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Urs Frauenfelder

Dedicated to Vladimir Arnold

Abstract. The goal of this paper is to give a self-contained exposition of Gromov compactness for pseudoholomorphic disks in compact symplectic manifolds. The proof leads naturally to the concept of stable maps which was first introduced by M. Kontsevich. Our definition of stable maps for disks is based on the one given by D. McDuff and D. Salamon for spheres. We also generalize the notion of Gromov convergence to the case of disks. We show that the homotopy class is preserved under Gromov convergence.

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Keywords. Stable maps, holomorphic disks, Gromov convergence.

Contents

1. Introduction

This paper is concerned with Gromov compactness for pseudoholomorphic disks. It is an extract of the author's Diploma thesis written in 1999/2000 under the guidance of D. Salamon. The result is not fundamentally new but was considered as a "folk theorem" for many years. However, since in the meantime the results of this paper were used in different important works the author decided to publish this paper despite the long time which elapsed since its first appearance.

Kontsevich's notion of stable maps extends naturally to Riemann surfaces with boundary. The purpose of the present paper is to give a selfcontained exposition of stable maps of genus zero with one boundary component, satisfying some Lagrangian boundary condition. We prove that every sequence of stable maps has a Gromov convergent subsequence, that limits are unique up to equivalence and that the homotopy class is preserved under Gromov convergence.

Other proofs of Gromov compactness for stable maps with boundary can be found in [Ye] and [IS1, IS2]. The definition of Gromov convergence given here differs from the one in [IS1, IS2] as follows. Firstly, we allow the boundary of the domains to degenerate in a sequence of J-holomorphic curves and show in some examples how this can happen. In this case the degenerate "boundary" of the limit curve takes the form of a single point on a holomorphic sphere. This case is needed to guarantee that the number of boundary components is preserved in the limit. Secondly, following [MS2] we work with sequences of holomorphic rescalings instead of nonholomorphic parametrizations of the nodal curve. In other words, in our definition each component of the limit stable map is obtained by convergence, modulo bubbling, from a sequence of the form $u^\nu \circ \phi^\nu$, where ϕ^ν is a sequence of Möbius transformations. This gives rise to an alternative definition of Gromov convergence. Thirdly, we prove uniqueness of the limit based on this definition. Namely, the relative behaviour of the sequences of Möbius transformations allows us to construct an isomorphism between the trees over which two given limit curves are modelled. Finally, we explicitly carry out the proof of Gromov compactness for stable maps in the presence of marked points.

There are many different definitions of Gromov convergence and various proofs of Gromov's compactness theorem in the literature. Gromov sketched in [Gr] a proof which used isoperimetric inequalities and the Schwarz Lemma for conformal maps and lead naturally to the concept of cusp curves. The full details of this proof for closed Riemann surfaces without boundary have been written up by C. Hummel [Hu]. Compare also the article by P. Pansu [Pan] and the papers by S. Ivashkovich and V. Shevchishin [IS1, IS2]. Another approach to Gromov compactness can be found in [PW], [Par1], and [Par2]. Following Sacks–Uhlenbeck's covering and rescaling scheme in [SaU], they proved that sequences of pseudoholomorphic maps from a closed Riemann surface converge to a bubble tree, which can be thought of as element of a quotient of a space of iterated S^2-bundles over the Riemann surface. In the spirit of Sacks–Uhlenbeck's scheme, Ye proved Gromov compactness for Riemann surfaces with boundary (see [Ye]). The present paper is based on the exposition of the Gromov compactness theorem for stable maps of genus zero without boundary given in [MS2].

Pseudoholomorphic curves were introduced into symplectic geometry by M. Gromov in 1985 (see [Gr]). In particular, he proved existence of pseudoholomorphic disks and found obstructions to Lagrangian embeddings. Since then there have been many other applications of pseudoholomorphic disks. For example, J-holomorphic disks play a crucial role in Floer homology for Lagrangian intersections and hence in the proof of the Arnold conjecture (see [Fl], [Oh], [FOOO], [La], [Ch]). The original motivation for the present paper was the work of [AS]. M. Akveld and D. Salamon used the fact that the homotopy class is preserved under Gromov convergence to define Gromov invariants for loops of Lagrangian submanifolds. The results of this paper were later used by P. Seidel [Se] and in the joint work of P. Biran and O. Cornea [BC].

1.1. Overview

In this first subsection we recall some well known facts about pseudoholomorphic curves and outline the content of this paper. Let (M, ω) be a compact symplectic manifold (possibly with boundary) and L a compact Lagrangian submanifold of M without boundary. Let J be an ω-tame almost complex structure, i.e. $g(v, w) = \frac{1}{2}(\omega(v, Jw) + \omega(Jv, w))$ is a Riemannian metric on M. Pseudoholomorphic curves (also called J-holomorphic curves) are smooth maps u from a compact Riemann surfaces (Σ, j) with boundary to M with complex linear differential, which map $\partial\Sigma$ to L. The energy of a J-holomorphic map $u : \Sigma \to M$ is defined by

$$E(u) = \frac{1}{2}\int_\Sigma |du|^2 = \int_\Sigma u^*\omega.$$

The Lagrangian boundary condition guarantees that the energy depends only on the relative homology class of u. This gives a uniform L^2-bound on the first derivatives of u. This bound does not guarantee compactness because it is a borderline case for Sobolev estimates. If instead one has a uniform bound on the L^p-norms for some $p > 2$, then such a bound guarantees compactness. The elliptic bootstrapping analysis fails in the case $p = 2$. The geometric reason lies in the conformal invariance of the energy. For example the noncompact group $G = PSL(2, \mathbb{C})$ acts on the space of holomorphic maps from S^2 to M by conformal reparametrizations. Hence this space is always noncompact, unless holomorphic spheres are constant.

In the case of holomorphic spheres with minimal positive energy the quotient by the reparametrization group $PSL(2, \mathbb{C})$ is compact (see [MS2]). But this is not the case in general as the following example shows (see [Hu]).

Example 1.1. Let $S^2 = \mathbb{C} \cup \{\infty\}$ denote the Riemann sphere. Then $u^\nu : S^2 \to S^2 \times S^2$ with $u^\nu(z) := (z, 1/(\nu^2 z))$, $\nu \geq 1$, is a sequence of holomorphic curves in $S^2 \times S^2$. As $\nu \to \infty$ the images $u^\nu(S^2)$ converge to $S^2 \times \{0\} \cup \{0\} \times S^2$.

In the example above a sphere bubbles off at the point $(0, 0)$. Obviously, this cannot occur in the case of minimal energy. The phenomenon of bubbling was first observed 1981 by J. Sacks and K. Uhlenbeck [SaU]. In [Gr], M. Gromov states his famous compactness theorem which says that every sequence of pseudoholomorphic curves $u^\nu : \Sigma \to M$ has a subsequence which converges smoothly away from a finite set of points at which "bubbles" develop.

It can happen that on a bubble an additional sphere bubbles off or that at some point several spheres bubble off as the following examples show (see [Hu]).

Example 1.2. Consider the sequence of pseudoholomorphic curves $u^\nu : S^2 \to S^2 \times S^2 \times S^2$ given by

$$u^\nu(z) = \left(z, \frac{1}{\nu z}, \frac{1}{\nu^2 z}\right).$$

As $\nu \to \infty$ the images $u^\nu(S^2)$ converge to $C_1 \cup C_2 \cup C_3$ where

$$C_1 = S^2 \times \{0\} \times \{0\}, \qquad C_2 = \{0\} \times S^2 \times \{0\}, \qquad C_3 = \{0\} \times \{0\} \times S^2.$$

C_1 is connected to C_2 at the point $\{0\} \times \{0\} \times \{0\}$ and C_2 is connected to C_3 at the point $\{0\} \times \{\infty\} \times \{0\}$, but C_1 and C_3 are disconnected.

Example 1.3. Consider the sequence of pseudoholomorphic curves $u^\nu : S^2 \to S^2 \times S^2 \times S^2$ given by

$$u^\nu(z) = \left(z, \frac{1}{\nu^2(z - 1/\nu)}, \frac{1}{\nu^2(z + 1/\nu)}\right).$$

As $\nu \to \infty$ the images $u^\nu(S^2)$ converge to $C_1 \cup C_2 \cup C_3$ where

$$C_1 = S^2 \times \{0\} \times \{0\}, \qquad C_2 = \{0\} \times S^2 \times \{0\}, \qquad C_3 = \{0\} \times \{\infty\} \times S^2.$$

The three spheres are connected at the point $\{0\} \times \{0\} \times \{0\}$.

Phenomena of this kind led M. Kontsevich to introduce the concept of stable maps (see [Ko]). A stable map is a connected graph whose vertices consist of pseudoholomorphic maps u_α from a closed Riemann surface Σ_α to M with marked points. Two vertices α and β are connected by an edge iff there exists an equivalent pair of nodal points $z_{\alpha\beta} \in \Sigma_\alpha$ and $z_{\beta\alpha} \in \Sigma_\beta$, i.e. $u_\alpha(z_{\alpha\beta}) = u_\beta(z_{\beta\alpha})$. Marked points and nodal points are not allowed to coincide. Note that one can always achieve this by introducing auxiliary constant maps. Moreover, the stability condition asserts that the set of automorphisms is finite. This means that constant spheres have at least three special points (marked or nodal points) and constant tori have at least one special point. If e denotes the number of edges of the graph and v the number of vertices, then the genus of the stable map is given by the formula

$$g = \sum g_\alpha + e - v + 1.$$

The introduction of auxiliary constant maps gives us some additional information. Consider the case of a sequence of constant spheres each of which has three marked points such that the marked points converge to the same point in the limit. After rescaling by conformal transformations, one gets a constant sphere on which the four marked points are separated. The crossratio of the three marked points and the nodal point is invariant under conformal transformations and carries the additional information on how the three marked points converge.

This notion of stable maps generalizes to J-holomorphic curves with boundary as follows. We distinguish between marked points on the boundary and marked points in the interior, as well as between nodal points on the boundary and nodal points in the interior. Two nodal points can only form an equivalent pair if either both of them are on the boundary or both of them are in the interior. The stability condition for constant disks means that there are either three special points on the boundary, one special point on the boundary and one special point in the interior, or two special points in the interior. For constant annuli the stability condition means that there exists a special point either in the interior or on the boundary. Moreover, it is important to consider the degenerate case in which the boundary consists of a single point on a holomorphic sphere. Indeed, it can happen that two marked points on a constant disk converge to a single point in the interior. After rescaling one gets a constant sphere on which the two points are separated. This sphere is connected to a constant disk with no marked point. Hence the disk is unstable. We collapse the disk to a point. Think of this point as the boundary of the sphere. Note that the introduction of this point guarantees that the number of boundary components is preserved under limits. Another example where the limit consists of a single sphere with one boundary point is a sequence of holomorphic curves of degree two from the disk to the sphere, which map the boundary of the disk to the equator. If these curves represent the canonical generator of $\pi_2(S^2)$ then it can happen that the slit, the image of the boundary of the disk in the equator, converges to a point. In this case a single sphere bubbles off and the constant disk has to be collapsed to a point. See Subsection 6.1 for a detailed discussion of this example. One of the difficulties in carrying out the details of the analysis lies in dealing with this degenerate limit case.

In this paper, we restrict ourselves to the case of genus zero and one boundary component. The first condition means that the graph is a tree and the Riemann surfaces are either spheres or disks. The second condition means that the subset of the tree consisting of the disks is in fact a subtree.

To prove Gromov convergence we follow the exposition given by D. McDuff and D. Salamon in their book [MS2]. We work out the analogue of their results in the case of one boundary component. We give a careful definition of the notion of Gromov convergence for sequences of stable maps of genus zero with one boundary component and show that every sequence of stable maps of bounded energy has a Gromov convergent subsequence. It follows from our definition that the distinction between marked points on the boundary and marked points in the interior is preserved under limits. We prove that the homotopy class of a stable map is preserved under Gromov convergence and that limits of Gromov convergent sequences are unique up to equivalence. Using Gromov convergence one can define Gromov topology on the space of stable maps with some fixed number of marked points in the interior and some fixed number of marked points on the boundary, which represent a fixed homotopy class. Convergence with respect to the Gromov topology is equivalent to Gromov convergence. The space of stable maps is a compact Hausdorff space with a countable basis.

Stable maps are defined in Section 2. In Section 3 we define Gromov convergence for sequences of J-holomorphic disks without marked points. This definition will be extended in Section 4 to the case of holomorphic disks with marked points and in Section 5 to the general case of stable maps. In Section 6 we give some examples and applications.

In Appendix A we state some theorems which are needed for the analysis and give references to their proofs. We prove that for every compact totally real submanifold of an almost complex manifold, a compatible metric exists such that the submanifold is totally geodesic. This result is needed for the proof of an a priori estimate at the boundary and I could not find it in the literature. In Appendix B we study sequences of Möbius transformations which do not have a uniformly convergent subsequence. These results are used to prove uniqueness of limits for stable maps and to deal with marked points.

1.2. Notation

In this paper we will use the following notation:

$$\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z \geq 0\}$$

denotes the closed upper half-plane,

$$B = \mathbb{H} \cup \{\infty\}$$

denotes the closed disk, and

$$S^2 = \mathbb{C} \cup \{\infty\}$$

denotes the two-sphere. We may think of B as a subset of S^2.

The sphere is endowed with a canonical symplectic structure

$$\omega_{FS} = \frac{dx \wedge dy}{(1 + x^2 + y^2)^2};$$

FS stands for Fubini–Study. This form agrees up to a factor with the area form on the unit sphere $S^2 \subset \mathbb{R}^3$. Moreover, ω_{FS} is compatible with the standard complex structure

$$J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

This means that $g_{FS}(v, w) = \omega_{FS}(v, J_0 w)$ defines a Riemannian metric on S^2 called the *Fubini–Study metric*. We will denote by G the group of orientation preserving conformal diffeomorphisms of (S^2, J_0). Its elements are given by the Möbius transformations

$$z \mapsto \frac{az + b}{cz + d}, \qquad ad - bc \neq 0.$$

G is isomorphic to $PSL(2, \mathbb{C}) = SL(2, \mathbb{C}) \setminus \{\mathrm{id}, -\mathrm{id}\}$ via the isomorphism

$$\left(z \mapsto \frac{az + b}{cz + d} \right) \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We denote by $G_0 \subset G$ the set of Möbius transformations which map B to itself. Note that $G_0 = PSL(2, \mathbb{R}) = SL(2, \mathbb{R}) \setminus \{\mathrm{id}, -\mathrm{id}\}$.

1.3. Acknowledgements

This paper is an outgrowth of my Diploma thesis which I wrote in the winter of 1999/2000 at ETH Zürich under the guidance of Prof. Dietmar Salamon. I would like to express hearty thanks to my teacher Prof. Dietmar Salamon for many useful discussions. I profited greatly from his advice and from his enthusiasm. I would like to thank also his students, especially Katrin Wehrheim, for all their help and encouragement during my work on this article.

Since the results of this paper were somehow folk knowledge before I did not publish it for a while. However, Paul Biran convinced me that due to growing interest in this paper it would be desirable to have a published version. I would like to thank Paul Biran for his encouragement.

2. Stable maps

2.1. J-holomorphic curves

Let (M, J) be an almost complex manifold and (Σ, j) be a Riemann surface. A smooth map $u : \Sigma \to M$ is called *J-holomorphic* if the differential du is a complex linear map with respect to j and J:

$$J \circ du = du \circ j.$$

One can rephrase this equation as $\bar{\partial}_J(u) = 0$. Here, the 1-form

$$\bar{\partial}_J(u) = \tfrac{1}{2}(du + J \circ u \circ j) \in \Omega^{0,1}(u^*TM)$$

is the complex antilinear part of du, and takes values in the complex vector bundle $u^*TM = \{(z, v) : z \in \Sigma, v \in T_{u(z)}M\}$.

If (M, ω) is a symplectic manifold then we define the space $\mathcal{J}_\tau(M, \omega)$ as the set of all almost complex structures J on M which *tame* ω. This means that

$$\omega(v, Jv) > 0, \qquad v \neq 0.$$

In particular,

$$\langle v, w\rangle_J := \tfrac{1}{2}(\omega(v, Jw) + \omega(w, Jv))$$

is a Riemannian metric on M. One can show that the space $\mathcal{J}_\tau(M, \omega)$ is a nonempty contractible space (see [MS1]). We will denote by $|v|_J^2 = \langle v, v\rangle_J$ the associated norm.

The *energy* of a smooth map $u : \Sigma \to M$ is the L^2-norm of the 1-form $du \in \Omega^1(u^*TM)$:

$$E(u) = \frac{1}{2}\int_\Sigma |du|_J^2 \,\mathrm{dvol} = \int_\Sigma u^*\omega.$$

If we assume that Σ has no boundary or that $\partial\Sigma$ is mapped to a fixed Lagrangian submanifold L of M, then the energy identity shows that the L^2-norm of the derivatives of a J-holomorphic curve is uniformly bounded. This bound only depends on the homology class relative to L represented by u.

2.2. Trees

A tree is a finite connected graph with no cycles. More precisely

Definition 2.1. A *tree* is a finite set T equipped with a relation $E \subset T \times T$ such that the following holds.

(*symmetric*) If $\alpha E \beta$ then $\beta E \alpha$.

(*connected*) For all $\alpha, \beta \in T$ with $\alpha \neq \beta$ there exist $\gamma_0, \dots, \gamma_m \in T$ with $\gamma_0 = \alpha$ and $\gamma_m = \beta$ such that $\gamma_i E \gamma_{i+1}$ for all i.

(*no cycles*) If $\gamma_0, \dots, \gamma_m \in T$ with $\gamma_i E \gamma_{i+1}$ and $\gamma_i \neq \gamma_{i+2}$ for all i then $\gamma_0 \neq \gamma_m$.

Elements of T are called *vertices* and elements of E are called *edges*. Two vertices α, β are called *adjacent* if they can be connected by an edge, i.e. $\alpha E \beta$. A tree isomorphism is a bijection between the set of vertices of two trees which maps adjacent vertices to adjacent ones.

Definition 2.2. A map $f : T \to \tilde{T}$ is called a *tree homomorphism* if $f^{-1}(\tilde{\alpha})$ is a tree for every $\tilde{\alpha} \in \tilde{T}$ and, for all $\alpha, \beta \in T$ with $\alpha E \beta$ and $f(\alpha) \neq f(\beta)$, we have $f(\alpha)\tilde{E}f(\beta)$. It is called a *tree isomorphism* if it is bijective and both f and f^{-1} are tree homomorphisms.

We will introduce some notation from [MS2]. Let (T,E) be a tree. Then for every pair $\alpha,\beta \in T$ with $\alpha \neq \beta$ there exists a unique ordered set of vertices $\gamma_0,\dots,\gamma_m \in T$ such that $\gamma_i E \gamma_{i+1}$, $\gamma_i \neq \gamma_{i+2}$, $\gamma_0 = \alpha$, and $\gamma_m = \beta$. This is called the *chain (of edges) running from* α *to* β, and the set of vertices belonging to this chain is denoted by

$$[\alpha,\beta] = [\beta,\alpha] = \{\gamma_i : i = 0,\dots,m\}.$$

Cutting any edge $\alpha E \beta$ decomposes the tree T into two components. The component containing β is given by

$$T_{\alpha\beta} = \{\gamma \in T : \beta \in [\alpha,\gamma]\}.$$

This set is called a *branch* of the tree T. Note that T is the disjoint union of $\{\alpha\}$ and the sets $T_{\alpha\beta}$ over all $\beta \in T$ with $\alpha E \beta$.

2.3. Stable maps

The concept of stable map was introduced by M. Kontsevich [Ko]. We generalize his notion to the case of genus zero with one boundary component.

Definition 2.3. Let (M,ω) be a compact symplectic manifold with boundary, L a compact Lagrangian submanifold of M without boundary and $J \in \mathcal{J}_\tau(M,\omega)$. A (k_1,k_2)*-marked, J-holomorphic stable map of genus zero with one boundary component in L* is a tuple

$$(\mathbf{u},\mathbf{z}) = (\{(\Sigma_\alpha,\Gamma_\alpha,u_\alpha)\}_{\alpha\in T},\{z_{\alpha\beta}\}_{\alpha E\beta},\{\alpha_i,z_i\}_{1\leq i\leq k})$$

modelled over a tree (T,E), consisting of a collection of J-holomorphic maps $u_\alpha : (\Sigma_\alpha,\Gamma_\alpha) \to (M,L)$ indexed by $\alpha \in T$, a collection of nodal points $z_{\alpha\beta} \in \Sigma_\alpha$, indexed by directed edges $\alpha E\beta$, and a collection of marked points $z_i \in \Sigma_{\alpha_i}$. They satisfy the following conditions:

(i) For every α either $\Sigma_\alpha = S^2$ or $\Sigma_\alpha = B$. Moreover, if $\Sigma_\alpha = B$ then $\Gamma_\alpha = \partial B$ and if $\Sigma_\alpha = S^2$ then $\#\Gamma_\alpha \leq 1$.

(ii) There are k_1 interior marked points and k_2 boundary marked points, i.e. $z_i \in \operatorname{Int}\Sigma_{\alpha_i}$ for $i \leq k_1 \leq k$ and $z_i \in \partial\Sigma_{\alpha_i}$ for $k_1 < i \leq k = k_1 + k_2$.

(iii) If $\alpha,\beta \in T$ with $\alpha E\beta$ then $u_\alpha(z_{\alpha\beta}) = u_\beta(z_{\beta\alpha})$.

(iv) If $\alpha E\beta$, $\alpha E\gamma$, and $\beta \neq \gamma$, then $z_{\alpha\beta} \neq z_{\alpha\gamma}$. If $\alpha_i = \alpha_j$ with $i \neq j$ then $z_i \neq z_j$. If $\alpha_i = \alpha$ and $\alpha E\beta$ then $z_i \neq z_{\alpha\beta}$. If $\Sigma_\alpha = S^2$ then $z_{\alpha\beta} \notin \Gamma_\alpha$ for $\alpha E\beta$ and $z_i \notin \Gamma_\alpha$ where $i = 1,\dots,k$.

(v) For $\alpha \in T$ define

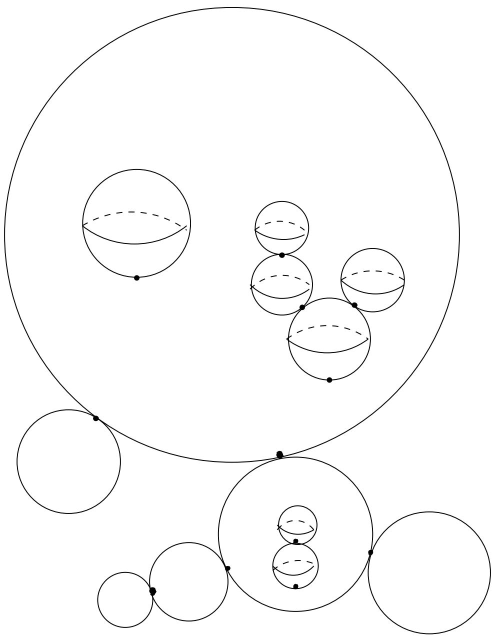
$$Z_\alpha = \begin{cases} \{z_{\alpha\beta} : \beta \in T, \alpha E\beta\} \cup \{z_i : 1 \leq i \leq k, \alpha_i = \alpha\} & \text{if } \Sigma_\alpha = B, \\ \Gamma_\alpha \cup \{z_{\alpha\beta} : \beta \in T, \alpha E\beta\} \cup \{z_i : 1 \leq i \leq k, \alpha_i = \alpha\} & \text{if } \Sigma_\alpha = S^2. \end{cases}$$

If Σ_α is the sphere and u_α is a constant function then Z_α consists of at least three elements. If Σ_α is the disk and u_α is a constant function then Z_α consists either of at least three elements or of two elements which do not both lie on the boundary.

(vi) If $\alpha E\beta$ then $z_{\alpha\beta} \in \partial\Sigma_\alpha$ iff $z_{\beta\alpha} \in \partial\Sigma_\beta$.

(vii) The set $\partial T := \{\alpha \in T : \Gamma_\alpha \neq \emptyset\}$ is a subtree called the *boundary tree*. The boundary tree is never empty. It consists either of disks or of one single sphere, i.e.

$$\alpha \in \partial T,\ \Sigma_\alpha = S^2 \ \Rightarrow\ \#\partial T = 1.$$



A stable map

The following remarks should clarify the meaning of this definition.

Remark 2.4. Item (v) is the stability condition of M. Kontsevich. If (v) is not satisfied, there are infinitely many orientation preserving conformal reparametrizations of Σ_α. Assume for example that $\Sigma_\alpha = B$ and $Z_\alpha = \{i\}$. Then the set of automorphisms is the circle.

Remark 2.5. Let $(\mathbf{u}, \mathbf{z})$ be a stable map with k marked points. The domain of the map can be represented as the quotient

$$\Sigma(\mathbf{z}) = \{(\alpha, w) : \alpha \in T,\ w \in \Sigma_\alpha\}/\!\sim$$

where the equivalence relation is given by $(\alpha, z) \sim (\beta, w)$ if and only if either $\alpha = \beta$ and $z = w$, or $\alpha E \beta$, $z = z_{\alpha\beta}$, and $w = z_{\beta\alpha}$. Denote by $[\alpha, z]$ the equivalence class of a pair (α, z), $\alpha \in T$, $z \in \Sigma_\alpha$. Then the collection $\{u_\alpha\}_{\alpha \in T}$ can be thought of as map

$$\Sigma(\mathbf{z}) \to M : [\alpha, z] \mapsto u_\alpha(z).$$

Call $[\alpha, z]$ a *nodal point* of $\Sigma(\mathbf{z})$ if $z = z_{\alpha\beta}$ for some $\beta \in T$ with $\alpha E \beta$. The *marked points* are the elements $x_i = [\alpha_i, z_i]$ of $\Sigma(\mathbf{z})$, and a *boundary point* is an element of the form $x = [\alpha, z_\alpha^\infty]$ where $\Gamma_\alpha = \{z_\alpha^\infty\}$. Note that nodal points, marked points and boundary points are mutually distinct. Moreover, there is at most one boundary point in $\Sigma(\mathbf{z})$. A point in $\Sigma(\mathbf{z})$ is called *special* if it is either a nodal, marked or boundary point.

Remark 2.6. Item (vii) in Definition 2.3 tells us that the boundary of the image of $\Sigma(\mathbf{z})$ consists of exactly one connected component. As was pointed out in the introduction, it is important to consider the degenerate case in which the boundary consists of a single point on a holomorphic sphere.

If $(\mathbf{u}, \mathbf{z})$ is a stable map then the tree T carries natural weights

$$m_\alpha(\mathbf{u}) = E(u_\alpha) = \int_{\Sigma_\alpha} u_\alpha^* \omega.$$

We will use the following notation from [MS2]:

$$m_{\alpha\beta}(\mathbf{u}) = \sum_{\gamma \in T_{\alpha\beta}} E(u_\gamma)$$

for $\alpha, \beta \in T$ with $\alpha E \beta$, and

$$E_\alpha(\mathbf{u}, \Omega) = \int_\Omega u_\alpha^* \omega + \sum_{\substack{\alpha E \beta \\ z_{\alpha\beta} \in \Omega}} m_{\alpha\beta}(\mathbf{u})$$

for $\alpha \in T$ and any open set $\Omega \subset \Sigma_\alpha$. Then the total energy

$$E(\mathbf{u}) = \sum_{\alpha \in T} E(u_\alpha)$$

of the stable map $(\mathbf{u}, \mathbf{z})$ is equal to $E_\alpha(\mathbf{u}, \Sigma_\alpha)$ for any $\alpha \in T$. The following lemma shows that the energy and the number of marked points give a uniform bound on the number of vertices of the tree.

