Reconstruction of Weak Quasi Hopf Algebras

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CONTENTS

1. INTRODUCTION

Semisimple braided tensor categories are the structure underlying the quantum invariants of links and 3-manifolds [19]. The most useful examples are derived from the (non-semisimple) representation categories of quantum groups by elimination of not fully decomposable objects and nilpotent morphisms.

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These cleaned up versions are no longer representation categories of usual quantum groups. It is the purpose of this paper to show that they nevertheless arise as representation categories of appropriate algebras. It is always possible to reconstruct a weak quasi Hopf algebras, as introduced by Mack and Schomerus [8], that has the given category as its representation category.

Semisimple braided tensor categories arise also as representation categories of the net of observable algebras in low dimensional quantum field theory. The weak quasi Hopf algebra we construct in this paper may therefore be considered to be a candidate to replace the gauge group in such systems [3].

This result is established in two steps. First we define the notion of a weak quasi tensor functor and show by construction that for any rational braided semisimple tensor category $\mathcal{C}$ such a functor F to the category of finite dimensional vector spaces exists.

If this functor was a tensor functor in the usual sense then Majid's reconstruction theorem [11, 12] would be applicable and would assert the existence of an associated quasi Hopf algebra. It is fairly easy to show [5, 16] that such tensor functors don't exist for large classes of rational semisimple braided tensor categories.

However, Majid's lines of thought can be applied even to the case of a weak quasi tensor functor $F\colon \mathcal{C} \to \mathrm{Vec}$. This generalized reconstruction theorem (Section 3) shows that the set $\mathrm{Nat}(F, F)$ of natural transformations from F to itself can be equipped with the structure of a weak quasi Hopf algebra $H = H(\mathcal{C}, F)$ such that F factors over $\mathrm{Rep}(H)$, i.e., there is a tensor functor $G\colon \mathcal{C} \to \mathrm{Rep}(H)$ such that $F = V \circ G$ where $V\colon \mathrm{Rep}(H) \to \mathrm{Vec}$ is the forgetful functor which assigns to any representation its underlying vector space.

Combining this reconstruction theorem with the construction of weak quasi tensor functors we conclude that every rational semisimple rigid braided tensor category is the representation category of some weak quasi Hopf algebra.

2. BRAIDED TENSOR CATEGORIES

2.1. *Definitions*

The objects of a category $\mathcal{C}$ are denoted by $X \in \mathrm{Obj}(\mathcal{C})$, the morphisms between $X, Y \in \mathrm{Obj}(\mathcal{C})$ with $\mathrm{Mor}(X, Y)$. We use the shorthand $\mathrm{End}(X) := \mathrm{Mor}(X, X)$. The identity functor of a category well be denoted by Id and the set of natural transformations between two functors by $\mathrm{Nat}(F, G)$.

A category $\mathcal{C}$ is called *monoidal* if there is a functor $\otimes\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ together with a functorial isomorphism $\Phi_{X,Y,Z}\colon X\otimes(Y\otimes Z)\tilde{\to}(X\otimes Y)\otimes Z$ satisfying the *pentagon identity* $\Phi\Phi=(\Phi\otimes 1)\Phi(1\otimes\Phi)$ and an *identity object* $1\in \mathrm{Ojb}(\mathcal{C})$, such that $r_X\colon X\mapsto 1\otimes X$ and $l_X\colon X\mapsto X\otimes 1$ are equivalences of categories compatible with Φ: $\Phi_{1,X,Y}\circ l_{X\otimes Y}=l_X\otimes \mathrm{id}_Y$. It is called *strict* if $1\otimes X=X\otimes 1=X$, $(X\otimes Y)\otimes Z=X\otimes(Y\otimes Z)$, $\Phi=\mathrm{id}$, $r=l=\mathrm{id}$. Thanks to MacLane's coherence theorem all equivalence classes of monoidal categories include strict ones. A monoidal category is called a *braided tensor category* if there is a functorial isomorphism $\Psi_{X,Y}\colon X\otimes Y\tilde{\to}Y\otimes X X,Y\in\mathrm{Obj}(\mathcal{C})$ satisfying the two *hexagon identities* $\Phi\Psi\Phi=(\Psi\otimes 1)\Phi(1\otimes\Psi)$ and $\Phi^{-1}\Psi\Phi^{-1}=(1\otimes\Psi)\Phi(\Psi\otimes 1)$ as well as $\Phi(r\otimes 1)=(1\otimes\Psi)(1\otimes r)$ and $l_X=\Psi\circ r_X$. In a *tensor category* or (in contrast to braided tensor categories) *symmetric tensor category* the identity $\Psi_{X,Y}\Psi_{Y,X}=\mathrm{id}_{X\otimes Y}$ should hold.

