mathbb{R}$, thus converges. We set (by abuse of notation)

$$\rho_Y(f) := \lim \rho_Y(f_n) \tag{4.39}$$

and note that for two sequences (f_n) and (f_n') with $||f_n - f||_{C^0}, ||f_n' - f||_{C^0} \to 0$ we can again by Corollary 4.9 estimate

$$\lim |\rho_Y(f_n') - \rho_Y(f_n)| \leq \lim ||f_n' - f_n||_{C^0} = 0 . \tag{4.40}$$

Thus, the extension ρ_Y is well-defined. A similar argument shows that the extension ρ_Y is 1-Lipschitz continuous.

In order to show that $\rho_Y(f)$ is a critical value of $f \in C^1(M)$ we first note that if f is in addition Morse then $\rho_Y(f)$ is a critical value by the very definition of ρ_Y. For the general case we point out that the space of Morse functions is in fact dense in the space of C^1-functions. Thus, for $f \in C^1(M)$ we can find a sequence f_n of smooth Morse functions such that $||f_n - f||_{C^1} \to 0$. In particular, also $||f_n - f||_{C^0} \to 0$ and therefore $\rho_Y(f) = \lim \rho_Y(f_n)$. Thus, there exists $x_n \in \mathrm{Crit}(f_n)$ such that $\rho_Y(f_n) = f_n(x_n)$. Because M is compact we can choose a convergent subsequence $x_{n_\nu} \to x \in M$. Since $||f_n - f||_{C^1} \to 0$ we conclude that $df(x) = \lim df_{n_\nu}(x_{n_\nu}) = 0$. Thus, $x \in \mathrm{Crit} f$. Finally,

$$f(x) = \lim f_{n_\nu}(x_{n_\nu}) = \lim \rho_Y(f_{n_\nu}) = \rho_Y(f) . \tag{4.41}$$

This proves the claim. $\qquad\square$

The following Theorem explains the term spectral invariant.

Theorem 4.12. *Let $\{f_r\}_{r \in [0,1]}$ be a continuous family of C^1 functions such that the spectrum $\mathfrak{S}(f_r) \subset \mathbb{R}$ is independent of r and nowhere dense. Then*

$$\rho_Y(f_0) = \rho_Y(f_1) \tag{4.42}$$

for all $Y \neq 0 \in \mathrm{HM}_*$.

Remark 4.13. The assumption that $\mathfrak{S}(f_r)$ is nowhere dense follows from Sard theorem if f_r is sufficiently differentiable.

PROOF. We consider the function

$$\begin{aligned} [0,1] &\longrightarrow \mathbb{R} \\ r &\mapsto \rho_Y(f_r) . \end{aligned} \tag{4.43}$$

By Corollary 4.11 this map is continuous and by assumption takes values in a nowhere dense subset of $\mathbb{R}$. Thus, it's constant. $\qquad\square$

5. Spectral Invariants in Rabinowitz Floer homology

Definition 5.1. Let $\mathfrak{M} \in \mathcal{MP}^{reg}(\Sigma)$. For $\xi = \sum_c \xi_c \, c \neq 0 \in \mathrm{RFC}_*(\mathfrak{M})$ we set

$$\mathcal{A}^{\mathfrak{M}}(\xi) := \max \left\{ \mathcal{A}^{\mathfrak{M}}(c) \mid \xi_c \neq 0 \right\} \tag{5.1}$$

and for $X \neq 0 \in \mathrm{RFH}_*(\mathfrak{M})$

$$\sigma_{\mathfrak{M}}(X) := \inf \left\{ \mathcal{A}^{\mathfrak{M}}(\xi) \mid [\xi] = X \right\} \in \mathbb{R} \cup \{-\infty\} . \tag{5.2}$$

We call $\sigma_{\mathfrak{M}}(X)$ the spectral value of X.

Remark 5.2. A priori the spectral value $\sigma_{\mathfrak{M}}(X)$ depends on the almost complex structure J used in the definition of the boundary operator in the Rabinowitz Floer complex. As in the warm-up (section 4) it is easy to show that $\sigma_{\mathfrak{M}}(X)$ is in fact independent of J.

Lemma 5.3. Let $\mathfrak{M} \in \mathcal{MP}^{reg}(\Sigma)$. If the spectral value satisfies $\sigma_{\mathfrak{M}}(X) \in \mathbb{R}$ then it is a critical value:

$$\sigma_{\mathfrak{M}}(X) \in \mathfrak{S}(\mathcal{A}^{\mathfrak{M}}) := \mathcal{A}^{\mathfrak{M}}\big(\mathrm{Crit}(\mathcal{A}^{\mathfrak{M}})\big) . \tag{5.3}$$

PROOF. Let $\xi_n \in \mathrm{RFC}_*(\mathfrak{M})$ be a sequence such that $X = [\xi_n]$ and

$$\lim_{n \to \infty} \mathcal{A}^{\mathfrak{M}}(\xi_n) = \sigma_{\mathfrak{M}}(X) . \tag{5.4}$$

By definition there exist $c_n = (u_n, \eta_n) \in \mathrm{Crit}(\mathcal{A}^{\mathfrak{M}})$ with the property

$$\mathcal{A}^{\mathfrak{M}}(c_n) = \mathcal{A}^{\mathfrak{M}}(\xi_n) . \tag{5.5}$$

From Lemma 3.15 we conclude that there exists a constant $C = C(H)$ such that

$$|\eta_n| \leq C(|\mathcal{A}^{\mathfrak{M}}(c_n)| + 1) \tag{5.6}$$

and since $\lim_{n \to \infty} \mathcal{A}^{\mathfrak{M}}(\xi_n) = \sigma_{\mathfrak{M}}(X)$ the Lagrange multipliers η_n are uniformly bounded. Thus, by Arzela-Ascoli and the critical point equation (2.3) there exists a convergent subsequence $c_{n_k} \to c^* \in \mathrm{Crit}(\mathcal{A}^{\mathfrak{M}})$ satisfying

$$\mathcal{A}^{\mathfrak{M}}(c^*) = \sigma_{\mathfrak{M}}(X) . \tag{5.7}$$

$\square$

The goal of this section is to compare the spectral invariants for different Moser pairs. This is established in Theorem 5.5. The main idea is to estimate how the action develops along the continuation homomorphisms.

For that let $\mathfrak{M}_{\pm} = (F_{f_{\pm}}, H_{\pm}) \in \mathcal{MP}^{reg}(\Sigma)$. We abbreviate $\Sigma_{\pm} := \Sigma_{f_{\pm}}$ and choose a smooth monotone function $\theta : \mathbb{R} \longrightarrow [0, 1]$ with $\theta(s) = 0$ for $s \leq 0$ and $\theta(s) = 1$ for $s \geq 1$ with $0 \leq \theta'(s) \leq 2$. We set

$$f_s := \theta(s)(f_+ - f_-) + f_- \tag{5.8}$$

and

$$F_s := F_{f_s} \quad \text{and} \quad H_s := \theta(s)(H_+ - H_-) + H_- . \tag{5.9}$$

For the definition of the function F_f we refer to equation (3.3). We consider the following family of Rabinowitz action functionals

$$\mathcal{A}_s(u, \eta) := -\int_0^1 u^*\lambda - \int_0^1 H_s(u(t), t)dt - \eta \int_0^1 F_s(u(t), t)dt \tag{5.10}$$

and set

$$\mathcal{A}_{\pm}(u, \eta) := -\int_0^1 u^*\lambda - \int_0^1 H_{\pm}(u(t), t)dt - \eta \int_0^1 F_{f_{\pm}}(u(t), t)dt . \tag{5.11}$$