Lemma 2.7. *Let $(\mathbf{u}, \mathbf{z})$ be a (k_1, k_2)-marked stable map, and let $T \to \mathbb{R} : \alpha \mapsto m_\alpha$ be the corresponding weights. Assume that the total energy of $\mathbf{u}$ is bounded by c, i.e.*

$$E(\mathbf{u}) = \sum_{\alpha \in T} m_\alpha(\mathbf{u}) \leq c.$$

Then the number of vertices is bounded by

$$\#T \leq 3c/\hbar + 2k_1 + k_2 - 2 \tag{1}$$

where $\hbar$ is defined as in Corollary A.4.

Proof. If $\#T = 1$, then (1) is satisfied. Hence assume $\#T \geq 2$. We first assume that $\#\partial T \geq 2$. Then it follows from (viii) of Definition 2.3 that for every $\alpha \in T$ the set Γ_α is either empty or consists of infinitely many points. (v) of Definition 2.3 implies that if u_α is constant and the set $Z_\alpha = \{z_{\alpha\beta} : \beta \in T, \alpha E \beta\} \cup \{z_i : 1 \leq i \leq k, \alpha_i = \alpha\}$ consists of fewer than three points, then $\Sigma_\alpha = B$ and Z_α has precisely two elements which do not both lie on the boundary of Σ_α. We can even assume that if u_α is constant and Z_α has fewer than three points, then Z_α consists of two elements, one in the interior and one on the boundary. To see this, observe that if Z_α has two points in the interior then we add an additional constant disk $\hat{\alpha}$ to our tree. α and $\hat{\alpha}$ are joined on the boundary and each of them has one special point in the interior. This procedure increases the number of vertices by one and leaves the energy and the number of marked points unchanged. Hence it is enough to show the formula above for the modified tree.

Now we estimate the number of constant disks having only one special point on the boundary and one special point in the interior. The point in the interior is either a marked point or a nodal point. If it is a node then a tree of spheres bubbles off on it. One of these spheres must be nonconstant or contain an interior marked point. This shows that the number of these disks is bounded by $k_1 + c/\hbar$.

Now we collapse each of the disks having only one special point on the boundary and one special point in the interior to a single point. This procedure decreases the number of vertices of the tree by at most $k_1 + c/\hbar$. Call the modified tree $\hat{T}$. We can assume without loss of generality that $\hat{T}$ is not empty. Otherwise T would consist of constant disks having one special point on the boundary and one in the interior. Because there are no spheres, the special points in the interior have to be interior marked points. Hence by collapsing the $\#T$ disks, one removes $\#T$ interior marked points. Then (1) follows from the inequality $\#T \leq 2\#T - 2$ for $\#T \geq 2$.

We have to show that

$$\#\hat{T} \leq 2c/\hbar + k_1 + k_2 - 2 = 2c/\hbar + k - 2.$$

To see this, note that each endpoint α of $\hat{T}$ either has weight $m_\alpha \geq \hbar$ or carries at least two marked points z_i and z_j with $\alpha_i = \alpha_j = \alpha$. Hence removing the endpoints either reduces the energy by $\hbar$ while increasing the number of marked points by no more than one, or it leaves the energy unchanged while decreasing the number of marked points by at least one. Hence the number $2E/\hbar + k + 1 - \#\hat{T}$ gets reduced by at least 1 whenever we remove an endpoint. After removing $\#\hat{T} - 1$ vertices we are left with $2E'/\hbar + k' \leq 2E/\hbar + k + 1 - \#\hat{T}$. Since $\#T \geq 2$ and $\hat{T} \neq \emptyset$ the tree $\hat{T}$ is either a single marked point or has more than one point. Hence $k' \geq 1$ and either $k' \geq 3$ or $E' \geq \hbar$. In either case $2E'/\hbar + k' \geq 3$, and this proves the inequality above.

It remains to consider the case where $\#\partial T = 1$. It follows from (vii) of Definition 2.3 that there exists $\alpha \in T$ such that $\Gamma_\alpha = \{z_\alpha^\infty\}$ and $\Sigma_\alpha = S^2$. Add a constant disk $\tilde{\alpha}$ to the tree having one additional boundary marked point z_{k+1}. Call the modified tree $\tilde{T}$. Because $\tilde{\alpha}$ is the only disk of $\tilde{T}$ it follows that

$$\#\tilde{T} \leq 2\tilde{c}/\hbar + \tilde{k} - 2.$$

Moreover, $\tilde{k}_1 = k_1$, $\tilde{k}_2 = k_2 + 1$, $\tilde{c} = c$ and $\#\tilde{T} = \#T + 1$. This shows that

$$\#T \leq 2c/\hbar + (k_1 + k_2 + 1) - 2 - 1 = 2c/\hbar + k_1 + k_2 - 2 \leq 3c/\hbar + 2k_1 + k_2 - 2.$$

Hence the lemma is proved. $\square$

Remark 2.8. Following [MS2] we extend the notation $z_{\alpha\gamma}$ to any pair of distinct points $\alpha, \gamma \in T$. Namely if $\alpha \neq \gamma$ then there exists a unique element $\beta \in T$ which is adjacent to α, i.e. $\alpha E \beta$, and satisfies $\gamma \in T_{\alpha\beta}$, or equivalently $\beta \in [\alpha, \gamma]$. In this case we define $z_{\alpha\gamma} = z_{\alpha\beta}$. In summary,

$$z_{\alpha\gamma} := z_{\alpha\beta}, \qquad \gamma \in T_{\alpha\beta}.$$

Thus $z_{\alpha\gamma} \in \Sigma_\alpha$ is the unique point on Σ_α through which Σ_γ can be reached. Similarly, define

$$z_{\alpha i} := \begin{cases} z_i & \text{if } \alpha_i = \alpha, \\ z_{\alpha\beta} & \text{if } \alpha_i \in T_{\alpha\beta}, \end{cases}$$

for $\alpha \in T$ and $i \in \{1, \ldots, n\}$. If $\alpha \neq \alpha_i$, then $z_{\alpha i} = z_{\alpha\alpha_i}$ is the unique nodal point on Σ_α through which it is connected to Σ_{α_i} which carries the marked point z_i. To deal with boundary points we define, for $\alpha \notin \partial T$,

$$z_\alpha^\infty := z_{\alpha\beta}, \qquad \partial T \subset T_{\alpha\beta}.$$

Hence z_α^∞ is the unique nodal point through which Σ_α is connected to the boundary of the stable map. If $\Sigma_\alpha = S^2$ and $\alpha \in \partial T$ let z_α^∞ be the unique element of Γ_α. We further introduce

$$\Xi_\alpha = \begin{cases} \{z_{\alpha\beta} : \beta \in T, \alpha E \beta\} & \text{if } \Sigma_\alpha = B, \\ \{z_\alpha^\infty\} \cup \{z_{\alpha\beta} : \beta \in T,\, \alpha E \beta\} & \text{if } \Sigma_\alpha = S^2, \end{cases}$$

the set consisting of nodal and boundary points. Note that $\Xi_\alpha = Z_\alpha \setminus \{z_i : 1 \leq i \leq k,\, \alpha_i = \alpha\}$.

There is a natural equivalence relation on the set of stable maps.[1] The equivalence relation is determined by complex diffeomorphisms of the domains of the curves which identify the nodal points, the marked points and the boundary points.

Definition 2.9. Two (k_1, k_2)-marked stable maps

$$(\mathbf{u}, \mathbf{z}) = (\{(\Sigma_\alpha, \Gamma_\alpha, u_\alpha)\}_{\alpha \in T}, \{z_{\alpha\beta}\}_{\alpha E \beta}, \{\alpha_i, z_i\}_{1 \leq i \leq k}),$$
$$(\tilde{\mathbf{u}}, \tilde{\mathbf{z}}) = (\{(\tilde\Sigma_\alpha, \tilde\Gamma_\alpha, \tilde u_\alpha)\}_{\alpha \in \tilde T}, \{\tilde z_{\alpha\beta}\}_{\alpha \tilde E \beta}, \{\tilde\alpha_i, \tilde z_i\}_{1 \leq i \leq k})$$

are called *equivalent* if there exists a tree isomorphism $f : T \to \tilde T$ and a collection $\varphi = \{\varphi_\alpha\}_{\alpha \in T}$ of Möbius transformations such that the following holds.

(i) For all $\alpha \in T$, $\tilde u_{f(\alpha)} = u_\alpha \circ \varphi_\alpha^{-1}$. Moreover, Σ_α and $\Sigma_{f(\alpha)}$ are either both spheres or both disks.
(ii) For all α,$\beta \in T$ with $\alpha E \beta$, $\tilde z_{f(\alpha) f(\beta)} = \varphi_\alpha(z_{\alpha\beta})$.
(iii) For $i = 1, \ldots, k$, $\tilde\alpha_i = f(\alpha_i)$ and $\tilde z_i = \varphi_{\alpha_i}(z_i)$.
(iv) $\Gamma_\alpha = \varphi_\alpha^{-1}(\Gamma_{f(\alpha)})$.

[1]Strictly speaking, the stable maps do not form a set but a class. However, if one thinks of a tree with m vertices as a relation on the set $\{1, \ldots, m\}$ then the stable maps form a set.

3. Gromov convergence for unmarked disks

3.1. Definition of convergence

In their book [MS2] D. McDuff and D. Salamon gave a careful definition of Gromov convergence for pseudoholomorphic spheres. We define the analogue for pseudoholomorphic disks. We will show that each sequence of J-holomorphic disks with bounded energy has a subsequence which Gromov converges to a stable map. Moreover, the homotopy class is preserved under Gromov convergence.

For simplicity we assume in this section that our disks have no marked points. We will define Gromov convergence for sequences of disks with marked points in the next section. In Section 5 the notion of Gromov convergence will be extended to sequences of stable maps.

Definition 3.1. Let M be a compact manifold with boundary, L a compact submanifold of M without boundary, ω^ν a sequence of symplectic structures on M such that L is Lagrangian for every ω^ν, and J^ν a sequence of ω^ν-tame almost complex structures. Assume that the ω^ν converge to some symplectic structure ω on M with respect to the C^∞-topology, and the J^ν converge in the C^∞-topology to some $J \in \mathcal{J}_\tau(M,\omega)$. A sequence of J^ν-holomorphic disks u^ν is said to *Gromov converge* to a J-holomorphic stable map

$$(\mathbf{u}) = (\{(\Sigma_\alpha, \Gamma_\alpha, u_\alpha)\}_{\alpha\in T}, \{z_{\alpha\beta}\}_{\alpha E\beta})$$

if there exists a collection $\{\varphi_\alpha^\nu\}_{\alpha\in T}$ of Möbius transformations such that the following holds.

(i) If $\Sigma_\alpha = B$, then $\varphi_\alpha^\nu \in G_0$.

(ii) If $\Sigma_\alpha = S^2$ then for every compact subset $K \subset S^2 \setminus \{z_\alpha^\infty\}$ there exists $\nu_0(K)$ such that $\varphi_\alpha^\nu(K) \subset B$ for every $\nu \geq \nu_0(K)$.

(iii) For every $\alpha \in T$ and for every compact subset K of $\Sigma_\alpha \setminus \Xi_\alpha$ (cf. Remark 2.8 for the definition of Ξ_α) the sequence[2] $u^\nu \circ \varphi_\alpha^\nu$ converges to u_α, uniformly with all derivatives on K.[3]

(iv) If $\beta \in T$ is such that $\alpha E\beta$, then

$$m_{\alpha\beta}(\mathbf{u}) = \lim_{\epsilon\to 0}\lim_{\nu\to\infty} E(u^\nu, \varphi_\alpha^\nu(B_\epsilon(z_{\alpha\beta})) \cap B). \tag{2}$$

(v) If $\Gamma_\alpha = \{z_\alpha^\infty\}$, then

$$\lim_{\epsilon\to 0}\lim_{\nu\to\infty} E(u^\nu \circ \varphi_\alpha^\nu, B_\epsilon(z_\alpha^\infty) \cap (\varphi_\alpha^\nu)^{-1}B) = 0.$$

(vi) $(\varphi_\alpha^\nu)^{-1} \circ \varphi_\beta^\nu$ converges to $z_{\alpha\beta}$, uniformly on compact subsets of $\Sigma_\beta \setminus \{z_{\beta\alpha}\}$.

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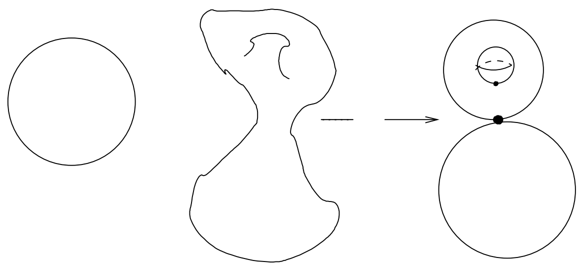
[2] The domain of $u^\nu \circ \varphi_\alpha^\nu$ is the set $(\varphi_\alpha^\nu)^{-1}(B)$.

[3] Note that according (ii) the expression $u^\nu \circ \varphi_\alpha^\nu$ is defined on the whole of K if $\nu \geq \nu_0(K)$. If f is a C^∞-function from a compact manifold X to a manifold Y and f^ν is a sequence of C^∞-functions from a submanifold $X^\nu \subset X$ to Y, and $Z \subset X$ is a finite set, then we say that f^ν converges to f uniformly with all derivatives on every compact subset K of $X \setminus Z$ if for every such K there exists a $\nu_0(K) \in \mathbb{N}$ such that for every $\nu \geq \nu_0(K)$ the set K is contained in X^ν and $f^\nu|_K$ converges to $f|_K$ in the C^∞-topology.



Gromov convergence

In the following let J^ν, M and L be as in Definition 3.1. The main results of this section are

Proposition 3.2. *Assume that $u^\nu : (B, \partial B) \to (M, L)$ is a sequence of J^ν-holomorphic disks which Gromov converges to a stable map $(\mathbf{u})$. Then the following holds.*

(i) *If $x^\nu \in u^\nu(B)$ converges to $x \in M$ then $x \in \bigcup_{\alpha \in T} u_\alpha(\Sigma_\alpha)$.*
(ii) *For large ν the disk u^ν is relative homotopic with respect to L to the connected sum $\#_{\alpha \in T} u_\alpha$.*

Theorem 3.3. *Let $u^\nu : (B, \partial B) \to (M, L)$ be a sequence of J^ν-holomorphic disks with bounded energy. Then u^ν has a Gromov convergent subsequence.*

If a sequence u^ν of J^ν-holomorphic disks Gromov converges to some stable map $\mathbf{u}$, then it converges to every equivalent map. The following theorem tells us that the converse also holds.

Theorem 3.4. *Let $u^\nu : (B, \partial B) \to (M, L)$ be a sequence of J^ν-holomorphic disks. Suppose that u^ν Gromov converges to two stable maps $\mathbf{u}$ and $\tilde{\mathbf{u}}$. Then $\mathbf{u}$ is equivalent to $\tilde{\mathbf{u}}$.*

We will prove this theorem in more generality, namely for disks with marked points, in the next section.

3.2. Gromov compactness

Let us first sketch the main ideas of the proof of Theorem 3.3. If there is a bound

$$\sup_\nu \|du^\nu\|_{L^p} < \infty$$

for some $p > 2$ then Theorem A.9 shows that a subsequence of u^ν converges to a J-holomorphic disk. However, the assumption of bounded energy gives us only an L^2-bound. This gives rise to the phenomenon of bubbling. We know from Theorem A.12 that after choosing a suitable subsequence there are only finitely many bubble points. If one rescales a small neighbourhood of such a point, then one gets a new J-holomorphic sphere or disk which is connected to the old one at the bubble point. The examples in the introduction show that it may happen that at some point more than one sphere or disk bubbles off, or that on a bubble new bubbling occurs. The subtle point is to find a reparametrization which detects all the bubbles. In the terminology of [HS] this is called soft rescaling.

In [MS2] Gromov compactness was proved for sequences of pseudoholomorphic spheres by induction on the number of vertices of the stable map. We work out the analogue of their proof for disks.

3.2.1. Soft rescaling

Theorem 3.5. *Let M be a compact manifold with boundary, L a compact submanifold of M without boundary, ω^ν a sequence of symplectic structures on M such that L is Lagrangian for every ω^ν, and J^ν a sequence of ω^ν-tame almost complex structures. Assume that the ω^ν converge to some symplectic structure ω of M with respect to the C^∞-topology, and the J^ν converge in the C^∞-topology to some $J \in \mathcal{J}_\tau(M,\omega)$. Let $S \subset \mathbb{H}$ be an open set, let $u_\nu : (S, \partial S) \to (M, L)$ be a sequence of J^ν-holomorphic curves with bounded energy, and let $u : (S, \partial S) \to (M, L)$ be a J-holomorphic curve and $w \in S$ such that*

(A) *u^ν converges to u uniformly with all derivatives on compact subsets of $S \setminus \{w\}$,*

(B) *the limit*
$$m_0 = \lim_{\epsilon\to 0}\lim_{\nu\to\infty} E(u^\nu, B_\epsilon(w))$$
exists and is positive.

Then there exists a sequence of Möbius transformations ψ^ν, a J-holomorphic curve $v : (\Sigma, \partial\Sigma) \to (M, L)$ where
$$\Sigma := \begin{cases} \mathbb{C} \cup \{\infty\} & \text{if } w \in S \setminus \partial S, \\ \mathbb{H} \cup \{\infty\} & \text{if } w \in \partial S, \end{cases}$$
and a finite set $Z = \{z_1, \ldots, z_\ell\} \subset \Sigma \setminus \{\infty\}$ such that:

(i) *If $\Sigma = B$ then $\psi^\nu \in G_0$.*

(ii) *A subsequence of $v^\nu = u^\nu \circ \psi^\nu$ (still denoted by v^ν) converges to a J-holomorphic curve $v : \Sigma \to M$, uniformly with all derivatives on compact subsets of $\Sigma \setminus (Z \cup \{\infty\})$ (cf. footnote 3 on page 228).*

(iii) *The limit*
$$m_j := m(z_j) := \lim_{\epsilon\to 0}\lim_{\nu\to\infty} E(v^\nu, (\psi^\nu)^{-1}(\Sigma) \cap B_\epsilon(z_j))$$
exists and is positive for $1 \le j \le \ell$.

(iv) *If v is constant and $\#Z < 2$, then $\Sigma = B$ and Z consists of precisely one point, which lies in the interior of Σ.*

(v) *Define*
$$m_{\ell+1} := m(\infty) := \lim_{R\to\infty}\lim_{\nu\to\infty} E(v^\nu, (\mathbb{C} \cap \Sigma) \setminus (B_R \cap \Sigma)).$$
Then
$$E := \lim_{\nu\to\infty} E(u^\nu) = E(v) + \sum_{j=1}^{\ell+1} m_j$$

and

$$m_0 = E(v) + \sum_{j=1}^{\ell} m_j.$$

(vi) $v(\infty) = u(w)$.

(vii) ψ^ν *converges to* w*, uniformly on compact subsets of* $\Sigma \setminus \{\infty\}$.

Proof. We only consider the case where $w \in \partial S$. The case where w lies in the interior was proved in [MS2]. Let $\Omega_\epsilon(z_0) := \{z \in \mathbb{C} : |z - z_0| \le \epsilon, \operatorname{Im} z \ge 0\}$ where $\epsilon \ge 0$ and $z_0 \in \mathbb{R}$. We abbreviate further $\Omega_\epsilon := \Omega_\epsilon(0)$ and $\Omega := \Omega_1$. We may assume without loss of generality that $w = 0$ and $S = \Omega$.

By (A), $|du^\nu|$ is uniformly bounded in $\Omega \setminus \Omega_\epsilon$ for every $\epsilon > 0$ and, by (B), $\sup_{\Omega_\epsilon} |du^\nu| \to \infty$ for every $\epsilon > 0$. Hence every sequence $z^\nu \in \Omega$ with

$$|du^\nu(z^\nu)| = \sup_\Omega |du^\nu|$$

must converge to zero. Let us fix such a sequence. Consider the function

$$\rho_\nu : \delta \mapsto E(u^\nu, B_\delta(z^\nu) \cap \Omega).$$

It is continuous, $\rho_\nu(0) = 0$ and, by definition of m_0, there are values for which ρ_ν approaches m_0 as closely as desired if ν is large enough. Hence it follows from the intermediate value theorem that for ν sufficiently large there exists $\delta^\nu > 0$ such that

$$E(u^\nu, B_{\delta^\nu}(z^\nu) \cap \Omega) = m_0 - \hbar/2$$

where $\hbar$ is as in Corollary A.4.

We have to distinguish two cases:

(I) There is a subsequence of ν (still denoted by ν) such that $\operatorname{Im} z^\nu/\delta^\nu$ converges to some finite number.

(II) There is a subsequence of ν (still denoted by ν) such that $\operatorname{Im} z^\nu/\delta^\nu$ converges to infinity.

Define

$$^\nu := \begin{cases} \operatorname{Re} z^\nu + \delta^\nu z & \text{in case (I),} \\ \operatorname{Re} z^\nu + \operatorname{Im}(z^\nu) z & \text{in case (II).} \end{cases}$$

Note that $\psi^\nu \in G_0$ and that ψ^ν converges to 0, uniformly on compact subsets of $\mathbb{C}$.

Step 1. *The sequences* $v^\nu = u^\nu \circ \psi^\nu$ *have subsequences (still denoted by* v^ν*) which satisfy the following: There are finitely many points* $z_1, \ldots, z_\ell \in \mathbb{H}$ *and a J-holomorphic disk* $v : (\mathbb{H} \cup \{\infty\}, \mathbb{R} \cup \{\infty\}) \to (M, L)$ *such that* v^ν *converges to* v*, uniformly with all derivatives on compact subsets of* $\mathbb{H} \setminus \{z_1, \ldots, z_\ell\}$.

We only consider the first case. $v^\nu = u^\nu \circ \psi^\nu$ is defined on Ω_{R_ν} where $R_\nu = (1 - |\operatorname{Re} z^\nu|)/\delta^\nu$. Because δ^ν and z^ν converge to zero, R^ν converges to infinity. Hence

$$v_n^\nu := v^\nu|_{\Omega_n}$$

for $n \in \mathbb{N}$ is defined for ν sufficiently large. It follows from conformal invariance that the energy of v^ν in Ω_{R_ν} is bounded by the energy of u^ν in Ω_1. Hence by Theorem A.12 there exists a subsequence of v_n^ν (still denoted by v_n^ν), a J-holomorphic curve $v_n : (\Omega_n, \Omega_n \cap \mathbb{R}) \to (M, L)$ and a finite set $Z_n = \{z_n^1, \dots, z_n^{\ell_n}\} \subset \Omega_n$ such that v_n^ν converges to v_n uniformly with all derivatives on compact subsets of $\Omega_n \setminus Z_n$. We can further assume that $v_{n+1}^\nu|_{\Omega_n}$ is a subsequence of v_n^ν. Hence $v_{n+1}|_{\Omega_n} = v_n$ and $Z_n \subset Z_{n+1}$.

For every $z \in \mathbb{H}$ there exists $n(z)$ such that for every $n \geq n(z)$ the diagonal sequence $w_n(z) := v_n^n(z)$ is defined and w_n converges to a J-holomorphic curve $w : (\mathbb{H}, \mathbb{R}) \to (M, L)$ uniformly on compact subsets of $\mathbb{H} \setminus Z$ where $Z = \bigcup_{n \in \mathbb{N}} Z_n$. Because the energy of w is bounded by the energy of u^ν in B_1 and the mass of every nodal point is bounded below, (iii) of Theorem A.12 shows that Z is a finite set.

The function $\tilde{w}(z) := w(1/z)$ is a J-holomorphic curve $(B_r \cap \mathbb{H}) \setminus \{0\} \to M$ with $\tilde{w}(B_r \cap \mathbb{R} \setminus \{0\}) \subset L$ and finite energy. It follows from Theorem A.11 that $\tilde{w}$ extends to a smooth map on $B_r \cap \mathbb{H}$. Because L is compact, $\tilde{w}(B_r \cap \mathbb{R}) \subset L$. Defining $w(\infty) := \tilde{w}(0)$ extends w to a J-holomorphic curve on $\mathbb{H} \cup \{\infty\}$ which is biholomorphic to the disk. Furthermore, $w(\mathbb{R} \cup \{\infty\}) \subset L$.

Step 2. *The limits*

$$m_j = \lim_{\epsilon \to 0} \lim_{\nu \to \infty} E(v^\nu, B_\epsilon(z_j))$$

exist and are greater than or equal to $\hbar$ where $\hbar$ is as in Corollary A.4.

This follows immediately from Theorem A.12.

We introduce the following notation:

$$a^\nu := \operatorname{Re} z^\nu \quad \text{and} \quad b^\nu := \begin{cases} \delta^\nu & \text{in case (I),} \\ \operatorname{Im} z^\nu & \text{in case (II).} \end{cases}$$

In both cases a^ν and b^ν converge to 0. Let

$$\kappa^\nu := \begin{cases} i \cdot \operatorname{Im} z^\nu / \delta^\nu & \text{in case (I),} \\ i & \text{in case (II).} \end{cases}$$

Note that in both cases κ^ν converges to some $\kappa \in \mathbb{H}$. We define further

$$\varrho^\nu := \begin{cases} 1 & \text{in case (I),} \\ \delta^\nu / \operatorname{Im} z^\nu & \text{in case (II),} \end{cases}$$

and

$$\varrho := \begin{cases} 1 & \text{in case (I),} \\ 0 & \text{in case (II).} \end{cases}$$

In both cases ϱ^ν converges to ρ.

Fix some $R > 0$ large enough. Then it follows from the definition of z^ν that

$$|dv^\nu(\kappa_\nu)| = \max_{\Omega_R} |dv^\nu| \tag{3}$$

if ν is large enough, such that v^ν is defined on the whole of Ω_R.