We assume all categories to be abelian (and all functors to be additive) with direct sum $\oplus$ and zero element 0. Then End(1) is a ring and we assume it in addition to be a field which we denote by $\mathbb{K}$.

An object $X\in\mathrm{Obj}(\mathcal{C})$ is called *indecomposable* if $\mathrm{End}(X)=\mathrm{span}\,\mathrm{id}_X\oplus\mathcal{N}$ where $\mathcal{N}$ consists only of nilpotent elements, and X is called *irreducible*, if $\mathcal{N}=0$. The set of irreducible objects is denoted by Obj_{irr}. In a *fully reducible* category all $X\in\mathrm{Obj}(\mathcal{C})$ are isomorphic to sums of irreducible objects.

Let $\nabla\subset\mathrm{Obj}(\mathcal{C})$ denote a set containing one object out of every equivalence class of irreducible objects.

In a *quasi rational category* every object is isomorphic to a finite sum of indecomposable objects. A *rational category* is a quasi rational category with only finitely many equivalence classes of indecomposable objects. $\mathcal{C}$ is called *irredundant*, if $X\cong Y\Rightarrow X=Y$ and it is called *locally finite* if all $\mathrm{Mor}(X,Y)$ are finite dimensional vector spaces. (Note that quasi rational categories are locally finite.) A locally finite abelian braided tensor category is called *semisimple* if all $\mathrm{End}(X)$ are semisimple algebras. By Wedderburn's theorem we get for locally finite categories the equivalence of semisimplicity and full reducibility.

In a *C^*-category* $\mathcal{C}$ all $\mathrm{Mor}(X,Y)$ are Banach spaces with an anti-linear involution $\dagger\colon \mathrm{Mor}(X,Y)\to\mathrm{Mor}(Y,X)$ such that $(fg)^\dagger=g^\dagger f^\dagger$, $\|f^\dagger f\|=\|f\|^2$ (this implies that $\mathrm{End}(X)$ is a unital C^*-algebra) and $\Psi^\dagger=\Psi^{-1}$, $\Phi^\dagger=\Phi^{-1}$.

A functor $F\colon\mathcal{C}_1\to\mathcal{C}_2$ is called *faithful* if $F\colon \mathrm{Mor}(X,Y)\to\mathrm{Mor}(F(X),F(Y))$ is injective for all $X,Y\in\mathrm{Obj}(\mathcal{C}_1)$.

A tensor category is *rigid* if it has dual objects, i.e., there is a map of objects $X\mapsto X^*$ and morphisms $\mathrm{ev}_X\in\mathrm{Mor}(X^*\otimes X,1)$ (*evaluation*) and

$\mathrm{coev}_X \in \mathrm{Mor}(1, X \otimes X^*)$ (*coevaluation*) with the properties

$$(\mathrm{id} \otimes \mathrm{ev}_X) \circ (\mathrm{coev}_X \otimes \mathrm{id}) = \mathrm{id}_X, \qquad (\mathrm{ev}_X \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{coev}_X) = \mathrm{id}_{X^*}. \tag{1}$$

The object mapping $*$ extends naturally to an involutive (in the sense that $* \circ *$ is equivalent to Id) contravariant functor by the definition $f^* := (\mathrm{ev}_Y \otimes \mathrm{id}) \circ (\mathrm{id} \otimes f \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{coev}_X) \in \mathrm{Mor}(Y^*, X^*)$ for $f \in \mathrm{Mor}(X, Y)$.

Evaluation and coevaluation are unique up to a unique isomorphism. It is always assumed that coev is an isometry if $\mathcal{C}$ is a C^*-category. The duality map $*$ induces an involution $X \mapsto \hat{X}$ of ∇ such that $X^* \cong \hat{X}$ for all $X \in \nabla$.

