Cyclotomic Birman–Murakami–Wenzl algebras

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1. Introduction

The theory of quantum invariants of links nowaday rests on a broad theory that includes quantum groups, their centralizer algebras and tensor categories. It is the ultimate goal of the 'Knot Theory and Root Systems' program initiated in [2] to carry over this theory to the braid groups associated to the other root systems. The greatest progress so far has been taken for the braid group of Coxeter type B where the notions of quasi-triangular Hopf algebra and monoidal categories have been defined and nontrivial examples have been found [5,7]. Furthermore, Temperley–Lieb algebras [2] and Hecke algebras [3,11] have been studied intensively for this root system. In the present paper we continue the study of generalizations of the Birman–Murakami–Wenzl algebra [16,9].

Every Coxeter diagram defines a braid group that is an infinite covering of its Coxeter group. The braid group ZB_n of Coxeter type B has generators $\tau_i, i = 0, 1, \ldots, n-1$. Generators $\tau_i, i \geq 1$ satisfy the relations of Artin's braid group (which is the braid

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group of Coxeter type A): $\tau_i\tau_j = \tau_j\tau_i$ if $|i-j|>1$, and $\tau_i\tau_j\tau_i = \tau_j\tau_i\tau_j$ if $|i-j|=1$. The generator τ_0 has relations

$$\tau_0\tau_1\tau_0\tau_1 = \tau_1\tau_0\tau_1\tau_0, \tag{1}$$

$$\tau_0\tau_i = \tau_i\tau_0, \quad i \geq 2. \tag{2}$$

The braid group ZB_n may be graphically interpreted (cf. Fig. 1) as symmetric braids or cylinder braids [4]: The symmetric picture shows it as the group of braids with $2n$ strings (numbered $-n,\ldots,-1,1,\ldots,n$) which are fixed under a 180° rotation about the middle axis. In the cylinder picture (Fig. 2) one adds a single fixed line (indexed 0) on the left and obtains ZB_n as the group of braids with n strings that may surround this fixed line. The generators $\tau_i, i \geq 0$ are mapped to the diagrams $X_i^{(G)}$ given in Fig. 1. More generally, tangles of B-type may be defined. The special case of tangles without crossings is the B-type Temperley–Lieb algebra TB_n that has been introduced by tom Dieck in [2].

The Ariki–Koike Algebra is the quotient of the group algebra of ZB_n where the images X_i of the generators τ_i for $i \geq 1$ fulfill quadratic relations while X_0 satisfies a

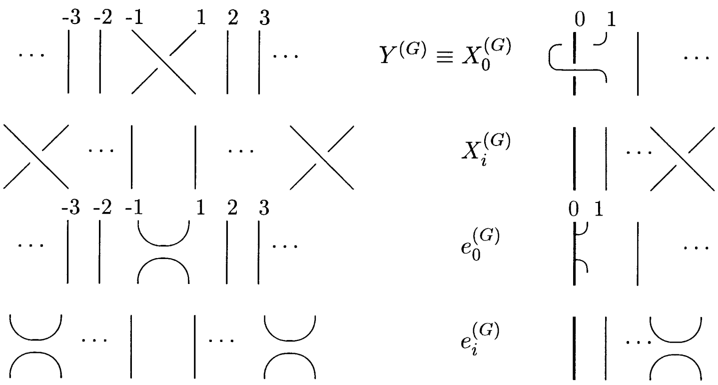

Fig. 1. The graphical interpretation of the generators as symmetric tangles (on the left) and as cylinder tangles (on the right).

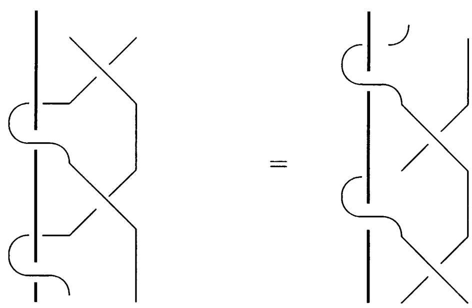

Fig. 2. Relation (1) in the cylinder picture

cyclotomic polynomial of arbitrary degree. The Hecke Algebra of B type is a special case where X_0 satisfies also a quadratic relation.

The standard Birman–Murakami–Wenzl algebra BA_n of type A imposes cubic relations on its generators in a way that enables its interpretation as an algebra of tangles with a skein relation that comes from the Kauffman polynomial.

Thus, it is natural to define an Ariki–Koike like extension of the BMW algebra $\mathscr{B}_n^k$ that contains a generator Y as image of τ_0 that satisfies $\prod_{i=0}^{k-1}(Y - p_i) = 0$. We call these algebras cyclotomic Birman–Murakami–Wenzl algebras. The special case $k = 2$ has been called restricted B-type BMW algebra and has been studied in [9]. We call k the height of the algebra and consider as well the case $k = \infty$ where Y does not satisfy a polynomial identity.

The current interest in the study of B type braid groups has several origins. Closing B type braids yields links that can be interpreted as links in a solid torus [11] and Markov traces on group algebras of ZB_n hence allow the calculations of invariants of such links (cf. end of Section 11). Markov traces on Hecke algebras of B-type have been studied by Lambropoulou [11] and lead to invariants of links in the solid torus as well. Note however, that the methods used in the Hecke and BMW cases as well as the resulting invariants are completely different in nature. Braid groups of all finite root systems further act as symmetries on the corresponding quantum groups [13]. The B braid group occurs furthermore in several physical situations [6,8]. The general idea is that the B type braids allow to treat with knot theoretic methods also physical models with a boundary. The τ_0 generator is interpreted as a reflection at the boundary.

We now outline the structure of the paper and point out the main results. After a short review of the Birman–Wenzl algebra of A-type we go on to define the cyclotomic Birman–Murakami–Wenzl algebra $\mathscr{B}_n^k$ in Section 2 and list a number of fundamental relations. Section 6 shows how to obtain the Ariki–Koike algebra as a quotient of $\mathscr{B}_n^k$. Furthermore, it investigates the B type Temperley-Lieb sub-algebra.

The knot theoretic point of view is elaborated in Section 3 where graphical counterparts of the algebras are introduced. This is not only an application but serves as a tool in proving that certain left ideals of the algebras of infinite height over suitable ground rings are free modules. This is essential for our construction of a Markov trace in Section 10. In the case of finite height algebras one has to work harder to see that the relevant left ideal is free (and hence that a Markov trace exists). The problem is solved in Section 7 algebraically by module construction and introduces a set of relations between the parameters. The knot theoretic origin of the difficulty is that we are dealing with un-oriented links. In the oriented setting of type B Hecke algebras no such relations occur [12].

Section 8 determines a partial normal form of words in $\mathscr{B}_n^k$ which shows that the algebras of finite height are finite dimensional. The classical limit is studied in Section 9.

The main theorem of this paper is contained in Section 11. We prove that $\mathscr{B}_n^k$ is semi-simple in the generic case and show how its simple components can be enumerated in terms of the Young diagrams. The Bratteli diagram is given and we show that the Markov trace is faithful. As an application a generalization of the Kauffman polynomial to links in the solid torus is defined.

The material of this paper is part of the author's Ph.D. thesis [10] and profited much from discussions with Professor tom Dieck.

2. Definition of the cyclotomic BMW Algebra

This section introduces a generalization of the Birman–Murakami–Wenzl that is related to the B-type braid group. Because the cyclotomic Hecke algebras of Ariki and Koike appear as quotients we call our algebras cyclotomic BMW algebras. We set off by recalling the definition of the ordinary BMW algebra.

Definition 1. Let R denote an integral domain with units $x, q, \lambda \in R$ such that with $\delta := q - q^{-1}$ the relation $(1-x)\delta = \lambda - \lambda^{-1}$ holds. The Birman–Murakami–Wenzl (BMW) algebra $\mathrm{BA}_n(R)$ is generated by $X_1^{\pm}, \ldots, X_{n-1}^{\pm}, e_1, \ldots, e_{n-1}$ and relations

$$X_iX_j = X_jX_i, \quad |i-j| > 1, \tag{3}$$

$$X_iX_jX_i = X_jX_iX_j, \quad |i-j| = 1, \tag{4}$$

$$X_ie_i = e_iX_i = \lambda e_i, \tag{5}$$

$$e_iX_j^{\pm 1}e_i = \lambda^{\mp 1}e_i, \quad |i-j| = 1, \tag{6}$$

$$e_i^2 = xe_i, \tag{7}$$

$$X_i^{-1} = X_i - \delta + \delta e_i, \tag{8}$$

$$e_ie_j = e_je_i, \quad |i-j| > 1, \tag{9}$$

$$e_iX_jX_i = X_j^{\pm}X_i^{\pm}e_j, \quad |i-j| = 1, \tag{10}$$

$$e_ie_je_i = e_i, \quad |i-j| = 1. \tag{11}$$

The next two lemmas recall results from the theory of Birman–Murakami–Wenzl algebras. The proofs are easy and standard.

Lemma 2.

$$X_i^2 = 1 + \delta X_i - \delta\lambda e_i, \tag{12}$$

$$X_i^{-1}X_j^{\pm 1}X_i = X_jX_i^{\pm 1}X_j^{-1}, \quad |i-j| = 1, \tag{13}$$

$$X_i^{\pm 1}e_je_i = X_j^{\mp 1}e_i, \quad |i-j| = 1, \tag{14}$$

$$e_ie_jX_i^{\pm 1}=e_iX_j^{\mp 1}, \quad |i-j|=1, \tag{15}$$

$$e_iX_j^{\pm}X_i^{\pm}=e_ie_j, \quad |i-j|=1, \tag{16}$$

$$X_i^{\pm}X_j^{\pm}e_i=e_je_i, \quad |i-j|=1, \tag{17}$$

$$X_ie_jX_i^{-1}=X_j^{-1}e_iX_j, \quad |i-j|=1, \tag{18}$$

$$X_ie_jX_i=X_j^{-1}e_iX_j^{-1}, \quad |i-j|=1. \tag{19}$$

Lemma 3. *If δ is a unit in R the algebra $\mathrm{BA}_n(R)$ is isomorphic to the algebra generated by invertible $X_1,\dots,X_{n-1}$ and relations* (3)–(6). *The element e_i is now defined by*

$$e_i:=1-\frac{X_i-X_i^{-1}}{\delta}, \quad i=1,\dots,n-1. \tag{20}$$

Now, we define our generalized algebra.

Definition 4. Let $q,x,\lambda \in R$ be units and let $A_1,A_2,\dots \in R$ be some further elements. Assume that the relation $(1-x)(q-q^{-1})=\lambda-\lambda^{-1}$ holds. The cyclotomic BWM-Algebra on n strings of infinite height $\mathscr{B}_n^{\infty}(R)$ is defined as R algebra generated by $Y,X_1,\dots,X_{n-1},e_1,\dots,e_{n-1}$ and the relations of the Birman–Murakami–Wenzl-Algebra BA_n and

$$X_1YX_1Y=YX_1YX_1, \tag{21}$$

$$YX_i=X_iY, \quad i>1, \tag{22}$$

$$YX_1Ye_1=\lambda^{-1}e_1=e_1YX_1Y, \tag{23}$$

$$e_1Y^ie_1=A_ie_1, \quad i\geq 1. \tag{24}$$

The cyclotomic BWM-Algebra on n strings of height $k\in\mathbb{N}$ is denoted by $\mathscr{B}_n^k(R)$. It is the quotient of the height ∞ algebra by the relation

$$0=\prod_{i=0}^{k-1}(Y-p_i). \tag{25}$$

Here $p_i\in R^*$, $i=0,\dots,k-1$ are further invertible parameters.

Relation (24) suggests to define $A_0:=x$.

The generic ground ring for our algebra is a quotient of a Laurent polynomial ring. We denote by $R[x]$ the polynomial ring and by $R\{x\}$ the Laurent ring in x over R. The generic ground ring for $\mathscr{B}_n^{\infty}$ is

$$R_0^{\infty}:=\mathbb{C}[A_1,A_2,\dots]\{q,x,\lambda\}/(x\delta-\delta-\lambda^{-1}+\lambda). \tag{26}$$

Here we used, as above, $\delta = q - q^{-1}$. The following definition will be used for the algebras of finite height:

$$R^{\infty}_{0,k} := R^{\infty}_0 \bigotimes\nolimits_{\mathbb{C}} \mathbb{C}[p_0^{\pm}, \ldots, p_{k-1}^{\pm}], \quad k \in \mathbb{N}. \tag{27}$$

Much of the following analysis can be done as well if $\mathbb{C}$ in these definitions is replaced by the integers. However, in connection with the classical limit we have to require that some equations have solutions. Note that R^{∞}_0 is an integral domain since the generator of the principal ideal which is divided out is an irreducible polynomial.

3. Graphical interpretation

The very definition of $\mathscr{B}^k_n$ is motivated by knot theory as was vaguely explained in the introduction. Here, we fill in the details.