Step 3. *We have*

$$|z_j - \kappa| \le \varrho \qquad \forall z_j \in Z. \tag{4}$$

This can be seen in the following way. Assume the contrary. Let z_ℓ be such that $R > |z_\ell - \kappa| > \varrho$. It follows from the definition of m_0 that there exists $\epsilon > 0$ such that

$$\lim_{\nu\to\infty} E(u^\nu, \Omega_\epsilon) \le m_0 + \hbar/8.$$

This implies that there exist $\nu_0 \in \mathbb{N}$ such that, for every integer $\nu \ge \nu_0$,

$$E(u^\nu, \Omega_\epsilon) \le m_0 + \hbar/4.$$

In particular, this shows that for every $0 < \epsilon' \le \epsilon$ we have

$$E(u^\nu, \Omega_{\epsilon'}) \le E(u^\nu, \Omega_\epsilon) \le m_0 + \hbar/4.$$

Because z^ν and δ^ν converge to zero we can assume (perhaps after enlarging ν_0) that

$$E(u^\nu, B_{2R\delta^\nu}(z^\nu) \cap \Omega) \le m_0 + \hbar/4.$$

Because κ^ν converges to κ we have, for ν large enough,

$$E(v^\nu, B_R(\kappa) \cap \mathbb{H}) \le E(v^\nu, B_{2R}(\kappa^\nu) \cap \mathbb{H}) = E(u^\nu, B_{2R\delta^\nu}(z^\nu) \cap \Omega) \le m_0 + \hbar/4.$$

On the other hand, $E(v^\nu, B_{\varrho^\nu}(\kappa^\nu) \cap \Omega) = m_0 - \hbar/2$. Choose $\epsilon > 0$ such that $\varrho + \epsilon < |z_\ell - \kappa|$. Let $z_1, \dots, z_k$ for $k < \ell$ be the bubbles in $B_{\varrho+\epsilon}(\kappa) \cap \mathbb{H}$. Then for ν large enough it follows from (iii) of Theorem A.12 that

$$E(v, B_{\varrho+\epsilon}(\kappa) \cap \mathbb{H}) + \sum_{j=1}^{k} m(z_j) = \lim_{\nu\to\infty} E(v^\nu, B_{\varrho+\epsilon}(\kappa) \cap \mathbb{H}) \ge m_0 - \hbar/2.$$

Choose $R > 0$ so large that $|z_k - \kappa| < R$ for $1 \le k \le \ell$. Because $m(z_\ell) \ge \hbar$ we obtain using Theorem A.12 again

$$\begin{aligned} m_0 + \hbar/4 &\ge \lim_{\nu\to\infty} E(v^\nu, B_R(\kappa) \cap \mathbb{H}) = E(v, B_R(\kappa) \cap \mathbb{H}) + \sum_{j=1}^{\ell} m(z_j) \\ &\ge E(v, B_{\varrho+\epsilon}(\kappa) \cap \mathbb{H}) + \sum_{j=1}^{k} m(z_j) + m(z_\ell) \\ &\ge m_0 - \hbar/2 + \hbar = m_0 + \hbar/2. \end{aligned}$$

This contradiction shows that $|z_j - \kappa| \le \varrho$ for every $z_j \in Z$.

Step 4. *We have*

$$\lim_{R\to\infty} \lim_{\nu\to\infty} E(u^\nu, B_{R\delta^\nu}(z^\nu) \cap \Omega) = m_0. \tag{5}$$

To see this, note that, by definition of m_0 and δ^ν, for every $R \ge 1$ and every $\epsilon > 0$ there exists a $\nu_0 = \nu_0(R, \epsilon) \in \mathbb{N}$ such that, for every integer $\nu \ge \nu_0$,

$$m_0 - \hbar/2 = E(u^\nu, B_{\delta^\nu}(z^\nu) \cap \Omega) \le E(u^\nu, B_{R\delta^\nu}(z^\nu) \cap \Omega) \le m_0 + \epsilon.$$

This shows that, for every $R \geq 1$,

$$m_0 - \hbar/2 \leq \lim_{\nu\to\infty} E(u^\nu, B_{R\delta^\nu}(z^\nu) \cap \Omega) \leq m_0.$$

Suppose that (5) is false. Then there exists a constant $0 < \rho \leq \hbar/2$ such that, for every $R \geq 1$,

$$\lim_{\nu\to\infty} E(u^\nu, B_{R\delta^\nu}(z^\nu) \cap \Omega) \leq m_0 - \rho. \tag{6}$$

Hence, by definition of δ^ν, we obtain, for every $R \geq 1$,

$$\lim_{\nu\to\infty} E(u^\nu, (B_{R\delta^\nu}(z^\nu) \setminus B_{\delta^\nu}(z^\nu)) \cap \Omega) \leq \hbar/2 - \rho. \tag{7}$$

This leads to a contradiction as follows.

For every $\ell \in \mathbb{N}$, there exists an $\epsilon_\ell \in (0, 1/\ell)$ and a $\nu_\ell \in \mathbb{N}$ such that $|E(u^\nu, B_{\epsilon_\ell}(z^\nu) \cap \Omega) - m_0| \leq 1/\ell$ for $\nu \geq \nu_\ell$. Suppose, without loss of generality, that $\epsilon_{\ell+1} < \epsilon_\ell$ and $\nu_{\ell+1} > \nu_\ell$ for every ℓ. Then the sequence ϵ^ν defined by $\epsilon^\nu = \epsilon_\ell$ for $\nu_\ell \leq \nu < \nu_{\ell+1}$ satisfies

$$\lim_{\nu\to\infty} E(u^\nu, B_{\epsilon^\nu}(z^\nu)) = m_0, \qquad \lim_{\nu\to\infty} \epsilon^\nu = 0. \tag{8}$$

Hence for every $R \geq 1$,

$$\lim_{\nu\to\infty} E(u^\nu, B_{R\epsilon^\nu}(z^\nu) \cap \Omega) = m_0, \qquad \lim_{\nu\to\infty} \delta^\nu/\epsilon^\nu = 0. \tag{9}$$

The first equality follows from (8) and the definition of m_0. The second equality follows from (6) and (8). Again we have to distinguish two cases:

(a) There is a subsequence of ν (still denoted by ν) such that $\operatorname{Im} z^\nu/\epsilon^\nu$ converges to some finite number.

(b) There is a subsequence of ν (still denoted by ν) such that $\operatorname{Im} z^\nu/\epsilon^\nu$ converges to infinity.

In case (a) we consider the sequence $w^\nu : \Omega_{R^\nu} \to M$ defined by

$$w^\nu(z) = u^\nu(\operatorname{Re} z^\nu + \epsilon^\nu z)$$

where $R^\nu = (1 - |\operatorname{Re} z^\nu|)/\epsilon^\nu$. In case (b) let $w^\nu : B_{R^\nu} \to M$ be defined by

$$w^\nu(z) = u^\nu(z^\nu + \epsilon^\nu z)$$

where $R^\nu = \operatorname{Im} z^\nu/\epsilon^\nu$. Note that R^ν converges to infinity in both cases.

In case (a), w^ν converges modulo bubbling to a J-holomorphic disk. Moreover, by (9), for every $R \geq 1$ and every $\delta > 0$,

$$\begin{aligned}\lim_{\nu\to\infty} E(w^\nu, (B_R(i \operatorname{Im} z^\nu/\epsilon^\nu) \setminus B_\delta(i \operatorname{Im} z^\nu/\epsilon^\nu) \cap \mathbb{H}) \\ &= \lim_{\nu\to\infty} E(u^\nu, (B_{R\epsilon^\nu}(z^\nu) \setminus B_{\delta\epsilon^\nu}(z^\nu)) \cap \Omega) \\ &\leq \lim_{\nu\to\infty} E(u^\nu, (B_{R\epsilon^\nu}(z^\nu) \setminus B_{\delta^\nu}(z^\nu) \cap \Omega) = \hbar/2.\end{aligned}$$

This implies that w^ν converges to a constant, uniformly with all derivatives on compact subsets of $\mathbb{H} \setminus \{0\}$.

In case (b), w^ν converges modulo bubbling to a J-holomorphic sphere. By the same argument this sphere is constant too.

Hence, for every $T > 0$,

$$\lim_{\nu\to\infty} E(u^\nu, (B_{\epsilon^\nu}(z^\nu) \setminus B_{e^{-T}\epsilon^\nu}(z^\nu)) \cap \Omega) = 0.$$

Moreover, by definition of ϵ^ν and δ^ν,

$$\lim_{\nu\to\infty} E(u^\nu, (B_{\epsilon^\nu}(z^\nu) \setminus B_{\delta^\nu}(z^\nu))) = \hbar/2.$$

We can assume that $\hbar$ is chosen so small that the conclusion of Lemma A.6 holds true. It follows that there exists a constant $c > 0$ such that, for every $T > 0$,

$$\lim_{\nu\to\infty} E(u^\nu, (B_{e^{-T}\epsilon^\nu}(z^\nu) \setminus B_{e^{T}\delta^\nu}(z^\nu)) \cap \Omega) \le \frac{c}{T}\frac{\hbar}{2}.$$

The last three formulas together show that

$$\lim_{\nu\to\infty} E(u^\nu, (B_{e^{T}\delta^\nu}(z^\nu) \setminus B_{\delta^\nu}(z^\nu)) \cap \Omega) \ge \left(1 - \frac{c}{T}\right)\frac{\hbar}{2}.$$

For $c\hbar/2T < \rho$ this contradicts (7). Thus we have proved (5).

Step 5. *Let v^ν be as in case* (I)*. If v is constant then $\#Z \ge 2$.*

Because v is constant and $|z_j - \kappa| \le \varrho = 1$ it follows that for $1 < r < R$,

$$\lim_{\nu\to\infty} E(v^\nu, (B_R(\kappa) \setminus B_r(\kappa)) \cap \mathbb{H}) = E(v, (B_R(\kappa) \setminus B_r(\kappa)) \cap \mathbb{H}) = 0.$$

This shows that the limit of $E(v^\nu, B_r(\kappa) \cap \mathbb{H})$ as $\nu \to \infty$ is independent of $r > 1$.

Fix $r > 1$. Because κ^ν converges to κ there exist $1 < \rho_1 < r < \rho_2$ such that $B_{\rho_1}(\kappa) \subset B_r(\kappa^\nu) \subset B_{\rho_2}(\kappa)$ for ν large enough. Therefore $E(u^\nu, B_{r\delta^\nu}(z^\nu) \cap \Omega)$ $= E(v^\nu, B_r(\kappa^\nu) \cap \mathbb{H})$ is also independent of $r > 1$ as $\nu \to \infty$. Hence by (5),

$$\lim_{\nu\to\infty} E(v^\nu, B_r(\kappa^\nu) \cap \mathbb{H}) = m_0$$

for every $r > 1$. Since $E(v^\nu, B_1(\kappa^\nu) \cap \mathbb{H}) = m_0 - \hbar/2$, this is only possible if bubbling occurs on $\partial B_1(\kappa) \cap \mathbb{H}$. This means that the singular set Z of v^ν contains a point z_j with $|z_j - \kappa| = 1$. But it follows from equation (3) that, if $Z \ne \emptyset$, then $\kappa \in Z$. Hence $\#Z \ge 2$.

Step 6. $E(v) + \sum_{j=1}^{\ell} m_j = m_0$.

Fix any number $r > 1$. Then, by (5),

$$\begin{aligned}
m_0 &= \lim_{R\to\infty}\lim_{\nu\to\infty} E(u^\nu, B_{R\delta^\nu}(z^\nu) \cap \Omega) \\
&= \lim_{R\to\infty}\lim_{\nu\to\infty} E(v^\nu, B_R(\kappa^\nu) \cap \mathbb{H}) \\
&= \lim_{R\to\infty}\lim_{\nu\to\infty} E(v^\nu, (B_R(\kappa^\nu) \setminus B_r(\kappa^\nu)) \cap \mathbb{H}) + \lim_{\nu\to\infty} E(v^\nu, B_r(\kappa) \cap \mathbb{H}) \\
&= E(v, \mathbb{H} \setminus (B_r(\kappa) \cap \mathbb{H})) + \lim_{\epsilon\to 0}\lim_{\nu\to\infty} E\Big(v^\nu, \Big(B_r \setminus \bigcup_{j=1}^{\ell} B_\epsilon(z_j)\Big) \cap \mathbb{H}\Big) + \sum_{j=1}^{\ell} m_j \\
&= E(v) + \sum_{j=1}^{\ell} m_j.
\end{aligned}$$

Step 7. $E = E(v) + \sum_{j=1}^{\ell+1} m_j$.

Using conformal invariance of the energy we get

$$E = \lim_{\nu\to\infty} E(u^\nu) = \lim_{\nu\to\infty} E(v^\nu) = E(v) + \sum_{j=1}^{\ell+1} m_j.$$

Step 8. *Let v^ν be as in case* (II). *Then v^ν converges to a constant having one bubble of mass m_0 at $\kappa = i$.*

(4) implies that v can have at most one bubble at $\kappa = i$. It follows from the definition of v^ν and δ^ν that

$$E(v^\nu, B_{\delta^\nu/\operatorname{Im} z^\nu}(i)) = m_0 - \hbar/2.$$

Because $\delta^\nu/\operatorname{Im} z^\nu$ converges to zero it follows for every $\epsilon > 0$ that

$$E(v^\nu, B_\epsilon(i)) \geq E(v^\nu, B_{\delta^\nu/\operatorname{Im} z^\nu}(i))$$

for ν large enough. Hence

$$m_1 := \lim_{\epsilon\to 0}\lim_{\nu\to\infty} E(v^\nu, B_\epsilon(i)) \geq \lim_{\epsilon\to 0}\lim_{\nu\to\infty} E(v^\nu, B_{\delta^\nu/\operatorname{Im} z^\nu}(i)) = m_0 - \hbar/2.$$

Step 6 implies that $E(v) = m_0 - m_1 \leq \hbar/2$. Hence the disk v is constant, $E(v) = 0$ and $m_1 = m_0$. This proves Step 8.

We have proved assertions (i)–(v) and (vii) of Theorem 3.5. Assertion (vi) will follow from Lemma 3.6. This proves the theorem. □

Lemma 3.6. *Let u^ν, v^ν, u, v, and $\psi^\nu = a^\nu + b^\nu z$ be as in the proof of Theorem* 3.5. *Then u and v are connected. More exactly,*

$$u(0) = v(\infty).$$

Moreover, if $\zeta^\nu \in \Omega$ is any sequence such that

$$\lim_{\nu\to\infty} \zeta^\nu = 0, \qquad \lim_{\nu\to\infty} (\psi^\nu)^{-1}(\zeta^\nu) = \infty,$$

then $u^\nu(\zeta^\nu)$ converges to $u(z_0)$.

Proof. We first show that we can apply Lemma A.6 to

$$G^\nu(\epsilon, Rb^\nu) = (B_\epsilon(a^\nu) \setminus B_{Rb^\nu}(a^\nu)) \cap \mathbb{H}$$

for small $\epsilon > 0$ and large R and ν. To see this, note that

$$\lim_{\nu\to\infty} E(u^\nu, B_\epsilon(a^\nu) \cap \mathbb{H}) = m_0 + E(u, \Omega_\epsilon)$$

and

$$\begin{aligned}\lim_{\nu\to\infty} E(u^\nu, B_{Rb^\nu}(a^\nu) \cap \mathbb{H}) &= \lim_{\nu\to\infty} E(v^\nu, B_R(\kappa) \cap \mathbb{H}) = E(v, B_R(\kappa) \cap \mathbb{H}) + \sum_{j=1}^{\ell} m_j \\ &= m_0 - E(v, \mathbb{H} \setminus (B_R(\kappa) \cap \mathbb{H}))\end{aligned}$$

where for the last equality we have used Step 6 of Theorem 3.5. Subtracting these two identities we obtain

$$\lim_{\nu\to\infty} E(u^\nu, (B_\epsilon(a^\nu) \setminus B_{Rb^\nu}(a^\nu)) \cap \mathbb{H}) = E(u, \Omega_\epsilon) + E(v, \mathbb{H} \setminus (B_R(\kappa) \cap \mathbb{H})).$$

Because the right hand side converges to zero as R converges to infinity and ϵ converges to zero, we see that there exist $R_0 > 0$, $\epsilon_0 > 0$ and $\nu_0 = \nu_0(\epsilon, R) > 0$ such that

$$E(u^\nu, (B_\epsilon(a^\nu) \setminus B_{Rb^\nu}(a^\nu)) \cap \mathbb{H}) \le \hbar$$

if $R > R_0$, $0 < \epsilon < \epsilon_0$, $\nu \ge \nu_0(\epsilon, R)$ and $\hbar$ is as in Lemma A.6.[4] We define

$$E(\epsilon, R) := \epsilon + 1/R + \lim_{\nu\to\infty} E(u^\nu, (B_\epsilon(a^\nu) \setminus B_{Rb^\nu}(a^\nu)) \cap \mathbb{H}).$$

Perhaps after enlarging $\nu_0(\epsilon, R)$ we can assume that

$$E(u^\nu, (B_\epsilon(a^\nu) \setminus B_{Rb^\nu}(a^\nu)) \cap \mathbb{H}) \le E(\epsilon, R).$$

Now Lemma A.6 shows

$$\sup_{\zeta\in G^\nu[\epsilon/2, 2Rb^\nu]} d(u^\nu(\zeta), u^\nu(a^\nu + \epsilon/2)) = \mathcal{O}\big(\sqrt{E(\epsilon, R)}\big).$$

Next observe that $u^\nu(a^\nu + \epsilon/2)$ converges to $u(\epsilon/2)$ as $\nu \to \infty$ and, by the a priori estimates, the distance between $u(\epsilon/2)$ and $u(0)$ can be estimated by a constant times $\sqrt{E(\epsilon, R)}$. It follows that

$$d(u^\nu(a^\nu + \epsilon/2), u(0)) = \mathcal{O}\big(\sqrt{E(\epsilon, R)}\big).$$

Using the triangle inequality, we find

$$\sup_{\zeta\in G^\nu[\epsilon/2, 2Rb^\nu]} d(u^\nu(\zeta), u(0)) = \mathcal{O}\big(\sqrt{E(\epsilon, R)}\big). \tag{10}$$

Taking the limit $\nu \to \infty$ in (10) with $\zeta = 2Rb^\nu + a^\nu$ we obtain

$$d(v(2R), u(0)) = \mathcal{O}\big(\sqrt{E(\epsilon, R)}\big).$$

Since $E(\epsilon, R) \to 0$ for $\epsilon \to 0$ and $R \to \infty$ we obtain $v(\infty) = u(0)$ as required.

Now suppose that

$$\zeta^\nu \to 0, \qquad (\varphi^\nu)^{-1}(\zeta^\nu) = \frac{\zeta^\nu - a^\nu}{b^\nu} \to \infty.$$

Fix constants $R > R_0$ and $\epsilon < \epsilon_0$. Then for ν sufficiently large, we have

$$|a^\nu| + |\zeta^\nu| < \epsilon/2, \qquad |\zeta^\nu - a^\nu| > 2Rb^\nu.$$

This implies that ζ^ν lies in $G^\nu(\epsilon/2, 2R\delta^\nu)$. Hence it follows from (10) that

$$d(u^\nu(\zeta^\nu), u(0)) = \mathcal{O}\big(\sqrt{E(\epsilon, R)}\big)$$

for $\nu \ge \nu_0(\epsilon, R)$. Hence $u^\nu(\zeta^\nu)$ converges to $u(0)$, and this proves the lemma. □

[4]There is some additional subtlety because $\hbar$ can depend on J. But because J^ν converges to J and $\hbar$ can be chosen to depend continuously on J the above inequality implies Lemma A.6 if one shrinks $\hbar$ a little.

3.2.2. Proof of Gromov compactness. The proof goes via induction on the number of vertices. We will use the notion of weak Gromov convergence which does not yet detect all the bubbles but which takes track of the energy at the bubbling points.

Definition 3.7. Let M, L, ω^ν, J^ν, ω and J be as in Definition 3.1. A *weighted stable map*

$$((\mathbf{u},\mathbf{z}),\mathbf{m}) = ((\{(\Sigma_\alpha,\Gamma_\alpha,u_\alpha)\}_{\alpha\in T},\{z_{\alpha\beta}\}_{\alpha E\beta},\{\alpha_i,z_i\}_{1\le i\le k}),(m_i)_{1\le i\le k})$$

consists of a stable map $(\mathbf{u},\mathbf{z})$ and positive real numbers m_i. We say that a sequence of J^ν-holomorphic maps u^ν *weakly Gromov converges* to a weighted stable map $((\mathbf{u},\mathbf{z}),\mathbf{m})$ if there exists a sequence $\{\varphi_\alpha^\nu\}_{\alpha\in T}$ of Möbius transformations such that u^ν, $\{\varphi_\alpha^\nu\}_{\alpha\in T}$, and $\mathbf{u}$ satisfy conditions (i), (ii), (v), (vi) of Definition 3.1 with (iii) and (iv) replaced by

(iii′) For every $\alpha\in T$ and for every compact subset $K\subset\Sigma_\alpha\setminus Z_\alpha$ the sequence $u^\nu\circ\varphi_\alpha^\nu$ converges to u_α, uniformly with all derivatives on K.

(iv′) For $\beta\in T$ such that $\alpha E\beta$,

$$\lim_{\epsilon\to 0}\lim_{\nu\to\infty}E(u^\nu,\varphi_\alpha^\nu(B_\epsilon(z_{\alpha\beta}))\cap B)=\sum_{\gamma\in T_{\alpha\beta}}\Big(E(u_\gamma)+\sum_{\alpha_j=\gamma}m_j\Big).$$

Moreover,

$$\lim_{\epsilon\to 0}\lim_{\nu\to\infty}E(u^\nu,\varphi_\alpha^\nu(B_\epsilon(z_i))\cap B)=m_i.$$

Note that if $k=0$ then a weighted stable map is just an unmarked stable map and weak Gromov convergence coincides with Gromov convergence.

Proof of Theorem 3.3. It follows from Theorem A.12 that there exists a subsequence (still denoted by u^ν), a J-holomorphic curve $u:B\to M$, and a finite set of points $Z=\{z_1,\dots,z_\ell\}\subset B$ such that u^ν converges to u uniformly with all derivatives on compact subsets of $B\setminus Z$. Moreover, the limit $m_j=\lim_{\epsilon\to 0}\lim_{\nu\to\infty}E(u^\nu,B_\epsilon(z_j))$ exists and is positive for $1\le j\le\ell$. We first claim that after a suitable reparametrization we can assume without loss of generality that one of the following four cases holds.

(I) u is not constant.

(II) u is constant and Z consists of at least three points.

(III) u is constant and Z consists of two points which do not lie both on the boundary.

(IV) u is constant and Z consists of precisely one point in the interior of B.

We have to exclude the cases where u is constant and Z consists either of one boundary point or of two boundary points. In the first case we replace u^ν by $u^\nu\circ\psi^\nu$ where $\psi^\nu\in G_0$ are the soft rescaling transformations provided by Theorem 3.5. It follows that for some finite set $\tilde Z\subset\mathbb{H}$ the sequence u^ν converges to some $v:(B,\partial B)\to(M,L)$ uniformly with all derivatives on compact subsets of $B\setminus(\tilde Z\cup\{\infty\})$. Because u is constant and Z consists of only one point, the mass of u^ν at ∞ vanishes and u^ν converges to v even uniformly with all derivatives on compact subsets of $B\setminus\tilde Z$. Replace u by v and Z by $\tilde Z$. Then it follows from

Theorem 3.5 that either u and Z satisfy one of conditions (I)–(IV), or Z consists of precisely two points on the boundary of B. In the latter case choose $z \in Z$ and apply Theorem 3.5 to the sequence u^ν and z. As in the case where Z consists of one point we get a soft rescaling sequence $\psi^\nu \in G_0$, $v : (B, \partial B) \to (M, L)$ and a finite set $\tilde{Z} \subset \mathbb{H}$ such that $u^\nu \circ \psi^\nu$ converges to v uniformly with all derivatives on compact subsets of $B \setminus (\tilde{Z} \cup \{\infty\})$. But because Z consists of two points, the mass of $u^\nu \circ \psi^\nu$ at ∞ does not vanish. Replace u^ν by $u^\nu \circ \psi^\nu$, u by v, and Z by $\tilde{Z} \cup \{\infty\}$. Then u and Z satisfy one of conditions (I)–(IV).

For some $n_0 \in \mathbb{N}$ we construct, for each $n \in \{1, \dots, n_0\}$,

(a) a subsequence of u^ν still denoted by u^ν,
(b) a weighted stable map

$$((\mathbf{u}_n, \mathbf{z}_n), \mathbf{m}_n) \\ = ((\{(\Sigma_\alpha, \Gamma_\alpha, u_\alpha)\}_{\alpha \in T^n}, \{z_{\alpha\beta}\}_{\alpha E^n \beta}, \{\alpha_i, z_i^n\}_{1 \le i \le k(n)}), (m_i^n)_{1 \le i \le k(n)})$$

such that $\# T^n = n$ and $\mathbf{m}_{n_0} = \mathbf{0}$.

To start the induction we set, in cases (I)–(III),

$$((\mathbf{u}_1, \mathbf{z}_1), \mathbf{m}_1) = (((B, \partial B, u), \{z_j\}_{1 \le j \le \ell}), \{m_j\}_{1 \le j \le \ell}).$$

If case (IV) holds, we replace u^ν by $u^\nu \circ \psi^\nu$ where $\psi^\nu \in G$ are the soft rescaling transformations from Theorem 3.5. Then there exist $v : S^2 \to M$ and $\tilde{Z} \subset B$ such that u^ν converges to v uniformly on compact subsets of $B \setminus (\tilde{Z} \cup \{\infty\})$. We replace u by v and Z by $\tilde{Z}$ and set

$$((\mathbf{u}_1, \mathbf{z}_1), \mathbf{m}_1) = (((S^2, \{\infty\}, u), \{z_j\}_{1 \le j \le \ell}), \{m_j\}_{1 \le j \le \ell}).$$

Suppose that $((\mathbf{u}_n, \mathbf{z}_n), \mathbf{m}_n)$ has been constructed for some $n \in \mathbb{N}$ and $\mathbf{m} \neq \mathbf{0}$. Pick some marked point of $(\mathbf{u}_n, \mathbf{z}_n)$, say $z_{k(n)}^n$. Theorem 3.5 applied to $u^\nu \circ \varphi^\nu_{\alpha^n_{k(n)}}$ and $z^n_{k(n)}$ provides us with a sequence of Möbius transformations ψ^ν, a J-holomorphic map $v : (\Sigma, \partial\Sigma) \to (M, L)$ where Σ is either the sphere or the disk, a finite set $Z = \{z_1, \dots, z_\ell\} \subset \Sigma$, and weights m_i for $1 \le i \le \ell$, such that $u^\nu \circ \varphi^\nu_{\alpha^n_{k(n)}} \circ \psi^\nu$ converges to v uniformly with all derivatives on compact subsets of $\Sigma \setminus (Z \cup \{\infty\})$, with mass m_i at $z_i \in Z$. To define the tree T^{n+1} add to T^n an additional vertex γ, which is only connected to $\alpha^n_{k(n)}$. Define

$$(\Sigma_\gamma, \Gamma_\gamma, u_\gamma) = (\Sigma, \partial\Sigma, v), \qquad z_{\alpha^n_{k(n)}\gamma} = z^n_{k(n)}, \qquad z_{\gamma\alpha^n_{k(n)}} = \infty.$$

Set $k(n+1) = k(n) - 1 + \ell$. For $1 \le i \le k(n) - 1$ define $\alpha_i^{n+1} = \alpha_i^n$, $z_i^{n+1} = z_i^n$, and $m_i^{n+1} = m_i^n$. For $k(n) \le i \le k(n) - 1 + \ell$ define $\alpha_i^{n+1} = \gamma$, $z_i^{n+1} = z_{i-k(n)+1}$, and $m_i^{n+1} = m_{i-k(n)+1}$. This defines $((\mathbf{u}_{n+1}, \mathbf{z}_{n+1}), \mathbf{m}_{n+1})$. It remains to show that this process stabilizes, i.e. there exists $n_0 \in \mathbb{N}$ such that $k(n_0) = 0$. If such an n_0 exists then weak Gromov convergence of u^ν to $((\mathbf{u}_{n_0}, \mathbf{z}_{n_0}), \mathbf{m}_{n_0})$ is equivalent to Gromov convergence of u^ν to $(\mathbf{u}_{n_0})$. The existence of the desired n_0 follows from the assumption of bounded energy. More precisely, Lemma 2.7 shows that the process stabilizes for some $n_0 \le 3E/\hbar - 2$. This proves the theorem. □

3.3. Proof that the homotopy class converges

The goal of this subsection is to prove Proposition 3.2. We need the following lemma.