The continuation homomorphism $\zeta_{\mathfrak{M}_-}^{\mathfrak{M}_+} : \mathrm{RFH}_*(\mathfrak{M}_-) \longrightarrow \mathrm{RFH}_*(\mathfrak{M}_+)$ is defined by counting solutions of

$$\partial_s w(s) + \nabla \mathcal{A}_s(w(s)) = 0 \tag{5.12}$$

that is, $w = (u, \eta)$ solves the problem

$$\left.\begin{aligned} \partial_s u + J(u)\big(\partial_t u - X_{H_s}(u,t) - \eta X_{F_s}(u,t)\big) &= 0 \\[1em] \partial_s \eta - \int_0^1 F_s(u,t)dt &= 0. \end{aligned}\right\} \tag{5.13}$$

Proposition 5.4. *Let w be a solution of* (5.12) *with* $\lim_{s\to\pm\infty} = w_\pm \in \mathrm{Crit}\mathcal{A}_\pm$. *If*

$$||f_+ - f_-||_{C^0(\Sigma)} \leq \frac{\delta(2-\delta)}{128 - 56\delta} \tag{5.14}$$

then

$$\mathcal{A}_+(w_+) \leq \max\left\{\left(1 + \frac{8\Delta_1}{2-\delta}\right)\mathcal{A}_-(w_-), 0\right\} + \Delta_0 + 2\Delta_1\left(\frac{64 - 28\delta}{\delta(2-\delta)}\Delta_0 + \frac{(\delta + 4\Delta_2)}{2-\delta}\right) \tag{5.15}$$

where we use the abbreviations

$$\Delta_0 := \int_0^1 ||H_-(\cdot, t) - H_+(\cdot, t)||_{C^0(M)}dt, \quad \Delta_1 := ||f_+ - f_-||_{C^0(\Sigma)}, \quad \Delta_2 := \max\{\kappa(H_+), \kappa(H_-)\}. \tag{5.16}$$

PROOF. Since by definition

$$F_s(y, t) = \begin{cases} \rho(t)\beta\big(r - f_s(x)\big) & \text{for } y = (x, r) \in \Sigma \times \mathbb{R} \\ -\delta\,\rho(t) & \text{for } y \in M \setminus (\Sigma \times \mathbb{R}) \end{cases} \tag{5.17}$$

we have

$$F_s'(y, t) := \frac{d}{ds}F_s(y, t) = \begin{cases} -\rho(t)\beta'\big(r - f_s(x)\big)\theta'(s)\big(f_1(x) - f_0(x)\big) & \text{for } y = (x, r) \in \Sigma \times \mathbb{R} \\ 0 & \text{for } y \in M \setminus (\Sigma \times \mathbb{R}) \end{cases} \tag{5.18}$$

and thus

$$|F_s'(y, t)| \leq 2\rho(t)\theta'(s)||f_+ - f_-||_{C^0(\Sigma)} \tag{5.19}$$

using that $0 \leq \beta' \leq 2$. For $v = (u, \eta) \in \mathscr{L}_M \times \mathbb{R}$ we compute

$$\mathcal{A}_s'(v) := \frac{\partial \mathcal{A}_s}{\partial s}(v) = -\int_0^1 H_s'(u(t), t)dt - \eta\int_0^1 F_s'(u(t), t)dt. \tag{5.20}$$

We set

$$0 \leq E^\sigma(w) := \int_{-\infty}^\sigma ||\partial_s w(s)||^2 ds, \qquad 0 \leq E_\sigma(w) := \int_\sigma^\infty ||\partial_s w(s)||^2 ds. \tag{5.21}$$

We recall that w solves (5.12) and estimate

$$
\mathcal{A}_\sigma(w(\sigma)) = \mathcal{A}_-(w_-) + \int_{-\infty}^\sigma \frac{d}{ds}\mathcal{A}_s(w(s))ds
$$

$$
= \mathcal{A}_-(w_-) + \int_{-\infty}^\sigma \mathcal{A}_s'(w(s)) + d\mathcal{A}_s(w(s))[\partial_s w(s)]ds
$$

$$
= \mathcal{A}_-(w_-) + \int_{-\infty}^\sigma \mathcal{A}_s'(w(s)) + \langle \nabla\mathcal{A}_s(w(s)), \partial_s w(s)\rangle ds
$$

$$
= \mathcal{A}_-(w_-) + \int_{-\infty}^\sigma \mathcal{A}_s'(w(s)) + \langle -\partial_s w(s), \partial_s w(s)\rangle ds
$$

$$
= \mathcal{A}_-(w_-) + \int_{-\infty}^\sigma \mathcal{A}_s'(w(s))ds - E^\sigma(w)
$$

$$
= \mathcal{A}_-(w_-) - E^\sigma(w) - \int_{-\infty}^\sigma \int_0^1 \left(H_s'(u(t),t) + \eta(s)F_s'(u(t),t) \right) dt ds
$$

$$
\leq \mathcal{A}_-(w_-) - E^\sigma(w)
$$
$$
+ \int_{-\infty}^\sigma \int_0^1 \left(-\theta'(s)\min_M\{H_+(\cdot,t) - H_-(\cdot,t)\} + 2\eta(s)\rho(t)\theta'(s)\|f_+ - f_-\|_{C^0(\Sigma)} \right) dt ds
$$

$$
= \mathcal{A}_-(w_-) - E^\sigma(w)
$$
$$
+ \int_{-\infty}^\sigma \theta'(s)\int_0^1 \left(-\min_M\{H_+(\cdot,t) - H_-(\cdot,t)\} + 2\eta(s)\rho(t)\|f_+ - f_-\|_{C^0(\Sigma)} \right) dt ds
$$

$$
= \mathcal{A}_-(w_-) - E^\sigma(w) + \int_{-\infty}^\sigma \theta'(s)\big(\|H_+ - H_-\|_- + 2\eta(s)\|f_+ - f_-\|_{C^0(\Sigma)}\big)ds
$$

$$
\leq \mathcal{A}_-(w_-) - E^\sigma(w) + \|H_+ - H_-\|_- + 2\|\eta\|_{C^0(\mathbb{R})}\|f_+ - f_-\|_{C^0(\Sigma)}
$$
$$
(5.22)
$$

where

$$
\|H_+ - H_-\|_+ = \int_0^1 \max_M\{(H_+ - H_-)(\cdot,t)\}dt, \quad \|H_+ - H_-\|_- = -\int_0^1 \min_M\{(H_+ - H_-)(\cdot,t)\}dt
$$

and similarly it holds

$$
\mathcal{A}_\sigma(w(\sigma)) \geq \mathcal{A}_+(w_+) + E_\sigma(w) - \|H_+ - H_-\|_+ - 2\|\eta\|_{C^0(\mathbb{R})}\|f_+ - f_-\|_{C^0(\Sigma)} . \tag{5.23}
$$

In particular, we have

$$
|\mathcal{A}_\sigma(w(\sigma))| \leq \max\{\mathcal{A}_-(w_-), -\mathcal{A}_+(w_+)\}
$$
$$
+ \int_0^1 \|H_-(\cdot,t) - H_+(\cdot,t)\|_{C^0(M)}dt + 2\|\eta\|_{C^0(\mathbb{R})}\|f_+ - f_-\|_{C^0(\Sigma)} . \tag{5.24}
$$

Moreover, we obtain for $\sigma = +\infty$

$$
\mathcal{A}_+(w_+) \leq \mathcal{A}_-(w_-) - E(w) + \|H_+ - H_-\|_- + 2\|\eta\|_{C^0(\mathbb{R})}\|f_+ - f_-\|_{C^0(\Sigma)} . \tag{5.25}
$$