Consider the set of ambient isotopy classes of tame embeddings of unoriented ribbons into the cylinder $(\mathbb{R}^2 - \{0\}) \times [0,1]$ between n upper and m lower intervals embedded in the half-rays $\mathbb{R}^+ \times 0 \times \{0,1\}$. Closed bands are allowed as well. For any commutative ring R let $C_{n,m} = C_{n,m}(R)$ denote the free R module generated by these isotopy classes. Stacking graphs defines a multiplication $C_{n,m} \times C_{l,n} \to C_{l,m}$ that turns $C_{n,m}$ into the morphism space of the category of unoriented ribbon tangles in the cylinder.

Now, let q, λ, A_0 denote units in R and $A_i, i \in \mathbb{N}$ further elements. The submodule $S^{\infty}_{n,m} \subset C_{n,m}(R)$ shall be spanned by all elements that are linear combinations of classes that have representations that differ only locally in the way shown in Fig. 3. Thus $S^{\infty}_{n,m}$ is the module of skein relations of the Kauffman polynomial enriched by the rule for eliminating certain closed bands. We define

$$K^{\infty}_{n,m}(R) := C_{n,m}(R)/S^{\infty}_{n,m}. \tag{28}$$

For $s \in S^{\infty}_{n,m}$ and $a \in C_{l,n}, b \in C_{m,j}$ one has $bsa \in S^{\infty}_{l,j}$ and hence the above given multiplication carries over and defines a skein category.

Elementary tangles are defined as in Fig. 1. The total number of strings is added as a second subscript: $X^{(G)}_{i,n}$, $e^{(G)}_{i,n}$, $Y^{(G)}_{i,n}$.

Definition 5. Let the graphical cyclotomic BMW algebra of infinite height $G\mathscr{B}^{\infty}_n(R)$ be the sub-algebra of $K^{\infty}_{n,n}(R)$ that is generated by $X^{(G)}_{i,n}, e^{(G)}_{i,n}, Y^{(G)}_{1,n}, 1 \le i \le n-1$.

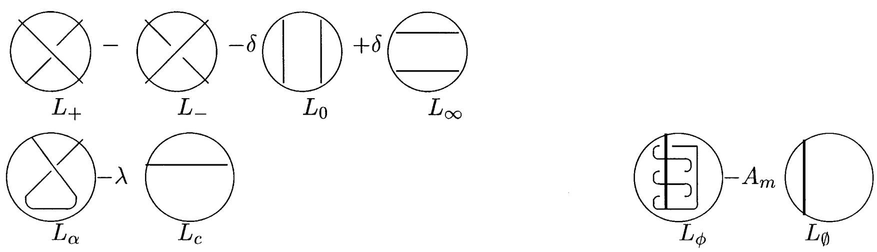

Fig. 3. Skein relations of $K^{\infty}_{n,m}(R)$.

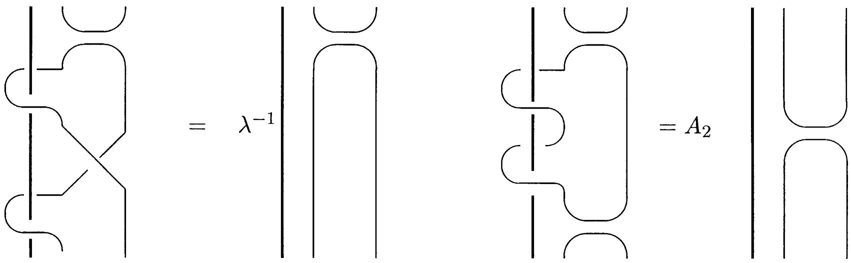

Fig. 4. Relation (23) and relation (24) (on the right).

Similarly, the cyclotomic BWM-Algebra of finite height $G\mathscr{B}_n^k(R)$ is the quotient by

$$0 = \prod_{i=0}^{k-1}(Y^{(G)} - p_i).$$

Lemma 6. *$\Psi_n : \mathscr{B}_n^k(R) \to G\mathscr{B}_n^k(R), X_i \mapsto X_{i,n}^{(G)}, e_i \mapsto e_{i,n}^{(G)}, Y \mapsto Y_{1,n}^{(G)}$ defines an algebra epimorphism. Here k may be finite of infinite.*

Surjectivity is clear from the definitions. It remains to understand the graphical meaning of the algebraic relations. Eq. (21) is the four-braid relation (1) illustrated in Fig. 2. (22) comes from the braid group as well. Relation (23) is illustrated in Fig. 4. $Y^{(G)}$ has to be interpreted as a band that is oriented always towards the cylinder axis.

4. The algebras of height ∞

This section studies some relations in the algebras of infinite height and discusses the relation between the graphical and algebraic versions in this case.

The following definitions will prove useful later on. Their graphical counterpart is given in Fig. 5.

$$Y_i := X_{i-1}X_{i-2}\cdots X_1 Y X_1^{-1}\cdots X_{i-2}^{-1}X_{i-1}^{-1}, \tag{29}$$

$$Y_i' := X_{i-1}X_{i-2}\cdots X_1 Y X_1\cdots X_{i-2}X_{i-1}. \tag{30}$$

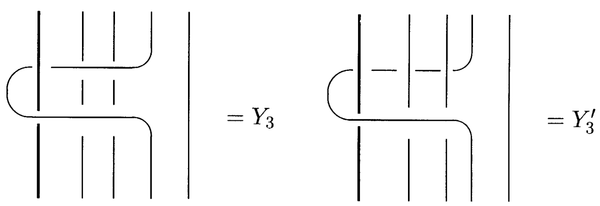

Fig. 5. Interpretation of Y_i and Y_i'.

The next lemma collects a stock of relations that show among other things that the most important properties of Y can be shifted to other strings.

Lemma 7. *The following relations hold in $\mathscr{B}_n^k$ for finite or infinite k*:

$$0 = [X_1 Y X_1 Y, \{Y, e_1, X_1\}], \tag{31}$$

$$0 = [Y_i, X_j] = [Y_i, e_j], \quad j \neq i, i-1, \tag{32}$$

$$0 = [Y_i', X_j] = [Y_i', e_j], \quad j \neq i, i-1, \tag{33}$$

$$Y_{i+1}' X_i^{-1} = X_i Y_i', \tag{34}$$

$$Y_{i+1} X_i = X_i Y_i, \tag{35}$$

$$X_i Y_i X_i Y_i = Y_i X_i Y_i X_i, \tag{36}$$

$$X_i Y_i' X_i Y_i' = Y_i' X_i Y_i' X_i, \tag{37}$$

$$\lambda^{-1} e_i = e_i Y_i X_i Y_i = Y_i X_i Y_i e_i, \tag{38}$$

$$\lambda^{-1} e_i = e_i Y_i' X_i Y_i' = Y_i' X_i Y_i' e_i, \tag{39}$$

$$e_i Y_i^m e_i = A_m e_i, \tag{40}$$

$$Y_i' Y_j' = Y_j' Y_i', \tag{41}$$

$$Y_i Y_{i-1}^{-1} = Y_{i-1}^{-1} X_{i-1}^{-1} Y_{i-1} X_{i-1}, \tag{42}$$

$$Y_i e_{i-1} = \lambda^{-2} Y_{i-1}^{-1} e_{i-1}, \tag{43}$$

$$e_{i-1} Y_i = e_{i-1} Y_{i-1}^{-1} - \lambda\delta e_{i-1} Y_{i-1} + \delta\lambda A_1 e_1, \tag{44}$$

$$e_{i-1} Y_i' = e_{i-1} Y_{i-1}'^{-1}, \tag{45}$$

$$Y_i' e_{i-1} = Y_{i-1}'^{-1} e_{i-1}, \tag{46}$$

$$X_i Y_{i+1} = X_i Y_i - \delta Y_i + \delta Y_i e_i + \delta Y_{i+1} - \delta e_i Y_i^{-1} + \delta^2 \lambda e_i Y_i - \delta^2 \lambda A_1 e_i, \tag{47}$$

$$Y_{i+1}^l X_i = X_i Y_i^l, \tag{48}$$

$$e_i Y_i^l X_i = e_i Y_i^{l-1} X_i Y_i^{-1} - \delta e_i Y_i^{l-2} + \delta A_{l-1} e_i Y_i^{-1}, \tag{49}$$

$$X_i Y_i^l e_i = Y_i^{-1} X_i Y_i^{l-1} e_i - \delta Y_i^{l-2} e_i + \delta A_{l-1} Y_i^{-1} e_i, \tag{50}$$

$$X_1 Y^m e_1 = \lambda^{-1} Y^{-m} e_1 + \sum_{s=1}^{m-1} \delta(A_{m-s} Y^{-s} - Y^{m-2s}) e_1, \tag{51}$$

$$X_1 Y^{-m} e_1 = \lambda Y^m e_1 + \sum_{s=0}^{m-1} \delta(Y^{-m+2s} - A_{s-m} Y^s) e_1. \tag{52}$$

These relations are shown by induction and similar proofs have already been published in [9] so that we restrict here to an example that gives the taste: In the induction step for (38) Eq. (10) is used to eliminate e_{i+1} in terms of e_i.

$$\begin{aligned}Y'_{i+1}X_{i+1}Y'_{i+1}e_{i+1} &= X_iY'_iX_iX_{i+1}X_iY'_iX_iX_i^{-1}X_{i+1}^{-1}e_iX_{i+1}X_i\\ &= X_iY'_iX_{i+1}X_iX_{i+1}Y'_iX_{i+1}^{-1}e_iX_{i+1}X_i\\ &= X_iX_{i+1}Y'_iX_iX_{i+1}X_{i+1}^{-1}Y'_ie_iX_{i+1}X_i\\ &= X_iX_{i+1}Y'_iX_iY'_ie_iX_{i+1}X_i = \lambda^{-1}X_iX_{i+1}e_iX_{i+1}X_i = \lambda^{-1}e_{i+1}.\end{aligned}$$

The last two relations allow to define A_m for negative m if the annihilator ideal of e_1 vanishes. These coefficient shall satisfy $e_1Y^{-m}e_1 = A_{-m}e_1$. Multiplying (52) by e_1 from the left yields

$$\lambda A_{-m} = \lambda A_m + \sum_{s=0}^{m-1} \delta(A_{2s-m} - A_{s-m}A_s)$$

and hence (note the change in the summation range)

$$A_{-m} = \lambda^2 A_m + \lambda \sum_{s=1}^{m-1} \delta(A_{2s-m} - A_{s-m}A_s). \tag{53}$$

This allows a recursive calculation of A_{-m}.

Remark 8. There is an anti-involution of the $\mathbb{C}$-*algebra* $\mathscr{B}_n^\alpha(R_0^\infty)$ such that

$$X_i^* = X_i^{-1}, \quad e_i^* = e_i, \quad Y^* = Y^{-1}, \quad q^* = q^{-1}, \quad \lambda^* = \lambda^{-1}, \quad A_i^* = A_{-i}. \tag{54}$$

Remark 9. $X_i^\dagger := X_{n-i}, Y^\dagger := Y_n$ defines an involution $\dagger$ on $\mathscr{B}_n^k$.

Proof. All relations that depend only on one index or on the absolute difference of two indices are obviously compatible. We check (24)

$$\begin{aligned}(e_1Y^ie_1 - A_ie_1)^\dagger &= e_{n-1}Y_n^ie_{n-1} - A_ie_{n-1}\\ &= e_{n-1}X_{n-1}Y_{n-1}^iX_{n-1}^{-1}e_{n-1} - A_ie_{n-1} = 0.\end{aligned}$$

Relation (23) is preserved as well:

$$\begin{aligned}(YX_1Ye_1 - \lambda^{-1}e_1)^\dagger &= e_{n-1}Y_nX_{n-1}Y_n - \lambda^{-1}e_{n-1}\\ &= e_{n-1}X_{n-1}Y_{n-1}X_{n-1}^{-1}X_{n-1}X_{n-1}Y_{n-1}X_{n-1}^{-1} - \lambda^{-1}e_{n-1}\\ &= \lambda e_{n-1}Y_{n-1}X_{n-1}Y_{n-1}X_{n-1}^{-1} - \lambda^{-1}e_{n-1}\\ &= \lambda\lambda^{-1}e_{n-1}X_{n-1}^{-1} - \lambda^{-1}e_{n-1} = 0. \qquad \square\end{aligned}$$

Remark 10. The relations show that there is a further anti-involution $a \mapsto \bar{a}$ which fixes all generators.

Later on we will show that there exists a Markov trace if the annihilator ideal of $E_n := e_1e_3\cdots e_{2n-1}$ vanishes. For the algebra of infinite height the generic ground ring has this important property:

Proposition 11. $\operatorname{ann}(E_n) = \{0\}$ *in* $\mathscr{B}_{2n}^{\infty}(R_0^{\infty})$.