Lemma 3.8. *Let u^ν be a sequence of J-holomorphic disks which Gromov converges to the stable map*

$$(\mathbf{u}) = (\{(\Sigma_\alpha, \Gamma_\alpha, u_\alpha)\}_{\alpha \in T}, \{z_{\alpha\beta}\}_{\alpha E \beta})$$

with corresponding sequences $\varphi_\alpha^\nu \in G$ respectively $\varphi_\alpha^\nu \in G_0$. Moreover, let $\alpha, \beta \in T$ with $\alpha E \beta$. Assume that $\zeta^\nu \in (\varphi_\alpha^\nu)^{-1}(B)$ is a sequence such that

$$\lim_{\nu \to \infty} \zeta^\nu = z_{\alpha\beta}, \qquad \lim_{\nu \to \infty} (\varphi^\nu)^{-1}(\zeta^\nu) = z_{\beta\alpha}$$

where $\varphi^\nu = (\varphi_\alpha^\nu)^{-1} \circ \varphi_\beta^\nu$. Then $u^\nu \circ \varphi_\alpha^\nu(\zeta^\nu)$ converges to $u_\alpha(z_{\alpha\beta})$.

Proof. We show that one may assume without loss of generality that

$$\varphi^\nu(z) = z^\nu + \delta^\nu z, \qquad z_{\alpha\beta} = 0, \qquad z_{\beta\alpha} = \infty$$

where $z^\nu \in \mathbb{C}$ and $\delta^\nu > 0$ both converge to zero. Then the lemma follows in the same way as Lemma 3.6. We may assume without loss of generality that $z_{\alpha\beta} = 0$ and $z_{\beta\alpha} = \infty$. Moreover, there exists a sequence $\rho^\nu \in G$ which converges uniformly to the identity, such that $\varphi^\nu \circ \rho^\nu$ maps ∞ to ∞. If $\Sigma_\alpha = \Sigma_\beta = B$, then ρ^ν can be chosen in G_0. By replacing φ^ν by $\varphi^\nu \circ \rho^\nu$ we see that φ^ν is of the form $\varphi^\nu(z) = z^\nu + \delta^\nu z$. Because φ^ν converges to 0 uniformly on compact subsets of $\Sigma_\beta \setminus \{\infty\}$, we see that z^ν and δ^ν converge to 0. □

Proof of Proposition 3.2. The proof of [MS2] for the case of spheres translates nearly word for word to the case of disks. We will include it here for the reader's convenience.

To prove (i) suppose that $x^\nu = u^\nu(z^\nu)$ converges to x. Passing to a subsequence if necessary, we may assume without loss of generality that $(\varphi_\alpha^\nu)^{-1}(z^\nu)$ converges to some point $z_\alpha \in B$ for every α. If there exists an $\alpha \in T$ such that $z_\alpha \neq z_{\alpha\beta}$ for all $\beta \in T$ with $\alpha E \beta$, then

$$x = \lim_{\nu \to \infty} u^\nu(z^\nu) = \lim_{\nu \to \infty} u^\nu \circ \varphi_\alpha^\nu((\varphi_\alpha^\nu)^{-1}(z^\nu)) = u_\alpha(z_\alpha)$$

and we are done. If there is no such α then there exists an edge $\alpha E \beta$ in T such that

$$z_\alpha = z_{\alpha\beta}, \qquad z_\beta = z_{\beta\alpha}. \tag{11}$$

To see this, let us begin with any vertex α_0 and note that there is a unique sequence of vertices $\alpha_0, \alpha_1, \alpha_2, \dots$ such that $\alpha_i E \alpha_{i+1}$ and $z_{\alpha_i} = z_{\alpha_i \alpha_{i+1}}$. There must be some j with $\alpha_j = \alpha_{j+2}$ since otherwise the α_i would form an infinite sequence of pairwise distinct vertices. Hence the vertices $\alpha = \alpha_j$ and $\beta = \alpha_{j+1}$ satisfy (11). Now Lemma 3.8 with $\zeta^\nu = (\varphi_\alpha^\nu)^{-1}(z^\nu)$ shows

$$u_\alpha(z_{\alpha\beta}) = \lim_{\nu \to \infty} u^\nu \circ \varphi_\alpha^\nu((\varphi_\alpha^\nu)^{-1}(z^\nu)) = \lim_{\nu \to \infty} u^\nu(z^\nu) = x.$$

To prove (ii) we rephrase the conclusion of Lemma 3.8. For $\alpha, \beta \in T$ with $\alpha E \beta$, $\nu \in \mathbb{N}$, and $r > 0$ consider the set

$$\begin{aligned} A_r^\nu(\alpha,\beta) &= \varphi_\alpha^\nu(B_r(z_{\alpha\beta}) \cap (\varphi_\alpha^\nu)^{-1}(B)) \cap \varphi_\beta^\nu(B_r(z_{\beta\alpha}) \cap (\varphi_\beta^\nu)^{-1}(B)) \\ &= \{z \in B : d((\varphi_\alpha^\nu)^{-1}(z), z_{\alpha\beta}) < r,\, d((\varphi_\beta^\nu)^{-1}(z), z_{\beta\alpha}) < r\}. \end{aligned}$$

Since $(\varphi_\beta^\nu)^{-1} \circ \varphi_\alpha^\nu$ converges to $z_{\beta\alpha}$, uniformly on compact subsets of $\Sigma_\alpha \setminus \{z_{\alpha\beta}\}$, it follows that this set is a half-annulus respectively an annulus whenever $r > 0$ is sufficiently small and $\nu \geq \nu_0(r)$ is sufficiently large. Now the assertion of Lemma 3.8 can be rephrased in the form that, for every $\epsilon > 0$, there exists an $r > 0$ and a $\nu_0 \in \mathbb{N}$ such that, for all $\nu \in \mathbb{N}$ and all $\alpha, \beta \in T$ with $\alpha E \beta$,

$$\nu \geq \nu_0 \;\Rightarrow\; \sup_{z \in A_r^\nu(\alpha,\beta)} d(u^\nu(z), u_\alpha(z_{\alpha\beta})) < \epsilon.$$

This shows that the image of $A_r^\nu(\alpha,\beta)$ under u^ν is arbitrarily close to $u_\alpha(z_{\alpha\beta}) = u_\beta(z_{\beta\alpha})$, provided that r is sufficiently small and ν is sufficiently large. Fixing a small number $r > 0$, we deduce that u^ν is homotopic to the map $\tilde{u}^\nu : B \to M$ in which the images of $A_r^\nu(\alpha,\beta)$ are replaced by sufficiently short curves connecting the two boundary circles, respectively half-circles. These curves are chosen in such a way that $\tilde{u}^\nu$ maps ∂B to L. In the limit $\nu \to \infty$, these deformed maps converge to the connected sum of the u_α, which is obtained by removing disks, respectively half-disks, of radius r around the points $z_{\alpha\beta}$ and replacing the union of $u_\alpha(B_r(z_{\alpha\beta}) \cap (\varphi_\alpha^\nu)^{-1}B)$ and $u_\beta(B_r(z_{\beta\alpha}) \cap (\varphi_\beta^\nu)^{-1}B)$ by the geodesics connecting $u_\alpha(\partial B_r(z_{\beta\alpha}) \cap (\varphi_\alpha^\nu)^{-1}B)$ and $u_\beta(\partial B_r(z_{\alpha\beta}) \cap (\varphi_\beta^\nu)^{-1}B)$. Hence u^ν is homotopic to this connected sum for large ν. This proves the proposition. □

4. Gromov convergence for marked disks

In this section we generalize Definition 3.1 to the case where the disks are allowed to have marked points. We will see how Proposition 3.2 and Theorem 3.3 continue to hold in this case. We will also prove a generalization of Theorem 3.4.

4.1. Gromov convergence for marked disks

Definition 4.1. Let M be a compact manifold with boundary, L a compact submanifold of M without boundary, ω^ν a sequence of symplectic structures on M such that L is Lagrangian for every ω^ν, J^ν a sequence of ω^ν-tame almost complex structures, and k_1 and k_2 two positive integers. Assume that the ω^ν converge to some symplectic structure ω on M with respect to the C^∞-topology and the J^ν converge in the C^∞-topology to some $J \in \mathcal{J}_\tau(M,\omega)$. A sequence $(u^\nu, \mathbf{z}^\nu)$ with $\mathbf{z}^\nu = (z_1^\nu, \ldots, z_k^\nu)$ of stable J^ν-holomorphic disks with k_1 marked points in the interior and k_2 marked points on the boundary is said to *Gromov converge* to a stable map

$$(\mathbf{u}, \mathbf{z}) = (\{(\Sigma_\alpha, \Gamma_\alpha, u_\alpha)\}_{\alpha \in T}, \{z_{\alpha\beta}\}_{\alpha E \beta}, \{\alpha_i, z_i\}_{1 \leq i \leq k})$$

if there exists a collection $\{\varphi_\alpha^\nu\}_{\alpha\in T}$ of Möbius transformations such that conditions (i)–(vi) of Definition 3.1 are satisfied and

(vii) $z_i = \lim_{\nu\to\infty}(\varphi_{\alpha_i})^{-1}(z_i^\nu)$ for $i = 1, \dots, k$. Moreover, $z_i \in \partial\Sigma_{\alpha_i}$ iff $z_i^\nu \in \partial B$ for every ν.

Remark 4.2. Note that the set of marked points naturally splits into two subsets: the set of interior marked points and the set of boundary marked points. (vii) of Definition 4.1 tells us that this splitting is preserved under limits.

In the same way as Proposition 3.2 one proves

Proposition 4.3. *Let J^ν be as in Definition* 4.1. *Fix some $k_1, k_2 \in \mathbb{N}$. Suppose that $(u^\nu, \mathbf{z}^\nu)$ is a sequence of (k_1, k_2)-marked J^ν-holomorphic disks with boundary in L which Gromov converges to a stable map $(\mathbf{u}, \mathbf{z})$. Then the following holds.*

(i) *If $x^\nu \in u^\nu(B)$ converges to $x \in M$ then $x \in \bigcup_{\alpha\in T} u_\alpha(\Sigma_\alpha)$.*
(ii) *u^ν is relative homotopic to $\#_{\alpha\in T}\, u_\alpha$ with respect to L for large ν.*

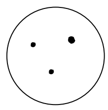
4.2. Gromov compactness

A natural generalization of Theorem 3.3 is

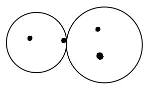
Theorem 4.4. *Let J^ν be as in Definition* 4.1 *and fix some $k_1, k_2 \in \mathbb{N}$. Assume that $(u^\nu, \mathbf{z}^\nu)$ is a sequence of (k_1, k_2)-marked J^ν-holomorphic disks with bounded energy. Then $(u^\nu, \mathbf{z}^\nu)$ has a Gromov convergent subsequence.*

Theorem 3.3 proves the case with no marked points. The difficulty in proving the general case lies in the fact that it may happen that two special points coincide in the limit, or that an interior marked point converges to the boundary. To deal with these cases one introduces auxiliary constant maps. The following example may illustrate this.

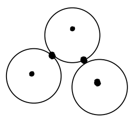
Example 4.5. Let $(u^\nu, z_1^\nu, z_2^\nu, z_3^\nu)$ be a sequence of constant maps from the disk to a fixed point on the Lagrangian with three interior marked points. Assume that the three marked points converge to points $z_1, z_2, z_3 \in B$. There are eight cases one has to deal with. In the first case the three limit points lie in the interior and do not coincide. In the second case one of the limit points lies on the boundary, while the other two are separated and lie in the interior. In this case the limit consists of two disks connected by a node, the other having two interior marked points and one disk having one interior marked point. In the third case two of the limit points are different points on the boundary, while the third lies in the interior. The limit stable map will be a chain consisting of three connected disks with one marked point in the interior. In the fourth case all three limit points are different points at the boundary. In this case one gets a star consisting of a disk with no marked point whose boundary is connected to three disks having one interior marked point. In the fifth case two of the marked points converge to the same limit point in the interior, while the third marked point converges to a different point in the interior. In this case the limit stable map is a sphere with two marked points connected to a disk with one interior marked point. In the sixth case two of the marked points converge again to the same limit point, while the third marked point converges to the boundary. The limit stable map is a chain consisting of a sphere with two marked points connected to a disk with no marked points connected to a disk with one marked point. In the seventh case all three marked points converge to the same limit point in the interior in such a way that the cross ratio of the three marked points with the limit point converges to some nonzero finite number. In this case one gets a sphere having three marked points and one boundary point. In the eighth case the marked points converge again to the same limit point in the interior, but their cross ratio with the limit point converges to zero or infinity. The limit stable map consists of two spheres connected by a node, one sphere having two marked points and one sphere having one marked point and one boundary point.



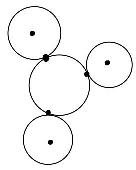
Case 1



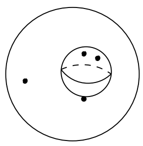
Case 2



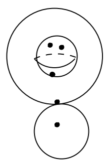
Case 3



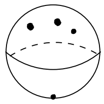
Case 4



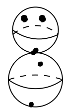
Case 5



Case 6



Case 7



Case 8

4.2.1. Marked points. The lemmas in this section are needed to deal with marked points. We will use some results of this subsubsection again to prove uniqueness of limits.

Lemma 4.6. *Suppose that* $(u^\nu, \mathbf{z}^\nu)$ *Gromov converges to* $(\mathbf{u}, \mathbf{z})$ *with corresponding Möbius transformations* $\varphi_\alpha^\nu \in G$. *Then the following holds.*

(i) *If* $\alpha \neq \beta$*, then* $(\varphi^\nu_\alpha)^{-1} \circ \varphi^\nu_\beta$ *converges to* $z_{\alpha\beta}$*, uniformly on compact subsets of* $\Sigma_\beta \setminus \{z_{\beta\alpha}\}$*.*
(ii) *For every* $\alpha \in T$ *and every* $i \in \{1, \ldots, k\}$*,* $z_{\alpha i} = \lim_{j \to \infty} (\varphi^\nu_\alpha)^{-1}(z^\nu_i)$*.*
(iii) *If* $\Gamma_\alpha = \{z^\infty_\alpha\}$*, then* $z^\infty_\alpha \neq \lim_{\nu \to \infty} (\varphi^\nu_\alpha)^{-1}(z^\nu_i)$ *for every* $i \in \{1, \ldots, k\}$*.*

Proof. To show (i) let us denote by $\gamma_0, \ldots, \gamma_m$ the chain of edges in T running from $\gamma_0 = \alpha$ to $\gamma_m = \beta$. It follows from the definition of Gromov convergence that $(\varphi^\nu_{\gamma_{i-1}})^{-1} \circ \varphi^\nu_{\gamma_i}$ converges to $z_{\gamma_{i-1}\gamma_i}$, uniformly on compact subsets of $\Sigma_{\gamma_i} \setminus \{z_{\gamma_i\gamma_{i-1}}\}$. Since $z_{\gamma_i\gamma_{i-1}} \neq z_{\gamma_i\gamma_{i+1}}$ we deduce, by induction, that $(\varphi^\nu_{\gamma_0})^{-1} \circ \varphi^\nu_{\gamma_i}$ converges to $z_{\gamma_0\gamma_i}$, uniformly on compact subsets of $\Sigma_{\gamma_i} \setminus \{z_{\gamma_i\gamma_0}\}$. (i) follows with $i = m$.

We show (ii). If $\alpha = \alpha_i$, then $z_{\alpha i} = z_i$. Hence in this case the assertion is equivalent to (vi) of Definition 4.1. If $\alpha \neq \alpha_i$, then (i) yields

$$z_{\alpha i} = z_{\alpha\alpha_i} = \lim_{j \to \infty} (\varphi^{\nu_j}_\alpha)^{-1} \circ \varphi^{\nu_j}_{\alpha_i}(z)$$

for $z \neq z_{\alpha_i\alpha}$. It follows from (vi) of Definition 4.1 that $z_i = \lim_{\nu \to \infty} (\varphi^\nu_{\alpha_i})^{-1}(z^\nu_i)$. Because $z_i \neq z_{\alpha_i\alpha}$ we have

$$z_{\alpha i} = \lim_{j \to \infty} (\varphi^{\nu_j}_\alpha)^{-1} \circ \varphi^{\nu_j}_{\alpha_i}((\varphi^{\nu_j}_{\alpha_i})^{-1}(z^\nu_i)) = \lim_{j \to \infty} (\varphi^{\nu_j}_\alpha)^{-1}(z^\nu_i).$$

This proves (ii).

To show (iii), assume that $z^\infty_\alpha = \lim_{\nu \to \infty} (\varphi^\nu_\alpha)^{-1}(z^\nu_i)$ for some $i \in \{1, \ldots, k\}$. Then (ii) shows that $z_{\alpha i} = z^\infty_\alpha$. It follows from (iii) of Definition 2.3 that $z^\infty_\alpha \neq z_{\alpha\beta}$ for $\alpha E \beta$. Hence $z_i = z_{\alpha i} = z^\infty_\alpha$ and $\alpha = \alpha_i$. But Definition 2.3 tells us also that $z_i \notin \Gamma_{\alpha_i} = \Gamma_\alpha$. This contradiction proves (iii). □

Lemma 4.7. *Let* $\varphi^\nu \in G_0$ *be a sequence of Möbius transformations which converges to* $y \in B$*, uniformly on compact subsets of* $B \setminus \{x\}$*. Moreover, let* $\xi^\nu \in B$ *be a sequence such that*

$$\lim_{\nu \to \infty} \xi^\nu = x, \qquad \lim_{\nu \to \infty} \varphi^\nu(\xi^\nu) = y.$$

Then there exists a sequence $\rho^\nu \in G_0$ *such that*

(a) ρ^ν *converges to* 0*, uniformly on compact subsets of* $B \setminus \{y\}$*.*
(b) $\rho^\nu \circ \varphi^\nu$ *converges to* ∞*, uniformly on compact subsets of* $B \setminus \{x\}$*.*
(c) $|\rho^\nu \circ \varphi^\nu(\xi^\nu)| = 1$ *for all* ν*.*

Proof. We can assume without loss of generality that

$$\varphi^\nu(z) = \epsilon^\nu z + \delta^\nu, \qquad x = \infty, \qquad y = 0$$

where $\delta^\nu \in \mathbb{R}$ and $\epsilon^\nu > 0$ both converge to zero (cf. the proof of Lemma 3.8). By assumption $\lim_{\nu \to \infty} \xi^\nu = \infty$ and $\lim_{\nu \to \infty} \epsilon^\nu \xi^\nu = 0$. Hence the sequence $\rho^\nu \in PSL(2, \mathbb{R})$ defined by

$$\rho^\nu(w) := -\frac{\epsilon^\nu |\xi^\nu|}{w - \delta^\nu}$$

for large ν satisfies (a), (b) and (c). □

The following two lemmas can be proved exactly as the corresponding lemmas in [MS2].

Lemma 4.8. *Let J^ν and J be as in Theorem* 3.3. *Let $(u^\nu, z_1^\nu, \ldots, z_k^\nu)$ be a sequence of stable J^ν-holomorphic disks which Gromov converges to a stable map*

$$(\mathbf{u}, \mathbf{z}) = (\{(\Sigma_\alpha, \Gamma_\alpha, u_\alpha)\}_{\alpha\in T}, \{z_{\alpha\beta}\}_{\alpha E \beta}, \{\alpha_i, z_i\}_{1\le i\le k})$$

with corresponding sequences $\varphi_\alpha^\nu \in G$ for $\alpha \in T$. Moreover, let $\zeta^\nu \in B \backslash \{z_1^\nu, \ldots, z_k^\nu\}$ and suppose that the limit

$$\zeta_\alpha = \lim_{\nu\to\infty} (\varphi_\alpha^\nu)^{-1}(\zeta^\nu) \tag{12}$$

exists for all $\alpha \in T$. Then exactly one of the following conditions is satisfied.

(I) *There exists a (unique) $\alpha \in T$ such that $\zeta_\alpha \notin Z_\alpha$.*
(II) *There exists a (unique) $\alpha \in T$ such that $\Gamma_\alpha = \{z_\alpha^\infty\}$ and $\zeta_\alpha = z_\alpha^\infty$.*
(III) *There exists a (unique) $i \in \{1, \ldots, k\}$ such that $\zeta_{\alpha_i} = z_i$.*
(IV) *There exists a (unique) edge $\alpha E \beta$ in T such that $\zeta_\alpha = z_{\alpha\beta}$ and $\zeta_\beta = z_{\beta\alpha}$.*

Proof. See Lemma 5.3.3 in [MS2]. □

Lemma 4.9. *Let $(u^\nu, z_1^\nu, \ldots, z_k^\nu)$ be a sequence of stable J-holomorphic disks with marked points which Gromov converges to the stable map*

$$(\mathbf{u}, \mathbf{z}) = (\{(\Sigma_\alpha, \Gamma_\alpha, u_\alpha)\}_{\alpha\in T}, \{z_{\alpha\beta}\}_{\alpha E \beta}, \{\alpha_i, z_i\}_{1\le i\le k})$$

with corresponding sequences $\varphi_\alpha^\nu \in G$ respectively G_0. Moreover, let $\alpha, \beta \in T$ with $\alpha E \beta$. Suppose that $\psi^\nu \in G$ is a sequence such that the following holds.

(i) *$(\varphi_\alpha^\nu)^{-1} \circ \psi^\nu$ converges to $z_{\alpha\beta}$, uniformly on compact subsets of $\tilde{\Sigma}_\gamma \setminus \{x\}$,*
(ii) *$(\varphi_\beta^\nu)^{-1} \circ \psi^\nu$ converges to $z_{\beta\alpha}$, uniformly on compact subsets of $\tilde{\Sigma}_\gamma \setminus \{y\}$,*

where x and $y \in \tilde{\Sigma}_\gamma$. Then $u^\nu \circ \psi^\nu$ converges to the constant $u_\alpha(z_{\alpha\beta}) = u_\beta(z_{\beta\alpha})$, uniformly on compact subsets of $\tilde{\Sigma}_\gamma \backslash \{x, y\}$. Moreover, if $\alpha_i \in T_{\alpha\beta}$ then $(\psi^\nu)^{-1}(z_i^\nu)$ converges to y, and if $\alpha_i \in T_{\beta\alpha}$ then $(\psi^\nu)^{-1}(z_i^\nu)$ converges to x.

Proof. See proof of Theorem 5.3.1 in [MS2]. □

4.2.2. Proof of Gromov compactness. We prove by induction on ℓ that a subsequence of $(u^\nu, z_1^\nu, \ldots, z_{\ell-1}^\nu)$ Gromov converges to a stable map

$$(\mathbf{u}, \mathbf{z}) = (\{(\Sigma_\alpha, \Gamma_\alpha, u_\alpha)\}_{\alpha\in T}, \{z_{\alpha\beta}\}_{\alpha E \beta}, \{\alpha_i, z_i\}_{1\le i\le \ell-1})$$

with corresponding Möbius transformations φ_α^ν. For $\ell = 1$ this follows from Theorem 3.3. Hence suppose this has been proved for some $\ell \in \{1, \ldots, k\}$. Passing to a further subsequence if necessary, we may assume that the limit

$$z_{\alpha\ell} = \lim_{\nu\to\infty} (\varphi_\alpha^\nu)^{-1}(z_\ell^\nu) \tag{13}$$

exists for all α and all i. By Lemma 4.8, for each i, one of the following three cases is satisfied.

(I) There exists a (unique) $\alpha \in T$ such that $z_{\alpha\ell} \notin Z_\alpha$.
(II) There exists a (unique) $\alpha \in T$ such that $\Gamma_\alpha = \{z_\alpha^\infty\}$ and $z_{\alpha\ell} = z_\alpha^\infty$.
(III) There exists a (unique) $i \in \{1, \ldots, \ell - 1\}$ such that $z_{\alpha\ell} = z_i$.
(IV) There exists a (unique) edge $\alpha E \beta$ in T such that $z_{\alpha\ell} = z_{\alpha\beta}$ and $z_{\beta\ell} = z_{\beta\alpha}$.

If (I) holds and $z_{\alpha\ell} \notin \Gamma_\alpha$ we define $z_\ell = z_{\alpha_\ell \ell}$, where α_ℓ is the unique element of T with $z_{\alpha\ell} \notin Z_\alpha$. In the same way we proceed if $z_{\alpha\ell} \in \partial\Sigma_\alpha$ and z_ℓ^ν is a boundary marked point.

If (I) holds, $z_{\alpha\ell} \in \partial\Sigma_\alpha$ and z_ℓ^ν is an interior marked point, we add an additional vertex γ to the tree. Because $\partial\Sigma_\alpha \neq \emptyset$ we have $\Sigma_\alpha = B$ and hence $\varphi_\alpha^\nu \in G_0$. This shows that $w^\nu := (\varphi_\alpha^\nu)^{-1}(z_\ell^\nu) \in B \setminus \partial B$. Let $\rho^\nu \in G_0$ be a sequence of Möbius transformations which map w^ν to i. Because w^ν converges to a boundary point we may assume without loss of generality that ρ^ν converges to a point $\tilde{w} \in \partial B$ uniformly on compact subsets of $B \setminus \{w\}$, where $w = \lim_{\nu\to\infty} w^\nu = z_{\alpha\ell}$. Now Lemma B.2 shows that $(\rho^\nu)^{-1}$ converges to w uniformly on compact subsets of $B \setminus \{\tilde{w}\}$. We define further $\psi^\nu := \varphi_\alpha^\nu \circ (\rho^\nu)^{-1}$. Then $(\varphi_\alpha^\nu)^{-1} \circ \psi^\nu = (\rho^\nu)^{-1}$, $(\psi^\nu)^{-1}(z_\ell^\nu) = 0$ and because $w = z_{\alpha\ell} \notin Z_\alpha$ we see that the sequence $u^\nu \circ \psi^\nu = u^\nu \circ \varphi_\alpha^\nu \circ (\rho^\nu)^{-1}$ converges to the constant $u_\alpha(z_{\alpha\ell})$. Now choose the sequence φ_γ^ν which corresponds to the new vertex γ to be ψ^ν. Then in the new stable map

$$z_{\gamma\alpha}^{\text{new}} = \tilde{w}, \qquad z_{\alpha\gamma}^{\text{new}} = w = z_{\alpha\ell}, \qquad \alpha_\ell^{\text{new}} = \gamma, \qquad z_\ell^{\text{new}} = 0.$$

If (I) holds, let $\psi^\nu \in G_0$ with $\psi^\nu(z_\ell^\nu) = 0$. Then it follows from Theorem A.12 that there exists a subsequence of $u^\nu \circ \psi^\nu$ (still denoted by $u^\nu \circ \psi^\nu$), a finite set of points $W = \{w_1, \ldots, w_h\}$ and a J-holomorphic curve $u_\psi : (B, \partial B) \to (M, L)$ such that $u^\nu \circ \psi^\nu$ converges to u_ψ uniformly on compact subsets of $B \setminus W$. Moreover,

$$E(u_\psi) + \sum_{j=1}^{h} m(w_j) = \lim_{\nu\to\infty} E(u^\nu)$$

where

$$m(w_j) = \lim_{\epsilon\to 0} \lim_{\nu\to\infty} E(u^\nu, B_\epsilon(w_j)) \geq \hbar.$$

On the other hand, there exists $y \in B$ such that $(\psi^\nu)^{-1} \circ \varphi_\alpha^\nu$ converges to y uniformly on compact subsets of $S^2 \setminus \{z_\alpha^\infty\}$. Now Lemma B.2 implies that $(\varphi_\alpha^\nu)^{-1} \circ \psi^\nu$ converges to z_α^∞ uniformly on compact subsets of $B \setminus \{y\}$. Using (v) of Definition 4.1 we see that for every $\epsilon > 0$ there exists $\nu_0(\epsilon) \in \mathbb{N}$ such that for every $\nu \geq \nu_0(\epsilon)$ we have

$$E(u^\nu \circ \psi^\nu, B \setminus B_\epsilon(y)) = E(u^\nu \circ \varphi_\alpha^\nu, (\varphi_\alpha^\nu)^{-1} \circ \psi^\nu(B \setminus B_\epsilon(y))) \leq \hbar/2.$$

Now it follows from the definition of $\hbar$ that $h = 1$, $w_1 = y$ and $E(u_\psi) = 0$.