For $\sigma \in \mathbb{R}$ we define

$$
\tau(\sigma) := \inf\left\{ \tau \geq 0 \mid \|\nabla\mathcal{A}_{\sigma+\tau}(w(\sigma+\tau))\| \leq \frac{\delta}{4} \right\} \tag{5.26}
$$

where δ is as in Lemma 3.15. Then we compute for the energy

$$E(w) = \int_{-\infty}^{\infty} ||\partial_s w(s)||^2 ds \geq \int_{\sigma}^{\sigma+\tau(\sigma)} \underbrace{||\nabla \mathcal{A}_s(w(s))||^2}_{\geq \frac{\delta^2}{16}} ds \geq \tau(\sigma)\frac{\delta^2}{16} \ . \qquad (5.27)$$

Combining the estimates (5.25) and (5.27) we obtain

$$\tau(\sigma) \leq \frac{16}{\delta^2} E(w) \leq \frac{16}{\delta^2}\Big(\mathcal{A}_-(w_-) - \mathcal{A}_+(w+) + ||H_+ - H_-||_- + 2||\eta||_{C^0(\mathbb{R})}||f_+ - f_-||_{C^0(\Sigma)}\Big) \ . \tag{5.28}$$

From the definition of F_s (see equation (5.9)) it follows

$$\left|\int_0^1 F_s(t, u(t))dt\right| \leq \delta \int_0^1 \rho(t)dt = \delta \ . \qquad (5.29)$$

From these estimates and the gradient flow equation (5.13)

$$\partial_s \eta(s) = \int_0^1 F_s(t, u(t))dt \qquad (5.30)$$

we obtain

$$\begin{aligned}
|\eta(\sigma)| &\leq |\eta(\sigma + \tau(\sigma))| + \int_{\sigma}^{\tau(\sigma)+\sigma} |\partial_s \eta(s)|ds \\
&= |\eta(\sigma + \tau(\sigma))| + \int_{\sigma}^{\tau(\sigma)+\sigma} \left|\int_0^1 F_s(t, u(t))dt\right| ds \\
&\leq |\eta(\sigma + \tau(\sigma))| + \tau(\sigma)\delta \\
&\leq |\eta(\sigma + \tau(\sigma))| \\
&\quad + \frac{16}{\delta}\Big(\mathcal{A}_-(w_-) - \mathcal{A}_+(w+) + ||H_+ - H_-||_- + 2||\eta||_{C^0(\mathbb{R})}||f_+ - f_-||_{C^0(\Sigma)}\Big)
\end{aligned} \qquad (5.31)$$

Using Lemma 3.15, the definition of $\tau(\sigma)$, and estimate (5.24) we get

$$\begin{aligned}
|\eta(\sigma + \tau(\sigma))| &\leq \frac{2}{2-\delta}\Big(|\mathcal{A}_\sigma(w(\sigma))| + \delta/4 + \max\{\kappa(H_+), \kappa(H_-)\}\Big) \\
&\leq \frac{2}{2-\delta}\bigg(\max\{\mathcal{A}_-(w_-), -\mathcal{A}_+(w_+)\} + \int_0^1 ||H_-(\cdot, t) - H_+(\cdot, t)||_{C^0(M)}dt \\
&\quad + 2||\eta||_{C^0(\mathbb{R})}||f_+ - f_-||_{C^0(\Sigma)} + \delta/4 + \kappa(H)\bigg)
\end{aligned} \qquad (5.32)$$

where we used that $\kappa(H)$ is a semi-norm, in particular,

$$\begin{aligned}
\kappa(H_s) &= \kappa\big(\theta(s)H_+ + (1-\theta(s))H_-\big) \\
&\leq \theta(s)\kappa\big(H_+\big) + (1-\theta(s))\kappa\big(H_-\big) \\
&\leq \max\{\kappa(H_+), \kappa(H_-)\} \ .
\end{aligned} \qquad (5.33)$$

We recall the abbreviation

$$\Delta_2 = \max\{\kappa(H_+), \kappa(H_-)\} \ . \qquad (5.34)$$

Combining the previous two inequalities we obtain

$$
\begin{aligned}
|\eta(\sigma)| \leq \frac{2}{2-\delta} \bigg(& \max\{\mathcal{A}_-(w_-), -\mathcal{A}_+(w_+)\} + \int_0^1 ||H_-(\cdot,t) - H_+(\cdot,t)||_{C^0(M)} dt \\
& + 2||\eta||_{C^0(\mathbb{R})} ||f_+ - f_-||_{C^0(\Sigma)} + \delta/4 + \Delta_2 \bigg) \\
& + \frac{16}{\delta} \Big(\mathcal{A}_-(w_-) - \mathcal{A}_+(w_+) + ||H_+ - H_-||_- + 2||\eta||_{C^0(\mathbb{R})} ||f_+ - f_-||_{C^0(\Sigma)} \Big)
\end{aligned}
$$
(5.35)

$$
\begin{aligned}
\leq & \frac{32 - 14\delta}{\delta(2-\delta)} \bigg(\int_0^1 ||H_-(\cdot,t) - H_+(\cdot,t)||_{C^0(M)} dt + 2||\eta||_{C^0(\mathbb{R})} ||f_+ - f_-||_{C^0(\Sigma)} \bigg) \\
& + \frac{2}{2-\delta} \max\{\mathcal{A}_-(w_-), -\mathcal{A}_+(w_+)\} + \frac{16}{\delta} \big(\mathcal{A}_-(w_-) - \mathcal{A}_+(w_+) \big) \\
& + \frac{(\delta + 4\Delta_2)}{4 - 2\delta} \ .
\end{aligned}
$$

We recall the abbreviation

$$
\Delta_0 = \int_0^1 ||H_-(\cdot,t) - H_+(\cdot,t)||_{C^0(M)} dt \ .
$$
(5.36)

Since the right hand side of (5.35) is independent of σ we conclude that

$$
\begin{aligned}
||\eta||_{C^0(\mathbb{R})} \leq & \frac{32 - 14\delta}{\delta(2-\delta)} \Big(\Delta_0 + 2||\eta||_{C^0(\mathbb{R})} ||f_+ - f_-||_{C^0(\Sigma)} \Big) + \frac{(\delta + 4\Delta_2)}{4 - 2\delta} \\
& + \frac{2}{2-\delta} \max\{\mathcal{A}_-(w_-), -\mathcal{A}_+(w_+)\} + \frac{16}{\delta} \big(\mathcal{A}_-(w_-) - \mathcal{A}_+(w_+) \big)
\end{aligned}
$$
(5.37)

thus

$$
\begin{aligned}
\Big(1 - \frac{64 - 28\delta}{\delta(2-\delta)} ||f_+ - f_-||_{C^0(\Sigma)} \Big) ||\eta||_{C^0(\mathbb{R})} \leq & \frac{32 - 14\delta}{\delta(2-\delta)} \Delta_0 + \frac{(\delta + 4\Delta_2)}{4 - 2\delta} \\
& + \frac{2}{2-\delta} \max\{\mathcal{A}_-(w_-), -\mathcal{A}_+(w_+)\} \\
& + \frac{16}{\delta} \big(\mathcal{A}_-(w_-) - \mathcal{A}_+(w_+) \big) \ .
\end{aligned}
$$
(5.38)

Now we recall our assumption

$$
||f_+ - f_-||_{C^0(\Sigma)} \leq \frac{\delta(2-\delta)}{128 - 56\delta}
$$
(5.39)

and therefore

$$
\begin{aligned}
||\eta||_{C^0(\mathbb{R})} \leq & \frac{64 - 28\delta}{\delta(2-\delta)} \Delta_0 + \frac{(\delta + 4\Delta_2)}{2 - \delta} \\
& + \frac{4}{2-\delta} \max\{\mathcal{A}_-(w_-), -\mathcal{A}_+(w_+)\} + \frac{32}{\delta} \big(\mathcal{A}_-(w_-) - \mathcal{A}_+(w_+) \big) \ .
\end{aligned}
$$
(5.40)