Proof. Assume $0 \neq \mu \in R_0^{\infty}$ to be an element of the annihilator ideal $\mu E_n = 0$, that is $0 = \mu \Psi_{2n}(E_n)$. We define two morphisms in the category of isotopy skein classes of cylinder ribbon tangles: $U_n \in K_{2n,0}^{\infty}(R_0^{\infty})$ is the class that consists of n minima and $U_n' \in K_{0,2n}^{\infty}(R_0^{\infty})$ contains n maxima. Multiplying from both sides with these elements one obtains

$$0 = \mu U_n \Psi_{2n}(E_n) U_n' = \mu A_0^{2n}[\emptyset].$$

Here $[\emptyset]$ is the class of the empty knot and we have derived that it has non-vanishing annihilator ideal. But this contradicts with Turaev's theorem [14] on the freeness of the Kauffman skein module of the solid torus which states (in our notation) that

$$K_{0,0}^{\infty}(R_0^{\infty}) \cong R_0^{\infty}. \qquad \square \tag{55}$$

Define the following submodules of the two string algebras:

$$N^{\infty} = N^{\infty}(R) = \operatorname{span}_R(Y^i e_1 \mid i \in \mathbb{Z}) \subset \mathscr{B}_2^{\infty}(R), \tag{56}$$

$$GN^{\infty} = GN^{\infty}(R) = \operatorname{span}_R((Y^{(G)})^i e_1 \mid i \in \mathbb{Z}) \subset G\mathscr{B}_2^{\infty}(R). \tag{57}$$

Proposition 12. N^{∞} (*resp.* GN^{∞}) *is the left ideal generated by* e_1 (*resp.* $e_1^{(G)}$). *For any ground ring that is a quotient of* R_0^{∞} *the above given sets are bases. Moreover, the sets* $\{Y^i e_1 Y^j \mid i,j \in \mathbb{Z}\}$ *and* $\{(Y^{(G)})^i e_1^{(G)} (Y^{(G)})^j \mid i,j \in \mathbb{Z}\}$ *are bases of the two-sided ideals generated by* e_1, *respectively* $e_1^{(G)}$.

For any ground ring we have if the determinant of (A_{i+j}), $i,j = 0,\ldots,m$ *is not a zero divisor, then* $e_1, Ye_1, \ldots, Y^m e_1$ (*resp.* $e_1^{(G)}, Y^{(G)} e_1, \ldots, (Y^{(G)})^m e_1$) *are linearly independent.*

Proof. The first claim follows from some of the relations derived above. To prove the second claim we first consider $R = R_0^{\infty}$. Assume there is a linear relation of the $(Y^{(G)})^i e_1^{(G)}$. Since $Y^{(G)}$ is invertible, we can assume that $i = 0$ is the lowest term with non-vanishing coefficient, thus

$$0 = \sum_{i=0}^{m} \alpha_i (Y^{(G)})^i e_1^{(G)}, \quad \alpha_i \in R_0^{\infty}.$$

Multiplying with $U_1 (Y^{(G)})^j$ and U_1' (these elements are defined in the proof of Proposition 11) yields

$$0 = \sum_i \alpha_i A_{i+j} A_0 [\emptyset]$$

and thus, using (55) and invertibility of A_0, $0 = \sum_i \alpha_i A_{i+j}$. The determinant of the matrix A_{i+j}, $i,j = 0,\ldots,m$ does not vanish and hence all α_i have to be zero. By standard ring change arguments we have the same basis for any quotient ring.

We show similarly that the sets $\{Y^i e_1 Y^j \mid i,j \in \mathbb{Z}\}$ and $\{(Y^{(G)})^i e_1^{(G)} (Y^{(G)})^j \mid i,j \in \mathbb{Z}\}$ are linearly independent. Recalling that Ψ_2 is an epimorphism it suffices to consider the graphical situation. Multiply a supposed linear dependency $0 = \sum_{i,j\geq 0} \alpha_{i,j}(Y^{(G)})^i e_1^{(G)} (Y^{(G)})^j$ from the left with $U_1(Y^{(G)})^s$ and from the right with $(Y^{(G)})^t U_1'$ to obtain

$$0 = \sum_{i,j\geq 0} \alpha_{i,j} A_{s+i} A_{j+t} A_0[\emptyset].$$

Written as matrices, we have $0 = A\alpha A$ and hence, by the above argumentation, $\alpha = 0$.

To prove the last claim one can argue along the same lines. Since the determinant is assumed to be not a zero divisor it may be made invertible in a suitable localization. □

Proposition 13. *Over $R_0^\infty, R_{0,k}^\infty$ or quotients thereof $G\mathscr{B}_2^\infty$ and $\mathscr{B}_2^\infty$ are isomorphic.*

Proof. We know from Proposition 12 that the two-sided ideals I_2 and $I_2^{(G)}$ generated by e_1 and $e_1^{(G)}$ are isomorphic. Now, consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_2 & \longrightarrow & \mathscr{B}_2^\infty & \longrightarrow & \mathscr{B}_2^\infty/I_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \Psi_2|_{I_2}} & & \downarrow{\scriptstyle \Psi_2} & & \downarrow{\scriptstyle \widehat{\Psi_2}} & & \\ 0 & \longrightarrow & I_2^{(G)} & \longrightarrow & G\mathscr{B}_2^\infty & \longrightarrow & G\mathscr{B}_2^\infty/I_2^{(G)} & \longrightarrow & 0 \end{array}$$

Here $\widehat{\Psi_2}$ denotes the induced mapping. By the five lemma it suffices to show that this map is an isomorphism. The quotient $\mathscr{B}_2^\infty/I_2$ is isomorphic to the generalized B-type Hecke algebra $\mathrm{H}^\infty\mathrm{B}_2$. In [4] tom Dieck has shown that this algebra is isomorphic to the algebra of braids in the cylinder modulo Hecke-type skein relations. On the other hand, by the definition of the graphical algebra, a tangle class from $G\mathscr{B}^k i_2/I_2^{(G)}$ vanishes iff it has horizontal parts. Hence, the quotient consists of the braids in the cylinder. For these tangles, the skein relations reduce to Hecke-type relations. Therefore, both algebras must be isomorphic. □

5. The algebras of finite height

This section is concerned with the algebras of finite height $k < \infty$. Its main purpose is to introduce expansion coefficients.

Definition 14. Let $q_{k-1}, \ldots, q_0$ be the signed elementary symmetric polynomials in $p_0, \ldots, p_{k-1}$ such that

$$Y^k = \sum_{i=0}^{k-1} q_i Y^i. \tag{58}$$

Note that $q_0 = (-1)^{k-1}\prod_i p_i$ is invertible. We calculate Y^{-1},

$$Y^{-1} = \sum_{i=0}^{k-1} \bar{q}_i Y^i \quad \text{with } \bar{q}_{k-1} = q_0^{-1},\ \bar{q}_{i-1} = -q_i q_0^{-1}. \tag{59}$$

The coefficients are determined uniquely if the Y^i are linearly independent.

By iteration one obtains elements $\bar{Q}_{i,j}$ that satisfy

$$Y^{-i} = \sum_{j=0}^{k-1} \bar{Q}_{i,j} Y^j. \tag{60}$$

We also introduce the following expansion coefficients:

$$Y^{k+m} = \sum_{i=0}^{k-1} \widehat{q}_{m,i} Y^i. \tag{61}$$

A simple calculations shows

$$\widehat{q}_{m,k-1} = \sum_{s=1}^{m+1} \sum_{i_1,\ldots,i_s}{}' q_{k-i_1} \cdots q_{k-i_s} \quad \text{with } \sum_{v} i_v = m+1, i_j \geq 1. \tag{62}$$

The prime indicates that the sum is restricted by the condition on the right. Multiplying (61) with Y^{2-k} and substituting $m \mapsto m-2$ yields

$$Y^m = \sum_{j=-(k-2)}^{1} \widehat{q}_{m-2,j+k-2} Y^j =: \sum_{j=-(k-2)}^{1} \widetilde{q}_{m,j} Y^j. \tag{63}$$

In terms of these coefficients the above relation reads

$$\widetilde{q}_{m+1,1} = \sum_{s=1}^{m} \sum_{i_1,\ldots,i_s}{}' q_{k-i_1} \cdots q_{k-i_s} \quad \text{with } \sum_{v} i_v = m, i_j \geq 1. \tag{64}$$

Assuming vanishing annihilator ideal of e_1 we obtain relations for $A_m, m \geq k$

$$A_m e_1 = e_1 Y^m e_1 = e_1 Y^k Y^{m-k} e_1 = \sum_{i=0}^{k-1} q_i A_{m-k+i} e - 1.$$

Thus, we demand

$$A_m = \sum_{i=0}^{k-1} q_i A_{m-k+i} \quad \forall m \geq k \tag{65}$$

and define as a first step towards a generic ground ring for $\mathscr{B}_n^k$

$$R_0^k := (R_0^\infty \otimes_{\mathbb{C}} \mathbb{C}[p_0^\pm, \ldots, p_{k-1}^\pm])/(65). \tag{66}$$

Remark 15. There is an anti-involution of $\mathscr{B}_n^k(R_0^k)$, as a $\mathbb{C}$ algebra such that

$$\begin{aligned} &X_i^* = X_i^{-1}, \qquad e_i^* = e_i, \qquad Y^* = Y^{-1}, \qquad q^* = q^{-1}, \qquad \lambda^* = \lambda^{-1}, \\ &p_i^* = p_i^{-1}, \qquad A_i^* = A_{-i}. \end{aligned} \tag{67}$$

The proof is trivial.

6. Relations to other Knot algebras

The e_i together with a projector e_0 on the p_0 eigenvalue of Y generate a sub-algebra that is a homomorphic image of a type-B-Temperley–Lieb algebra. The quotient by the ideal generated by e_1 is isomorphic to the Ariki–Koike algebra. For specific parameter values one may also obtain the A-type BMW algebra as a quotient.

Lemma 16. *Let J_n be the ideal generated by $Y_n - p_0$. Every other $Y_i - p_0,\ i = 1, \ldots, n$ generates the same ideal and the quotient $R/(\lambda^{-1} - \lambda p_0^2,\ x p_0^i - A_i) \otimes_R \mathscr{B}_n^k(R)/J_n$ is isomorphic to the A-type BMW algebra* $\mathrm{BA}_n(R/(\lambda^{-1} - \lambda p_0^2, x p_0^i - A_i))$.

Proof. The first claim is a consequence of the definition of the Y_i. The specialization of the ground ring is necessary since in the quotient one obtains $0 = e_1 Y X_1 Y - \lambda^{-1} e_1 = e_1 p_0 X_1 p_0 - \lambda^{-1} e_1 = e_1(p_0^2 \lambda - \lambda^{-1})$ and $A_i e_1 = e_1 Y^i e_1 = p_0^i e_1$. The remaining relations present no further restrictions. □

Definition 17. I_n denotes the ideal generated by e_{n-1} in $\mathscr{B}_n^k$.

As we shall see, the quotient by this ideal is an Ariki–Koike algebra.

Definition 18. AK_n^k denotes the Ariki–Koike algebra [1] with generators $X_0, X_1, \ldots, X_{n-1}$ and parameters $\delta, p_i,\ i = 0, \ldots, k-1$ and relations

$$
\begin{aligned}
&X_0 X_1 X_0 X_1 = X_1 X_0 X_1 X_0,\\
&X_i X_j = X_j X_i, \quad |i - j| > 1,\\
&X_i X_j X_i = X_j X_i X_j, \quad |i - j| = 1,\\
&X_i^2 = \delta X_i + 1, \quad i \geq 0,\\
&0 = \prod_{i=0}^{k-1} (X_0 - p_i).
\end{aligned}
$$

We use a slightly different normalization of the parameters than Ariki and Koike did. From their work we need later on the result that AK_n^k for the generic ground ring is semi-simple. The proof of the following lemma is now trivial.

Lemma 19. *I_n is generated by any of the e_i and the quotient by it is isomorphic to AK_n^k.*

Of some interest in knot theoretical applications is the projector on the eigenvalue p_0 of Y. Such a projector is given by $\prod_{i=1}^{k-1}(Y - p_i)$. It fulfills the modified B-Temperley–Lieb algebra ([2], [8]) TB_n' with relations $e_0^2 = c e_0,\ e_i^2 = d e_i,\ e_j e_l = e_l e_j,\ e_i e_j e_i = e_i,\ e_1 e_0 e_1 = c' e_1,\ 1 \leq i,\ j \leq n-1,\ 0 \leq l \leq n-1,\ |i - j| = 1,\ |j - l| > 1$. Obviously, we have a morphism $\mathrm{TB}_n' \to \mathscr{B}_n^k$.

7. Ground rings for finite height

The algebra $\mathscr{B}_n^k(R)$ is in general not semi-simple. For the proof of semi-simplicity over suitable ground rings we need a Markov trace and this demand in turn requires that the annihilator ideal of a certain element vanishes. This section studies this condition. We begin with the two string case $n = 2$ and simplify our notation by writing $B(R) := \mathscr{B}_2^k(R)$ and omitting the index 1 of e_1 and X_1.