We have to distinguish several cases. If $y \neq 0$ and z_ℓ^ν are boundary marked points, or if $y \neq 0, y \notin \partial B$ and z_ℓ^ν are interior marked points, then we add a vertex γ to our tree such that γ is only connected to α and contains the marked point z_ℓ. In the new stable map we have

$$z_{\gamma\alpha}^{\text{new}} = y, \qquad z_{\alpha\gamma}^{\text{new}} = z_\alpha^\infty, \qquad \alpha_\ell^{\text{new}} = \gamma, \qquad z_\ell^{\text{new}} = 0.$$

Now assume that z_ℓ^ν are interior marked points. If $y \in \partial B$ then we construct as in Theorem 3.5 a sequence of Möbius transformations $\rho^\nu \in G_0$ such that $u^\nu \circ$

$^\nu \circ \rho^\nu$ converges uniformly on compact subsets of the complement of a finite set to a J-holomorphic disk. As above one shows that $u^\nu \circ \psi^\nu \circ \rho^\nu$ converges to a constant uniformly on compact set of the complement of a single point $\tilde{y} \in B$. Moreover, $(\rho^\nu)^{-1} \circ (\psi^\nu)^{-1} \circ \varphi_\alpha^\nu$ converges to $\tilde{y}$ uniformly on compact subsets of $S^2 \setminus \{z_\alpha^\infty\}$. It follows from (iv) of Theorem 3.5 that $\tilde{y} \notin \partial B$. Hence we can add two additional vertices γ and δ to our tree T such that γ is connected to α and δ is only connected to γ but contains the marked point z_ℓ. In the new stable map we have

$$z_{\gamma\alpha}^{\text{new}} = \tilde{y}, \quad z_{\alpha\gamma}^{\text{new}} = z_\alpha^\infty, \quad z_{\delta\gamma}^{\text{new}} = y, \quad z_{\gamma\delta}^{\text{new}} = \infty, \quad \alpha_\ell^{\text{new}} = \delta, \quad z_\ell^{\text{new}} = 0.$$

If finally $y = 0$, then we add as in (IV) below a sphere to our tree, which is connected to α and contains the marked point z_ℓ. In this case the disk is not stable any more and we collapse it. If γ denotes again the new vertex then in the new stable map

$$z_{\gamma\alpha}^{\text{new}} = 0, \quad z_{\alpha\gamma}^{\text{new}} = z_\alpha^\infty, \quad \alpha_\ell^{\text{new}} = \gamma, \quad z_\ell^{\text{new}} = 1, \quad \Gamma_\gamma^{\text{new}} = \{\infty\}.$$

In a similar way we proceed if $\Gamma_\alpha = \{z_\alpha^\infty\}$, $z_{\alpha\ell} = z_\alpha^\infty$, and $z_\ell^\nu \in \partial B$ for ν large enough. First choose $\psi^\nu \in G_0$ such that $u^\nu \circ \psi^\nu$ converges to a constant up to one nodal point v in the interior of B. Because $z_\ell^\nu \in \partial B$ there exists $w \in \partial B$ such that $\psi^\nu(z_\ell^\nu)$ converges to w. Let γ be the new vertex. Then

$$z_{\gamma\alpha}^{\text{new}} = v, \quad z_{\alpha\gamma}^{\text{new}} = z_\alpha^\infty, \quad \alpha_\ell^{\text{new}} = \gamma, \quad z_\ell^{\text{new}} = w.$$

If (III) is satisfied for some i and $z_i \notin \partial\Sigma_\alpha$, choose ψ^ν such that

$$^\nu(0) = z_i^\nu, \quad \psi^\nu(1) = z_\ell^\nu, \quad \psi^\nu(\infty) = \varphi_{\alpha_i}^\nu(w),$$

where $w \in \Sigma_i \setminus \{z_i\}$ is chosen such that $w \neq (\varphi_{\alpha_i}^\nu)^{-1}(z_i^\nu)$ and $w \neq (\varphi_{\alpha_i}^\nu)^{-1}(z_\ell^\nu)$ for all ν. Now we can use Lemma B.6 replacing x_1^ν by 0, x_2^ν by 1, y^ν by w, x_0 by ∞ and $(\varphi_{\alpha_i}^\nu)^{-1} \circ \psi^\nu$ by φ^ν. Hence $(\varphi_{\alpha_i}^\nu)^{-1} \circ \psi^\nu$ converges to z_i, uniformly on compact subsets of $\mathbb{C} = S^2 \setminus \{\infty\}$. Now add an additional vertex γ to the tree, defining $\varphi_\gamma^\nu = \psi^\nu$. Note that $u^\nu \circ \psi^\nu$ converges to $u_{\alpha_i}(z_i)$, uniformly with all derivatives on compact subsets of $\mathbb{C} = S^2 \setminus \{\infty\}$. Note also that in the new stable map we have

$$z_{\gamma\alpha_i}^{\text{new}} = \infty, \quad z_{\alpha_i\gamma}^{\text{new}} = z_i, \quad \alpha_\ell^{\text{new}} = \gamma, \quad z_\ell^{\text{new}} = 1, \quad \alpha_i^{\text{new}} = \gamma, \quad z_i^{\text{new}} = 0.$$

Now assume that (III) is satisfied for some i but $z_i \in \partial\Sigma_\alpha$. Then $(\varphi_{\alpha_i}^\nu)^{-1}(z_i^\nu) \in \partial B$ for ν large enough and we may assume without loss of generality that this is the case for every ν. Choose $w \in \partial B \setminus \{z_i\}$ such that $w \neq (\varphi_{\alpha_i}^\nu)^{-1}(z_i^\nu)$ and $w \neq (\varphi_{\alpha_i}^\nu)^{-1}(z_\ell^\nu)$. There exists $\rho^\nu \in G_0$ such that $\rho^\nu \circ (\varphi_{\alpha_i}^\nu)^{-1}(z_i^\nu) = 0$, $\rho^\nu(w) = \infty$ and $|\rho^\nu \circ (\varphi_{\alpha_i}^\nu)^{-1}(z_\ell^\nu)| = 1$. Now we add an additional vertex to our tree by defining $\varphi_\gamma^\nu = \varphi_{\alpha_i}^\nu \circ (\rho^\nu)^{-1}$. Passing to a subsequence we may assume that $(\varphi_\gamma^\nu)^{-1}(z_\ell^\nu)$ converges to some $e^{i\phi}$ where $\phi \in [0, \pi]$. The new relations are

$$z_{\gamma\alpha_i}^{\text{new}} = \infty, \quad z_{\alpha_i\gamma}^{\text{new}} = z_i, \quad \alpha_\ell^{\text{new}} = \gamma, \quad z_\ell^{\text{new}} = e^{i\phi}, \quad \alpha_i^{\text{new}} = \gamma, \quad z_i^{\text{new}} = 0.$$

Note that if $z_\ell^\nu \in \partial B$ for ν large enough, then $\phi = 0$ or $\phi = \pi$ and we are done. If $z_\ell^\nu \notin \partial B$ for ν large enough then nevertheless it may happen that $\phi = 0$ or $\phi = \pi$.

In this case we add a further vertex δ to T as in (I), such that δ is only connected to γ at the point $e^{i\phi}$.

If (IV) holds we extend the tree T by introducing an additional vertex γ which is only connected to α and β and corresponds to the constant map

$$u_\gamma(z) = u_\alpha(z_{\alpha\beta}) = u_\beta(z_{\beta\alpha}).$$

To see that this is possible we have to construct the sequence $\varphi^\nu_\gamma = \psi^\nu$. We assume that $\Sigma_\alpha = \Sigma_\beta = B$ (the other cases are similar). Applying Lemma 4.7 to the sequences $\varphi^\nu = (\varphi^\nu_\beta)^{-1} \circ \varphi^\nu_\alpha$ and $\xi^\nu = (\varphi^\nu_\alpha)^{-1}(z^\nu_\ell)$, with $x = z_{\alpha\beta}$ and $y = z_{\beta\alpha}$, we see that there exists a sequence $\rho^\nu \in G_0$ which satisfies the following conditions.

(a) ρ^ν converges to 0, uniformly on compact subsets of $B \setminus \{z_{\beta\alpha}\}$.
(b) $\rho^\nu \circ (\varphi^\nu_\beta)^{-1} \circ \varphi^\nu_\alpha$ converges to ∞, uniformly on compact subsets of $B \setminus \{z_{\alpha\beta}\}$.
(c) $|\rho^\nu \circ (\varphi^\nu_\beta)^{-1}(z^\nu_\ell)| = 1$ for all ν.

Set $\psi^\nu = \varphi^\nu_\beta \circ (\rho^\nu)^{-1}$. We claim that a suitable subsequence of $(u^\nu, \{\varphi^\nu_\alpha\}_{\alpha \in T}, \psi^\nu)$ (which will be denoted by the same symbols) satisfies the following conditions.

(i) $|(\psi^\nu)^{-1}(z^\nu_\ell)| = 1$ for all ν.
(ii) $(\varphi^\nu_\alpha)^{-1} \circ \psi^\nu$ converges to $z_{\alpha\beta}$ uniformly on compact subsets of $B \setminus \{\infty\}$, and $(\varphi^\nu_\beta)^{-1} \circ \psi^\nu$ converges to $z_{\beta\alpha}$ uniformly on compact subsets of $B \setminus \{0\}$.
(iii) $u^\nu \circ \psi^\nu$ converges to $u_\alpha(z_{\alpha\beta}) = u_\beta(z_{\beta\alpha})$ uniformly with all derivatives on compact subsets of $B \setminus \{0, \infty\}$.
(iv) For every $r > 0$,

$$\lim_{\nu\to\infty} E(u^\nu \circ \psi^\nu, \Omega_r) = m_{\alpha\beta}(\mathbf{u}), \qquad \lim_{\nu\to\infty} E(u^\nu \circ \psi^\nu, \mathbb{H} \setminus \Omega_r) = m_{\beta\alpha}(\mathbf{u}).$$

(c) implies (i), and (a) and (b) imply (ii). (iii) follows from Lemma 4.9. To prove the first equation in (iv) note that

$$E(u^\nu \circ \psi^\nu, \Omega_r) = E(u^\nu \circ \varphi^\nu_\alpha, (\varphi^\nu_\alpha)^{-1} \circ \varphi^\nu_\beta \circ (\rho^\nu)^{-1}(\Omega_r)).$$

Now the formula follows from (b) and (iv) of Definition 4.1. The second equation in (iv) follows similarly.

We may assume that $(\psi^\nu)^{-1}(z^\nu_\ell)$ converges to $e^{i\phi}$ where $\phi \in [0, \pi]$. We add a vertex γ to our tree T with $\varphi^\nu_\gamma = \psi^\nu$. In this new stable map, α and β are no longer adjacent, but are separated by γ. The new relations are

$$z^{\mathrm{new}}_{\alpha\gamma} = z_{\alpha\beta}, \quad z^{\mathrm{new}}_{\beta\gamma} = z_{\beta\alpha}, \quad z^{\mathrm{new}}_{\gamma\alpha} = \infty, \quad z^{\mathrm{new}}_{\gamma\beta} = 0, \quad \alpha^{\mathrm{new}}_i = \gamma, \quad z^{\mathrm{new}}_i = e^{i\phi}.$$

If $z^\nu_\ell \in \partial B$ for ν large enough we have $\phi = 0$ or $\phi = \pi$ and we are done. If $z^\nu_\ell \notin \partial B$ for ν large enough then it may nevertheless happen that $\phi = 0$ or $\phi = \pi$. In this case we add a further vertex δ to T as in (I) such that δ is only connected to γ at the point $e^{i\phi}$. This completes the induction argument and the proof of Theorem 4.4. □

4.3. Uniqueness of limits

The main result of this subsection concerns the uniqueness of limits of Gromov convergent sequences. It follows immediately from the definition that if a sequence of J-holomorphic disks $(u^\nu, \mathbf{z})$ Gromov converges to $(\mathbf{u}, \mathbf{z})$, then it converges to every equivalent map. The following theorem asserts that the converse also holds.

Theorem 4.10. *Fix some $k \in \mathbb{N}$. Let $(u^\nu, \mathbf{z}^\nu)$ be a sequence of k-marked J-holomorphic disks with boundary in L. Suppose that $(u^\nu, \mathbf{z}^\nu)$ Gromov converges to two stable maps $(\mathbf{u}, \mathbf{z})$ and $(\tilde{\mathbf{u}}, \tilde{\mathbf{z}})$. Then $(\mathbf{u}, \mathbf{z})$ is equivalent to $(\tilde{\mathbf{u}}, \tilde{\mathbf{z}})$.*

Let us first sketch the main ideas of the proof. We will construct a bijection between the trees T of $\mathbf{u}$ and $\tilde{T}$ of $\tilde{\mathbf{u}}$. Using an idea of [MS2] we will define a map $f : T \to \tilde{T}$ by setting $f(\alpha) = \gamma$ if and only if $(\psi_\gamma^\nu)^{-1} \circ \varphi_\alpha^\nu$ converges. Here $\alpha \in T$, $\gamma \in \tilde{T}$, and φ_α^ν and ψ_γ^ν are the corresponding sequences of Möbius transformations. We show that f is well defined and is actually a tree isomorphism.

Lemma 4.11. *Let $(u^\nu, z_1^\nu, \ldots, z_k^\nu)$ be a sequence of marked stable J-holomorphic disks which Gromov converges to the stable maps*

$$(\mathbf{u}, \mathbf{z}) = (\{(\Sigma_\alpha, \Gamma_\alpha, u_\alpha)\}_{\alpha \in T}, \{z_{\alpha\beta}\}_{\alpha E \beta}, \{\alpha_i, z_i\}_{1 \le i \le k})$$

and

$$(\tilde{\mathbf{u}}, \tilde{\mathbf{z}}) = (\{(\tilde{\Sigma}_\alpha, \tilde{\Gamma}_\alpha, \tilde{u}_\alpha)\}_{\alpha \in \tilde{T}}, \{\tilde{z}_{\alpha\beta}\}_{\alpha \tilde{E} \beta}, \{\alpha_i, \tilde{z}_i\}_{1 \le i \le k})$$

with corresponding sequences φ_α^ν and ψ_γ^ν in G respectively G_0. Assume that there exists $\alpha \in T$ and $\gamma \in \tilde{T}$ such that $(\psi_\gamma^\nu)^{-1} \circ \varphi_\alpha^\nu$ converges to some $z \in \tilde{\Sigma}_\gamma$ uniformly on compact subsets of $\Sigma_\alpha \setminus \{z_\alpha^\infty\}$. Then $\Gamma_\alpha \neq \{z_\alpha^\infty\}$.

Proof. Suppose that $\Gamma_\alpha = \{z_\alpha^\infty\}$. We show that this implies:

(i) If $\tilde{\Sigma}_\gamma = S^2$, then $\tilde{u}_\gamma$ is constant and $\tilde{\Xi}_\gamma \subset \{z, \tilde{z}_\gamma^\infty\}$. If $\tilde{\Sigma}_\gamma = B$ then $\tilde{u}_\gamma$ is constant and $\tilde{\Xi}_\gamma \subset \{z\}$ (cf. Remark 2.8 for the definition of $\tilde{\Xi}_\gamma$).
(ii) $(\psi_\gamma^\nu)^{-1}(z_j^\nu)$ converges to z for $1 \le j \le k$.

(i) and (ii) contradict (v) of Definition 2.3 and this implies the lemma.

We prove (i). It follows from (v) of Definition 4.1 that for $\epsilon > 0$ sufficiently small

$$\lim_{\nu \to \infty} E(u^\nu \circ \varphi_\alpha^\nu, B_\epsilon(z_\alpha^\infty) \cap (\varphi_\alpha^\nu)^{-1}(B)) \le \hbar/4$$

where $\hbar$ is as in Corollary A.4. It follows that

$$\lim_{\nu \to \infty} E(u^\nu \circ \psi_\gamma^\nu, (\psi_\gamma^\nu)^{-1} \circ \varphi_\alpha^\nu(B_\epsilon(z_\alpha^\infty)) \cap (\psi_\gamma^\nu)^{-1}(B)) \le \hbar/4.$$

As $(\psi_\gamma^\nu)^{-1} \circ \varphi_\alpha^\nu$ converges to $z \in \tilde{\Sigma}_\gamma$ uniformly on compact subsets of $\Sigma_\alpha \setminus \{z_\alpha^\infty\}$, it follows that for every $\delta > 0$ there exists a $\nu_0(\delta)$ such that for every $\nu \ge \nu_0(\delta)$,

$$E(u^\nu \circ \psi_\gamma^\nu, (\tilde{\Sigma}_\gamma \setminus B_\delta(z)) \cap (\psi_\gamma^\nu)^{-1}(B)) \le \hbar/2.$$

Hence $E(\tilde{u}_\gamma) \le \hbar/2$. Now it follows from the definition of $\hbar$ that $E(\tilde{u}_\gamma) = 0$. Moreover, $\tilde{\Xi}_\gamma \subset \{z, \tilde{z}_\gamma^\infty\}$ if $\tilde{\Sigma}_\gamma = S^2$, and $\tilde{\Xi}_\gamma \subset \{z\}$ if $\tilde{\Sigma}_\gamma = B$. This proves (i).

To prove (ii) observe that it follows from Lemma B.2 that $(\varphi_\alpha^\nu)^{-1} \circ \psi_\gamma^\nu$ converges to z_α^∞ uniformly on compact subsets of $\Sigma\backslash\{z\}$. Assume by contradiction that $(\psi^\nu)^{-1}(z^\nu)$ does not converge to z. Then $(\varphi_\alpha^\nu)^{-1}(z^\nu) = (\varphi_\alpha^\nu)^{-1} \circ \psi_\gamma^\nu \circ (\psi_\gamma^\nu)^{-1}(z^\nu)$ converges to z_α^∞. But Lemma 4.6 shows that this cannot be true. □

Lemma 4.12. *Let $(u^\nu, z_1^\nu, \ldots, z_k^\nu)$ be a sequence of marked stable J-holomorphic disks which Gromov converges to the stable maps*

$$(\mathbf{u}, \mathbf{z}) = (\{(\Sigma_\alpha, \Gamma_\alpha, u_\alpha)\}_{\alpha \in T}, \{z_{\alpha\beta}\}_{\alpha E \beta}, \{\alpha_i, z_i\}_{1 \le i \le k})$$

and

$$(\tilde{\mathbf{u}}, \tilde{\mathbf{z}}) = (\{(\tilde\Sigma_\alpha, \tilde\Gamma_\alpha, \tilde u_\alpha)\}_{\alpha \in \tilde T}, \{\tilde z_{\alpha\beta}\}_{\alpha \tilde E \beta}, \{\alpha_i, \tilde z_i\}_{1 \le i \le k})$$

with corresponding sequences φ_α^ν and ψ_γ^ν in G respectively G_0. Let

$$\rho^\nu = (\psi_\gamma^\nu)^{-1} \circ \varphi_\alpha^\nu$$

where $\alpha \in T$ and $\gamma \in \tilde T$. Assume that ρ^ν has no convergent subsequence. Then there exists a subsequence of ρ^ν (still denoted by ρ^ν) and nodal points $x_0 \in Z_\alpha$, $y_0 \in \tilde Z_\gamma$ such that ρ^ν converges to y_0 uniformly on compact subsets of $\Sigma_\alpha \setminus \{x_0\}$.

Proof. By Lemma B.1 there exist points $x_0, y_0 \in S^2$ such that, after passing to a subsequence of ρ^ν if necessary, we have $\lim_{\nu\to\infty} \rho^\nu(x) = y_0$, uniformly on compact subsets of $S^2 \setminus \{x_0\}$. We claim that y_0 is either a marked or a nodal point. Suppose this is not the case. Then $u^\nu \circ \psi_\gamma^\nu$ converges to $\tilde u_\gamma$ uniformly in a neighbourhood of y_0. Hence $u^\nu \circ \varphi_\alpha^\nu = u^\nu \circ \psi_\gamma^\nu \circ \rho^\nu$ converges to $\tilde u_\gamma(y_0)$ uniformly on every compact subset of $\Sigma_\alpha \setminus \{x_0\}$. Moreover $y_0 \neq \tilde z_{\gamma i}$ for $i \in \{1, \ldots, k\}$. Lemma 4.6 shows that $\tilde z_{\gamma i} = \lim_{\nu\to\infty} (\psi_\gamma^\nu)^{-1}(z_i^\nu)$. By Lemma B.2 we see that $(\rho^\nu)^{-1}$ converges to x_0 uniformly on compact subsets of $\tilde\Sigma_\gamma \setminus \{y_0\}$. All this shows that

$$z_{\alpha i} = \lim_{\nu\to\infty} (\varphi_\alpha^\nu)^{-1}(z_i^\nu) = \lim_{\nu\to\infty} (\rho^\nu)^{-1} \circ (\psi_\gamma^\nu)^{-1}(z_i^\nu) = x_0.$$

Hence u_α is constant and the union of nodal and marked points of α is contained in $\{x_0\}$; this contradicts (v) of Definition 2.3.

It remains to show that y_0 is not a marked point. Suppose the contrary. Hence there exists $i \in \{1, \ldots, k\}$ such that $\tilde z_i = \tilde z_{\gamma i} = y_0$. In particular, y_0 is not a nodal point. Hence it follows as above that $u^\nu \circ \varphi_\alpha^\nu$ converges to $\tilde u_\gamma(y_0)$ uniformly on compact subsets of $\Sigma_\alpha \setminus \{x_0\}$ and $z_{\alpha j} = x_0$ for every $j \neq i$. Hence the union of marked and nodal points of α is contained in $\{x_0, z_{\alpha i}\}$. To deduce a contradiction we have to distinguish different cases.

Case 1: $\Sigma_\alpha = \tilde\Sigma_\gamma = B$. In this case ψ_γ^ν, φ_α^ν and $\rho^\nu \in G_0$. Lemma B.1 shows that x_0 and $y_0 \in \partial B$. It follows that $\tilde z_i = y_0 \in \partial B$. (vii) of Definition 4.1 shows that $z_i^\nu \in \partial B$. This implies that $z_{\alpha i} \in \partial\Sigma_\alpha$. Now $Z_\alpha \subset \{x_0, z_{\alpha i}\}$ and $x_0, z_{\alpha i} \in \partial\Sigma_\alpha$. But this contradicts (v) of Definition 2.3.

Case 2: $\Sigma_\alpha = B$, $\tilde\Sigma_\gamma = S^2$. It follows from Lemma B.2 that $y_0 = \tilde z_\gamma^\infty$. (iv) of Definition 2.3 shows that $\tilde z_\gamma^\infty$ is never a marked point. Hence this case does not occur either.

Case 3: $\Sigma_\alpha = S^2$, $\Gamma_\alpha = \emptyset$. (v) of Definition 2.3 shows that $Z_\alpha \subset \{x_0, z_{\alpha i}\}$ cannot happen.

Case 4: $\Sigma_\alpha = S^2$, $\Gamma_\alpha = \{z_\alpha^\infty\}$. Lemma 4.11 shows that $x_0 \neq z_\alpha^\infty$. Hence Lemma B.4 implies that $y_0 = \tilde{z}_\gamma^\infty$. But we have already seen that $\tilde{z}_\gamma^\infty$ is never a marked point. Hence this case does not occur either. This shows that $y_0 \in \tilde{Z}_\gamma$ is a nodal point.

By Lemma B.2 we see that $(\rho^\nu)^{-1}$ converges to x_0, uniformly on compact subsets of $\tilde{\Sigma}_\gamma \setminus \{y_0\}$. Applying the above argument to the sequence $u^\nu \circ \psi_\gamma^\nu = u^\nu \circ \varphi_\alpha^\nu \circ (\rho^\nu)^{-1}$ we see that $x_0 \in Z_\alpha$ is a nodal point. $\square$

Lemma 4.13. *Let $(u^\nu, z_1^\nu, \ldots, z_k^\nu)$ be a sequence of stable J-holomorphic disks with marked points which Gromov converges to the stable maps*

$$(\mathbf{u}, \mathbf{z}) = (\{(\Sigma_\alpha, \Gamma_\alpha, u_\alpha)\}_{\alpha \in T}, \{z_{\alpha\beta}\}_{\alpha E \beta}, \{\alpha_i, z_i\}_{1 \leq i \leq k})$$

and

$$(\tilde{\mathbf{u}}, \tilde{\mathbf{z}}) = (\{(\tilde{\Sigma}_\alpha, \tilde{\Gamma}_\alpha, \tilde{u}_\alpha)\}_{\alpha \in \tilde{T}}, \{\tilde{z}_{\alpha\beta}\}_{\alpha \tilde{E} \beta}, \{\tilde{\alpha}_i, \tilde{z}_i\}_{1 \leq i \leq k})$$

with corresponding sequences φ_α^ν and ψ_α^ν in G respectively G_0. Moreover, let $\alpha, \beta \in T$ with $\alpha E \beta$. Let $\psi^\nu = \psi_\gamma^\nu$ for $\gamma \in \tilde{T}$. Then there exists a unique $\alpha \in T$ such that the sequence $(\varphi_\alpha^\nu)^{-1} \circ \psi^\nu$ has a uniformly convergent subsequence. Moreover, $\tilde{\Sigma}_\gamma = \Sigma_\alpha$.