Using the abbreviation

$$
\Delta_1 = ||f_+ - f_-||_{C^0(\Sigma)}
$$
(5.41)

and combining the inequalities (5.25) and (5.40) we obtain

$$
\begin{aligned}
\mathcal{A}_+(w_+) \leq \mathcal{A}_-(w_-) + \Delta_0 + 2\Delta_1 \bigg(& \frac{64 - 28\delta}{\delta(2-\delta)}\Delta_0 + \frac{(\delta + 4\Delta_2)}{2-\delta} \\
& + \frac{4}{2-\delta}\max\{\mathcal{A}_-(w_-), -\mathcal{A}_+(w_+)\} + \frac{32}{\delta}\big(\mathcal{A}_-(w_-) - \mathcal{A}_+(w_+)\big) \bigg).
\end{aligned}
\tag{5.42}
$$

In the case $\mathcal{A}_+(w_+) \leq \mathcal{A}_-(w_-)$ or $0 \geq \mathcal{A}_+(w_+)$ the assertion of the Proposition to be proved follows trivially. Therefore, from now on we assume that $\mathcal{A}_+(w_+) \geq \mathcal{A}_-(w_-)$ and $\mathcal{A}_+(w_+) \geq 0$. Then we can simplify the above estimate to

$$
\begin{aligned}
\mathcal{A}_+(w_+) \leq \mathcal{A}_-(w_-) &+ \Delta_0 \\
&+ 2\Delta_1 \bigg(\frac{64 - 28\delta}{\delta(2-\delta)}\Delta_0 + \frac{(\delta + 4\Delta_2)}{2-\delta} + \frac{4}{2-\delta}\max\{\mathcal{A}_-(w_-), -\mathcal{A}_+(w_+)\} \bigg).
\end{aligned}
\tag{5.43}
$$

Next we distinguish two cases. If $\mathcal{A}_+(w_+) \geq \mathcal{A}_-(w_-) \geq 0$ then

$$
\begin{aligned}
\mathcal{A}_+(w_+) &\leq \mathcal{A}_-(w_-) + \Delta_0 + 2\Delta_1 \bigg(\frac{64 - 28\delta}{\delta(2-\delta)}\Delta_0 + \frac{(\delta + 4\Delta_2)}{2-\delta} + \frac{4}{2-\delta}\mathcal{A}_-(w_-) \bigg) \\
&= \mathcal{A}_-(w_-)\Big(1 + \frac{8\Delta_1}{2-\delta}\Big) + \Delta_0 + 2\Delta_1 \bigg(\frac{64 - 28\delta}{\delta(2-\delta)}\Delta_0 + \frac{(\delta + 4\Delta_2)}{2-\delta} \bigg).
\end{aligned}
\tag{5.44}
$$

If $\mathcal{A}_+(w_+) \geq 0 \geq \mathcal{A}_-(w_-)$ then

$$
\mathcal{A}_+(w_+) \leq \Delta_0 + 2\Delta_1 \bigg(\frac{64 - 28\delta}{\delta(2-\delta)}\Delta_0 + \frac{(\delta + 4\Delta_2)}{2-\delta} \bigg).
\tag{5.45}
$$

The Proposition follows from the last two inequalities. $\qquad\square$

Theorem 5.5. *Let $\mathfrak{M}_\pm = (F_{f_\pm}, H_\pm) \in \mathcal{MP}^{reg}(\Sigma)$. We abbreviate $\Sigma_\pm := \Sigma_{f_\pm}$. Then*

$$
\sigma_{\mathfrak{M}_-}(X_-) \leq e^{\frac{16\Delta_1}{2-\delta}} \max\left\{\sigma_{\mathfrak{M}_-}(X_-), 0\right\} + \Big(\frac{(2-\delta)\Delta_0}{\Delta_1} + 2(\delta + 4\Delta_2)\Big)\Big(e^{\frac{16\Delta_1}{2-\delta}} - 1\Big)
\tag{5.46}
$$

where $X_+ = \zeta_{\mathfrak{M}_-}^{\mathfrak{M}_+}(X_-) \neq 0 \in \mathrm{RFH}_*(\mathfrak{M}_+)$ and Δ_0, Δ_1, and Δ_2 are as in Proposition 5.4.

PROOF. Under the assumption

$$
||f_+ - f_-||_{C^0(\Sigma)} \leq \frac{\delta(2-\delta)}{128 - 56\delta}
\tag{5.47}
$$

Proposition 5.4 implies as in Corollary 4.6 that

$$
\sigma_{\mathfrak{M}_+}(X_+) \leq \max\left\{\Big(1 + \frac{8\Delta_1}{2-\delta}\Big)\sigma_{\mathfrak{M}_-}(X_-), 0\right\} + \Delta_0 + 2\Delta_1\bigg(\frac{64 - 28\delta}{\delta(2-\delta)}\Delta_0 + \frac{(\delta + 4\Delta_2)}{2-\delta}\bigg).
\tag{5.48}
$$

where $X_1 = \zeta_{\mathfrak{M}_0}^{\mathfrak{M}_1}(X_0) \neq 0 \in \mathrm{RFH}_*(\mathfrak{M}_1)$. In general this assumption is not satisfied. But we can always split the homotopy from f_- to f_+ into many small homotopies each of which satisfies the above inequality. To obtain the statement of the theorem we eventually take an adiabatic limit. We again define

$$
f_s := \theta(s)(f_+ - f_-) + f_-, \quad s \in \mathbb{R}
\tag{5.49}
$$

and

$$
F_s := F_{f_s} \quad \text{and} \quad H_s := \theta(s)(H_+ - H_-) + H_-.
\tag{5.50}
$$

where θ is the cut-off function defined above Proposition 5.4. We choose $N \in \mathbb{N}$ such that

$$N \geq \frac{256 - 112\delta}{\delta(2 - \delta)} ||f_+ - f_-||_{C^0(\Sigma)} \tag{5.51}$$

and set for $k = 0, \ldots, N$

$$f^k := f_{\frac{k}{N}}, \qquad H^k := H_{\frac{k}{N}}, \quad \text{and} \quad \mathfrak{M}^k := (F_{f^k}, H^k) . \tag{5.52}$$

For convenience we proceed with the proof under the assumption that $\mathfrak{M}^k$ is a regular Moser pair. Otherwise, in the following arguments $\mathfrak{M}^k$ has to be replaced by an arbitrarily small regular perturbation. By taking the limit this does not influence the action estimates. We recall that $0 \leq \theta' \leq 2$ and observe that by the choice of N

$$\begin{aligned}
||f^{k+1} - f^k||_{C^0(\Sigma)} &= ||\big(\theta(\tfrac{k+1}{N}) - \theta(\tfrac{k}{N})\big)(f_+ - f_-)||_{C^0(\Sigma)} \\
&\leq 2\big(\tfrac{k+1}{N} - \tfrac{k}{N}\big)||f_+ - f_-||_{C^0(\Sigma)} \\
&\leq \tfrac{2}{N}||f_+ - f_-||_{C^0(\Sigma)} \\
&\leq \frac{\delta(2 - \delta)}{128 - 56\delta} .
\end{aligned} \tag{5.53}$$

In particular,

$$\Delta_1^k := ||f^{k+1} - f^k||_{C^0(\Sigma)} \leq \tfrac{2}{N}\Delta_1 . \tag{5.54}$$

Similarly,

$$||H_{\frac{k+1}{N}}(\cdot, t) - H_{\frac{k}{N}}(\cdot, t)||_{C^0(M)} \leq \tfrac{2}{N}||H_+(\cdot, t) - H_-(\cdot, t)||_{C^0(M)} \tag{5.55}$$

and therefore

$$\Delta_0^k := \int_0^1 ||H_{\frac{k+1}{N}}(\cdot, t) - H_{\frac{k}{N}}(\cdot, t)||_{C^0(M)} dt \leq \tfrac{2}{N}\Delta_0 . \tag{5.56}$$