The parameters of the algebra cannot be chosen independently. Note, for example, that both $e = \lambda^{-1}eYXY$ and $Y^k = \sum_i q_i Y^i$ fix the length of Y.

Definition 20. Define the ideal $c \subset R_0^k$ to be generated by k Laurent polynomials that are obtained by the following procedure: Expand $Ye - \lambda^{-1}X_1^{-1}Y^{-1}e$ using (24), (8), (58) and (50) into a linear combination $\sum_{i=0}^{k-1} h_i Y^i e$. The coefficient h_i of this sum are the generators of $c = (h_0, \ldots, h_{k-1})$.

Define a ring R_1^k as a quotient $R_1^k := R_0^k/c$ of R_0^k.

To shed some light on the ideal c we note that (51) allows to write the defining relations in the form

$$\begin{aligned} &Ye + \delta\lambda^{-1}Y^{-1}e - \lambda^{-1}\delta \sum_m \bar{q}_m A_m e_1 \\ &= \lambda^{-1} \sum_m \bar{q}_m \left(\lambda^{-1}Y^{-m}e + \sum_{s=1}^{m-1} \delta(A_{m-s}Y^{-s}e - Y^{m-2s}e) \right). \end{aligned} \tag{68}$$

We now introduce a ring that will become relevant later on as the ring of the classical limit of the algebra. At this stage we need it purely as a tool.

Definition 21. The ideal $J_c^k \subset R_1^k$ is given by $J_c^k := (\lambda - 1, q - 1, q_0 - 1, q_1, \ldots, q_{k-1})$. Set $R_c^k := R_1^k/J_c^k$.

According to results of the theory of symmetric polynomials the equations for the q_i are solvable. Hence the ring R_c^k is non-trivial.

The same polynomials $(\lambda - 1, q - 1, q_0 - 1, q_1, \ldots, q_{k-1})$ define an ideal in R_0^k. It contains c since after dividing by J_c^k we have $Y^{-1} = Y^{k-1}$, $\bar{q}_{k-1} = 1$, $\bar{q}_i = 0$ and hence (68) becomes trivial. It follows that R_c^k is the quotient of R_0^k by J_c^k.

Now, we are in a position to single out ground rings with good properties.

Definition 22. A ring R is called admissible, if
1. R is an integral domain.
2. R has R_c^k as a quotient.
3. R is a quotient of R_1^k.
4. The Ariki–Koike algebra is semi-simple over R.

The ring is called potentially admissible if it has the second and third properties.

The first point is motivated by the intended application of Jones–Wenzl theory which needs passing to the field of quotients. The second allows to use the classical limit as a tool. The third point plays a role in the construction of certain modules as we shall see now.

Lemma 23. *If R is potentially admissible and $k<\infty$, then the determinant of $(A_{i+j})_{i,j}, 0 \leq i,j < k$ does not vanish.*

Proof. First map $d := \det(A_{i+j})$ to R^k_c. Then we have $A_i = A_{i+k}$. In a second step map the image of d into $R^k_c/(A_1 = 0, \ldots, A_{k-1} = 0)$. The result is $(-1)^{k-1}A_0^k$. As this is non-zero, d itself cannot be zero. □

Using (24), (58) and (50) we see that the ideal I_2 is spanned $Y^i e Y^j$, $i,j = 0,1,\ldots,$ $k-1$.

Definition 24. Let R be as in the definition of $\mathscr{B}^k_n$. Let $V := V(R)$ be the free R-module of dimension k. The basis is denoted by $b_i, 0 \leq i < k$. V is turned into a module of the free algebra generated by e, X, Y by the following definitions:

$$\begin{aligned}
&e.b_i := A_i b_0 \quad \text{with } A_0 = x,\\
&Y.b_{k-1} := \sum_{j=0}^{k-1} q_j b_j, \quad Y.b_i := b_{i+1},\\
&X.b_0 := \lambda b_0, \quad X.b_1 := \lambda^{-1} Y^{-1}.b_0,\\
&X.b_i := (Y^{-1}.X.b_{i-1} - \delta b_{i-2} + \delta A_{i-1} Y^{-1}.b_0), \quad i \geq 2.
\end{aligned}$$

The definition of this action is guided by the desire that it should factor over $B(R)$. Y^{-1} and X^{-1} shall act by their expansions in terms of Y^i (implying $(Y^{-1}).{=}(Y.)^{-1}$), resp. $X, e, 1$. It turns out, however, that V is not in general a B-module. Most relations are easy to check but two of them may not hold: (a) $XYXY = YXYX$ and (b) $X^2 = 1 + \delta X - \delta\lambda e$. Relation (b) is equivalent to $(X^{-1}). = (X.)^{-1}$.

Lemma 25. *For any potentially admissible ring R the module $V(R)$ is a $B(R)$-module.*

Proof. By the very definition of a potentially admissible ring we have

$$b_1 = Y.b_0 = \lambda^{-1} X^{-1}.Y^{-1}.b_0. \tag{69}$$

On b_0 relation (b) holds trivially. We check (a):

$$\begin{aligned}
X.Y.X.Y.b_0 &= X.Y.X.b_1 = \lambda^{-1} X.Y.Y^{-1}.b_0 = \lambda^{-1} X.b_0 = b_0\\
&= Y.Y^{-1}.b_0 = \lambda Y.X.b_1 = Y.X.Y.X.b_0.
\end{aligned}$$

Furthermore, we check the inverse of (a):

$$X^{-1}.Y^{-1}.X^{-1}.Y^{-1}.b_0 \overset{(69)}{=} \lambda^{-1^{-1}} X^{-1}.Y^{-1}.Y.b_0 = \lambda^{-1^{-1}} X^{-1}.b_0 = \lambda^{-1^{-1}} \lambda^{-1} b_0,$$

$$
\begin{aligned}
Y^{-1}.X^{-1}.Y^{-1}.X^{-1}.b_0 &= \lambda^{-1}Y^{-1}.X^{-1}.Y^{-1}.b_0 \overset{(69)}{=} \lambda^{-1^{-1}}\lambda^{-1}Y^{-1}.Y.b_0 \\
&= \lambda^{-1}\lambda^{-1^{-1}}b_0.
\end{aligned}
$$

Eq. (69) enables us to write for all $i = 0,\ldots,k-1$,

$$X.Y.b_{i-1} = X.b_i = Y^{-1}.X^{-1}.b_{i-1}. \tag{70}$$

Here we used the convention that $b_{-1} = Y^{-1}.b_0$. The case $i = 0$ follows from (69), the case $i = 1$ is trivial, and the cases $i > 1$ are simple rewritings of the action of X.

Now, we can start the inductive proof that (a) and (b) hold on all basis vectors. The induction assumption H_i is relations (a) and (b) hold on b_{i-1}. We show that the inverse of relation (a) holds on b_{i-1},

$$
\begin{aligned}
Y^{-1}.X^{-1}.Y^{-1}.X^{-1}.b_{i-1} &\overset{(70)}{=} \lambda^{-1^{-1}}\lambda^{-1}X.Y.\lambda^{-1^{-1}}\lambda^{-1}X.Y.b_{i-1} \\
&= \lambda^{-1^{-2}}\lambda^{-2}X.Y.X.Y.b_{i-1} \overset{H_i}{=} \lambda^{-1^{-1}}\lambda^{-1}b_{i-1}, \\
X^{-1}.Y^{-1}.X^{-1}.Y^{-1}.b_{i-1} &= X^{-1}.Y^{-1}.X^{-1}.b_{i-2} \overset{(70)}{=} \lambda^{-1^{-1}}\lambda^{-1}X^{-1}.X.b_{i-1} \\
&\overset{H_i}{=} \lambda^{-1^{-1}}\lambda^{-1}b_{i-1}.
\end{aligned}
$$

We now check (b):

$$
\begin{aligned}
X^{-1}.X.b_i &\overset{(70)}{=} X^{-1}.Y^{-1}.X^{-1}.b_{i-1} = Y.Y^{-1}.X^{-1}.Y^{-1}.X^{-1}.b_{i-1} \\
&= Y.X^{-1}.Y^{-1}.X^{-1}.Y^{-1}.b_{i-1} \overset{(70)}{=} Y.X^{-1}.X.b_{i-1} \overset{H_i}{=} b_i.
\end{aligned}
$$

Finally, we look at (a):

$$
\begin{aligned}
Y.X.Y.X.b_i &= Y.X.Y.Y^{-1}.X^{-1}.b_{i-1} = Y.X.X^{-1}.b_{i-1} \overset{H_i}{=} Y.b_{i-1} = b_i, \\
X.Y.X.Y.b_i &\overset{(70)}{=} X.Y.Y^{-1}.X^{-1}.b_i = X.X^{-1}.b_i = b_i. \qquad \square
\end{aligned}
$$

Lemma 26. *Assume R to be potentially admissible and define $U_m := \mathrm{span}_{R_1}\{Y^i e Y^m | i = 0,\ldots,k-1\}$. Each U_m is a $B(R)$-module isomorphic to $V = V(R)$. If R is admissible, then this is a direct sum decomposition and I_2 is thus a free R-module.*

Proof. We show that the map $\varrho : V(R) \to B(R)$, $b_i \mapsto Y^i e$ defines a module isomorphism of V and U_0. It is a surjection of R-modules, and, by the above lemma, a morphism of $B(R)$-modules. It remains to check injectivity. Suppose we had $0 = \sum_i \alpha_i Y^i e, \alpha_i \in R$. Applying this to b_0 we obtain $0 = x\sum_i \alpha_i b_i$. Now, x is invertible, and hence all the α_i have to vanish. Thus we have shown that $\mathrm{span}_{R_1}\{Y^i e_1\}$ is a free R module. The same is true for the isomorphic $B(R_1)$ modules U_m.

We prove the last statement by showing that the ideal $I_2 := \mathrm{span}\{Y^i e_1 Y^j\}$ is R-free with the given basis. Suppose that there were a linear dependency $0 = \sum_{i,j} \alpha_{i,j} Y^i e_1 Y^j$. Multiply this with $Y^r e_1$ from the right to obtain $0 = \sum_{i,j} \alpha_{i,j} Y^i e_1 A_{j+r}$. The $Y^i e_1$ are linearly independent and hence we have for every i, $0 = \sum_j \alpha_{i,j} A_{j+r}$. Sine R is supposed to be an integral domain, this means that the determinant of the matrix A is zero. But this contradicts Lemma 23. $\square$

Proposition 27. *The annihilator ideal of* $E_n := e_1 e_3 \cdots e_{2n-1} \in \mathscr{B}^k_{2n}(R)$ *vanishes if* R *is potentially admissible.*

Proof. We use the map $\Psi_{2n} : \mathscr{B}^k_{2n} \to G\mathscr{B}^k_{2n}$ to pass to the graphical situation. Assume $\mu E_n = 0$ with some $0 \neq \mu \in R$. Then $0 = \mu \Psi_{2n}(E_n)$ as well.

Let $V \in K^k_{2n,2}(R)$ be the image of the tangle that contains $n-1$ minimae between two vertical strings and similarly $V' \in K^k_{2,2n}(R)$ with maximae (cf. Fig. 6). From

$$0 = \mu V \Psi_{2n}(E_n) V' = \mu e_1^{(G)} \in G\mathscr{B}^k_2(R),$$

we conclude that it suffices to show that $e_1^{(G)} \subset G\mathscr{B}^k_2(R)$ has vanishing annihilator ideal.

Denote by J^k the ideal generated by $\prod_{i=0}^{k-1}(Y - p_i)$ in $\mathscr{B}^\infty_2(R)$ and by $GJ^k := \Psi_2(J^k)$ the corresponding ideal in $G\mathscr{B}^\infty_2(R)$.

$$\begin{array}{ccc}
\mathscr{B}^\infty_2(R) & \xrightarrow{\Psi_2} & G\mathscr{B}^\infty_2(R) \\
\downarrow \pi & & \downarrow \pi' \\
\mathscr{B}^\infty_2(R)/J^k & \xrightarrow{\widehat{\Psi_2}} & G\mathscr{B}^\infty_2(R)/GJ^k.
\end{array}$$

In this diagram π, π' are the natural projections and $\widehat{\Psi_2}$ is the induced map. The quotients $\mathscr{B}^\infty_2(R)/J^k$ and $G\mathscr{B}^\infty_2(R)/GJ^k$ are the algebras $\mathscr{B}^k_2(R)$ and $G\mathscr{B}^k_2(R)$.