Proof. We first show uniqueness. It follows from Lemma 4.6 that the sequence $(\varphi_\alpha^\nu)^{-1} \circ \varphi_\beta^\nu$ does not have a uniformly convergent subsequence for any two distinct elements $\alpha, \beta \in T$. Assume that $(\varphi_\alpha^\nu)^{-1} \circ \psi^\nu$ and $(\varphi_\beta^\nu)^{-1} \circ \psi^\nu$ both converge uniformly. It follows that $((\varphi_\beta^\nu)^{-1} \circ \psi^\nu)^{-1} = (\psi^\nu)^{-1} \circ \varphi_\beta^\nu$ converges uniformly and hence also $(\varphi_\alpha^\nu)^{-1} \circ \psi^\nu \circ (\psi^\nu)^{-1} \circ \varphi_\beta^\nu = (\varphi_\alpha^\nu)^{-1} \circ \varphi_\beta^\nu$. This shows $\alpha = \beta$.

To show existence we have to distinguish different cases.

Case 1: $\tilde{\Sigma}_\gamma = B$. If Σ_α is a sphere, then there exists a point $w \in B$ such that $(\psi^\nu)^{-1} \circ \varphi_\alpha^\nu$ converges[5] to w uniformly with all derivatives on compact subsets of $S^2 \setminus \{z_\alpha^\infty\}$. It follows from Lemma 4.11 that $\Gamma_\alpha \neq \{z_\alpha^\infty\}$. This contradicts (iv) of Definition 2.3.

Hence there exists $\beta \in T$ such that $\alpha E \beta$ and $z_\alpha^\infty = z_{\alpha\beta}$. If Σ_β is a sphere again, we obtain in the same manner a point $z_\beta^\infty = z_{\beta\gamma}$, where $\beta E \gamma$. In particular, $\gamma \neq \alpha$. Because $\#T < \infty$ we arrive after finitely many steps at a $\delta \in T$ such that $\Sigma_\delta = B$.

If $(\varphi_\delta^\nu)^{-1} \circ \psi^\nu$ has a convergent subsequence we are done. Otherwise, it follows from Lemma 4.12 that there exist nodal points $x \in \tilde{\Sigma}_\gamma$ and $y \in \Sigma_\delta$ such that $(\varphi_\delta^\nu)^{-1} \circ \psi^\nu$ converges to y uniformly on compact subsets of $\tilde{\Sigma}_\gamma \setminus \{x\}$. Moreover, Lemma B.1 shows that $x \in \partial\tilde{\Sigma}_\gamma$ and $y \in \partial\Sigma_\delta$. Hence there exists $\epsilon \in T$ such that $\delta E \epsilon$, $y = z_{\delta\epsilon}$ and $\Sigma_\epsilon = B$. The last equality follows from (vi) of Definition 2.3.

[5]Strictly speaking, a subsequence of $(\psi^\nu)^{-1} \circ \varphi_\alpha^\nu$ does. In the following we will not explicitly mention every transition to a subsequence.

If $(\varphi_\epsilon^\nu)^{-1} \circ \psi^\nu$ has a convergent subsequence we are done. Otherwise, we obtain as above an $\eta \in T$ with $\epsilon E \eta$ and $\tilde{x} \in \partial\tilde{\Sigma}_\gamma$ such that $(\varphi_\epsilon^\nu)^{-1} \circ \psi^\nu$ converges to $z_{\epsilon\eta}$ uniformly on compact subsets of $\tilde{\Sigma}_\gamma \setminus \{\tilde{x}\}$. We want to show that $\eta \neq \delta$. Otherwise we could apply Lemma 4.9 to δ and ϵ with $\delta E \epsilon$. It would then follow that $u^\nu \circ \psi^\nu$ converges to the constant $u_\delta(z_{\delta\epsilon})$, uniformly on compact subsets of $\tilde{\Sigma}_\gamma \setminus \{x, \tilde{x}\}$, and that $(\psi^\nu)^{-1}(z_i^\nu)$ converges to either x or $\tilde{x}$ for all i. But this would imply that $\tilde{u}_\gamma$ is constant and $\tilde{Z}_\gamma \subset \{x, \tilde{x}\}$, where $x, \tilde{x} \in \partial\tilde{\Sigma}_\gamma$, which contradicts (v) of Definition 2.3.

Using again the fact that $\#T < \infty$ we see that after finitely many steps we arrive at some $\zeta \in T$ such that $(\varphi_\zeta^\nu)^{-1} \circ \psi^\nu$ has a convergent subsequence.

Case 2: $\tilde{\Sigma}_\gamma = S^2$, $\tilde{\Gamma}_\gamma = \emptyset$. Fix an element $\alpha \in T$. If $(\varphi_\alpha^\nu)^{-1} \circ \psi^\nu$ has a convergent subsequence we are done. Otherwise Lemma 4.12 shows that there exist $x \in \tilde{\Sigma}_\gamma$ and $\beta \in T$ with $\alpha E \beta$ such that $(\varphi_\alpha^\nu)^{-1} \circ \psi^\nu$ converges to $z_{\alpha\beta}$ uniformly on compact subsets of $\tilde{\Sigma}_\gamma \setminus \{x\}$.

If $(\varphi_\beta^\nu)^{-1} \circ \psi^\nu$ has a convergent subsequence we are done. Otherwise there exists $\delta \in T$ with $\beta E \delta$ and $\tilde{x} \in \tilde{\Sigma}_\gamma$ such that $(\varphi_\beta^\nu)^{-1} \circ \psi^\nu$ converges to $z_{\beta\delta}$ uniformly on compact subsets of $\tilde{\Sigma}_\gamma \setminus \{\tilde{x}\}$. We want to show that $\alpha \neq \delta$. Otherwise it would follow from Lemma 4.9 that $u^\nu \circ \psi^\nu$ converges to the constant $u_\alpha(z_{\alpha\delta})$, uniformly on compact subsets of $\tilde{\Sigma}_\gamma \setminus \{x, \tilde{x}\}$, and that $(\psi^\nu)^{-1}(z_i^\nu)$ converges to either x or $\tilde{x}$ for all i. But this would imply that $\tilde{u}_\gamma$ is constant and $\tilde{Z}_\gamma \subset \{x, \tilde{x}\}$. This contradicts (v) of Definition 2.3.

Case 3: $\tilde{\Sigma}_\gamma = S^2$, $\tilde{\Gamma}_\gamma \neq \emptyset$. Let $\alpha \in T$. We first show that $\Sigma_\alpha = S^2$. Otherwise there exists $w \in B$ such that $(\varphi_\alpha^\nu)^{-1} \circ \psi^\nu$ converges to w uniformly on compact subsets of $\tilde{\Sigma}_\gamma \setminus \{\tilde{z}_\gamma^\infty\}$. This contradicts Lemma 4.11.

Now proceed as in the second case. We can assume that we have found $\alpha, \beta, \delta \in T$ with $\alpha E \beta$, $\beta E \delta$ and $x, \tilde{x} \in \tilde{\Sigma}_\gamma$ such that $(\varphi_\alpha^\nu)^{-1} \circ \psi^\nu$ converges to $z_{\alpha\beta}$ uniformly on compact subsets of $\tilde{\Sigma}_\gamma \setminus \{x\}$, and $(\varphi_\beta^\nu)^{-1} \circ \psi^\nu$ converges to $z_{\beta\delta}$ uniformly on compact subsets of $\tilde{\Sigma}_\gamma \setminus \{\tilde{x}\}$. We claim that $\alpha \neq \delta$. Suppose the contrary. To finish the proof, it remains to show that this implies that $\tilde{z}_\gamma^\infty \in \{x, \tilde{x}\}$. Because $z_{\alpha\beta} \neq z_\alpha^\infty$ or $z_{\beta\alpha} \neq z_\beta^\infty$, we can assume without loss of generality that $z_{\alpha\beta} \neq z_\alpha^\infty$. Hence there exist points $x_1, x_2 \in B$ such that

$$\lim_{\nu \to \infty} (\varphi_\alpha^\nu)^{-1}(x_i) \neq z_{\alpha\beta}, \qquad i \in \{1, 2\}.$$

Using Lemma B.2 we see that $(\psi^\nu)^{-1} \circ \varphi_\alpha^\nu$ converges to x uniformly on compact subsets of $\Sigma_\alpha \setminus \{z_{\alpha\beta}\}$. This shows that

$$\lim_{\nu \to \infty} (\psi^\nu)^{-1}(x_i) = x, \qquad i \in \{1, 2\}.$$

But this implies that $x = \tilde{z}_\gamma^\infty$. Hence if $\alpha = \delta$, then $\tilde{u}_\gamma$ is constant and $\tilde{Z}_\gamma \subset \{x, \tilde{x}\}$. This contradicts (v) of Definition 2.3. Hence the lemma is proved. $\square$

Proof of Theorem 4.10. Given Lemma 4.13 the corresponding proof in [MS2] translates nearly word for word.

Consider the case $T^\nu = \{(u^\nu, z_1^\nu, \ldots, z_k^\nu)\}$ and suppose that $(\mathbf{u}, \mathbf{z}^\nu) = (u^\nu, z_1^\nu, \ldots, z_k^\nu)$ Gromov converges to two stable maps $(\mathbf{u}, \mathbf{z})$ and $(\tilde{\mathbf{u}}, \tilde{\mathbf{z}})$ with corresponding sequences $\{\varphi_\alpha\}_{\alpha\in T}$ and $\{\tilde\varphi_\alpha\}_{\alpha\in\tilde T}$ of Möbius transformations. We prove in four steps that $(\mathbf{u}, \mathbf{z})$ is equivalent to $(\tilde{\mathbf{u}}, \tilde{\mathbf{z}})$.

Step 1. *There is a unique bijection $f : T \to \tilde T$ and a subsequence (still denoted by $(u^\nu, z_i^\nu, \varphi_\alpha^\nu)$) such that the limit*

$$ {}_{\alpha} = \lim_{\nu\to\infty} (\varphi_\alpha^\nu)^{-1} \circ \tilde\varphi^\nu_{f(\alpha)} \tag{14}$$

exists for every $\alpha \in T$. Here the convergence is uniform on all of S^2.

Lemma 4.13 tells us that for every $\alpha \in T$ there exists a unique element $\tilde\alpha \in \tilde T$ such that the sequence $(\varphi_\alpha^\nu)^{-1} \circ \tilde\varphi_{\tilde\alpha}^\nu$ has a uniformly convergent subsequence. Now we can apply Lemma 4.13 again to this subsequence and any other element $\beta \in T$, and proceed by induction. This gives rise to a map $f : T \to \tilde T$ and a collection $= \{\psi_\alpha\}_{\alpha\in T}$ of Möbius transformations which satisfy the requirements of Step 1. Reversing the roles of $\mathbf{u}$ and $\tilde{\mathbf{u}}$ we find that f is bijective.

Step 2. *Let f and ψ_α be as in Step 1. Then*

$$\tilde u_{f(\alpha)} = u_\alpha \circ \psi_\alpha, \qquad \tilde z_{f(\alpha)i} = \psi_\alpha^{-1}(z_{\alpha i}), \qquad \tilde z_{f(\alpha)f(\beta)} = \psi_\alpha^{-1}(z_{\alpha\beta}) \tag{15}$$

for $\alpha, \beta \in T$ with $\alpha \neq \beta$ and $i = 1, \ldots, k$. Moreover,

$$\lim_{\epsilon\to 0}\lim_{\nu\to\infty} E(u^\nu \circ \tilde\varphi^\nu_{f(\alpha)}, B_\epsilon(\tilde z_{f(\alpha)f(\beta)})) = m_{\alpha\beta}(\mathbf{u}) \tag{16}$$

whenever $\alpha E \beta$.

The first equality in (15) follows from (14), namely

$$u_\alpha \circ \psi_\alpha = \lim_{\nu\to\infty} u^\nu \circ \varphi_\alpha^\nu \circ \psi_\alpha = \lim_{\nu\to\infty} u^\nu \circ \tilde\varphi^\nu_{f(\alpha)} = \tilde u_{f(\alpha)}.$$

The second equality in (15) also uses (14) and Lemma 4.6:

$${}_\alpha^{-1}(z_{\alpha i}) = \psi_\alpha^{-1}\big(\lim_{\nu\to\infty} (\varphi_\alpha^\nu)^{-1}(z_i^\nu)\big) = \lim_{\nu\to\infty} (\tilde\varphi^\nu_{f(\alpha)})^{-1}(z_i^\nu) = \tilde z_{f(\alpha)i}.$$

To prove the third equality in (15), suppose that $\alpha, \beta \in T$ with $\alpha \neq \beta$. Then $(\varphi_\alpha^\nu)^{-1} \circ \varphi_\beta^\nu$ converges to $z_{\alpha\beta}$ uniformly on compact subsets of $\Sigma_\beta \setminus \{z_{\beta\alpha}\}$. Hence, for $z \neq z_{\beta\alpha}$ we obtain, by (14),

$$\begin{aligned} {}_\alpha^{-1}(z_{\alpha\beta}) &= \psi_\alpha^{-1}(\lim_{\nu\to\infty} (\varphi_\alpha^\nu)^{-1} \circ \varphi_\beta^\nu(z)) = \lim_{\nu\to\infty} (\tilde\varphi^\nu{}_{f(\alpha)})^{-1} \circ \varphi_\beta^\nu(z) \\ &= \lim_{\nu\to\infty} (\tilde\varphi^\nu_{f(\alpha)})^{-1} \circ \tilde\varphi^\nu_{f(\beta)}(\psi_\beta^{-1}(z)). \end{aligned}$$

In other words, $(\tilde\varphi^\nu_{f(\alpha)})^{-1} \circ \tilde\varphi^\nu_{f(\beta)}$ converges to $\psi_\alpha^{-1}(z_{\alpha\beta})$, uniformly on compact subsets of $\Sigma_{f(\beta)} \setminus \{\psi_\beta^{-1}(z_{\beta\alpha})\}$. This proves (15).

To prove (16), note that the limit remains the same if the ball

$$B_\epsilon(\tilde z_{f(\alpha)f(\beta)}) = B_\epsilon(\psi_\alpha^{-1}(z_{\alpha\beta}))$$

is replaced by $\psi_\alpha^{-1}(B_\epsilon(z_{\alpha\beta}))$. We have

$$\begin{aligned}
&\lim_{\nu\to\infty} E(u^\nu \circ \tilde{\varphi}^\nu_{f(\alpha)}, \psi_\alpha^{-1}(B_\epsilon(z_{\alpha\beta})) \cap (\tilde{\varphi}^\nu_{f(\alpha)})^{-1}(B)) \\
&\quad = \lim_{\nu\to\infty} E(u^\nu \circ \tilde{\varphi}^\nu_{f(\alpha)} \circ \psi_\alpha^{-1}, B_\epsilon(z_{\alpha\beta}) \cap (\varphi^\nu_\alpha)^{-1}(B)) \\
&\quad = \lim_{\nu\to\infty} E((u^\nu \circ \varphi^\nu_\alpha) \circ ((\varphi^\nu_\alpha)^{-1} \circ \tilde{\varphi}^\nu_{f(\alpha)} \circ \psi_\alpha^{-1}), B_\epsilon(z_{\alpha\beta}) \cap (\varphi^\nu_\alpha)^{-1}(B)) \\
&\quad = \lim_{\nu\to\infty} E(u^\nu \circ \varphi^\nu_\alpha, B_\epsilon(z_{\alpha\beta}) \cap (\varphi^\nu_\alpha)^{-1}(B)).
\end{aligned}$$

The last equality follows from the fact that $(\varphi^\nu_\alpha)^{-1} \circ \tilde{\varphi}^\nu_{f(\alpha)} \circ \psi_\alpha^{-1}$ converges uniformly to the identity. Now (16) follows by taking the limit $\epsilon \to 0$. This proves Step 2.

Step 3. *Let $\alpha, \beta \in T$. Then $\alpha E \beta$ if and only if*

$$\begin{aligned}
\lim_{\epsilon\to 0}\lim_{\nu\to\infty} (E(u^\nu \circ \varphi^\nu_\alpha, B_\epsilon(z_{\alpha\beta}) \cap (\varphi^\nu_\alpha)^{-1}(B)) + E(u^\nu \circ \varphi^\nu_\beta, B_\epsilon(z_{\beta\alpha}) \cap (\varphi^\nu_\beta)^{-1}(B))) \\
= E(\mathbf{u}) \qquad (17)
\end{aligned}$$

and there is no $i \in \{1, \dots, k\}$ which satisfies both $z_{\alpha i} = z_{\alpha\beta}$ and $z_{\beta i} = z_{\beta\alpha}$.

If $\alpha E \beta$ then it follows from the definition of Gromov convergence that these conditions are satisfied. Conversely, suppose that $\alpha, \beta \in T$ satisfy (17) and either $z_{\alpha i} \neq z_{\alpha\beta}$ or $z_{\beta i} \neq z_{\beta\alpha}$ for all i. Choose a chain of edges $\gamma_0, \dots, \gamma_m \in T$ running from $\gamma_0 = \alpha$ to $\gamma_m = \beta$. Then (17) is equivalent to $m_{\gamma_0\gamma_1}(\mathbf{u}) + m_{\gamma_m\gamma_{m-1}}(\mathbf{u}) = E(\mathbf{u})$. This implies

$$m_{\gamma_0\gamma_1}(\mathbf{u}) = m_{\gamma_{m-1}\gamma_m}(\mathbf{u}).$$

If $m \neq 1$ then it follows that $E(u_\gamma) = 0$ for every $\gamma \in T_{\gamma_0\gamma_1} \setminus T_{\gamma_{m-1}\gamma_m}$ and this set is nonempty. Hence there must be a sphere or a disk in $T_{\gamma_0\gamma_1} \setminus T_{\gamma_{m-1}\gamma_m}$ which carries a marked point z_i. For this i we obtain $z_{\alpha i} = z_{\alpha\beta}$ and $z_{\beta i} = z_{\beta\alpha}$, in contradiction to our assumption. This proves that $m = 1$ and hence $\alpha E \beta$. Thus we have proved Step 3.

Step 4. $(\mathbf{u}, \mathbf{z})$ *is equivalent to* $(\tilde{\mathbf{u}}, \tilde{\mathbf{z}})$.

Step 2 shows that if $\alpha E \beta$ then

$$\begin{aligned}
E(\tilde{\mathbf{u}}) = \lim_{\epsilon\to 0}\lim_{\nu\to\infty} &(E(u^\nu \circ \tilde{\varphi}^\nu_{f(\alpha)}, B_\epsilon(\tilde{z}_{f(\alpha)f(\beta)}) \cap (\tilde{\varphi}^\nu_{f(\alpha)})^{-1}(B)) \\
&+ E(u^\nu \circ \tilde{\varphi}^\nu_{f(\beta)}, B_\epsilon(\tilde{z}_{f(\beta)f(\alpha)}) \cap (\tilde{\varphi}^\nu_{f(\beta)})^{-1}(B)))
\end{aligned}$$

and that either $\tilde{z}_{f(\alpha)i} \neq \tilde{z}_{f(\alpha)f(\beta)}$ or $\tilde{z}_{f(\beta)i} \neq \tilde{z}_{f(\beta)f(\alpha)}$ for all i. Hence Step 3 shows that $f(\alpha)\tilde{E}f(\beta)$. Replacing f by f^{-1} we find that $f(\alpha)\tilde{E}f(\beta)$ implies $\alpha E \beta$. Hence f is a tree isomorphism.

By (15) it only remains to verify that $\Gamma_\alpha = \psi_\alpha(\Gamma_{f(\alpha)})$. If $\Sigma_{f(\alpha)} = B$ then $\Sigma_\alpha = B$ and $\Gamma_{f(\alpha)} = \Gamma_\alpha = \partial B$. Moreover, $\tilde{\varphi}^\nu_{f(\alpha)} \in G_0$ and $\varphi^\nu_\alpha \in G_0$ and hence ${}_\alpha \in G_0$. Hence

$$\Gamma_\alpha = \partial B = \psi_\alpha(\partial B) = \psi_\alpha(\Gamma_{f(\alpha)}).$$

If $\Gamma_{f(\alpha)} = \{\tilde{z}^\infty_{f(\alpha)}\}$, then it follows from Lemma B.2 that $\psi_\alpha(\tilde{z}^\infty_{f(\alpha)}) = z^\infty_\alpha$. Because according Step 2, $\tilde{z}_{f(\alpha)f(\beta)} = \psi_\alpha^{-1}(z_{\alpha\beta})$, it follows that there exists no $\beta \in T$ with $\alpha E \beta$ such that $z^\infty_\alpha = z_{\alpha\beta}$. This implies that

$$\Gamma_\alpha = \{z^\infty_\alpha\} = \{\psi_\alpha(\tilde{z}^\infty_{f(\alpha)})\} = \psi_\alpha(\Gamma_{f(\alpha)}).$$

If finally $\Gamma_{f(\alpha)} = \emptyset$, then $\Sigma_{f(\alpha)} = \Sigma_\alpha = S^2$. We have to show that $\Gamma_\alpha = \emptyset$. Otherwise $\Gamma_\alpha = \{z^\infty_\alpha\}$ and it follows from the considerations above that $\Gamma_{f(\alpha)} = (\psi_\alpha)^{-1}(\Gamma_\alpha) \neq \emptyset$. This contradiction proves $\Gamma_\alpha = \psi_\alpha(\Gamma_{f(\alpha)})$ in this case.

5. The space of stable maps

In this section we define Gromov convergence for sequences of stable maps. We will see that generalizations of Proposition 4.3, Theorem 4.4 and Theorem 4.10 continue to hold.

Definition 5.1. Let M be a compact manifold with boundary, L a compact submanifold of M without boundary, ω^ν a sequence of symplectic structures on M such that L is Lagrangian for every ω^ν, J^ν a sequence of ω^ν-tame almost complex structures, and k_1 and k_2 two positive integers. Assume that the ω^ν converge to some symplectic structure ω on M with respect to the C^∞-topology, and the J^ν converge in the C^∞-topology to some $J \in \mathcal{J}_\tau(M, \omega)$. A sequence

$$(\mathbf{u}^\nu, \mathbf{z}^\nu) = (\{(\Sigma^\nu_\alpha, \Gamma^\nu_\alpha, u^\nu_\alpha)\}_{\alpha \in T^\nu}, \{z^\nu_{\alpha\beta}\}_{\alpha E^\nu \beta}, \{\alpha^\nu_i, z^\nu_i\}_{1 \le i \le k})$$

of (k_1, k_2)-marked, J^ν-holomorphic stable maps of genus zero with one boundary component in L is said to *Gromov converge* to a (k_1, k_2)-marked stable map

$$(\mathbf{u}, \mathbf{z}) = (\{(\Sigma_\alpha, \Gamma_\alpha, u_\alpha)\}_{\alpha \in T}, \{z_{\alpha\beta}\}_{\alpha E \beta}, \{\alpha_i, z_i\}_{1 \le i \le k})$$

if, for ν sufficiently large, there exists a surjective tree homomorphism $f^\nu : T \to T^\nu$ and a collection $\{\varphi^\nu_\alpha\}_{\alpha \in T}$ of Möbius transformations such that the following holds.

(i) If $\Sigma_{f^\nu(\alpha)} = B$ and $\Sigma_\alpha = B$, then $\varphi^\nu_\alpha \in G_0$.

(ii) If $\Sigma_\alpha = S^2$, then for every subsequence ν_j of ν with $\Sigma_{f^{\nu_j}(\alpha)} = B$ and for every compact subset K of $S^2 \backslash \{z^\infty_\alpha\}$ there exists $j_0(K)$ such that $\varphi^{\nu_j}_\alpha(K) \subset \Sigma_{f^{\nu_j}(\alpha)}$ for every $j \ge j_0(K)$.

(iii) For every $\alpha \in T$ and for every compact subset K of $\Sigma_\alpha \setminus \Xi_\alpha$ the sequence $u^\nu_{f^\nu(\alpha)} \circ \varphi^\nu_\alpha$ converges to u_α, uniformly with all derivatives on K.

(iv) If $\beta \in T$ with $\alpha E \beta$, then

$$m_{\alpha\beta}(\mathbf{u}) = \lim_{\epsilon \to 0} \lim_{\nu \to \infty} E_{f^\nu(\alpha)}(\mathbf{u}^\nu, \varphi^\nu_\alpha(B_\epsilon(z_{\alpha\beta})) \cap \Sigma_{f^\nu(\alpha)}). \tag{18}$$

(v) If $\Gamma_\alpha = \{z^\infty_\alpha\}$, then

$$\lim_{\epsilon \to 0} \lim_{\nu \to \infty} E(u^\nu_{f^\nu(\alpha)} \circ \varphi^\nu_\alpha, B_\epsilon(z^\infty_\alpha) \cap (\varphi^\nu_\alpha)^{-1} \Sigma_{f^\nu(\alpha)}) = 0.$$

(vi) Let $\alpha, \beta \in T$ with $\alpha E \beta$. If $f^{\nu_j}(\alpha) \neq f^{\nu_j}(\beta)$ for some subsequence ν_j then

$$z_{\alpha\beta} = \lim_{j\to\infty} (\varphi_\alpha^{\nu_j})^{-1}(z^{\nu_j}_{f^{\nu_j}(\alpha) f^{\nu_j}(\beta)}).$$

If $f^{\nu_j}(\alpha) = f^{\nu_j}(\beta)$ for some subsequence ν_j then $(\varphi_\alpha^{\nu_j})^{-1} \circ \varphi_\beta^{\nu_j}$ converges to $z_{\alpha\beta}$, uniformly on compact subsets of $\Sigma_\beta \setminus \{z_{\beta\alpha}\}$.

(vii) $\alpha_i^\nu = f^\nu(\alpha_i)$ and $z_i = \lim_{\nu\to\infty} (\varphi_{\alpha_i}^\nu)^{-1}(z_i^\nu)$ for $i = 1, \ldots, k$. Moreover, $z_i \in \partial\Sigma_{\alpha_i}$ iff $z_i^\nu \in \partial\Sigma_{\alpha_i^\nu}$.

(viii) If ν_j is some subsequence such that $\Gamma_{f^{\nu_j}(\alpha)} = \{z^{\nu_j,\infty}_{f^{\nu_j}(\alpha)}\}$ for $z^{\nu_j,\infty}_{f^{\nu_j}(\alpha)} \in \Sigma_{f^{\nu_j}(\alpha)} = S^2$, then $\Gamma_\alpha = \{z_\alpha^\infty\}$ for $z_\alpha^\infty \in \Sigma_\alpha = S^2$ and

$$z_\alpha^\infty = \lim_{j\to\infty} (\varphi_\alpha^{\nu_j})^{-1}(z^{\nu_j,\infty}_{f^{\nu_j}(\alpha)}).$$

Remark 5.2. If Σ_α is a disk it follows from (iii) of Definition 5.1 that for ν sufficiently large, $\varphi_\alpha^\nu(B) \subset \Sigma_{f^\nu(\alpha)}$. This is only possible if $\Sigma_{f^\nu(\alpha)} = B$.

We will show that the homotopy class converges under Gromov convergence, that every sequence of stable maps with bounded energy has a Gromov convergent subsequence and that the limits are unique up to equivalence. In the following let J^ν, M and L be as in Definition 5.1.