Finally, since κ is a semi-norm

$$\kappa(H^k) \leq \max\{\kappa(H_+), \kappa(H_-)\} = \Delta_2 . \tag{5.57}$$

Thus, we conclude from Proposition 5.4 as explained at the beginning of the proof

$$\begin{aligned}
\sigma_{\mathfrak{M}^{k+1}}(X^{k+1}) &\leq \max\left\{\left(1 + \frac{8\Delta_1^k}{2 - \delta}\right)\sigma_{\mathfrak{M}^k}(X^k), 0\right\} + \Delta_0^k + 2\Delta_1^k\left(\frac{64 - 28\delta}{\delta(2 - \delta)}\Delta_0^k + \frac{(\delta + 4\Delta_2)}{2 - \delta}\right) \\
&\leq \max\left\{\left(1 + \frac{16\Delta_1}{2 - \delta}\frac{1}{N}\right)\sigma_{\mathfrak{M}^k}(X^k), 0\right\} + \frac{2}{N}\Delta_0 + \frac{4}{N}\Delta_1\left(\frac{64 - 28\delta}{\delta(2 - \delta)}\frac{2}{N}\Delta_0 + \frac{(\delta + 4\Delta_2)}{2 - \delta}\right)
\end{aligned} \tag{5.58}$$

where $X^{k+1} = \zeta_{\mathfrak{M}^k}^{\mathfrak{M}^{k+1}}(X^k) = \zeta_{\mathfrak{M}^k}^{\mathfrak{M}^{k+1}}(X_-)$. Lemma A.1 implies that

$$\sigma_{\mathfrak{M}_+}(X_+) \leq \left(1 + \frac{16\Delta_1}{2-\delta}\frac{1}{N}\right)^N \max\left\{\sigma_{\mathfrak{M}_-}(X_-), \frac{2}{N}\Delta_0 + \frac{4}{N}\Delta_1\left(\frac{64-28\delta}{\delta(2-\delta)}\frac{2}{N}\Delta_0 + \frac{(\delta+4\Delta_2)}{2-\delta}\right)\right\}$$

$$+ \left(\frac{2}{N}\Delta_0 + \frac{4}{N}\Delta_1\left(\frac{64-28\delta}{\delta(2-\delta)}\frac{2}{N}\Delta_0 + \frac{(\delta+4\Delta_2)}{2-\delta}\right)\right)\frac{\left(1 + \frac{16\Delta_1}{2-\delta}\frac{1}{N}\right)^N - 1}{\left(1 + \frac{16\Delta_1}{2-\delta}\frac{1}{N}\right) - 1}$$

$$= \left(1 + \frac{16\Delta_1}{2-\delta}\frac{1}{N}\right)^N \max\left\{\sigma_{\mathfrak{M}_-}(X_-), \frac{2}{N}\Delta_0 + \frac{4}{N}\Delta_1\left(\frac{64-28\delta}{\delta(2-\delta)}\frac{2}{N}\Delta_0 + \frac{(\delta+4\Delta_2)}{2-\delta}\right)\right\}$$

$$+ N\frac{2-\delta}{16\Delta_1}\left(\frac{2}{N}\Delta_0 + \frac{4}{N}\Delta_1\left(\frac{64-28\delta}{\delta(2-\delta)}\frac{2}{N}\Delta_0 + \frac{(\delta+4\Delta_2)}{2-\delta}\right)\right)\left(\left(1 + \frac{16\Delta_1}{2-\delta}\frac{1}{N}\right)^N - 1\right)$$

$$= \left(1 + \frac{16\Delta_1}{2-\delta}\frac{1}{N}\right)^N \max\left\{\sigma_{\mathfrak{M}_-}(X_-), \frac{2}{N}\Delta_0 + \frac{4}{N}\Delta_1\left(\frac{64-28\delta}{\delta(2-\delta)}\frac{2}{N}\Delta_0 + \frac{(\delta+4\Delta_2)}{2-\delta}\right)\right\}$$

$$+ \frac{2-\delta}{16\Delta_1}\left(2\Delta_0 + 4\Delta_1\left(\frac{64-28\delta}{\delta(2-\delta)}\frac{2}{N}\Delta_0 + \frac{(\delta+4\Delta_2)}{2-\delta}\right)\right)\left(\left(1 + \frac{16\Delta_1}{2-\delta}\frac{1}{N}\right)^N - 1\right).$$

In the limit $N \to \infty$

$$\sigma_{\mathfrak{M}_+}(X_+) \leq e^{\frac{16\Delta_1}{2-\delta}} \max\left\{\sigma_{\mathfrak{M}_-}(X_-), 0\right\} + \frac{2-\delta}{8\Delta_1}\left(\Delta_0 + 2\Delta_1\frac{(\delta+4\Delta_2)}{2-\delta}\right)\left(e^{\frac{16\Delta_1}{2-\delta}} - 1\right)$$

$$= e^{\frac{16\Delta_1}{2-\delta}} \max\left\{\sigma_{\mathfrak{M}_-}(X_-), 0\right\} + \frac{1}{8}\left(\frac{(2-\delta)\Delta_0}{\Delta_1} + 2(\delta+4\Delta_2)\right)\left(e^{\frac{16\Delta_1}{2-\delta}} - 1\right). \tag{5.59}$$

$\square$

Definition 5.6. We define the norm of an adapted Moser pair $\mathfrak{M} = (F_f, H) \in \mathcal{MP}(\Sigma)$ by

$$||\mathfrak{M}|| := ||f||_{C^0(\Sigma)} + \int_0^1 ||H(\cdot, t)||_{C^0(M)}dt + \kappa(H). \tag{5.60}$$

We denote by

$$\mathcal{D}(\overline{\mathfrak{M}}) := \{\mathfrak{M}' = (F', H') \mid ||\overline{\mathfrak{M}} - \mathfrak{M}'|| < 1\} \tag{5.61}$$

the open 1-ball around $\overline{\mathfrak{M}}$ in $\mathcal{MP}(\Sigma)$.

Estimating $\Delta_0, \Delta_1 \leq ||\mathfrak{M}_+ - \mathfrak{M}_-||$ and $\Delta_2 \leq \max\{||\mathfrak{M}_+||, ||\mathfrak{M}_-||\}$ and using of the monotonicity of $x \mapsto \frac{e^x - 1}{x}$ for $x \geq 0$ we immediately obtain from Theorem 5.5 the following corollary.

Corollary 5.7. *Under the assumptions of Theorem 5.5 we have*

$$\sigma_{\mathfrak{M}_+}(X_+) \leq e^{\frac{16||\mathfrak{M}_+ - \mathfrak{M}_-||}{2-\delta}} \max\left\{\sigma_{\mathfrak{M}_-}(X_-), 0\right\} + \frac{1}{8}\left(2+\delta+8\max\{||\mathfrak{M}_+||, ||\mathfrak{M}_-||\}\right)\left(e^{\frac{16||\mathfrak{M}_+ - \mathfrak{M}_-||}{2-\delta}} - 1\right) \tag{5.62}$$

Definition 5.8. For a weakly regular Moser pair $\mathfrak{M} \in \mathcal{MP}^{reg}(\Sigma)$ and $X \neq 0 \in \mathrm{RFH}_*$ we set

$$\sigma_{\mathfrak{M}}(X) := \sigma\left(\zeta^{\mathfrak{M}}(X)\right) \tag{5.63}$$

where RFH_* and $\zeta^{\mathfrak{M}}$ are defined in Definition 3.12. Moreover, for fixed $X \neq 0 \in \mathrm{RFH}_*$ we define

$$\rho_X : \mathcal{MP}^{reg}(\Sigma) \longrightarrow \mathbb{R}$$
$$\mathfrak{M} \mapsto \sigma_{\mathfrak{M}}(X). \tag{5.64}$$

Convention 5.9. From now on we fix a weakly regular Moser pair $\mathfrak{M}_0 = (F_{f_0}, H_0) \in \mathcal{MP}^{reg}(\Sigma)$.