We know from Proposition 13 that Ψ_2 is an isomorphism. Now, $e_1 \in U_0 \subset \mathscr{B}^\infty_2(R)/J^k$ has vanishing annihilator ideal. Hence, we are done if we can show that $\widehat{\Psi_2}$ is injective. Suppose, we have $a + J^k \in \mathscr{B}^\infty_2(R)/J^k$ such that $0 = \widehat{\Psi_2}(a + J^k) = \pi'(\Psi_2(a)) = \Psi_2(a) + GJ^k$. Then $\Psi_2(a) \in GJ^k$. Since Ψ_2 is bijective this implies $a \in J^k$ and hence $a + J^k \in \mathscr{B}^\infty_2(R)/J^k$ is zero. $\square$

The above proof shows furthermore

Corollary 28. $G\mathscr{B}^k_2(R)$ *and* $\mathscr{B}^k_2(R)$ *are isomorphic if the ground ring is potentially admissible.*

Proposition 29. *Let* R *be admissible. Then* $\mathscr{B}^k_2(R)$ *is* R*-free of dimension* $k^2 + k^2 \cdot 2! = k^2 \cdot (2 \cdot 2 - 1)!! = 3k^2$ *with basis*

$$S_k := \{Y^i e_1 Y^j\} \cup \{Y^i X_1 Y^j X_1\} \cup \{Y^i X_1 Y^j\}, \quad i, j \in \{0, \ldots, k-1\}$$

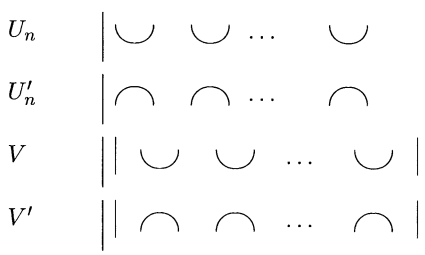

Fig. 6. Some auxiliary morphisms.

Proof. The set $\{Y^i e_1 Y^j\}$ is a basis of I_2 according to Lemma 26. The quotient is the Ariki–Koike algebra AK_2 and the quotient map sends generators to generators. The union of the second and third subset of S_k is mapped to a basis of the Ariki–Koike algebra and hence must be free. Therefore, the following exact sequence of R-modules splits $0 \to I_2 \to \mathscr{B}_2^k(R) \to \mathrm{AK}_2(R) \to 0$. $\square$

Next, we display the relations $h_i = 0$ explicitly for $k \leq 3$. In the case $k = 2$ they read

$$0 = h_0 q_0^2 = q^{-1}\lambda^{-2}(-qq_0q_1 + \lambda(q^2-1)A_1q_0 - qq_1),$$
$$0 = h_1 q_0^2 = \lambda^{-2} + \lambda^{-1}q_0q^{-1} - \lambda^{-1}qq_0 - q_0^2.$$

The second equation factorizes

$$q_0 = \pm\lambda^{-1}q^{\mp 1}.$$

The first equation may then be solved uniquely

$$A_1 = q_1\lambda^{-1}\delta^{-1}(1 \mp \lambda q^{\mp 1}).$$

Hence we set

$$L_2^{\pm} := R_0/(h_0, \pm\lambda^{-1}q^{\mp 1} - q_0). \tag{71}$$

Both rings are integral domains and its easy to check by Ariki's criterion that $\mathrm{AK}_n^2(L_2^{\pm})$ is semi-simple. Hence they are admissible.

In the case $k = 3$ we use the basis $\{Y^{-1}e_1, e_1, Ye_1\}$ to obtain equivalent but somewhat simpler formulas. The relations read

$$0 = h_0 q_0^2 = -(q_0q_1 + \lambda^{-2}q_2) - \lambda^{-1}\delta q_0(1 - A_2 + xq_1 + A_1q_2)),$$
$$0 = h_{-1} q_0^2 = -\lambda^{-1}(\delta q_0(q_0 - A_1) + \lambda^{-1}(q_1 + q_0q_2)),$$
$$0 = h_1 = \lambda^{-2}q_0^{-2} - 1.$$

Again, the last equation factorizes

$$q_0 = \pm\lambda^{-1}.$$

We set

$$L_3^{\pm} := R_0/(h_0, h_{-1}, \pm\lambda^{-1} - q_0). \tag{72}$$

$L_3^{\pm}$ are integral domains. The defining relations can be solved uniquely:

$$A_1 = q_2\lambda^{-1}\delta^{-1} \pm \lambda^{-1} \pm q_1\delta^{-1},$$

$$A_2 = \lambda^{-1}(q^2-1)^{-1}(\ -\lambda q\delta + \lambda q_1 - qq_1 - \lambda q^2 q_1 \pm q_2 \mp \lambda qq_2 \mp q^2q_2 \mp \lambda qq_1q_2 - qq_2^2).$$

$L_3^{\pm}$ are admissible.

We now return to the general case of finite k. Using (51) we can rewrite the defining relation from Definition 20 in the following way:

$$\sum_{i=0}^{k-1} h_i Y^i e = Ye + \delta\lambda^{-1} Y^{-1} e - \lambda^{-1}\delta \sum_m \bar{q}_m A_m e_1$$
$$- \lambda^{-1} \sum_m \bar{q}_m \left(\lambda^{-1} Y^{-m} e + \sum_{s=1}^{m-1} \delta(A_{m-s} Y^{-s} e - Y^{m-2s} e) \right). \qquad (73)$$

This suggests to switch to the basis $Y^{-(k-2)}e_1, Y^{-(k-2)+1}e_1, \ldots, Ye_1$,

$$\sum_{i=-(k-2)}^{1} h_i' Y^i e = Ye + \delta\lambda^{-1} Y^{-1} e - \lambda^{-1}\delta \sum_m \bar{q}_m A_m e_1$$
$$- \lambda^{-2} \sum_{m=0}^{k-2} \bar{q}_m Y^{-m} e - \lambda^{-2} \bar{q}_{k-1} \sum_{i=0}^{k-1} \bar{q}_i Y^{i-(k-2)} e$$
$$- \lambda^{-1}\delta \sum_{m=0}^{k-1} \bar{q}_m \sum_{s=1}^{m-1} A_{m-s} Y^{-s} e + \lambda^{-1}\delta \sum_{m=0}^{k-1} \bar{q}_m \sum_{s=1}^{m-1} Y^{m-2s} e. \qquad (74)$$

Of course, the coefficients of the defining relation with respect to these two bases generate the same ideal $(h_0, \ldots, h_{k-1}) = (h'_{-(k-2)}, \ldots, h'_1)$. Only the last term in (74) is not yet fully converted to the new basis. We take a closer look at the coefficient of Ye_1,

$$h_1' = 1 - \lambda^{-2}\bar{q}_{k-1}^2 + \delta\lambda^{-1} \sum_{m=0}^{k-1} \bar{q}_m \sum_{s=1}^{m-1} \tilde{q}_{m-2s,1}.$$

Let B denote the sum on the right (without the factor $\delta\lambda^{-1}$) and split it into $B = B_1 + B_2$ such that B_1 contains the terms with $m - 2s = 1$. Now, distinguish between even $k = 2\mu$ and odd $k = 2\mu + 1$ values of k. In the odd case we have

$$B_1 = \sum_{m=3,5,\ldots}^{k-2} \bar{q}_m = \sum_{s=1}^{\mu-1} \bar{q}_{2s+1} = -q_0^{-1} \sum_{s=2}^{\mu} q_{2s}.$$

In the even case similarly

$$B_1 = \sum_{m=3,5,\ldots}^{k-1} \bar{q}_m = \sum_{s=1}^{\mu-1} \bar{q}_{2s+1} = -q_0^{-1} \sum_{s=1}^{\mu-2} q_{2+2s} + q_0^{-1} = -q_0^{-1} \sum_{s=2}^{\mu-1} q_{2s} + q_0^{-1}.$$

Now, let us calculate B_2 using (64).

$$B_2 = \sum_{m=4}^{k-1} \sum_{s=1}^{\lfloor (m-2)/2 \rfloor} \bar{q}_m \widetilde{q}_{m-2s,1}$$

$$= -q_0^{-1} \sum_{m=4}^{k-2} \sum_{s=1}^{\lfloor (m-2)/2 \rfloor} q_{m+1} \sum_{t=1}^{m-2s-1} \sum_{i_1+\cdots i_t = m-2s-1} q_{k-i_1} \cdots q_{k-i_t}$$

$$+ q_0^{-1} \sum_{s=1}^{\lfloor (k-3)/2 \rfloor} \sum_{t=1}^{k-2s-2} \sum_{i_1+\cdots i_t = k-2s-2} q_{k-i_1} \cdots q_{k-i_t}.$$

Both lines cancel almost completely. Only the summands with only one term q_{k-i_j} survive.

$$B_2 = q_0^{-1} \sum_{s=1}^{\lfloor (k-3)/2 \rfloor} q_{k-(k-2-2s)} = q_0^{-1} \sum_{s=1}^{\mu-2+\varepsilon} q_{2+2s} = q_0^{-1} \sum_{s=2}^{\mu-1+\varepsilon} q_{2s}.$$

Here $\varepsilon = 0$ for k even and $\varepsilon = 1$ otherwise. Upon adding B_1 and B_2 a lot of terms cancel: $B = 0$ for k odd and $B = q_0^{-1}$ for k even. Put together,

$$h'_1 = 1 - \lambda^{-2} q_0^{-2} = (1 - \lambda^{-1} q_0^{-1})(1 + \lambda^{-1} q_0^{-1}), \quad k \text{ odd},$$
$$h'_1 = 1 - \lambda^{-2} q_0^{-2} + \delta\lambda^{-1} q_0^{-1} = \lambda^{-2} q_0^{-2} (q + \lambda q_0)(\lambda q_0 - q^{-1}), \quad k \text{ even}.$$

At this point we notice that $h'_{-(k-2)} = 0, \ldots, h'_{-1} = 0,\ h'_0 = 0$ is a triangular system of equations for $A_{k-2}, \ldots, A_1, A_{k-1}$. The denominator is up to units always δ. The numerators are irreducible. Using this information one can easily prove that the ideals given in the next definition are prime. Moreover, the classical limit exists as a quotient because the relations $h'_j = 0$ are fulfilled trivially for $q = q_0 = 1, q_i = 0$.

Definition and Proposition 30. *For odd k define*

$$L_k^{\pm} := R_0^k / (\pm\lambda^{-1} - q_0, h'_0, \ldots, h'_{-(k-2)}) \tag{75}$$

and for even k define

$$L_k^{\pm} := R_0^k / (\pm\lambda^{-1} q^{\mp 1} - q_0, h'_0, \ldots, h'_{-(k-2)}). \tag{76}$$

Then L_k^+ and L_k^- are admissible rings for $\mathscr{B}_n^k$.

8. Normal forms of words problem in $\mathscr{B}_n^k$

In this section we single out a set of words in standard form that linearly generate $\mathscr{B}_n^k$. This is fundamental to the following analysis. However, it does not yet lead to a linear basis of $\mathscr{B}_n^k$. Only in the classical limit it will be possible to strengthen the results to obtain a tight upper bound for the dimension.

Proposition 31. *Every element in $\mathscr{B}_n^k$ is a linear combination of words of the form $w_1 \gamma w_2$ where $w_i \in \mathscr{B}_{n-1}^k$ and $\gamma \in \Gamma_n := \{1, e_{n-1}, X_{n-1}, Y_n^j, j = 1, \ldots, k-1\}$ The same is true if in Γ_n the generators X_{n-1} or Y_n or both are replaced by their inverses.*

Proof. We prove the proposition by induction. The case $n = 1$ is trivial and $n = 2$ can also be verified easily.

Let $w_0\gamma_0 w_1\gamma_1\cdots w_m\gamma_m w_{m+1}\in\mathscr{B}_n^k, w_i\in\mathscr{B}_{n-1}^k,\ \gamma_i\in\Gamma_n$ be an arbitrary word. It suffices to show that any two neighboring γ_i can be combined together. Hence the situation we have to investigate is $w=\gamma_1 w_1\gamma_2, w_1\in\mathscr{B}_{n-1}^k,\gamma_1,\gamma_2\in\Gamma_n$. By induction hypothesis we have $w_1=u_1\alpha u_2, u_i\in\mathscr{B}_{n-2}^k,\alpha\in\Gamma_{n-1}$ and hence $w=\gamma_1 u_1\alpha u_2\gamma_2=u_1\gamma_1\alpha\gamma_2 u_2$. Thus it suffices to investigate $w'=\gamma_1\alpha\gamma_2$. The cases $\gamma_1=1$ or $\gamma_2=1$ are trivial. We now investigate in turn the four possible values of α.

Case 1. $\alpha=1$. The following table gives the relation that allows to reduce the product $\gamma_1\gamma_2$ to the standard form of the proposition.