Proposition 5.3. *Suppose that $(\mathbf{u}^\nu, \mathbf{z}^\nu)$ is a sequence of (k_1, k_2)-marked, J^ν-holomorphic stable maps of genus zero with one boundary component in L which Gromov converges to a stable map $(\mathbf{u}, \mathbf{z})$. Then the following holds.*

(i) *If $x^\nu \in \bigcup_{\alpha\in T^\nu} u_\alpha^\nu(\Sigma_\alpha^\nu)$ converges to $x \in M$ then $x \in \bigcup_{\alpha \in T} u_\alpha(\Sigma_\alpha)$.*

(ii) *For large ν the connected sum $\#_{\alpha\in T^\nu} u_\alpha^\nu$ is relative homotopic to $\#_{\alpha\in T} u_\alpha$ with respect to L.*

Theorem 5.4. *Fix some $k_1, k_2 \in \mathbb{N}$. Assume that $(\mathbf{u}^\nu, \mathbf{z}^\nu)$ is a sequence of (k_1, k_2)-marked, J^ν-holomorphic stable maps of genus zero with one boundary component in L whose energy is uniformly bounded. Then $(\mathbf{u}^\nu, \mathbf{z}^\nu)$ has a Gromov convergent subsequence.*

As an immediate corollary of these two results we obtain

Corollary 5.5. *For every $J \in \mathcal{J}_\tau(M, \omega)$ and every constant $c > 0$ there exist only finitely many homotopy classes $A \in \pi_2(M, L)$ with $\omega(A) \leq c$ which can be represented by a J-holomorphic stable map of genus zero with one boundary component in L.*

Theorem 5.6. *Fix some $k_1, k_2 \in \mathbb{N}$. Let $(\mathbf{u}^\nu, \mathbf{z}^\nu)$ be a sequence of (k_1, k_2)-marked, J-holomorphic stable maps of genus zero with one boundary component in L. Suppose that $(\mathbf{u}^\nu, \mathbf{z}^\nu)$ Gromov converges to two stable maps $(\mathbf{u}, \mathbf{z})$ and $(\tilde{\mathbf{u}}, \tilde{\mathbf{z}})$. Then $(\mathbf{u}, \mathbf{z})$ is equivalent to $(\tilde{\mathbf{u}}, \tilde{\mathbf{z}})$.*

The idea of the proof is the same for these three results. We show that we can assume without loss of generality that the sequence of stable maps is modelled over the same tree. Then the results follow from Proposition 4.3, Theorem 4.4, Theorem 4.10 and their analogues for spheres which are proved in [MS2]. Our main problem will be to show that the energy of a Gromov convergent sequence converges.

Lemma 5.7. *Let $(\mathbf{u}^\nu, \mathbf{z}^\nu)$ be a sequence of stable maps which Gromov converges to $(\mathbf{u}, \mathbf{z})$ with corresponding surjective tree homomorphisms $f^\nu : T \to T^\nu$ and Möbius transformations $\varphi_\alpha^\nu \in G$. Then $E(\mathbf{u}^\nu)$ converges to $E(\mathbf{u})$.*

Proof. The proof is in four steps.

Step 1. *If $\alpha \neq \beta$ and ν_j is a subsequence such that $f^{\nu_j}(\alpha) \neq f^{\nu_j}(\beta)$ then*

$$z_{\alpha\beta} = \lim_{j\to\infty} (\varphi_\alpha^{\nu_j})^{-1}(z^{\nu_j}_{f^{\nu_j}(\alpha)f^{\nu_j}(\beta)}).$$

Let $\gamma_0, \ldots, \gamma_m$ be the chain of edges from α to β. Passing to a further subsequence, we may assume that, for every i, either $f^{\nu_j}(\gamma_i) = f^{\nu_j}(\gamma_{i+1})$ for all j or $f^{\nu_j}(\gamma_i) \neq f^{\nu_j}(\gamma_{i+1})$ for all j. If $f^{\nu_j}(\alpha) = f^{\nu_j}(\gamma_{i_0})$ then it follows that $f^{\nu_j}(\alpha) = f^{\nu_j}(\gamma_i)$ for all $i \leq i_0$. This is because $\gamma_0, \ldots, \gamma_{i_0}$ is a chain of edges in T running from α to γ_{i_0}. Hence there exists $\ell < m$ such that $f^{\nu_j}(\alpha) = f^{\nu_j}(\gamma_i)$ for $i \leq \ell$ and $f^{\nu_j}(\alpha) \neq f^{\nu_j}(\gamma_i)$ for $i > \ell$. Then

$$z_{\alpha\beta} = z_{\gamma_0\gamma_1} = z_{\gamma_0\gamma_\ell}, \qquad z^{\nu_j}_{f^{\nu_j}(\alpha)f^{\nu_j}(\beta)} = z^{\nu_j}_{f^{\nu_j}(\gamma_\ell)f^{\nu_j}(\gamma_{\ell+1})}.$$

By (v) in Definition 5.1, this implies that $(\varphi_{\gamma_\ell}^{\nu_j})^{-1}(z^{\nu_j}_{f^{\nu_j}(\gamma_\ell)f^{\nu_j}(\gamma_{\ell+1})})$ converges to $z_{\gamma_\ell\gamma_{\ell+1}} \neq z_{\gamma_\ell\gamma_0}$. Hence it follows from Lemma 4.6 that

$$\begin{aligned} z_{\alpha\beta} = z_{\gamma_0\gamma_\ell} &= \lim_{j\to\infty} (\varphi_{\gamma_0}^{\nu_j})^{-1} \circ \varphi_{\gamma_\ell}^{\nu_j}((\varphi_{\gamma_\ell}^{\nu_j})^{-1}(z^{\nu_j}_{f^{\nu_j}(\gamma_\ell)f^{\nu_j}(\gamma_{\ell+1})})) \\ &= \lim_{j\to\infty} (\varphi_{\gamma_0}^{\nu_j})^{-1}(z^{\nu_j}_{f^{\nu_j}(\gamma_\ell)f^{\nu_j}(\gamma_{\ell+1})}). \end{aligned}$$

Thus we have proved that every subsequence of ν_j has a further subsequence which satisfies the assertions of the lemma. Hence the sequence ν_j itself satisfies the claim.

Step 2. *For every $\epsilon > 0$ there exists a $\nu_0 \in \mathbb{N}$ such that*

$$\nu \geq \nu_0,\ f^\nu(\alpha)E^\nu\gamma^\nu \ \Rightarrow\ (\varphi_\alpha^\nu)^{-1}(z^\nu_{f^\nu(\alpha)\gamma^\nu}) \in \bigcup_{\substack{\beta\in T\\ \alpha E\beta}} B_\epsilon(z_{\alpha\beta}). \tag{19}$$

Suppose otherwise that there exist sequences $\nu_j \to \infty$ and $\gamma^{\nu_j} \in T^{\nu_j}$ such that $f^{\nu_j}(\alpha)E^{\nu_j}\gamma^{\nu_j}$ and $z^{\nu_j}_{f^{\nu_j}(\alpha)\gamma^{\nu_j}} \notin \varphi_\alpha^{\nu_j}(B_\epsilon(z_{\alpha\beta}))$ for all $\beta \neq \alpha$. Choose $\gamma_j \in T$ such that $f^{\nu_j}(\gamma_j) = \gamma^{\nu_j}$. Passing to a further subsequence we may assume that $\gamma_j = \gamma$ is independent of j. By Step 1, the sequence $(\varphi_\alpha^{\nu_j})^{-1}(z^{\nu_j}_{f^{\nu_j}(\alpha)f^{\nu_j}(\gamma)})$ converges to $z_{\alpha\gamma}$, contradicting our assumption. This proves (19).

Step 3. *For every compact set $K \subset \Sigma_\alpha \setminus \{z_{\alpha\beta} : \beta \in T,\, \alpha E\beta\}$,*

$$E(u_\alpha, K) = \lim_{\nu\to\infty} E(u^\nu_{f^\nu(\alpha)} \circ \varphi_\alpha^\nu, K \cap (\varphi_\alpha^\nu)^{-1}(\Sigma_{f_\alpha^\nu})).$$

This follows immediately from (iii) and (v) of Definition 5.1.

Step 4. We can now prove the lemma. It follows directly from the definitions that

$$E(\mathbf{u}) = E(u_\alpha, \Sigma_\alpha) + \sum_{\substack{\beta \in T \\ \alpha E \beta}} m_{\alpha\beta}(\mathbf{u}).$$

Because the $z_{\alpha\beta}$ are a discrete set, the sets $B_\epsilon(z_{\alpha\beta})$ are distinct for small ϵ. Hence

$$E\Big(u_\alpha, \bigcup_{\alpha E \beta} B_\epsilon(z_{\alpha\beta})\Big) = \sum_{\substack{\beta \in T \\ \alpha E \beta}} E(u_\alpha, B_\epsilon(z_{\alpha\beta})).$$

This two formulas together show

$$E(\mathbf{u}) = E\Big(u_\alpha, \Sigma_\alpha \setminus \bigcup_{\alpha E \beta} B_\epsilon(z_{\alpha\beta})\Big) + \sum_{\substack{\beta \in T \\ \alpha E \beta}} (m_{\alpha\beta}(\mathbf{u}) + E(u_\alpha, B_\epsilon(z_{\alpha\beta}))).$$

From Step 3 it follows that

$$\begin{aligned} &E\Big(u_\alpha, \Sigma_\alpha \setminus \bigcup_{\alpha E \beta} B_\epsilon(z_{\alpha\beta})\Big) \\ &\qquad = \lim_{\nu\to\infty} E\Big(u^\nu_{f^\nu(\alpha)} \circ \varphi^\nu_\alpha, \Big(\Sigma_\alpha \setminus \bigcup_{\alpha E \beta} B_\epsilon(z_{\alpha\beta})\Big) \cap (\varphi^\nu_\alpha)^{-1}(\Sigma_{f^\nu_\alpha})\Big). \end{aligned}$$

For the second term we observe that

$$E(u_\alpha, B_\epsilon(z_{\alpha\beta})) = E(u_\alpha, B_\epsilon(z_{\alpha\beta}) \setminus B_\delta(z_{\alpha\beta})) + E(u^\alpha, B_\delta(z_{\alpha\beta}))$$

for $\delta < \epsilon$. Because of (19) we can use Step 3 to obtain

$$\begin{aligned} E(u_\alpha, B_\epsilon(z_{\alpha\beta})) = &\lim_{\nu\to\infty} E(u^\nu_{f^\nu(\alpha)} \circ \varphi^\nu_\alpha, (B_\epsilon(z_{\alpha\beta}) \setminus B_\delta(z_{\alpha\beta})) \cap (\varphi^\nu_\alpha)^{-1}(\Sigma_{f^\nu(\alpha)})) \\ &+ E(u_\alpha, B_\delta(z_{\alpha\beta})). \end{aligned}$$

Using (iii) of Definition 5.1 we have

$$\begin{aligned} m_{\alpha\beta}(\mathbf{u}) + E(u_\alpha, B_\epsilon(z_{\alpha\beta})) = &\lim_{\delta\to 0}\lim_{\nu\to\infty} E_{f^\nu(\alpha)}(\mathbf{u}^\nu, \varphi^\nu_\alpha(B_\delta(z_{\alpha\beta})) \cap \Sigma_{f^\nu(\alpha)}) \\ &+ \lim_{\delta\to 0}\lim_{\nu\to\infty} E(u^\nu_{f^\nu(\alpha)} \circ \varphi^\nu_\alpha, (B_\epsilon(z_{\alpha\beta}) \setminus B_\delta(z_{\alpha\beta})) \\ &\qquad\qquad \cap (\varphi^\nu_\alpha)^{-1}(\Sigma_{f^\nu(\alpha)})) \\ = &\lim_{\nu\to\infty} E_{f^\nu(\alpha)}(\mathbf{u}^\nu, \varphi^\nu_\alpha(B_\epsilon(z_{\alpha\beta})) \cap \Sigma_{f^\nu(\alpha)}). \end{aligned}$$

Altogether

$$\begin{aligned} E(\mathbf{u}) = &\lim_{\nu\to\infty} E\Big(u^\nu_{f^\nu(\alpha)} \circ \varphi^\nu_\alpha, \Big(\Sigma_\alpha \setminus \bigcup_{\alpha E \beta} B_\epsilon(z_{\alpha\beta})\Big) \cap (\varphi^\nu_\alpha)^{-1}(\Sigma_{f^\nu_\alpha})\Big) \\ &+ \sum_{\substack{\beta \in T \\ \alpha E \beta}} \lim_{\nu\to\infty} E_{f^\nu(\alpha)}(\mathbf{u}^\nu, \varphi^\nu_\alpha(B_\epsilon(z_{\alpha\beta})) \cap \Sigma_{f^\nu(\alpha)}). \end{aligned}$$

It follows directly from the definition that

$$E_{f^\nu(\alpha)}(\mathbf{u}^\nu, \varphi^\nu_\alpha(B_\epsilon(z_{\alpha\beta})) \cap \Sigma_{f^\nu(\alpha)}) = E(u^\nu_{f^\nu(\alpha)} \circ \varphi^\nu_\alpha, B_\epsilon(z_{\alpha\beta}) \cap (\varphi^\nu_\alpha)^{-1}(\Sigma_{f^\nu(\alpha)})) + \sum_{\substack{f^\nu(\alpha)E^\nu\gamma^\nu \\ z_{f^\nu(\alpha)\gamma^\nu} \in \varphi^\nu_\alpha(B_\epsilon(z_{\alpha\beta}))}} m_{f^\nu(\alpha)\gamma^\nu}(\mathbf{u}^\nu).$$

It follows from (19) that

$$\begin{aligned} &\lim_{\nu\to\infty} \sum_{\substack{\beta\in T \\ \alpha E\beta}} E_{f^\nu(\alpha)}(\mathbf{u}^\nu, \varphi^\nu_\alpha(B_\epsilon(z_{\alpha\beta})) \cap \Sigma_{f^\nu(\alpha)}) \\ &= \lim_{\nu\to\infty} \Big(E(u^\nu_{f^\nu(\alpha)} \circ \varphi^\nu_\alpha, \bigcup_{\alpha E\beta} B_\epsilon(z_{\alpha\beta}) \cap (\varphi^\nu_\alpha)^{-1}(\Sigma_{f^\nu(\alpha)})) + \sum_{\substack{\gamma^\nu\in T^\nu \\ f^\nu(\alpha)E^\nu\gamma^\nu}} m_{f^\nu(\alpha)\gamma^\nu}(\mathbf{u}^\nu) \Big). \end{aligned}$$

Hence

$$\begin{aligned} E(\mathbf{u}) &= \lim_{\nu\to\infty} \Big(E(u^\nu_{f^\nu(\alpha)} \circ \varphi^\nu_\alpha, (\varphi^\nu_\alpha)^{-1}(\Sigma_{f^\nu(\alpha)})) + \sum_{\substack{\gamma^\nu\in T^\nu \\ f^\nu(\alpha)E^\nu\gamma^\nu}} m_{f^\nu(\alpha)\gamma^\nu}(\mathbf{u}^\nu) \Big) \\ &= \lim_{\nu\to\infty} E(\mathbf{u}^\nu). \end{aligned}$$ □

Proof of Proposition 5.3. We prove (i). By Lemma 5.7, the energy $E(\mathbf{u}^\nu)$ is uniformly bounded. Since the constant $\hbar = \hbar(M, \omega, J)$ of Corollary A.4 depend continuously on ω and J (cf. Remark A.5), we have

$$\inf_\nu \{\hbar(\omega^\nu, J^\nu)\} > 0.$$

By Lemma 2.7, this implies that there exists a finite set of isomorphism classes of trees which contains all the T^ν. Passing to a subsequence if necessary, we may assume that the curves $(\mathbf{u}^\nu, \mathbf{z}^\nu)$ are all modelled over the same tree $T' = T^\nu$. Perhaps passing to a further subsequence we may even assume that $x^\nu \in u^\nu_\alpha(\Sigma'_\alpha)$ for some fixed $\alpha \in T'$. If $\Sigma'_\alpha = B$ then (i) follows from Proposition 3.2. The case $\Sigma'_\alpha = S^2$ is proved in [MS2]. A similar argument shows (ii). □

Proof of Theorem 5.4. Let

$$(\mathbf{u}^\nu, \mathbf{z}^\nu) = (\{(\Sigma^\nu_\alpha, \Gamma^\nu_\alpha, u^\nu_\alpha)\}_{\alpha\in T^\nu}, \{z^\nu_{\alpha\beta}\}_{\alpha E^\nu\beta}, \{\alpha_i, z^\nu_i\}_{1\le i\le k})$$

be a sequence of stable J^ν-holomorphic curves which satisfies $E(\mathbf{u}^\nu) \le c$ for all ν and some constant $c > 0$. As in the proof of Proposition 5.3 one shows that there are only finitely many trees, up to isomorphism, which correspond to stable J^ν-holomorphic curves with energy bounded by c. Passing to a subsequence, we may assume that the trees T^ν are all isomorphic. We choose an isomorphism $T' = T^1 \to T^\nu$ for each ν.

Interpret the points $z^\nu_{\alpha'\beta'}$ and $z^{\infty,\nu}_{\alpha'}$ for $\alpha', \beta' \in T$ with $\alpha' E' \beta'$ as marked points. One can generalize Theorem 4.4 to the case of spheres (see [MS2]). Hence $(u^\nu_{\alpha'}, Z^\nu_{\alpha'})$ has a subsequence which Gromov converges to a stable J-holomorphic curve with marked points for each $\alpha' \in T'$. Then connect the limit curves to a tree by introducing an additional edge for each pair $\alpha' E' \beta'$ connecting the points $z_{\alpha'\beta'}$ and $z_{\beta'\alpha'}$. This proves Theorem 5.4.

Proof of Theorem 5.6. Passing to a subsequence if necessary, we may as in the proof of Proposition 5.3 assume that the curves $(\mathbf{u}^\nu, \mathbf{z}^\nu)$ are all modelled over the same tree $T' = T^\nu$. We may also assume that the subtrees $(f^\nu)^{-1}(\alpha') = T_{\alpha'} \subset T$ and $(\tilde{f}^\nu)^{-1}(\alpha') = \tilde{T}_{\alpha'} \subset \tilde{T}$ are independent of ν for all $\alpha' \in T'$. Our considerations imply uniqueness for the sequences $(u^\nu_{\alpha'}, Z^\nu_{\alpha'})$ if $\Sigma_{\alpha'} = B$. An analogous result holds for the case $\Sigma_{\alpha'} = S^2$ (see [MS2]). This proves the theorem. □

Remark 5.8. One can use Gromov convergence to define a topology on the space $\mathcal{M}_{k_1,k_2,A} = \mathcal{M}_{0,1,k_1,k_2,A}(M, L, J)$ of equivalence classes of (k_1, k_2)-marked stable maps of genus zero having one boundary component in L, which represent the relative homology class A. The *Gromov topology* is the collection $\mathcal{U} \subset 2^{\mathcal{M}_{k_1,k_2,A}}$ of all subsets $U \subset \mathcal{M}_{k_1,k_2,A}$ such that for all $(\mathbf{u}, \mathbf{z}) \in U$ and all sequences $(\mathbf{u}^\nu, \mathbf{z}^\nu)$ which Gromov converge to $(\mathbf{u}, \mathbf{z})$ there exists an $N \in \mathbb{N}$ such that $\nu \geq N$ implies $(\mathbf{u}^\nu, \mathbf{z}^\nu) \in U$. As in [MS2] one can show that the Gromov topology has the following properties.

(i) A sequence in $\mathcal{M}_{k_1,k_2,A}$ converges with respect to the Gromov topology if and only if it Gromov converges.
(ii) The topology of $\mathcal{M}_{k_1,k_2,A}$ has a countable basis.
(iii) $\mathcal{M}_{k_1,k_2,A}$ is a compact Hausdorff space.

In particular, $\mathcal{M}_{k_1,k_2,A}$ is a compact metrizable space.

6. Examples and applications

6.1. Maps from the disk to the sphere

Let (M, ω) be (S^2, ω_{FS}) and let $L = S^1 = \mathbb{R} \cup \{\infty\}$ be the equator. Note that $\pi_2(S^2, S^1)$ is generated by two elements A_1 and A_2, where A_1 represents the northern hemisphere and A_2 represents the southern hemisphere. In particular, $A_1 + A_2$ is the canonical generator of $\pi_2(S^2)$. A J-holomorphic map from B to S^2 has to represent $k_1A_1 + k_2A_2$ where $k_1, k_2 \geq 0$. Proposition 3.2 implies that J-holomorphic spheres cannot bubble off from sequences of maps which represent a class kA_1 or kA_2 for $k > 0$. Hence there cannot be bubbling off in the interior. However, bubbling off may occur at the boundary such that the limit consists of a bubble tree consisting of several maps which represent multiples of the same relative homotopy class. An example of this case is the sequence of maps

$$\varphi_\nu : \mathbb{H} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}, \qquad z \mapsto \frac{z^2 - 1/\nu}{z}.$$

For sequences of maps $B \to S^2$ which represent a class $k_1A_1 + k_2A_2$ with $k_1, k_2 > 0$, bubbling in the interior is possible. To see this in the case $k_1 = k_2 = 1$, let $a \in \mathbb{H} \setminus \mathbb{R}$ and pick a sequence $0 \leq x_\nu < 1$ which converges to 1. Define

$$\varphi_\nu(z) = \frac{(z - ax_\nu)(z - \bar{a}x_\nu)}{(1 - \frac{x_\nu}{a}z)(1 - \frac{x_\nu}{\bar{a}}z)}.$$

One easily verifies that φ_ν converges to $|a|^2$ uniformly on compact subsets of $\mathbb{H} \cup \{\infty\} \setminus \{a\}$. Hence φ^ν Gromov converges to a stable map consisting of one sphere with one boundary point.

6.2. Calculation of some Gromov invariant

This application is due to M. Akveld and D. Salamon (see [AS]). They examine manifolds of the form $B \times \mathbb{C}P^n$ with the submanifold

$$\Lambda = \bigcup_{t \in [0,1]} \{e^{2\pi i t}\} \times \iota_t(\mathbb{R}P^n) \subset \partial B \times \mathbb{C}P^n$$

where the inclusion is defined by

$$\iota_t : \mathbb{R}P^n \to \mathbb{C}P^n;\ [x_0 : \ldots : x_n] \mapsto [e^{\pi i t}x_0 : \ldots : x_n].$$

These manifolds are endowed with the following almost complex structure:

$$\tilde{J}_{x,y} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ -J_{x,y}X_F + X_G & -X_F - J_{x,y}X_G & J_{x,y} \end{pmatrix}$$

where for $(x, y) \in B$, $J_{x,y}$ is a family of almost complex structures on $\mathbb{C}P^n$ compatible with ω_{FS}, and X_F and X_G are the Hamiltonian vector fields for the functions $F_{x,y}$ and $G_{x,y}$ on $\mathbb{C}P^n$.

A $\tilde{J}$-holomorphic curve defines a relative homotopy class $A \in \pi_2(B \times \mathbb{C}P^n, \Lambda)$. For such a relative homotopy class and for a triple $(F, G, J) \in \mathcal{I} = C^\infty(B \times \mathbb{C}P^n) \times C^\infty(B \times \mathbb{C}P^n) \times \mathcal{J}$ where $\mathcal{J}$ denotes the set of families $J_{x,y}$ of almost complex structures compatible with ω_{FS} we define

$$\mathcal{M}(A; F, G, J) = \left\{ u \in C^\infty(B, B \times \mathbb{C}P^n) : \frac{\partial u}{\partial x}(x, y) + \tilde{J}\frac{\partial u}{\partial y}(x, y) = 0,\ [u] = A \right\}.$$

One can show that for 'most' triples $(F, G, J) \in \mathcal{I}$, $\mathcal{M}(A; F, G, J)$ is a smooth manifold. Call such triples *regular*. Then for two regular triples $(F_0, G_0, J_0) \in \mathcal{I}$ and $(F_1, G_1, J_1) \in \mathcal{I}$ the manifolds $\mathcal{M}(A; F_0, G_0, J_0)$ and $\mathcal{M}(A; F_1, G_1, J_1)$ are cobordant via a manifold of the form $\mathcal{M}(A; \{(F_t, G_t, J_t)\}_t)$. Here (F_t, G_t, J_t) is a smooth path in $\mathcal{I}$ which connects (F_0, G_0, J_0) to (F_1, G_1, J_1) and satisfies some regularity conditions. This in itself is not very relevant information because every manifold V is cobordant to the empty manifold via the noncompact cobordism $V \times [0, 1)$. However, this fact will become useful if we are able to establish certain compactness criteria.

For a regular triple we know from Riemann–Roch that the dimension of the moduli space $\mathcal{M}(A; F, G, J)$ is given by the formula

$$\dim \mathcal{M}(A; F, G, J) = \mu_{B \times \mathbb{C}P^n}(A) + n - 2.$$

Here $\mu_{B\times\mathbb{C}P^n}$ is the Maslov index which associates an integer to a relative homotopy class in $\pi_2(B\times\mathbb{C}P^n,\Lambda)$. We will now consider the homotopy class $A=D_N$ which is represented by the map $u(x,y)=(x,y,[1:0:\ldots:0])$. The Maslov index of D_N can be calculated to be $2-n$ so that for a regular triple the moduli space is 0-dimensional.

M. Akveld and D. Salamon prove that for a regular homotopy (F_t,G_t,J_t) the moduli space $\mathcal{M}(D_N;\{(F_t,G_t,J_t)\}_t)$ is a compact one-dimensional manifold. They first show that bubbling can occur only fibrewise. Then it follows from Gromov's compactness theorem that every sequence in $\mathcal{M}(D_N;\{(F_t,G_t,J_t)\}_t)$ converges to a bubble-tree consisting of a $\tilde{J}$-holomorphic map $u:B\to B\times\mathbb{C}P^n$ and a finite number of maps

$$v_i:S^2\to\mathbb{C}P^n\quad\text{and}\quad w_j:B\to\mathbb{C}P^n$$

with $i=1,\ldots,k$ and $j=1,\ldots,l$ such that v_i is J_{ζ_i}-holomorphic for some $\zeta_i\in B$ and w_j is J_{ζ_j}-holomorphic for some $\zeta_j=e^{2\pi i t_j}\in\partial B$ and $w_j(\partial B)\subset\iota_{t_j}(\mathbb{R}P^n)$. Moreover, because the homotopy class converges, we get

$$[u]+\sum_{i=1}^{k}[v_i]+\sum_{j=1}^{\ell}[w_j]=D_N.$$

Because the homotopy class is preserved it follows that the Maslov class is preserved too. So this reads

$$\mu(D_N)=\mu(u)+\sum_{i=1}^{k}\mu(v_i(S^2))+\sum_{j=1}^{l}\mu(w_j(B)).$$

Now the spheres represent elements of $\pi_2(\mathbb{C}P^n)$ and the disks of $\pi_2(\mathbb{C}P^n,L)$ where $L=\iota_t(\mathbb{R}P^n)$ for some $t\in[0,1]$. The generator of $\pi_2(\mathbb{C}P^n,L)$ has Maslov index $n+1$ and the generator of $\pi_2(\mathbb{C}P^n)=\mathbb{Z}$ has Maslov index $2n+2$. In order to be holomorphic the disks and spheres must be positive multiples of the generator, so we get the following equation for the Maslov index:

$$2-n=\mu(u)+\kappa(n+1)+\lambda(2n+2),$$

where $\kappa\geq 0$ and $\lambda\geq 0$. If $\kappa>0$ or $\lambda>0$ then $\mu(u)<2-n$ and hence the corresponding moduli space is empty. Hence there cannot be fibrewise bubbling and therefore there is no bubbling at all.