Lemma 5.10. For a Moser pair $\overline{\mathfrak{M}} = (\overline{F}, \overline{H})$ we define

$$\mathcal{B}(\overline{\mathfrak{M}}) := \left\{ X \in \mathrm{RFH}_* \mid \sigma_{\mathfrak{M}_0}(X) > \frac{1}{8}\Big(2 + \delta + 8\max\left\{||\mathfrak{M}_0||, ||\overline{\mathfrak{M}}|| + 1\right\}\Big)\Big(e^{\frac{16(||\mathfrak{M}_0 - \overline{\mathfrak{M}}|| + 1)}{2 - \delta}} - 1\Big) \right\} \tag{5.65}$$

If $X \in \mathcal{B}(\overline{\mathfrak{M}})$ then the map

$$\rho_X : \mathcal{MP}^{reg} \longrightarrow \mathbb{R}, \qquad \mathfrak{M} \mapsto \sigma_{\mathfrak{M}}(X) \tag{5.66}$$

is locally Lipschitz continuous around $\overline{\mathfrak{M}}$ with respect to the norm on $\mathcal{MP}(\Sigma)$ introduced in Definition 5.6.

PROOF. We recall that $\mathcal{D}(\overline{\mathfrak{M}})$ denotes the open 1-ball around $\overline{\mathfrak{M}}$ in $\mathcal{MP}(\Sigma)$. We assume by contradiction $\sigma_{\mathfrak{M}'}(X) \leq 0$, $\forall \mathfrak{M}' \in \mathcal{D}(\overline{\mathfrak{M}}) \cap \mathfrak{M}^{reg}$. Then applying Corollary 5.7 to $\sigma_{\mathfrak{M}_-}(X) \equiv \sigma_{\mathfrak{M}'}(X)$ and $\sigma_{\mathfrak{M}_+}(X) \equiv \sigma_{\mathfrak{M}_0}(X)$ we obtain

$$\sigma_{\mathfrak{M}_0}(X) \leq \frac{1}{8}\Big(2 + \delta + 8\max\left\{||\mathfrak{M}_0||, ||\mathfrak{M}'||\right\}\Big)\Big(e^{\frac{16||\mathfrak{M}_0 - \mathfrak{M}'||}{2 - \delta}} - 1\Big) \tag{5.67}$$

From

$$||\mathfrak{M}'|| < ||\overline{\mathfrak{M}}|| + 1, \quad ||\mathfrak{M}_0 - \mathfrak{M}'|| < ||\mathfrak{M}_0 - \overline{\mathfrak{M}}|| + 1 \tag{5.68}$$

we get

$$\begin{aligned} \sigma_{\mathfrak{M}_0}(X) &\leq \frac{1}{8}\Big(2 + \delta + 8\max\left\{||\mathfrak{M}_0||, ||\mathfrak{M}'||\right\}\Big)\Big(e^{\frac{16||\mathfrak{M}_0 - \mathfrak{M}'||}{2 - \delta}} - 1\Big) \\ &\leq \frac{1}{8}\Big(2 + \delta + 8\max\left\{||\mathfrak{M}_0||, ||\overline{\mathfrak{M}}|| + 1\right\}\Big)\Big(e^{\frac{16(||\mathfrak{M}_0 - \overline{\mathfrak{M}}|| + 1)}{2 - \delta}} - 1\Big) \end{aligned} \tag{5.69}$$

This contradicts the assumption that $X \in \mathcal{B}(\overline{\mathfrak{M}})$. Thus, we conclude

$$\sigma_{\mathfrak{M}'}(X) \geq 0 \qquad \forall \mathfrak{M}' \in \mathcal{D}(\overline{\mathfrak{M}}) \cap \mathfrak{M}^{reg} . \tag{5.70}$$

We choose $\mathfrak{M}', \mathfrak{M}'' \in \mathcal{D}(\overline{\mathfrak{M}}) \cap \mathfrak{M}^{reg}$ and estimate for $X \in \mathcal{B}(\overline{\mathfrak{M}})$ using again Corollary 5.7 and $\mathfrak{M}', \mathfrak{M}'' \in \mathcal{D}(\overline{\mathfrak{M}})$ and employing the elementary estimate $e^t \leq 1 + e^C t$ or $e^t - 1 \leq e^C t$, $\forall t \in [0, C]$:

$$\begin{aligned} \rho_X(\mathfrak{M}') &\leq e^{\frac{16||\mathfrak{M}' - \mathfrak{M}''||}{2 - \delta}} \rho_X(\mathfrak{M}'') + \frac{1}{8}\Big(2 + \delta + 8\max\left\{||\mathfrak{M}'||, ||\mathfrak{M}''||\right\}\Big)\Big(e^{\frac{16||\mathfrak{M}' - \mathfrak{M}''||}{2 - \delta}} - 1\Big) \\ &\leq \Big(1 + e^{\frac{32}{2-\delta}}\frac{16}{2 - \delta}||\mathfrak{M}' - \mathfrak{M}''||\Big)\rho_X(\mathfrak{M}'') \\ &\quad + \frac{1}{8}\Big(10 + \delta + 8||\overline{\mathfrak{M}}||\Big)\Big(e^{\frac{32}{2-\delta}}\frac{16}{2 - \delta}||\mathfrak{M}' - \mathfrak{M}''||\Big) \\ &= \rho_X(\mathfrak{M}'') + \Big(e^{\frac{32}{2-\delta}}\frac{16}{2 - \delta}||\mathfrak{M}' - \mathfrak{M}''||\Big)\rho_X(\mathfrak{M}'') \\ &\quad + \frac{1}{8}\Big(10 + \delta + 8||\overline{\mathfrak{M}}||\Big)\Big(e^{\frac{32}{2-\delta}}\frac{16}{2 - \delta}||\mathfrak{M}' - \mathfrak{M}''||\Big) . \end{aligned}$$

Using again Corollary 5.7 we estimate

$$
\begin{aligned}
\rho_X(\mathfrak{M}'') &\leq e^{\frac{16||\mathfrak{M}''-\mathfrak{M}_0||}{2-\delta}} \max\left\{\rho_X(\mathfrak{M}_0), 0\right\} \\
&\quad + \frac{1}{8}\left(2+\delta+8\max\{||\mathfrak{M}''||,||\mathfrak{M}_0||\}\right)\left(e^{\frac{16||\mathfrak{M}''-\mathfrak{M}_0||}{2-\delta}}-1\right) \\
&\leq e^{\frac{16(||\overline{\mathfrak{M}}-\mathfrak{M}_0||+1)}{2-\delta}} \max\left\{\rho_X(\mathfrak{M}_0), 0\right\} \\
&\quad + \frac{1}{8}\left(2+\delta+8\max\{||\overline{\mathfrak{M}}||+1,||\mathfrak{M}_0||\}\right)\left(e^{\frac{16(||\overline{\mathfrak{M}}-\mathfrak{M}_0||+1)}{2-\delta}}-1\right) \\
&=: C(\mathfrak{M}_0, X)\,.
\end{aligned}
\tag{5.71}
$$