A *ribbon category* is a braided category with the additional structure of a twist, i.e., natural isomorphisms $\sigma \in \mathrm{Nat}(\mathrm{Id}, \mathrm{Id})$ obeying $\sigma(X)^* = \sigma(X^*)$ and $\Psi_{Y,X} \circ \Psi_{X,Y} \circ \sigma(X \otimes Y) = \sigma(X) \otimes \sigma(Y)$. Note that these axioms imply $(\sigma(X)^2 \otimes \mathrm{id}) \circ \mathrm{coev}_X = \Psi_{X^*,X} \circ \Psi_{X,X^*} \circ \mathrm{coev}_X$ which shows that in symmetric tensor categories one has $\sigma(X)^2 = \mathrm{id}_X$.

Every C^* category carries a natural ribbon structure: Define the *statistical parameter* by $\lambda(X) := (\mathrm{id}_X \otimes \mathrm{coev}_X^\dagger) \circ (\mathrm{id}_X \otimes \Psi_{X,X^*}^\dagger) \circ (\mathrm{coev}_X \otimes \mathrm{id}_X) \in \mathrm{End}(X)$ and write its polar decomposition as $\lambda(X) = \sigma(X)^{-1} \circ P(X)$ where the positive part is $P(X)$ and the unitary part $\sigma(X)^{-1}$ yields the desired ribbon structure.

A ribbon structure allows the definition of a trace on $\mathrm{End}(X)$ by $\mathrm{tr}_X(f) := \mathrm{ev}_X \circ (\mathrm{id} \otimes (f \circ \sigma(X)^{-1})) \circ \Psi_{X,X^*} \circ \mathrm{coev}_X$ and a dimension $d(X) := \mathrm{tr}(\mathrm{id}_X)$. One has $\mathrm{tr}(f \circ g) = \mathrm{tr}(g \circ f)$, $\mathrm{tr}(f) = \mathrm{tr}(f^*)$, $\mathrm{tr}(f \otimes g) = \mathrm{tr}(f)\mathrm{tr}(g)$, $\mathrm{tr}(f \oplus g) = \mathrm{tr}(f) + \mathrm{tr}(g)$, $\mathrm{tr}(f^\dagger) = \mathrm{tr}(f)^\dagger$ and hence $d(X \oplus Y) = d(X) + d(Y)$, $d(X \otimes Y) = d(X)d(Y)$, $d(X^*) = d(X)$. Related to the trace is the *conditional expectation* E_X: $\mathrm{End}(A \otimes X) \to \mathrm{End}(A)$, $f \mapsto (\mathrm{id}_A \otimes (\mathrm{ev}_X \circ \Psi_{X,X^*})) \circ (f \otimes \mathrm{id}_{X^*}) \circ (\mathrm{id}_A \otimes \sigma(X)^{-1} \otimes \mathrm{id}_{X^*}) \circ (\mathrm{id} \otimes \mathrm{coev}_X)$. Easy calculations show that it obeys $E_X((g \otimes \mathrm{id}) \circ h \circ (f \otimes \mathrm{id})) = g \circ E_X(h) \circ f$, $E_X(f \otimes \mathrm{id}_X) = d(X)f$, $E_X(f \otimes \Psi_{X,X}) = f \otimes \sigma(X)^{-1}$. On $\mathrm{End}(X^{\otimes n})$ the iterated expectation E_X^n coincides with tr.

In the application we have in mind, the categories are representation categories of algebras.

For an algebra A we let $\mathrm{Rep}(A)$ denote its *representation category*. The objects are the representations of A (it is common to consider only a special class of representations) and the morphisms are the intertwiners.

$\mathrm{Rep}(A)$ is a braided tensor category if A permits products of representations which are symmetric up to isomorphisms.

Examples of monoidal categories are superselection categories in algebraic quantum field theory, the Moore–Seiberg categories in conformal quantum field theory, and the representation categories of quantum groups. For any quasi triangular Hopf algebra H the class of its representations is a braided tensor category with braid isomorphism between two representations ϱ_1, ϱ_2 naturally given by

$$\Psi_{\varrho_1,\varrho_2}(v_1 \otimes v_2) := \tau \circ (\varrho_1 \otimes \varrho_2)(R)(v_1 \otimes v_2). \tag{2}$$

Here τ is the flip operator $a \otimes b \mapsto b \otimes a$ and R is the usual R-matrix, i.e., $\tau \circ \Delta(a) = R\Delta(a)R^{-1}$.

The subcategory $\mathrm{Rep}(H)^{fd}$ of finite dimensional representations is rigid thanks to the conjugate representation. Usually we will consider only this subcategory and hence omit the superscript *fd*.