$\gamma_1\setminus\gamma_2$	Y_n^j	e_{n-1}	X_{n-1}
Y_n^l	trivial	(77)	(48)
e_{n-1}	(78)	(7)	(5)
X_{n-1}	(79)	(5)	(12)

$$Y_n^l e_{n-1}=X_{n-1}Y_{n-1}^l X_{n-1}^{-1}e_{n-1}=\lambda^{-1}X_{n-1}Y_{n-1}^l e_{n-1}\ \text{apply (50) recursively} \tag{77}$$

$$\begin{aligned}e_{n-1}Y_n^j&=\lambda e_{n-1}Y_{n-1}^j X_{n-1}^{-1}\\&=\lambda e_{n-1}Y_{n-1}^j X_{n-1}-\delta\lambda e_{n-1}Y_{n-1}^j+\delta\lambda A_j e_{n-1}.\end{aligned}\tag{78}$$

The first term is reduced by applying (49) recursively:

$$\begin{aligned}X_{n-1}Y_n^j&=X_{n-1}^2Y_{n-1}^j X_{n-1}^{-1}\\&=Y_{n-1}^j X_{n-1}^{-1}-\delta\lambda e_{n-1}Y_{n-1}^j X_{n-1}^{-1}+\delta X_{n-1}Y_{n-1}^j X_{n-1}^{-1}\end{aligned}\tag{79}$$

$$\begin{aligned}&=Y_{n-1}^j X_{n-1}-\delta Y_{n-1}^j+\delta Y_{n-1}^j e_{n-1}\\&\quad-\delta\lambda(e_{n-1}Y_{n-1}^j X_{n-1}-\delta e_{n-1}Y_{n-1}^j+\delta A_j e_{n-1})+\delta Y_n^j.\end{aligned}\tag{80}$$

Again, one needs (49) for recursive reduction:

Case 2: $\alpha=X_{n-2}$:

$\gamma_1\setminus\gamma_2$	Y_n^j	e_{n-1}	X_{n-1}
Y_n^l	$=X_{n-2}Y_n^{j+l}$	$=X_{n-2}Y_n^l e_{n-1}$ (77)	$=X_{n-2}Y_n^l X_{n-1}$ (48)
e_{n-1}	$=e_{n-1}Y_n^j X_{n-2}$ (78)	(6)	(16)
X_{n-1}	$=X_{n-1}Y_n^j X_{n-2}$ (79)	(17)	(4)

Case 3: $\alpha = e_{n-2}$:

$\gamma_1 \setminus \gamma_2$	Y_n^j	e_{n-1}	X_{n-1}
Y_n^l	$=e_{n-2}Y_n^{l+j}$	$=e_{n-2}Y_n^l e_{n-1}$ (77)	$=e_{n-2}Y_n^l X_{n-1}$ (48)
e_{n-1}	$=e_{n-1}Y_n^j e_{n-2}$ (78)	(11)	(15)
X_{n-1}	$X_{n-1}Y_n^j e_{n-2}$ (79)	(14)	(19)

Case 4: $\alpha = Y_{n-1}^m$:

$\gamma_1 \setminus \gamma_2$	Y_n^j	e_{n-1}	X_{n-1}
Y_n^l	*	like (81)	*
e_{n-1}	(81)	(40)	(49)
X_{n-1}	*	(50)	(83)

$$\begin{aligned} e_{n-1}Y_{n-1}^m Y_n^j &= e_{n-1}Y_{n-1}^m X_{n-1}Y_{n-1}^j X_{n-1}^{-1} \\ &\overset{(49)}{\in} \operatorname{span}\{e_{n-1}Y_{n-1}^s \mid 0 \le s < k\} Y_{n-1}^j X_{n-1}^{-1} \end{aligned} \tag{81}$$

$$\overset{(49)}{\subseteq} \operatorname{span}\{e_{n-1}Y_{n-1}^s | 0 \le s < k\}, \tag{82}$$

$$\begin{aligned} X_{n-1}Y_{n-1}^m X_{n-1} &= Y_n^m X_{n-1}^2 = Y_n^m + \delta Y_n^m X_{n-1} - \delta\lambda Y_n^m e_{n-1} \\ &= Y_n^m + \delta X_{n-1}Y_{n-1}^m - \delta\lambda Y_n^m e_{n-1}. \end{aligned} \tag{83}$$

The last term can be reduced using (77).

The remaining cases (marked by $*$ in the table) are

$$Y_n^l Y_{n-1}^m Y_n^j = X_{n-1}Y_{n-1}X_{n-1}^{-1}Y_{n-1}^m X_{n-1}Y_{n-1}^j X_{n-1}^{-1},$$
$$X_{n-1}Y_{n-1}^m Y_n^j = X_{n-1}Y_{n-1}^m X_{n-1}Y_{n-1}^j X_{n-1}^{-1},$$
$$Y_n^l Y_{n-1}^m X_{n-1} = X_{n-1}Y_{n-1}X_{n-1}^{-1}Y_{n-1}^m X_{n-1}.$$

We note that we are dealing with sequences of generators where all indices are equal. Hence we will suppress the index in further calculations. Eqs. (49) and (50) imply that every such sequence containing e is reducible to $Y^t e Y^s$ and thus is of the standard form. This motivates the following notation: We write $a \sim b$ if $\exists c, \chi\ a = b + \chi c$, where c contains e and χ is some parameter. As a consequence the substitutions $X - \delta \leftrightarrow X^{-1}$ preserve this equivalence relation.

To complete the proof it suffices to show that any finite sequence of the kind $\cdots XY^{i_1}XY^{i_2}X\cdots$ is equivalent under $\sim$ to a sequence that contains at most two X

because if the sequence contains none or only one X it is in the standard form and if it contains exactly two X it is either $XY^lXY^m \sim XY^lX^{-1}Y^m + \delta XY^{l+m} = Y^l_{\cdot+1}Y^m + \delta XY^{l+m}$ or $Y^lXY^mX \sim Y^lXY^mX^{-1} + \delta Y^lXY^m = Y^lY^m_{\cdot+1} + \delta Y^lXY^m$.

The reducibility to sequences with at most two X follows by induction from the following lemma: There exists families of scalars α, β such that

$$XY^sXY^tX \sim \sum_{i,j} \alpha^{s,t}_{s,t} XY^iXY^j + \sum_{i,j} \beta^{s,t}_{i,j} Y^iXY^j. \tag{84}$$

We prove (84) by induction on s. For $s=1$ we have $XYXY^tX = Y^tXYX^2 \sim Y^tXY - \delta Y^tXYX = Y^tXY - \delta XYXY^t$. Assume that (84) holds for s. We show it for $s+1$:

$$\begin{aligned} XY^{s+1}XY^tX &= XYX^{-1}XY^sXY^t \\ &\sim \sum_{i,j} \alpha^{s,t}_{i,j} XYX^{-1}XY^iXY^j + \sum_{i,j} \beta^{s,t}_{i,j} XYX^{-1}Y^iXY^j \\ &\sim \sum_{i,j} \alpha^{s,t}_{i,j} XY^{i+1}XY^j + \sum_{i,j} \beta^{s,t}_{i,j} XYXY^iXY^j - \delta \sum_{i,j} \beta^{s,t}_{i,j} XY^{i+1}XY^j. \end{aligned}$$

The first and third summands are already in a form in which their contribution to $\alpha^{s+1,t}_{i+1,j}$ can be read off. In the second summand we apply the induction hypothesis once again

$$\sum_{i,j} \beta^{s,t}_{i,j} XYXY^iXY^j \sim \sum_{i,j} \beta^{s,t}_{i,j} \sum_{p,q} (\alpha^{1,i}_{p,q} XY^pXY^{q+j} + \beta^{1,i}_{p,q} Y^pXY^{q+j}).$$

We now establish the last statement of the proposition. Using the involution from Remark 15 we see that we may replace X and Y in Γ_n by their inverses. Since Y_n^{-1} is just a linear combination of powers of Y_n we also may replace Y_n by Y_n^{-1} alone. Combining both operations replaces just X_{n-1} by its inverse. $\square$

The proposition implies that $\mathscr{B}^k_n$ is finite dimensional.

Lemma 32. *There exist elements $R_{i,m} \in \mathscr{B}^k_{i-1}$ such that $e_iY_i'^m e_i = R_{i,m}e_i$,*

$$e_iY_i'^lX_i = e_iY_i'^{l-1}X_iY_i'^{-1} - \delta e_iY_i'^{l-2} + \delta R_{i,l-1}e_iY_i'^{-1}. \tag{85}$$

Proof. To prove the first statement one writes $Y_i'^m = \sum_j a_jb_jc_j$ according to Proposition 31 with $a_j, c_j \in \mathscr{B}^k_{i-1}, b_j \in \Gamma_i$. The claim is then obvious:

$$\begin{aligned} e_iY_i'^lX_i &= e_iY_i'^lX_iY_i'X_iX_i^{-1}Y_i'^{-1} = e_iX_iY_i'X_iY_i'^lX_i^{-1}Y_i'^{-1} \\ &= e_iY_i'^{l-1}X_i^{-1}Y_i'^{-1} \\ &= e_iY_i'^{l-1}X_iY_i'^{-1} - \delta e_iY_i'^{l-1}Y_i'^{-1} + \delta e_iY_i'^{l-1}e_iY_i'^{-1} \\ &= e_iY_i'^{l-1}X_iY_i'^{-1} - \delta e_iY_i'^{l-2} + \delta R_{i,l-1}e_iY_i'^{-1}. \quad \square \end{aligned} \tag{86}$$

This lemma implies that $I_n = \mathscr{B}^k_{n-1}e_{n-1}\mathscr{B}^k_{n-1}$.

Proposition 33. *In Proposition* 31 *one may replace* Γ_n *by* $\Gamma'_n := \{1, e_{n-1}, X_{n-1}, Y_n'^j, j = 1, \dots, k-1\}$.

Proof. We express an arbitrary element a in $\mathscr{B}_n^k$ as $a = \sum_j f_j h_j g_j$ with $f_j, g_j \in \mathscr{B}_{n-1}^k, h_j \in \Gamma_n$. We are finished if we can show that $Y_n^i = \sum_s l_s^{(n)} \gamma_s^{(n)} r_s^{(n)}$ with $\gamma_s^{(n)} \in \Gamma'_n, l_s^{(n)}, r_s^{(n)} \in \mathscr{B}_{n-1}^k$ since in this case we can simply substitute this expressions for the Y_i^n which appear among the h_j.

We show $Y_n^i = \sum_s l_s^{(n)} \gamma_s^{(n)} r_s^{(n)}$ by induction. The case $n = 1$ is trivial. Now assume that the formula holds for $n-1$.

$$\begin{aligned} Y_n^i &= X_{n-1} Y_{n-1}^i X_{n-1}^{-1} = \sum_s X_{n-1} l_s^{(n-1)} \gamma_s^{(n-1)} r_s^{(n-1)} X_{n-1}^{-1} \\ &= \sum_s l_s^{(n-1)} X_{n-1} \gamma_s^{(n-1)} X_{n-1}^{-1} r_s^{(n-1)}. \end{aligned}$$

The cases $\gamma_s^{(n-1)} \in \{1, e_{n-2}, X_{n-2}\}$ are easily reduced using Lemma 7. It remains to investigate the case $\gamma_s^{(n-1)} = Y_{n-1}'^j$.

$$\begin{aligned} X_{n-1} Y_{n-1}'^j X_{n-1}^{-1} &= X_{n-1} Y'_{n-1} X_{n-1} X_{n-1}^{-1} Y_{n-1}'^{j-1} X_{n-1}^{-1} \\ &= Y'_n (X_{n-1} - \delta + \delta e_{n-1}) Y_{n-1}'^{j-1} X_{n-1}^{-1} \\ &= Y'_n X_{n-1} Y_{n-1}'^{j-1} X_{n-1}^{-1} - \delta Y'_n Y_{n-1}'^{j-1} X_{n-1}^{-1} + \delta Y'_n e_{n-1} Y_{n-1}'^{j-1} X_{n-1}^{-1}. \quad (87) \end{aligned}$$

The second summand is $-\delta Y_{n-1}'^{j-1} X_{n-1} Y'_{n-1}$ which is already of the standard form. The third summand is

$$\begin{aligned} \delta Y'_n e_{n-1} Y_{n-1}'^{j-1} X_{n-1}^{-1} &= \delta \lambda X_{n-1} Y'_{n-1} e_{n-1} Y_{n-1}'^{j-1} (X_{n-1} - \delta + \delta e_{n-1}) \\ &= \delta Y_{n-1}'^{-1} e_{n-1} Y_{n-1}'^{j-1} X_{n-1} - \delta^2 Y_{n-1}'^{-1} e_{n-1} Y_{n-1}'^{j-1} \\ &\quad + \delta^2 Y_{n-1}'^{-1} e_{n-1} Y_{n-1}'^{j-1} e_{n-1}. \end{aligned}$$

Here the last summand is reduced using the formula for $e_i Y_i'^m e_i$ from Lemma 32 while the first summand needs (85). The middle summand is already of the standard form.

The first summand of (87) is reduced by iteration. □

We continue our study of words in $\mathscr{B}_n^k$ by cutting down the size of sets that linearly generate the algebra.