Combining the last two inequalities we see

$$
\begin{aligned}
\rho_X(\mathfrak{M}') &\leq \rho_X(\mathfrak{M}'') + C(\mathfrak{M}_0, X)e^{\frac{32}{2-\delta}}\frac{16}{2-\delta}||\mathfrak{M}'-\mathfrak{M}''|| \\
&\quad + \frac{1}{8}\left(10+\delta+8||\overline{\mathfrak{M}}||\right)\left(e^{\frac{32}{2-\delta}}\frac{16}{2-\delta}||\mathfrak{M}'-\mathfrak{M}''||\right)
\end{aligned}
$$

and thus

$$
\begin{aligned}
\rho_X(\mathfrak{M}'') - \rho_X(\mathfrak{M}') &\leq C(\mathfrak{M}_0, X)e^{\frac{32}{2-\delta}}\frac{16}{2-\delta}||\mathfrak{M}'-\mathfrak{M}''|| \\
&\quad + \frac{1}{8}\left(10+\delta+8||\overline{\mathfrak{M}}||\right)\left(e^{\frac{32}{2-\delta}}\frac{16}{2-\delta}||\mathfrak{M}'-\mathfrak{M}''||\right) \\
&\leq D(\overline{\mathfrak{M}}, \mathfrak{M}_0, X)||\mathfrak{M}'-\mathfrak{M}''||
\end{aligned}
\tag{5.72}
$$

where we abbriviate

$$
D(\overline{\mathfrak{M}}, \mathfrak{M}_0, X) := C(\mathfrak{M}_0, X)e^{\frac{32}{2-\delta}}\frac{16}{2-\delta} + \frac{1}{8}\left(10+\delta+8||\overline{\mathfrak{M}}||\right)\left(e^{\frac{32}{2-\delta}}\frac{16}{2-\delta}\right)\,.
\tag{5.73}
$$

By symmetry

$$
|\rho_X(\mathfrak{M}'') - \rho_X(\mathfrak{M}')| \leq D(\overline{\mathfrak{M}}, \mathfrak{M}_0, X)||\mathfrak{M}'-\mathfrak{M}''||\,.
\tag{5.74}
$$

This proves the Lemma.   □

We recall that we fixed $\mathfrak{M}_0 \in \mathcal{MP}(\Sigma)$. Similarly as in Corollary 4.11 we can extend ρ_X.

Corollary 5.11. *Let $\overline{\mathfrak{M}} \in \mathfrak{M}(\Sigma)$ and $X \in \mathcal{B}(\overline{\mathfrak{M}})$. Then there exists a Lipschitz continuous function*

$$
\overline{\rho}_X : \mathcal{D}(\overline{\mathfrak{M}}) \longrightarrow (0, \infty)
\tag{5.75}
$$

satisfying

$$
\overline{\rho}_X(\mathfrak{M}') = \rho_X(\mathfrak{M}') \qquad \forall \mathfrak{M}' \in \mathcal{D}(\overline{\mathfrak{M}}) \cap \mathcal{MP}^{reg}(\Sigma)\,.
\tag{5.76}
$$

Moreover, is a spectral value

$$
\overline{\rho}_X(\mathfrak{M}) \in \mathfrak{S}(\mathcal{A}^{\mathfrak{M}})\,.
\tag{5.77}
$$

Finally, we have

$$
\rho_X(\mathfrak{M}_0) \leq e^{\frac{16||\mathfrak{M}_0-\overline{\mathfrak{M}}||}{2-\delta}}\overline{\rho}_X(\overline{\mathfrak{M}}) + \frac{1}{8}\left(2+\delta+8\max\{||\mathfrak{M}_0||,||\overline{\mathfrak{M}}||\}\right)\left(e^{\frac{16||\mathfrak{M}_0-\overline{\mathfrak{M}}||}{2-\delta}}-1\right)\,.
\tag{5.78}
$$

PROOF. That ρ_X has an extension as a Lipschitz continuous function follows immediately from Lemma 5.10 and the fact that $\mathfrak{M}^{reg}(\Sigma)$ is dense in $\mathfrak{M}(\Sigma)$, see Proposition 3.9.

To prove that $\overline{\rho}_X(\mathfrak{M})$ is a critical value of $\mathcal{A}^{\mathfrak{M}}$ we choose a sequence $\mathfrak{M}_n \in \mathfrak{M}^{reg}(\Sigma) \cap \mathcal{D}(\overline{\mathfrak{M}})$ with $\mathfrak{M}_n \longrightarrow \mathfrak{M}$. Then by Lemma 5.3 there exist $w_n = (v_n, \eta_n) \in \mathrm{Crit} \mathcal{A}^{\mathfrak{M}_n}$ with

$$\rho_X(\mathfrak{M}_n) = \mathcal{A}^{\mathfrak{M}_n}(w_n) \ . \tag{5.79}$$

Moreover, by Lemma 3.15 we conclude

$$\begin{aligned}
|\eta_n| &\leq C\big(|\mathcal{A}^{\mathfrak{M}_n}(w_n)| + 1\big) \\
&= C\big(|\rho_X(\mathfrak{M}_n)| + 1\big) \\
&\leq C\big(|\overline{\rho}_X(\overline{\mathfrak{M}})| + D(\overline{\mathfrak{M}}, \mathfrak{M}_0, X)\|\overline{\mathfrak{M}} - \mathfrak{M}_n\| + 1\big) \\
&\leq C\big(|\overline{\rho}_X(\overline{\mathfrak{M}})| + D(\overline{\mathfrak{M}}, \mathfrak{M}_0, X) + 1\big)
\end{aligned} \tag{5.80}$$

by Lipschitz continuity and definition of $\mathbb{D}(\overline{\mathfrak{M}})$. In particular, the sequence η_n is uniformly bounded and applying the Theorem of Arzela-Ascoli $w_{n_\nu} \to w^* \in \mathrm{Crit} \mathcal{A}^{\mathfrak{M}}$ and

$$\overline{\rho}_X(\mathfrak{M}) = \mathcal{A}^{\mathfrak{M}}(w^*) \ . \tag{5.81}$$

The last inequality claimed in the statement of the Corollary follows from Corollary 5.7 together with the observation that $\overline{\rho}_X(\mathfrak{M}) \geq 0$. The latter follows from (5.70) by continuity of ρ_X. $\qquad\square$

Definition 5.12. For an adapted Moser pair $\mathfrak{M} \in \mathcal{MP}(\Sigma)$ and $X \in \mathcal{B}(\overline{\mathfrak{M}})$ we define

$$\overline{\sigma}_{\overline{\mathfrak{M}}}(X) := \overline{\rho}_X(\overline{\mathfrak{M}}) \ . \tag{5.82}$$

Corollary 5.13. *We recall that we fixed a weakly regular $\mathfrak{M}_0$. If*

$$\{\sigma_{\mathfrak{M}_0}(X) \mid X \in \mathrm{RFH}_*\} \subset \mathbb{R} \cup \{-\infty\} \tag{5.83}$$

is unbounded from above then

$$\{\overline{\sigma}_{\overline{\mathfrak{M}}}(X) \mid X \in \mathcal{B}(\overline{\mathfrak{M}})\} \subset (0, \infty) \tag{5.84}$$

is also unbounded from above for all $\overline{\mathfrak{M}} \in \mathcal{MP}(\Sigma)$.