The Gromov invariant of the class D_N is defined by

$$\mathrm{Gr}(D_N)=\#\mathcal{M}(D_n;F,G,J)\ \mathrm{mod}\ 2.$$

Since there is no bubbling, the moduli space for a regular homotopy $\{(F_t,G_t,J_t)\}_t$ connecting the regular points (F_0,G_0,J_0) and (F_1,G_1,J_1) is a one-dimensional compact manifold with boundary. Its boundary is the disjoint union of moduli spaces at $t=0$ and $t=1$. So we have found a compact cobordism between two zero-dimensional manifolds. Since the boundary of a one-dimensional manifold consists of an even number of points, we deduce that the parity of the number of points of $\mathcal{M}(D_N; F_0, G_0, J_0)$ and $\mathcal{M}(D_N; F_1, G_1, J_1)$ is the same. This shows that the Gromov invariant is independent of the regular triple (F, G, J).

M. Akveld and D. Salamon show in [AS] that $\mathrm{Gr}(D_N) = 1$. This is done by calculating the Gromov invariant for a special pair and then showing that this is actually a regular pair.

Appendix A. Analytical background

A.1. Mean value estimates

We always assume in this subsection that (M, ω) is a compact symplectic manifold (possibly with boundary), $L \subset M$ a compact Lagrangian submanifold without boundary and J an ω-tame almost complex structure on TM. The following two a priori estimates give an L^∞-bound for J-holomorphic curves whose energy is small enough.

Lemma A.1 (A priori estimate). *There exists a constant $\hbar_0 > 0$ such that the following holds. If $r > 0$ and $u : B_r \to M$ is a J-holomorphic curve such that*

$$\int_{B_r} |du|^2 \,\mathrm{dvol} < \hbar_0$$

then

$$|du(0)|^2 \leq \frac{8}{\pi r^2} \int_{B_r} |du|^2 \,\mathrm{dvol}.$$

The proof of Lemma A.1 relies on a result about the partial differential inequality $\Delta e \geq -Be^2$ for the energy density $e = |du|^2$ (see [Sa] for example).

Lemma A.2 (A priori estimate on the boundary). *There exist constants $\hbar_1 > 0$ and $c > 0$ such that the following holds. If $r > 0$ and $u : B_r \cap \mathbb{H} \to M$ is a J-holomorphic curve such that $u(B_{2r} \cap \mathbb{R}) \subset L$ and*

$$\int_{B_{2r} \cap \mathbb{H}} |du|^2 \,\mathrm{dvol} < \hbar_1$$

then

$$\sup_{B_r \cap \mathbb{H}} |du|^2 \leq \frac{c}{\pi r^2} \int_{B_{2r} \cap \mathbb{H}} |du|^2 \,\mathrm{dvol}.$$

The idea of the proof of Lemma A.2 is to extend u to B_{2r} by a reflection. Then the lemma follows as Lemma A.1. This works only for special metrics. Lemma A.3 is needed to construct such a metric.

Lemma A.3. *Let (M, J) be an almost complex manifold and $L \subset M$ be a totally real submanifold. Then there exists a Riemannian metric g on M such that*

(i) *$g(J(p)v, J(p)w) = g(v, w)$ for $p \in M$ and $v, w \in T_pM$,*
(ii) *$J(p)T_pL$ is the orthogonal complement of T_pL for every $p \in L$,*
(iii) *L is totally geodesic with respect to g.*

Proof. Choose coordinates $x_1, \dots, x_n$ on L and extend them to coordinates $x_1, \dots, x_n, y_1, \dots, y_n$ on M such that

$$J\frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i}, \qquad i = 1, \dots, n,$$

on L. In these coordinates the almost complex structure is represented by the matrices

$$J(x,y) = \begin{pmatrix} A(x,y) & B(x,y) \\ C(x,y) & D(x,y) \end{pmatrix}$$

where

$$A(x,0) = D(x,0) = 0, \qquad C(x,0) = -B(x,0) = \mathrm{id}.$$

A metric

$$g(x,y) = \begin{pmatrix} a(x,y) & b^T(x,y) \\ b(x,y) & c(x,y) \end{pmatrix}$$

satisfies (i)–(iii) iff

$$a(x,0) = c(x,0), \qquad b(x,0) = 0, \qquad \partial_{y_i} a(x,0) = 0, \qquad g = J^T g J. \tag{20}$$

To find this metric we first construct an auxiliary metric

$$\tilde{g}(x,y) = \begin{pmatrix} \tilde{a}(x,y) & \tilde{b}^T(x,y) \\ \tilde{b}(x,y) & \tilde{c}(x,y) \end{pmatrix}$$

where $\tilde{g}$ satisfies

$$\tilde{a}(x,0) = \tilde{c}(x,0), \qquad \tilde{b}(x,0) = 0,$$

$$\partial_{y_i}\tilde{a}(x,0) + \partial_{y_i} C^T(x,0)\tilde{a}(x,0) + \tilde{a}(x,0)\partial_{y_i} C(x,0) = 0.$$

Then $g(v_1, v_2) = \frac{1}{2}(\tilde{g}(Jv_1, Jv_2) + \tilde{g}(v_1, v_2))$ satisfies (20). Conditions (i)–(iii) are intrinsic and they continue to hold under convex combinations and under multiplication by cutoff functions $\beta = \beta(x,y)$ that satisfy

$$\partial_{y_i}\beta(x,0) = 0.$$

This condition on the cutoff function is intrinsic. It asserts that

$$q \in L,\ v \in T_q L \ \Rightarrow\ d\beta(q)J(q)v = 0. \tag{21}$$

Hence the result follows by choosing local metrics g which satisfy (20) and patching with a partition of unity consisting of finitely many cutoff functions that satisfy (21). □

Proof of Lemma A.2. By Lemma A.3 there exists $\tilde{g}$ such that L is totally geodesic with respect to $\tilde{g}$ and $JTL = TL^\perp$. Now extend the energy density $e : B_r \cap \mathbb{H} \to \mathbb{R}$ defined by $e(s,t) = \frac{1}{2}|\partial_s u(s,t)|^2_{\tilde{g}}$ to the ball B_r by

$$e(s,-t) = e(s,t).$$

We must prove that e is twice continuously differentiable. For this it suffices to show that $\partial_t e(s,0) = 0$. Now

$$\begin{aligned}\partial_t e &= \langle \nabla_t \partial_s u, \partial_s u\rangle_{\tilde g} = \langle \nabla_s \partial_t u, \partial_s u\rangle_{\tilde g} = \langle \nabla_s (J\partial_s u), \partial_s u\rangle_{\tilde g} \\ &= \langle (\nabla_s J)\partial_s u, \partial_s u\rangle_{\tilde g} + \langle J\nabla_s \partial_s u, \partial_s u\rangle_{\tilde g} = \langle J\nabla_s \partial_s u, \partial_s u\rangle_{\tilde g}\end{aligned}$$

where the last equality follows from (i) in Lemma A.3. Because L is totally geodesic we have $\nabla_s \partial_s u(s,0) \in T_{u(s)}L$. Since JTL is orthogonal to TL we obtain $\langle J\nabla_s\partial_s u, \partial_s u\rangle_{\tilde g} = 0$ for $t=0$. Hence e is of class C^2 as claimed. One can show that in this case e satisfies a partial differential inequality of the form $\Delta e \geq -Be^2$. It follows that there exist constants $\tilde\hbar_1 > 0$ and $\tilde c > 0$ such that

$$\int_{B_{2r}\cap\mathbb{H}} |du|^2_{\tilde g}\,\mathrm{dvol} < \tilde\hbar_1$$

implies

$$\sup_{B_r\cap\mathbb{H}} |du|^2_{\tilde g} \leq \frac{\tilde c}{\pi r^2}\int_{B_{2r}\cap\mathbb{H}} |du|^2_{\tilde g}\,\mathrm{dvol}.$$

Because M is compact there exist constants $c_1, c_2 > 0$ such that

$$c_1 g \leq \tilde g \leq c_2 g$$

where $g(v,w) = \frac{1}{2}(\omega(v,Jw)+\omega(Jv,w))$. Hence the lemma follows with $\hbar_1 = \tilde\hbar_1/c_2$ and $c = \tilde c c_2/c_1$. □

An immediate corollary from the a priori estimates is the following:

Corollary A.4. *Let Σ be the sphere or the disk. Then there exists a positive constant $\hbar$ such that $E(u) \geq \hbar$ for every nonconstant J-holomorphic curve $u : (\Sigma, \partial\Sigma) \to (M, L)$.*

Proof. Let $\hbar = \min(\hbar_0, \hbar_1)$. Then the claim follows from the a priori estimates as r goes to infinity. □

Remark A.5. By carefully checking the details of the proof of the a priori estimate one can show that $\hbar$ in Corollary A.4 depends continuously on J and ω.

Let

$$G(r,R) := G_{z_0}(r,R) := \{z \in \mathbb{H} : r \leq |z - z_0| \leq R\}$$

where $z_0 \in \mathbb{H}$ and $r < R$. Denote by $[\phi_\tau^-, \phi_\tau^+] \subset [-\pi/2, 3\pi/2]$ the maximal interval such that $z_0 + e^{\tau+i\theta} \in \mathbb{H}$ for every $\theta \in [\phi_\tau^-, \phi_\tau^+]$. Lemma A.6 below says that if the energy of a J-holomorphic curve on an arbitrarily long cylinder is sufficiently small then it cannot spread out uniformly but must be concentrated near the ends. It is proved as Lemma 4.7.3 in [MS2].

Lemma A.6. *Let $u : G(r,R) \to M$ be a J-holomorphic curve such that*

$$\mathrm{im}(u|_{G(r,R)\cap\mathbb{R}}) \subset L.$$

Then there exist constants $\hbar > 0$ and $c > 0$ such that if $E(u) < \hbar$ then

$$d(u(r_1 e^{i\theta_1}), u(r_2 e^{i\theta_2})) \leq c\sqrt{E(u, G(r,R))} \qquad \textit{for } re \leq r_1, r_2 \leq Re^{-1}$$

and

$$E(u, G(e^T r, e^{-T} R)) \le c \frac{E(u, G(r, R))}{T} \qquad \textit{whenever } e^2 \le e^{2T} \le R/r.$$

A.2. Compactness and regularity

J-holomorphic curves are smooth and if one has a uniform L^p-bound on the first derivatives for some $p > 2$, then one gets compactness. The proofs are based on elliptic bootstrapping. We assume that (M, J) is a compact almost complex manifold (perhaps with boundary) and $L \subset M$ is a compact totally real submanifold without boundary.

Definition A.7. Let (Σ, j) be a Riemann surface with complex structure j. We call j *standard near the boundary* if each boundary point has a neighbourhood which is biholomorphic to the intersection of the unit disk $U_1 = \{z \in \mathbb{C} : |z| < 1\}$ with the closed upper half-space $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z \ge 0\}$.

Theorem A.8 (Regularity). *Let (Σ, j) be a Riemann surface standard near the boundary. If $u : \Sigma \to M$ is a J-holomorphic curve of class $W^{k,p}$ with $kp > 2$ and $u(\partial\Sigma) \subset L$ then u is smooth.*

Proof. See [MS2, Theorem B.4.1]. □

Theorem A.9 (Compactness). *Let Σ be a Riemann surface with complex structure j. Let J_ν be a sequence of almost complex structures on M converging to J in the C^∞-topology*[6] *and j_ν be a sequence of complex structures on Σ, standard near the boundary, converging to j in the C^∞-topology. Let $U_\nu \subset \Sigma$ be an increasing sequence of open sets whose union is Σ and $u_\nu : U_\nu \to M$ be a sequence of (j_ν, J_ν)-holomorphic curves such that $u_\nu(U_\nu \cap \partial\Sigma) \subset L$. Assume that for every compact set $Q \subset \Sigma$ there exist constants $p > 2$ and $c > 0$ such that*

$$\|du_\nu\|_{L^p(Q)} \le c$$

for ν sufficiently large. Then a subsequence of u_ν converges uniformly with all derivatives on compact sets to a (j, J)-holomorphic curve $u : (\Sigma, \partial\Sigma) \to (M, L)$.

Proof. See [MS2, Theorem B.4.2]. □

A.3. Removable singularities

The removable singularity theorem says that any J-holomorphic curve $u : B \setminus \{0\} \to M$ on the punctured disk which has finite energy extends smoothly to the disk. An analogous result holds for the punctured half-disk $(B \cap \mathbb{H}) \setminus \{0\}$. The proof is an application of the a priori estimates.

We assume that (M, ω) is a compact symplectic manifold with boundary, L a compact Lagrangian submanifold of M without boundary, and $J \in \mathcal{J}_\tau(M, \omega)$. The metric is given by $g(v, w) = \frac{1}{2}(\omega(v, Jw) + \omega(Jv, w))$.

[6] Hence L is totally real with respect to J_ν for ν sufficiently large.

Theorem A.10 (Removable singularities in the interior). *Let $B_r = \{z \in \mathbb{C} : |z| < r\}$. Then every J-holomorphic curve $u : B_r \setminus \{0\} \to M$ with finite energy*

$$E(u) = \frac{1}{2}\int_{B_r} |du|^2 < \infty$$

extends to a smooth map on B_r.

Proof. See [MS2, Theorem 4.1.2(i)]. □

Theorem A.11 (Removable singularities on the boundary). *Let $u : B_r \cap \mathbb{H} \setminus \{0\} \to M$ be a J-holomorphic curve with $u(B_r \cap \mathbb{R} \setminus \{0\}) \subset L$ and finite energy $E(u) < \infty$. Then u extends to a smooth map on $B_r \cap \mathbb{H}$.*

Proof. See [MS2, Theorem 4.1.2(ii)]. □

A.4. Bubbling

The bubbling theorem asserts that a sequence of J-holomorphic curves with bounded energy converges to a J-holomorphic curve up to finitely many points where bubbles occur. For a proof of this theorem see for instance [MS2]. The idea of the proof is that at every point where the energy density tends to zero one rescales the sequence using conformal invariance. Then it follows from the compactness theorem for sequences of J-holomorphic curves with a uniform L^∞ bound on the first derivatives that a subsequence converges. Using removal of singularities one shows that the bubble is either a sphere or a disk. If the bubble point lies in the interior, then one always get spheres. However, if the bubble point lies on the boundary, then it depends on how fast the energy converges to the boundary. Because the energy of a J-holomorphic disk or a J-holomorphic sphere is bounded uniformly from below by a constant greater than zero, one sees that there cannot be infinitely many bubble points.

We assume that (M, ω) is a compact symplectic manifold with boundary, L a compact Lagrangian submanifold of M without boundary, and $J \in \mathcal{J}_\tau(M, \omega)$.

Theorem A.12. *Let (Σ, j) be a compact Riemann surface with complex structure j. Let J_ν be a sequence of almost complex structures on M converging to J in the C^∞-topology and j_ν be a sequence of complex structures on Σ, standard near the boundary, converging to j in the C^∞-topology. Let $u_\nu : (\Sigma, \partial\Sigma) \to (M, L)$ be a sequence of (j_ν, J_ν)-holomorphic curves whose energy is uniformly bounded. Then there exists a subsequence (still denoted by u_ν), a J-holomorphic curve $u : \Sigma \to M$ and a finite set of points $Z = \{z^1, \ldots, z^\ell\} \subset \Sigma$ such that the following holds.*

(i) *u_ν converges to u uniformly with all derivatives on compact subsets of $\Sigma \setminus Z$.*

(ii) *For every j and every $\epsilon > 0$ such that $B_\epsilon(z^j) \cap Z = \{z^j\}$, the limit*

$$m_\epsilon(z^j) = \lim_{\nu\to\infty} E(u_\nu, B_\epsilon(z^j))$$

exists. Moreover,

$$m(z^j) = \lim_{\epsilon\to 0} m_\epsilon(z^j) \geq \hbar$$

where $\hbar$ is as in Corollary A.4.

(iii) *For every compact subset $K \subset \Sigma$ with $Z \subset \mathrm{int}(K)$,*

$$E(u,K) + \sum_{j=1}^{\ell} m(z^j) = \lim_{\nu \to \infty} E(u_\nu, K).$$

Proof. See [MS2, Theorem 4.6.1]. □

Appendix B. Sequences of Möbius transformations

The group of Möbius transformations is noncompact. For example, the sequence $\varphi^\nu(z) = \nu z$ has no convergent subsequence. However, φ^ν converges to ∞ uniformly on compact subsets of $S^2 \setminus \{0\}$. More generally, the following holds.

Lemma B.1. *Let $\varphi^\nu : S^2 \to S^2$ be a sequence of Möbius transformations which does not have a uniformly convergent subsequence. Then the following holds.*

(i) *There exist points $x_0, y_0 \in S^2$ and a subsequence φ^{ν_i} which converges to y_0, uniformly on compact subsets of $S^2 \setminus \{x_0\}$.*
(ii) *If $\varphi^\nu \in G_0$ then $x_0, y_0 \in \partial B$.*

Proof. Assertion (i) follows immediately from Theorem A.12 with $E(\varphi) = \pi = \hbar$ for $\varphi \in G$.

To prove (ii) interpret φ as a J-holomorphic map $B \to B$ with $E(\varphi) = \pi/2 = \hbar$. □

The lemma above shows that every divergent sequence of Möbius transformations provides us with two exceptional points. The following lemma tells us how these exceptional points of two sequences of divergent Möbius transformations are related if one assumes that the product of the Möbius transformations converges.

Lemma B.2. *Let $\varphi^\nu : S^2 \to S^2$ and $\psi^\nu : S^2 \to S^2$ be sequences of Möbius transformations and $x_1, x_2, y_1, y_2 \in S^2$ be such that φ^ν converges to x_1 uniformly on compact subsets of $S^2 \setminus \{x_2\}$ and ψ^ν converges to y_1 uniformly on compact subsets of $S^2 \setminus \{y_2\}$. Assume that $\psi^\nu \circ \varphi^\nu$ converges to a Möbius transformation ρ. Then the following holds.*

(i) $x_1 = y_2$.
(ii) $\rho(x_2) = y_1$.

In particular, $(\varphi^\nu)^{-1}$ converges to x_2 uniformly on compact subsets of $S^2 \setminus \{x_1\}$.

Proof. We first prove (i). If $x_1 \neq y_2$ then $(\psi^\nu) \circ \varphi^\nu$ converges to y_1 uniformly on compact subsets of $S^2 \setminus \{x_2\}$. But this contradicts the assumption that $\psi^\nu \circ \varphi^\nu$ converges uniformly on all of S^2. Hence $x_1 = y_2$.

Let us prove (ii). Fix $\delta > 0$. Let $x \in S^2 \setminus B_\delta(\rho(x_2))$. Because $(\psi^\nu \circ \varphi^\nu)^{-1} \circ \rho$ converges uniformly to the identity, we see that there exists $\epsilon = \epsilon(\delta) > 0$ and $\nu_0(\epsilon) \in \mathbb{N}$ such that for all $\nu \geq \nu_0(\epsilon)$, $(\psi^\nu \circ \varphi^\nu)^{-1}(x) \in S^2 \setminus \{B_\epsilon(x_2)\}$. This shows that $(\psi^\nu)^{-1}(x) = \varphi^\nu \circ (\psi^\nu \circ \varphi^\nu)^{-1}(x)$ converges to $x_1 = y_2$ uniformly on $S^2 \setminus B_\delta(\rho(x_2))$. Because δ was arbitrary, (i) implies that $\rho(x_2) = y_1$. □

The following lemmas are rather technical. They are needed to prove uniqueness of limits for stable maps and to deal with marked points.

Definition B.3. Let X be a topological space. Let A_ν be a sequence of subsets of X. We say that A_ν converges to $Y \subset X$ if for every compact subset $K \subset Y$ and every open set U, $Y \subset U \subset X$, there exists a $\nu_0(K,U)$ such that

$$K \subset A_\nu \subset U, \qquad \nu \geq \nu_0(K,U).$$

Lemma B.4. *Assume that there exist $x, y \in S^2$ and sequences of sets $A_\nu, B_\nu \subset S^2$ and $\varphi^\nu \in G$ such that the following holds. A_ν converges to $S^2 \setminus \{x\}$, B_ν converges to $S^2 \setminus \{y\}$, and φ^ν is a bijection between A_ν and B_ν. Assume further that the sequence φ^ν has no convergent subsequence. Then the following is fulfilled.*

(i) *There exist points $x_0, y_0 \in S^2$ such that φ^ν converges to y_0 uniformly with all derivatives on compact subsets of $S^2 \setminus \{x_0\}$.*
(ii) *$x = x_0$ or $y = y_0$.*

Proof. The first assertion follows directly from Lemma B.1. To prove the second one, assume that $y \neq y_0$. We want to show that $x = x_0$. Otherwise for every $r > 0$ there exists $\nu_0(r)$ such that

$$\varphi^\nu(x) \in U_r(y_0), \qquad \nu \geq \nu_0(r).$$

Because $y \neq y_0$ there exists $r > 0$ such that

$$U_r(y_0) \subset S^2 \setminus U_r(y).$$

Perhaps after enlarging $\nu_0(r)$ we can assume that

$$S^2 \setminus U_r(y) \subset B_\nu, \qquad A_\nu \subset S^2 \setminus \{x\}, \qquad \nu \geq \nu_0(r).$$

Altogether

$$x \in (\varphi^\nu)^{-1}(B_\nu) = A_\nu \subset S^2 \setminus \{x\}.$$

This contradiction shows that $y \neq y_0$ implies $x = x_0$. Applying the same argument to $(\varphi^\nu)^{-1}$ one sees that $x \neq x_0$ implies $y = y_0$. $\square$

Example B.5. The following example shows that in Lemma B.4 one cannot assume that $x = x_0$ and $y = y_0$. Let $x = \infty$, $y = \infty$, $A_\nu = B_{1/\epsilon^\nu}$, $B_\nu = B_{1/\delta^\nu}$, where ϵ^ν, δ^ν and ϵ^ν/δ^ν converge to zero. It follows that A_ν converges to $S^2 \setminus \{x\}$ and B_ν converges to $S^2 \setminus \{y\}$. The sequence of Möbius transformations φ^ν defined by

$$\varphi^\nu(z) = \frac{\epsilon^\nu}{\delta^\nu}(z)$$

gives a bijection between A_ν and B_ν. Because ϵ^ν/δ^ν converges to zero we see that φ^ν converges uniformly to zero on compact subsets of $S^2 \setminus \{\infty\}$. Hence $x_0 = \infty = x$ but $y_0 = 0 \neq \infty = y$.

We finally mention the following result without proof.

Lemma B.6. *Let $\varphi^\nu : S^2 \to S^2$ be a sequence of Möbius transformations. Suppose that $x_0, y_0 \in S^2$ and x_1^ν, x_2^ν, y^ν are convergent sequences such that*

$$x_0 \neq \lim_{\nu\to\infty} x_1^\nu \neq \lim_{\nu\to\infty} x_2^\nu \neq x_0, \qquad y_0 \neq \lim_{\nu\to\infty} y^\nu,$$

and

$$\lim_{\nu\to\infty} \varphi^\nu(x_1^\nu) = \lim_{\nu\to\infty} \varphi^\nu(x_2^\nu) = y_0, \qquad \lim_{\nu\to\infty} (\varphi^\nu)^{-1}(y^\nu) = x_0.$$

Then φ^ν converges to y_0, uniformly on compact subsets of $S^2 \setminus \{x_0\}$.

References

[AS] M. Akveld and D. Salamon, *Loops of Lagrangian submanifolds and J-holomorphic disks.* Geom. Funct. Anal. **11** (2001), 609–650.

[AL] M. Audin and J. Lafontaine (eds.), *Holomorphic Curves in Symplectic Geometry.* Progr. Math. 117, Birkhäuser, Basel, 1994.

[BC] P. Biran and O. Cornea, *Quantum structures for Lagrangian submanifolds.* arXiv:0708.4221.

[Ch] Y. Chekanov, *Lagrangian intersections, symplectic energy, and areas of holomorphic curves.* Duke Math. J. **95** (1998), 213–226.

[Fl] A. Floer, *Morse theory for Lagrangian intersections.* J. Differential Geom. **28** (1988), 513–547.

[Flm] H. Hofer, C. Taubes, A. Weinstein and E. Zehnder (eds.), *The Floer Memorial Volume.* Birkhäuser, Basel, 1995.

[FOOO] K. Fukaya, Y. Oh, H. Ohta and K. Ono, *Lagrangian Intersection Floer Theory—Anomaly and Obstruction.* Kyoto Univ., 2006.

[Gr] M. Gromov, *Pseudo holomorphic curves in symplectic manifolds.* Invent. Math. **82** (1985), 307–347.

[HS] H. Hofer and D. Salamon, *Floer homology and Novikov rings.* In [Flm].

[Hu] C. Hummel, *Gromov's Compactness Theorem for Pseudoholomorphic Curves.* Progr. Math. 151, Birkhäuser, 1997.

[IS1] S. Ivashkovich and V. Shevchishin, *Gromov compactness theorem for J-complex curves with boundary.* Int. Math. Res. Notices **2000**, no. 22, 1167–1206.

[IS2] S. Ivashkovich and V. Shevchishin, *Reflection principle and J-complex curves with boundary on totally real immersions.* Comm. Contemp. Math. **4** (2002), 65–106.

[Ko] M. Kontsevich, *Enumeration of rational curves via torus actions.* In: The Moduli Space of Curves (Texel Island, 1994), Progr. Math. 129, Birkhäuser, 1995, 335–368.

[La] L. Lazzarini, *Existence of a somewhere injective pseudoholomorphic disc.* Geom. Funct. Anal. **10** (2000), 829–862.

[MS1] D. McDuff and D. Salamon, *Introduction to Symplectic Topology*, Oxford Math. Monogr., Oxford Univ. Press, 1995.

[MS2] D. McDuff and D. Salamon, *J-Holomorphic Curves and Symplectic Topology.* Colloq. Publ. 52, Amer. Math. Soc., Providence, RI, 2004.

[Oh] Y. Oh, *Floer cohomology of Lagrangian intersections and pseudoholomorphic disks I, II.* Comm. Pure Appl. Math. **46** (1993), 949–994, 995–1012; III. *Arnold-Givental conjecture*, in [Flm].

[Pan] P. Pansu, *Compactness.* In [AL].

[Par1] T. Parker, *Bubble tree convergence for harmonic maps*, J. Differential Geom. **44** (1996), 595–633.

[Par2] T. Parker, *Compactified Moduli Spaces of Pseudo-Holomorphic Curves.* AMS/IP Stud. Adv. Math. 10, Amer. Math. Soc., 1999, 77–113.

[PW] T. Parker and J. Wolfson, *Pseudoholomorphic maps and bubble trees.* J. Geom. Anal. **3** (1993), 63–98.

[SaU] J. Sacks and K. Uhlenbeck, *The existence of minimal immersions of 2-spheres*, Ann. of Math. **113** (1981), 1–24.

[Sa] D. Salamon, *Morse theory, the Conley index and Floer homology.* Bull. London Math. Soc. **22** (1990), 113–140.

[Se] P. Seidel, *A long exact sequence for symplectic Floer cohomology.* Topology **42** (2003), 1003–1063.

[Ye] R. Ye, *Gromov's compactness theorem for pseudoholomorphic curves.* Trans. Amer. Math. Soc. **342** (1994), 671–694.

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