Lemma 34. $\mathscr{B}_n^k$ *is linearly spanned by the set* S_n *which is recursively defined,*

$$\begin{aligned} S_1 &:= \{Y^i \mid i = 0, \dots, k-1\}, \\ S_n &:= \Gamma_1 \cdots \Gamma_n S_{n-1}. \end{aligned}$$

It suffices to take out of $\Gamma_1 \cdots \Gamma_n$ *those elements that are of the following form*:

$$Y_{l_1}^{m_1} \cdots Y_{l_s}^{m_s} X_i \cdots X_j e_{j+1} \cdots e_n, \quad m_t \in \{0, \dots, k-1\}, \ l_1 < \cdots < l_s = i.$$

Here we have $1 \le i \le n$ *and* $i - 1 \le j \le n - 1$ *so that the chains of* X *and* e *may be empty.*

Proof. Proposition 31 yields the following representation of $\mathscr{B}_n^k$:

$$\begin{aligned}\mathscr{B}_n^k &= \operatorname{span} \mathscr{B}_{n-1}^k \Gamma_n \mathscr{B}_{n-1}^k = \operatorname{span} \mathscr{B}_{n-2}^k \Gamma_{n-1} \mathscr{B}_{n-2}^k \Gamma_n \mathscr{B}_{n-1}^k \\ &= \operatorname{span} \mathscr{B}_{n-2}^k \Gamma_{n-1} \Gamma_n \mathscr{B}_{n-1}^k \\ &= \operatorname{span} \Gamma_1 \cdots \Gamma_n \mathscr{B}_{n-1}^k. \end{aligned} \tag{88}$$

To establish the second statement we consider the Y_j^m that appears at the leftmost position in a chain $Z_i \cdots Z_{j-1} Y_j^m Z_{j+1} \cdots Z_n$ of generators $Z_s \in \Gamma_s$. Then $Z_i \cdots Z_{j-1}$ consists only of e and X and hence it can be commuted to the right and be absorbed in $\mathscr{B}_{n-1}^k$. Similarly, e and X that appear between two $Y_.$ can be commuted to the right. Iterating this argument, we obtain only chains of the form $Y_{i_1}^{m_1} \cdots Y_{i_s}^{m_s} Z_{j+1} \cdots Z_n,\ i_1 < \cdots < i_s$.

If $e_i X_{i+1}$ appears in such a chain it may be converted to $e_i X_{i+1} = e_i e_{i+1} X_i^{-1}$. The X_i^{-1} can the be absorbed in $\mathscr{B}_{n-1}^k$. Hence all X have to appear to the left of all e. □

A similar proof establishes a related lemma using the Y_i' instead.

9. The classical limit

The classical limit of tangle algebra is a specialization in which braidings degenerate to permutations. We define $\mathrm{BP}_n^k(R)$ in its own right as algebra of Brauer graphs [15] where each arc carries an element of $\mathbb{Z}_k$. We visualize this as dotted Brauer graphs, i.e., $\mathrm{BP}_n^k(R)$ is the free R module of dimension $k^n(2n-1)!!$ that has as basis the set of Brauer graphs where each arc carries at most $k-1$ points. We require that vertical arcs have no extrema with respect to the height function and that horizontal arcs have exactly one extremum. Furthermore, we demand that the dots of vertical arcs are concentrated at the left endpoint.

Multiplication is given as for Brauer graphs. Dots may flow along an arc and may cross another arc. If a dot traverses an extremum it gets replaced by $k-1$ dots. Dot numbers are reduced modulo k. Using this we may isolate cycles and concentrate dots on their leftmost position. Such a cycle with i dots on it may be deleted at the expense of a factor A_i. Dots on vertical arcs may be brought to the lower endpoint and thereafter the arc may be straightened. Similarly, dots on horizontal arcs may be concentrated according to our convention. Just as in the case of ordinary Brauer graphs we see that $\mathrm{BP}_n^k(R)$ is generated by $X_{i,n}^{(G)}, e_{i,n}^{(G)}, Y_{1,n}^{(G)}$ (where $X_{i,n}^{(G)}$ is to be understood as a permutation two-cycle).

Let us compare BP_n^k with the classical limit of $\mathscr{B}_n^k$.

Definition 35. The classical limit of $\mathscr{B}_n^k$ is defined to be the algebra

$$C\mathscr{B}_n^k := \mathscr{B}_n^k(R_1^k) \otimes_{R_1^k} R_c^k \tag{89}$$

In $C\mathscr{B}_n^k$ we have $X_i = X_i^{-1}$ and hence $Y_i^j = Y_i^{(j)} = Y_i'^j$. An important consequence is that Y_i behaves natural with respect to the braidings X_i. This gives the next lemma.

Lemma 36. *$C\mathscr{B}_n^k$ is spanned by a set of elements of the form $\alpha\beta\gamma$, where α is a product of Y_i, γ is a product of Y^{-1} and β is an element of a basis of the A-type BMW algebra* BA_n.

For $k < \infty$ we have moreover: Each word of this basis contains at most k^n factors Y, Y^{-1} in α and γ combined. The number $k^n(2n-1)!!$ is therefore an upper bound for the dimension of $C\mathscr{B}_n^k$.

Proof. The proof is by induction on n. Assume it to be already established for $n-1$. We represent $C\mathscr{B}_n^k$ similarly to (88). It suffices to show that we can move all Y_i which appear on the left of the basis words of $C\mathscr{B}_{n-1}^k$ either through the outer chain to the left or, negated, to the right. We investigate the various cases that arise. First, assume that $e_{n-1}Y_{n-1}$ occurs. We rewrite it as

$$\begin{aligned} e_{n-1}Y_{n-1} &= e_{n-1}Y_{n-1}X_{n-1}Y_{n-1}Y_{n-1}^{-1}X_{n-1}^{-1} = \lambda^{-1}e_{n-1}Y_{n-1}^{-1}X_{n-1}^{-1} \\ &= e_{n-1}X_{n-1}^{-1}Y_{n-1}^{-1}X_{n-1}^{-1} = e_{n-1}Y_n^{-1} = e_{n-1}Y_n^{-1}. \end{aligned}$$

If $e_ie_{i+1}Y_i = e_iY_ie_{i+1}$ occurs, the same calculation applied twice yields that Y_{i+2} can be moved to the left. The only remaining case is resolved in the following way:

$$X_iY_i = Y_{i+1}X_i^{-1} = Y_{i+1}X_i - \delta Y_{i+1} + \delta Y_{i+1}e_i = Y_{i+1}X_i.$$

None of these rewritings did change the combined number of Y and Y^{-1}. Hence, there can be at most $(k-1)^n$ of them: Every chain from the recursive construction of the set S_n (Lemma 34) contributes at most $k-1$ of them. By induction assumption, the dimension of $C\mathscr{B}_{n-1}^k$ is less than $k^{n-1}(2n-3)!!$ The theory of the standard BW algebra BA_n shows that $2n-1$ chains $Z_i \cdots Z_n, Z_j \in \{e_{i-1}, X_{i-1}\}$ suffice to obtain a basis. Each of these can start with Y_i^m. Hence, the dimension of $C\mathscr{B}_n^k$ exceeds that of $\mathscr{B}_{n-1}^k$ by at most a factor $k(2n-1)$. The claim follows. □

Lemma 37. *The algebras $C\mathscr{B}_n^k$ and $\mathrm{BP}_n^k(R_c)$ are isomorphic.*

Proof. We define the morphism $\chi_n : C\mathscr{B}_n^k \to \mathrm{BP}_n^k(R_c)$ that maps $e_i \mapsto e_i^{(G)}, X_i \mapsto X_i^{(G)}$ and Y to a dot on the first string. It is easy to see that this is a morphism (it is relation (23) that requires the somewhat strange minimum/maximum rule). It is surjective. Injectivity may be seen by looking at the dimension of these algebras. □

10. The Markov trace

The graphical calculus as well as the relationship with the A-type BMW algebra suggest that there should exist a Markov trace on $\mathscr{B}_n^k$.

Definition and Proposition 38. *Let $a_n \subset R$ denote the annihilator ideal of $E_n = E(1, 2n-1) = e_1e_3 \cdots e_{2n-1}$ in $\mathscr{B}_{2n}^k(R)$. Then the equation*

$$x^n \mathrm{tr}(a)E(1, 2n-1) = \bar{H}_n a H_n \quad \forall a \in \mathscr{B}_n^k$$

(H_n is to be defined in (94)) defines a Markov trace $\mathrm{tr} : \mathscr{B}_n^k(R) \otimes_R (R/a_n) = \mathscr{B}_n^k(R/a_n) \to R/a_n$. *Here k may be finite or infinite.*

For potentially admissible rings a_n vanishes and the theorem yields the existence of a Markov trace (see Fig. 7).

The proof is split up into a sequence of lemmata. As a basic tool we need elements that model the concentric half-circles used in closing braids:

$$X(i,j) := X_i X_{i+1} \cdots X_j, \tag{90}$$

$$X^{-1}(i,j) := X_i^{-1} X_{i+1}^{-1} \cdots X_j^{-1}, \tag{91}$$

$$E(i,j) := e_i e_{i+2} \cdots e_j, \tag{92}$$

$$H_1 := e_1, \tag{93}$$

$$H_{n+1} := e_{n+1} X(n+2, 2n+1) X(n+1, 2n) H_n. \tag{94}$$

Lemma 39.

$$H_n = E(n,n)E(n-1,n+1)\cdots E(1,2n-1), \tag{95}$$

$$H_{n+1} = e_{n+1} X^{-1}(n+2, 2n+1) X^{-1}(n+1, 2n) H_n, \tag{96}$$

$$X_i^{\pm} H_n = X_{2n-i}^{\pm} H_n, \qquad e_i H_n = e_{2n-i} H_n, \tag{97}$$

$$e_{2n-1} = X(n, 2n-2)^{-1} X(n+1, 2n-1)^{-1} e_n X(n+1, 2n-1) X(n, 2n-2), \tag{98}$$

$$e_n X(n+1, n+m) X(n, n+m-1) = X(n+1, n+m) X(n, n+m-1) e_{n+m}, \tag{99}$$

$$H_{n+1} = e_{n+1} X(n+2, 2n) X(n+1, 2n-1) X_{2n+1}^{-1} X_{2n}^{-1} H_n, \tag{100}$$

$$Y^{\pm 1} H_n = Y_{2n}^{\prime \mp 1} H_n. \tag{101}$$

Proof. All these relations become pretty obvious upon drawing pictures but, of course, we are not yet in a position to take graphs for proofs. The algebraic proofs are all done by induction and we restrict ourselves to last one. Here, induction starts from

$$YH_1 = Ye_1 = X_1^{-1} Y^{-1} Y X_1 Y e_1 = (X_1 Y X_1)^{-1} e_1.$$

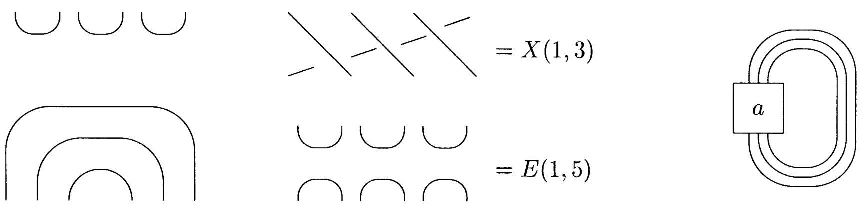

Fig. 7. The graphical interpretation of H_3 (on the left), of $X(1,3)$ and $E(1,5)$ (in the middle) and of the Markov trace.

The induction step is

$$\begin{aligned}
YH_{n+1} &= Ye_{n+1}X^{-1}(n+2,2n+1)X^{-1}(n+1,2n)H_n \\
&= e_{n+1}X^{-1}(n+2,2n+1)X^{-1}(n+1,2n)YH_n \\
&= e_{n+1}X^{-1}(n+2,2n+1)X^{-1}(n+1,2n-1)X_{2n}^{-1}Y_{2n}'^{-1}H_n \\
&= e_{n+1}X^{-1}(n+2,2n)X_{2n+1}^{-1}X^{-1}(n+1,2n-1)Y_{2n+1}'^{-1}X_{2n}H_n \\
&= e_{n+1}X^{-1}(n+2,2n)X_{2n+1}^{-1}Y_{2n+1}'^{-1}X^{-1}(n+1,2n-1)X_{2n}H_n \\
&= e_{n+1}X^{-1}(n+2,2n)Y_{2n+2}'^{-1}X_{2n+1}X^{-1}(n+1,2n-1)X_{2n}H_n \\
&= Y_{2n+2}'^{-1}e_{n+1}X^{-1}(n+2,2n)X^{-1}(n+1,2n-1)X_{2n+1}X_{2n}H_n \\
&= Y_{2(n+1)}'^{-1}H_{n+1}. \qquad \square
\end{aligned}$$

Using the involution from Remark 10 this yields

Lemma 40. *$\overline{H_n}X_i^{\pm}=\overline{H_n}X_{2n-i}^{\pm}, \overline{H_n}e_i=\overline{H_n}e_{2n-i}, \overline{H_n}Y^{\pm 1}=\overline{H_n}Y_{2n}'^{\mp 1}$. Hence $\overline{H_n}abH_n=\overline{H_n}baH_n$, $\forall a,b \in \mathscr{B}_n^k$*

Thus we have shown that tr is a trace.