PROOF. The assumption that the spectral values are unbounded together with the definition of $\mathcal{B}(\overline{\mathfrak{M}})$, see Lemma 5.10, implies that also the set

$$\{\sigma_{\mathfrak{M}_0}(X) \mid X \in \mathcal{B}(\overline{\mathfrak{M}})\} \subset (0, \infty) \tag{5.85}$$

is unbounded from above. Combining this with the estimate in Corollary 5.11 implies the assertion. $\qquad\square$

6. PROOF OF THEOREM 1

We recall that in Theorem 1 we assume that $(M = T^*B, \omega)$ where B is a closed manifold and $S \subset M$ is fiber-wise star-shaped hypersurface. We fix a a bumpy metric g in the sense of Abraham [Abr70] and set

$$\Sigma := \{(q, p) \in T^*B \mid \|p\|_g^2 = 1\} \ . \tag{6.1}$$

According to the Theorem of Abraham [Abr70] bumpy metrics exist (and are even dense). Since g is bumpy the Moser pair

$$\mathfrak{M}_0 := (F_{f_0}, 0) \in \mathfrak{M}(\Sigma) \tag{6.2}$$

is weakly-regular if we choose $f_0 = 0$. A hypersurface in T^*B is fiber-wise star-shaped if and only if it is of the form Σ_f for some $f : \Sigma \longrightarrow \mathbb{R}$. In particular, there exists a function $f_S : \Sigma \longrightarrow \mathbb{R}$ with

$$S = \Sigma_{f_S} . \tag{6.3}$$

Proposition 6.1. *With the above notation we have*

$$\mu_{\mathrm{CZ}}(X) \geq 0 \quad \Longrightarrow \quad \sigma_{\mathfrak{M}_0}(X) \geq 0 \qquad \forall X \in \mathrm{RFH}_*(\mathfrak{M}_0) . \tag{6.4}$$

PROOF. First we recall that critical points (u, η) of $\mathcal{A}^{\mathfrak{M}_0}$ with positive/negative η are positively/negatively parametrized geodesics for g and that the Conley-Zehnder index coincides with the negative of the Morse index. In particular, positive Conley-Zehnder index implies negatively parametrised geodesics. Let

$$\xi = \sum_{c \,:\, \mu_{\mathrm{CZ}}(c)=k} \xi_c\, c \in \mathrm{RFC}_{\geq 0}(\mathfrak{M}_0) \tag{6.5}$$

then since $\mathfrak{M}_0 = (F_{f_0}, 0)$ the action value $\mathcal{A}^{\mathfrak{M}_0}(u, \eta) = -\eta$ is the negative of the period of the geodesic. In particular, $\mathcal{A}^{\mathfrak{M}_0}(c) \geq 0$ if $\mu_{\mathrm{CZ}}(c) \geq 0$. $\qquad\square$

Lemma 6.2. Under the same assumptions as in Theorem 1 for each $\kappa > 0$ the set

$$\mathscr{R}_\kappa := \{X \in \mathrm{RFH}_{\geq 0} \mid 0 \leq \sigma_{\mathfrak{M}_0}(X) \leq \kappa\} \tag{6.6}$$

is finite.

PROOF. We fix an auxiliary Morse function f on the critical set $\mathrm{Crit}\mathcal{A}^{\mathfrak{M}_0}$. Then

$$\mathscr{C}_\kappa := \{c \in \mathrm{Crit}(f) \mid 0 \leq \mathcal{A}^{\mathfrak{M}_0}(c) \leq \kappa\} \tag{6.7}$$

is finite, see Remark 3.11 for notation. Indeed, this follows from the theorem of Arzela-Ascoli together with assumption that $\mathfrak{M}_0$ is weakly regular, see also the proof of Lemma 5.3. If $X \in \mathrm{RFH}_{\geq 0}(\mathfrak{M}_0)$ and $\sigma_{\mathfrak{M}_0}(X) \leq \kappa$ then X is of the form

$$X = \sum_{c \in \mathscr{C}_\kappa} \xi_c c \tag{6.8}$$

with $\xi_c \in \mathbb{Z}/2$, and therefore,

$$\#\mathscr{R}_\kappa \leq 2^{\#\mathscr{C}_\kappa} \tag{6.9}$$

is finite. $\qquad\square$

Proposition 6.3. *Under the same assumptions as in Theorem 1 the set*

$$\{\sigma_{\mathfrak{M}_0}(X) \mid X \in \mathrm{RFH}_{\geq 0}\} \tag{6.10}$$

is unbounded from above.

PROOF. Assume by contradiction that there exists $\kappa > 0$ such that

$$\sigma_{\mathfrak{M}_0}(X) \leq \kappa \tag{6.11}$$

for all $X \in \mathrm{RFH}_{\geq 0}(\mathfrak{M}_0)$. From the Proposition 6.1 we also know $0 \leq \sigma_{\mathfrak{M}_0}(X)$. We recall from [CFO09, AS09] that the assumption on $H_*(\mathscr{L}_B)$ implies the same for Rabinowitz Floer homology, that is,

$$\dim \mathrm{RFH}_*(\mathfrak{M}_0) = \infty . \tag{6.12}$$

Thus, the set

$$\mathscr{R}_\kappa = \{X \in \mathrm{RFH}_{\geq 0} \mid 0 \leq \sigma_{\mathfrak{M}_0}(X) \leq \kappa\} \tag{6.13}$$

is infinite. This directly contradicts Lemma 6.2. $\qquad\square$

To finish the proof of Theorem 1 we set $\mathfrak{M}_S = (F_{f_S}, \widehat{H})$ where f_S is as above and $\widehat{H} \in \mathcal{H}$ is such that $\phi^1_{\widehat{H}} = \phi^1_H$. We apply Corollary 5.13 to $\mathfrak{M}_0$ and conclude that

$$\{\overline{\sigma}_{\mathfrak{M}_S}(X) \mid X \in \mathcal{B}(\mathfrak{M}_S)\} \subset (0, \infty) \tag{6.14}$$

is unbounded from above. Thus, $\mathcal{A}^{\mathfrak{M}_S}$ has arbitrarily large critical values. At a critical point $(v, \eta) \in \mathrm{Crit}\mathcal{A}^{\mathfrak{M}_S}$ we compute

$$\mathcal{A}^{\mathfrak{M}_S}(v, \eta) = -\eta - \int \left[\lambda(X_{\widehat{H}}(v(t), t)) + \widehat{H}(t, v(t)) \right] dt \tag{6.15}$$

and thus

$$\eta \leq -\mathcal{A}^{\mathfrak{M}_S}(v, \eta) + \kappa(\widehat{H}) \tag{6.16}$$

where $\kappa(\widehat{H})$ is the seminorm defined in Definition 3.14. In particular, there exist critical points of $\mathcal{A}^{\mathfrak{M}_S}$ with arbitrarily negative η-value. This proves Theorem 1 for negative η-values.

Looking at Rabinowitz Floer co-homology the statement for positive η-values follows.

APPENDIX A. AN ITERATION INEQUALITY

Let x_n, $n \geq 0$ be a sequence of numbers satisfying

$$x_{n+1} \leq \max\{\alpha x_n, 0\} + \beta \tag{A.1}$$

for numbers $\alpha > 0$, and $\beta > 0$.

Lemma A.1.

$$x_n \leq \alpha^n \max\{x_0, \beta\} + \beta \sum_{j=0}^{n-1} \alpha^j = \alpha^n \max\{x_0, \beta\} + \beta \frac{\alpha^n - 1}{\alpha - 1} \tag{A.2}$$

PROOF. The proof goes by induction on n. For $n = 0$ we check

$$x_0 \leq \max\{x_0, \beta\} = \alpha^0 \max\{x_0, \beta\} + \beta \sum_{j=0}^{-1} \alpha^j . \tag{A.3}$$

For the induction step $n \to n+1$ we distinguish two cases.

Case 1: $x_n \leq 0$. Then

$$x_{n+1} \leq \max\{\alpha x_n, 0\} + \beta$$

$$= \beta$$

$$\leq \underbrace{\alpha^{n+1} \max\{x_0, \beta\}}_{\geq 0} + \beta \cdot \underbrace{\sum_{j=0}^{n} \alpha^j}_{\geq 1} . \tag{A.4}$$

Case 2: $x_n > 0$. Then

$$
\begin{aligned}
x_{n+1} &\leq \max\{\alpha x_n, 0\} + \beta \\
&\leq \alpha x_n + \beta \\
&\leq \alpha \Big(\alpha^n \max\{x_0, \beta\} + \beta \sum_{j=0}^{n-1} \alpha^j \Big) + \beta \\
&= \alpha^{n+1} \max\{x_0, \beta\} + \beta \sum_{j=0}^{n} \alpha^j
\end{aligned}
\tag{A.5}
$$

where we used the induction hypothesis in the third inequality. This proves the Lemma. $\square$

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