2.2. *Description of Semisimple Categories via Polynomial Equations*

Let $\mathcal{C}$ denote a semisimple braided tensor category. For each triple $X, Y, Z \in \nabla$ let $N^Z_{X,Y}$ denote the dimension of $\mathrm{Mor}(X \otimes Y, Z)$ and choose a basis $\phi(e) \in \mathrm{Mor}(X \otimes Y, Z)$ $(e =^{i} \left(\begin{smallmatrix} Z \\ XY \end{smallmatrix}\right)$ is a multi-index with $i \in 1, \ldots, N^Z_{X,Y}\})$. The composite morphisms $\phi^i\left(\begin{smallmatrix} Z \\ XY \end{smallmatrix}\right) \circ \Psi_{Y,X} \in \mathrm{Mor}(Y \otimes X, Z)$ and $\phi^i\left(\begin{smallmatrix} R \\ XM \end{smallmatrix}\right) \circ (\mathrm{id}_X \otimes \phi^j\left(\begin{smallmatrix} M \\ YZ \end{smallmatrix}\right)) \in \mathrm{Mor}(X \otimes (Y \otimes Z))$ can then be expanded in the basis via matrices

$$\phi(e) \circ \Psi = \sum_f \Omega_{e,f} \phi(f) \tag{3}$$

$$\phi(e_2)(\mathrm{id} \otimes \phi(e_1)) = \sum_{e,f} F_{e_1,e_2;f,e} \phi(e)(\phi(f) \otimes \mathrm{id}). \tag{4}$$

It follows straightforwardly from the axioms of braided tensor categories that these matrices satisfy the Moore–Seiberg polynomial equations [15].

Two semisimple rigid braided tensor categories are equivalent if they are equivalent as ordinary categories and they share the same structural data Ω, F.

Moore and Seiberg have shown [15] that in the opposite direction every solution to their equations yields such a category. Their construction is essentially the following: Take a set of irreducible objects X_i, $i \in /$ and set $\mathrm{Mor}(X_i, X_j) := \mathbb{K}\delta_{i,j}\mathrm{id}_{X_i}$. Tensor products are formally introduced via $X_i \otimes X_j := \bigoplus_l V^l_{i,j} \otimes X_l$ where the $V^l_{i,j}$ are $N^l_{i,j}$ dimensional vector spaces of morphisms $\mathrm{Mor}(X_i \otimes X_j, X_l)$. The braid isomorphism operates on this tensor product via the operation of Ω on $V^l_{i,j}$.

2.3. *Weak Tensor Functors*

A functor $F\colon \mathcal{C}_1 \to \mathcal{C}_2$ between two monoidal categories is called *monoidal* (resp. *weak monoidal*) if there is a functorial isomorphism (resp. epimorphism) $c_{X,Y}$

$$c_{X,Y}\colon F(X) \otimes_2 F(Y) \tilde{\to} F(X \otimes_1 Y) \tag{5}$$

such that F becomes compatible with the associator and the unit:

$$\begin{array}{ccccc} F(X)\otimes(F(Y)\otimes F(Z)) & \xrightarrow{1\otimes c} & F(X)\otimes F(Y\otimes Z) & \xrightarrow{c} & F(X\otimes(Y\otimes Z)) \\ \big\downarrow{\scriptstyle\Phi_2} & & & & \big\downarrow{\scriptstyle F(\Phi_1)} \\ (F(X)\otimes F(Y))\otimes F(Z) & \xrightarrow{c\otimes 1} & F(X\otimes Y)\otimes F(Z) & \xrightarrow{c} & F((X\otimes Y)\otimes Z) \end{array} \tag{6}$$

$$l_2|_{F(\mathrm{Obj}(\mathcal{C}_1))} = c^{-1}\circ F(l_1)\colon F(X) \mapsto F(1) \otimes_2 F(X) \cong 1 \otimes_2 F(X). \tag{7}$$

A functor between two (braided) tensor categories is called *symmetric* if it is compatible with the braid isomorphism, i.e., for all $X, Y \in \mathrm{Obj}(\mathcal{C})$ the diagram

$$\begin{array}{ccc} F(X)\otimes F(Y) & \xrightarrow{c} & F(X\otimes Y) \\ \big\downarrow{\scriptstyle\Psi_2} & & \big\downarrow{\scriptstyle F(\Psi_1)} \\ F(Y)\otimes F(X) & \xrightarrow{c} & F(Y\otimes X) \end{array} \tag{8}$$

is commutative. A monoidal functor between braided tensor categories is called a *tensor functor* if

$$F(\Psi(X\otimes Y)) \cong F(X\otimes Y). \tag{9}$$

(This property follows in all cases with exception of the ultra weak case from the other axioms. One could therefore formulate most of the present paper using only the term monoidal functor.)