Lemma 41. *tr is a Markov trace, i.e., for $a \in \mathscr{B}_n^k$ the following relations hold*:

$$\operatorname{tr}(1) = 1, \tag{102}$$

$$\operatorname{tr}(ae_n) = x^{-1}\operatorname{tr}(a), \tag{103}$$

$$\operatorname{tr}(aX_n^{\pm}) = x^{-1}\lambda^{\mp}\operatorname{tr}(a), \tag{104}$$

$$\operatorname{tr}(aY_{n+1}^m) = A_m x^{-1}\operatorname{tr}(a). \tag{105}$$

Proof. Let $a \in \mathscr{B}_n^k$ and $\gamma \in \Gamma_{n+1}$. Then $e_{n+1}a\gamma e_{n+1} = ae_{n+1}\gamma e_{n+1} = s(\gamma)ae_{n+1}$ with a factor s which assumes the values $s(\gamma)=x,1,\lambda^{-1},A_m$ when $\gamma=1,e_n,X_n,Y_{n+1}^m$. The proof is by induction on n. The induction start is trivial and the step is

$$\begin{aligned}
&\bar{H}_{n+1}a\gamma H_{n+1} \\
&\quad=\bar{H}_nX(n+1,2n)^{-1}X(n+2,2n+1)^{-1}e_{n+1} \\
&\qquad a\gamma e_{n+1}X(n+2,2n+1)X(n+1,2n)H_n \\
&\quad=\bar{H}_nX(n+1,2n)^{-1}X(n+2,2n+1)^{-1} \\
&\qquad as(\gamma)e_{n+1}X(n+2,2n+1)X(n+1,2n)H_n \\
&\quad=s(\gamma)\bar{H}_naX(n+1,2n)^{-1}X(n+2,2n+1)^{-1} \\
&\qquad e_{n+1}X(n+2,2n+1)X(n+1,2n)H_n \\
&\quad=s(\gamma)\bar{H}_nae_{2n+1}H_n = s(\gamma)\bar{H}_naH_ne_{2n+1} \\
&\quad=s(\gamma)x^n\operatorname{tr}(a)E(1,2n-1)e_{2n+1} = (s(\gamma)/x)x^{n+1}\operatorname{tr}(a)E(1,2n+1). \qquad \square
\end{aligned}$$

This proof moreover shows that tr is indeed well-defined independent of n. If we had given an index n to the trace in Definition 38 then the case $a \in \mathscr{B}_n^k, \gamma = 1$ in the above proof yields $x^{n+1}\operatorname{tr}_{n+1}(a)E(1,2n+1) = \bar{H}_{n+1}a\gamma H_{n+1} = (x/x)x^{n+1}\operatorname{tr}_n(a)E(1,2n+1)$, hence $\operatorname{tr}_{n+1}(a) = \operatorname{tr}_n(a)$.

In the classical limit the trace is given as well by closing the strings from the right. Hence, let $a \in \mathrm{BP}_n^k$ be a dotted Brauer graph and denote by $n_i(a)$ the number of cycles with i dots on its closure. Then we have

$$\operatorname{tr}(a) = x^{-n}\prod_{i=0}^{k-1} A_i^{n_i(a)}. \tag{106}$$

Lemma 42. *The trace is nondegenerate on $C\mathscr{B}_n^k = \mathrm{BP}_n^k$.*

Proof. Let $\{v_i \mid i = 1, \ldots, k^n(2n-1)!!\}$ be a linear basis of dotted Brauer graphs. It suffices to show $\det(\operatorname{tr}(v_i v_j^*)_{i,j}) \neq 0$.

The involution $a \mapsto a^\star$ maps graphs to their top-down mirror image and replaces each dot by $k-1$ dots. Hence the closure of aa^* is free of dots. Now assume that a has s upper (and hence s lower) horizontal arcs. Then there are s cycles in aa^*. Upon closing another s cycles are produced from the remaining horizontal arcs. The vertical arcs form a permutation and a^* contains the inverse permutation. Upon closing these $n-2s$ vertical arcs yield $n-2s$ cycles. The closure of aa^* has therefore a total of n cycles and $\operatorname{tr}(aa^*) = 1$.

We now specialize the ground ring: $A_1 := \ldots := A_{k-1} := x^{-1}$. The trace is then a Laurent polynomial in x. The choice for the A_i implies that additional dots on an arc decrease the degree (in x) of the trace. If β is an arc of a and b is any other graph which does not contain an arc which is the mirror image of β. By considering the cases that β is vertical and horizontal individually one easily sees that the cycle in the closure of ab which contains β consists of more than two arcs from a and b. The closure of ab has therfore less cycles than the closure of aa^*. We conclude that $b = a^*$ is the unique graph with highest x degree of $\operatorname{tr}(ab)$. We now consider the determinant of the trace

$$\det(\operatorname{tr}(v_i v_j^*)_{i,j}) = x^{-nk^n(2n-1)!!}\det\left(\left(x^{n_0(v_i v_j^*)}\prod_{s=1}^{k-1} x^{n_s(v_i v_j^*)/2}\right)_{i,j}\right).$$

In each row the element at the diagonal is the unique element with highest degree in x. Calculating the determinant thus yields a sum with a unique term of highest degree. Thus the determinant does not vanish. □

Up to now we know that there is a trace functional on our algebra, but its faithfulness is only established in the classical limit. This should not come as a surprise since this is the only specialization for which we know the dimension!

11. The structure theorem

In this section we determine the structure of $\mathscr{B}_n^k$ in the generic case. It will turn out to be semi-simple and of dimension $k^n(2n-1)!!$ We only need a few definitions on the Young diagrams before we can state the structure theorem.

A Young diagram λ of size n is a partition of the natural number n. $\lambda = (\lambda_1, \ldots, \lambda_k)$, $\sum_i \lambda_i = n, \lambda_i \geq \lambda_{i+1}$. In the following, we use ordered tuples of the Young diagrams $\underline{\lambda} = (\lambda^1, \ldots, \lambda^k)$ (cf. [1]). The (total) size of the a tuple of the Young diagrams is the sum of sizes of its components. Let $\hat{\Gamma}_n^k$ be the set of all k tuples of the Young diagrams that have total size $n, n-2, \ldots, 1$ or 0.

Proposition 43. *Let R be an admissible ring and denote by K its field of fractions. For the algebra $\mathscr{B}_n^k = \mathscr{B}_n^k(K)$ of finite height k the following statements hold*:

1. $\mathscr{B}_n^k$ *is semi-simple and its simple components are indexed by* $\hat{\Gamma}_n^k$:

$$\mathscr{B}_n^k = \bigoplus_{\underline{\lambda} \in \hat{\Gamma}_n^k} \mathscr{B}_{n,\underline{\lambda}}^k. \tag{107}$$

2. *The Bratteli rule for restrictions of modules*: *A simple $\mathscr{B}_{n,\underline{v}}^k$ module $V_{\underline{v}}, \underline{v} \in \hat{\Gamma}_n^k$ decomposes into $\mathscr{B}_{n-1}^k$ modules such that the $\mathscr{B}_{n-1}^k$ module $\underline{\lambda} \in \hat{\Gamma}_{n-1}^k$ occurs iff $\underline{\lambda}$ may be obtained from $\underline{v}$ by adding or removing a box.*
3. tr *is a faithful trace. To every tuple of Young diagrams $\underline{\lambda} \in \hat{\Gamma}_n^k$ there is an idempotent $p_{\underline{\lambda}}$ and a non vanishing, rational function $Q_{\underline{\lambda}}$ which does not depend on n and satisfies* $\mathrm{tr}(p_{\underline{\lambda}}) = Q_{\underline{\lambda}}/x^n$.

The proof uses the same techniques as [9,16].

Proof. $\mathscr{B}_0^k$ is simply the ground ring. Thus the proposition is true with $\mathrm{tr}(p_{(\cdot,\ldots,\cdot)}) = \mathrm{tr}(1) = Q_{(\cdot,\ldots,\cdot)}/x^0, Q_{(\cdot,\ldots,\cdot)} = 1$. The algebra $\mathscr{B}_1^k$ is of dimension k and has a basis $\{1, Y, \ldots, Y^{k-1}\}$. It is commutative and semi-simple (the minimal polynomial of Y is separable). The simple blocks are given by the eigenspaces of Y. Existence of idempotents is clear. The graphical version is isomorphic as a simple consequence of Turaev's result on the skein module of the solid torus [14]. Moreover, from $\mathrm{tr}(Y^iY^j) = A_{i+j}/A_0$ and Lemma 23 it follows that the trace on $\mathscr{B}_1^k$ is non-degenerate.

Assume the proposition is shown by induction for $\mathscr{B}_n^k$.

We apply Jones–Wenzl theory [15,16] to the following inclusion $\mathscr{B}_{n-1}^k \subset \mathscr{B}_n^k \subset \mathscr{B}_{n+1}^k$. The idempotent is $e = x^{-1}e_n$. This is possible because $\mathscr{B}_{n-1}^k, \mathscr{B}_n^k$ are semi-simple algebras with a faithful trace by induction assumption. All required properties needed for e have already been established. Jones–Wenzl theory asserts the semi-simplicity of the ideal generated by e. Thus I_{n+1} is semi-simple. The quotient algebra $\mathscr{B}_{n+1}^k/I_{n+1}$ is the Ariki–Koike algebra $\mathrm{AK}_{n+1}^{(k)}$ and is semi-simple according to [1]. Since we work over a field we can conclude (by looking at the radicals) that $\mathscr{B}_{n+1}^k$ is semi-simple and that it is isomorphic to the direct sum $\mathscr{B}_{n+1}^k = I_{n+1} \oplus \mathscr{B}_{n+1}^k/I_{n+1}$. Jones–Wenzl theory further implies that the simple components of I_{n+1} are indexed by $\hat{\Gamma}^k_{n-1}$. The simple components of $\mathrm{AK}^{(k)}_{n+1}$ are indexed by tuples of the Young diagrams of size $n+1$ (see [1]). This completes the proof of point 1 of the theorem.

The inclusion matrix for the part I_{n+1} is the transpose of the inclusion matrix of $\mathscr{B}^k_{n-1} \subset \mathscr{B}^k_n$. For the part $\mathrm{AK}^{(k)}_{n+1}$ the Bratteli rule follow from [1].

We have to show that tr is faithful, i.e., that the Q functions do not vanish. If $p_{\underline{\lambda}} \in \mathscr{B}^k_{n-1}$ is a minimal idempotent in $\mathscr{B}^k_{n-1,\underline{\lambda}}$ then $x^{-1}p_{(\mu,\lambda)}e_n$ is a minimal idempotent according to Jones–Wenzl theory. The trace of this idempotent is $\mathrm{tr}(x^{-1}p_{\underline{\lambda}}e_n) = x^{-2}\mathrm{tr}(p_{\underline{\lambda}}) = Q_{\underline{\lambda}}/x^{n-1+2}$. Obviously, this is non vanishing (using the induction assumption). The idempotents of this kind are those of I_{n+1}. For the other idempotents (which are those of $\mathscr{B}^k_{n+1}/I_{n+1}$) the function Q is defined by $\mathrm{tr}(p_{\underline{\lambda}}) = Q_{\underline{\lambda}}/x^n$.

To establish faithfulness of the trace we use the classical limit. A minimal idempotent $p_{\underline{\lambda}}$ of $\mathscr{B}^k_n$ yields an idempotent in the classical limit described in Section 9. We know already that on the classical limit algebra the trace is non-degenerate. Hence the function $Q_{\underline{\lambda}}$ has a non-vanishing classical limit and hence cannot be zero itself. □

The cyclotomic Birman–Wenzl algebra with Y satisfying a quadratic relation is of special interest and has been studied in [9].

Naturally, one would like to study the algebra not only over the generic field but also with complex parameters. The classical limit is then a point in parameter space and we know that at this point the algebra is semi-simple. The functions Q are apart from a finite set of poles continuous and hence there is a neighborhood of the classical point where the algebra is semi-simple. While some necessary conditions for semi-simplicity may be derived easily from the knowledge of the Ariki–Koike algebra the determination of sufficient conditions has to await further studies.

A natural application of the Markov trace is to define a Kauffman polynomial of links in the solid torus. This can be done exactly as described in [9], i.e., $L(\hat{\beta}, n) := x^{n-1}\lambda^{e(\beta)}\mathrm{tr}(\beta)$ $\beta \in \mathrm{ZB}_n$ defines an invariant of the link obtained by closing the braid β ($e : \mathrm{ZB}_n \to \mathbb{Z}$ is the exponential sum with $e(X_i) = 1, e(Y) = 0$).

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