If (6) is not required F is called a *quasi tensor functor* and if $c_{X,Y}$ is only an epimorphism (but with $c_{X,1}$ and $c_{1,X}$ remaining isomorphisms) with right inverse $c^{-1}_{X,Y}$ then F is only a *weak quasi tensor functor*. Finally F is called an *ultra weak quasi tensor functor* if (7) and $c_{1,X}, c_{X,1}$ are not postulated to be isomorphisms but $c_{1,X} = c_{X,1}\circ\Psi^{\mathcal{C}_2}$.

If $\mathcal{C}_1$ and $\mathcal{C}_2$ are rigid then we demand in addition the existence of functorial isomorphisms $d_X\colon F(X)^* \to F(X^*)$.

If both categories are C^* then F is called *isometric* if $F(f^\dagger) = F(f)^\dagger$. Consistency with the tensor product requires then c^{-1} to be an isometry (This is implied by the following calculation: $c\circ(F(f^\dagger)\otimes F(g^\dagger))\circ c^{-1} = (f^\dagger \otimes g^\dagger) = F((f\otimes g)^\dagger) = F(f\otimes g)^\dagger = (c\circ(F(f)\otimes F(g))\circ c^{-1})^\dagger = c^{-1\dagger}\circ F(f^\dagger)\otimes F(g^\dagger))\circ c^\dagger$).

$\mathcal{C}_1$ and $\mathcal{C}_2$ are *equivalent as braided tensor categories* if they are equivalent as usual categories with symmetric tensor functors.

2.4. *Construction of (Weak) Quasi Tensor Functors*

DEFINITION 1. A function defined on the irreducible objects of a semisimple, rigid braided tensor category D: $\mathrm{Obj}_{irr}(\mathcal{C}) \to \mathbb{N}_0$ which is constant on equivalence classes is called a *weak dimension function*, if

$$D(1) = 1, \qquad D(X) = D(X^*),$$
$$D(X)D(Y) \geq \sum_{Z \in \nabla} D(Z)\dim(\mathrm{Mor}(X \otimes Y, Z)). \tag{10}$$

D is called a *dimension function* if equality holds.

Dimension functions allow the construction of monoidal functors:

PROPOSITION 1. *Let* $\mathcal{C}$ *be a quasi rational semisimple, rigid, braided tensor category and* D: $\mathrm{Obj}(\mathcal{C}) \to \mathbb{N}$ *a (weak) dimension function. Then there is a faithful (weak) quasi tensor functor* F: $\mathcal{C} \to \mathrm{Vec}$ *into the category of finite dimensional vector spaces.*

The following lemma will be used in the proof of the proposition.

LEMMA 2. *Let* $X \in \mathrm{Obj}(\mathcal{C})$ *be an irreducible object in a semisimple category. Then for all* $Y \in \mathrm{Obj}(\mathcal{C})$ *we have* $\mathrm{Mor}(Y, X) \cong \mathrm{Mor}(X, Y)^{*\mathrm{Vec}}$.

Proof. Let $g \in \mathrm{Mor}(Y, X)$ and define $\lambda_g \in \mathrm{Mor}(X, Y)^{*\mathrm{Vec}}$ by $\lambda_g(f) := g \circ f \in \mathrm{Mor}(X, X)$. This pairing is non-degenerate: Assume $g \neq 0$. Then, by semisimplicity, $Y \cong X \oplus Y_1$. But this implies existence of a $f \in \mathrm{Mor}(Y, X)$ such that $g \circ f = \mathrm{id} \neq 0$. ∎

Proof. For $X \in \nabla$ let $F(X) := \mathbb{K}^{D(X)}$ and for arbitrary objects $Y \in \mathrm{Obj}(\mathcal{C})$ this is extended via $F(Y) := \oplus_{X \in \nabla} \mathrm{Mor}(X, Y) \otimes F(X)$. F has to map morphisms $f \in \mathrm{Mor}(Y_1, Y_2)$ to morphisms $F(f) \in \mathrm{Mor}(F(Y_1), F(Y_2))$. Because of linearity, $F(f)$ needs only to be defined on the summands of type $\mathrm{Mor}(X, Y_1) \otimes F(X)$. Let $F(f)(g \otimes x) := f \circ g \otimes x$, $x \in F(X)$ for $g \in \mathrm{Mor}(X, Y_1)$.

Assume $f_1, f_2 \in \mathrm{Mor}(Y_1, Y_2)$, $F(f_1) = F(f_2)$. By the definition of F this implies that for all $X \in \nabla$ and for all $g \in \mathrm{Mor}(X, Y_1)$ we have $f_1 \circ g = f_2 \circ g$. Since $\mathcal{C}$ is assumed to be semisimple we have an isomorphism $\phi \in \mathrm{Mor}(X_{i_1} \oplus \cdots \oplus X_{i_n}, Y_1)$ with $X_{i_l} \in \nabla$. From this we get $p_{i_l} \in \mathrm{Mor}(X_{i_1} \oplus \cdots \oplus X_{i_n}, X_{i_l})$, $q_{i_l} \in \mathrm{Mor}(X_{i_l}, Y_1)$ such that $\phi = \sum_l q_{i_l} \circ p_{i_l}$. Now ϕ is epi and we have $f_1 \circ q_{i_l} = f_2 \circ q_{i_l}$ by the above remark. Hence $f_1 \circ \phi = f_2 \circ \phi$ and by this $f_1 = f_2$: F is faithful.

F satisfies $F(Y^*) \cong F(Y)^*$:

$$F(Y^*) = \bigoplus_{X \in \nabla} \mathrm{Mor}(X, Y^*) \otimes F(X) \cong \bigoplus_{X \in \nabla} \mathrm{Mor}(X^*, Y^*) \otimes F(X^*)$$

$$\cong \bigoplus_{X \in \nabla} \mathrm{Mor}(X, Y)^* \otimes F(X^*)$$

$$\cong \bigoplus_{X \in \nabla} \mathrm{Mor}(X, Y)^* \otimes F(X)^* = F(Y)^*.$$

The lemma is used in the third step. The fourth step uses the fact that $F(X)$ and $F(X^*)$ are vector spaces of equal dimension.

For every pair of irreducible objects $X_1, X_2 \in \nabla$ we choose an arbitrary (epi/iso)morphism

$$C_{X_1, X_2} \colon F(X_1) \otimes F(X_2) \to F(X_1 \otimes X_2) = \bigoplus_{X \in \nabla} \mathrm{Mor}(X, X_1 \otimes X_2) \otimes F(X).$$

c is defined as an extension of C:

$$c_{Y_1, Y_2} : \quad F(Y_1) \otimes F(Y_2) \to F(Y_1 \otimes Y_2)$$

$$c_{Y_1, Y_2} : \quad \Big(\bigoplus_{X_1 \in \nabla} \mathrm{Mor}(X_1, Y_1) \otimes F(X_1) \Big) \otimes \Big(\bigoplus_{X_2 \in \nabla} \mathrm{Mor}(X_2, Y_2) \otimes F(X_2) \Big)$$

$$\to \bigoplus_{X \in \nabla} \mathrm{Mor}(X, Y_1 \otimes Y_2) \otimes F(X)$$

$$c_{Y_1, Y_2} := \bigoplus_{X_1, X_2 \in \nabla} (\Gamma \otimes \mathrm{id}) \circ \big(\mathrm{id} \otimes \mathrm{id} \otimes C_{X_1, X_2}\big) \circ \tau_{2,3}$$

$$\Gamma : \quad \mathrm{Mor}(X_1, Y_1) \otimes \mathrm{Mor}(X_2, Y_2) \otimes \mathrm{Mor}(X, X_1 \otimes X_2)$$

$$\to \mathrm{Mor}(X, Y_1 \otimes Y_2)$$

$$\Gamma(f_1 \otimes f_2 \otimes g) := (f_1 \otimes f_2) \circ g.$$

c behaves as a functorial, i.e., for $f_i \in \mathrm{Mor}(Y_i, \tilde{Y}_i)$, $i = 1, 2$, we have $F(f_1 \otimes f_2) \circ c_{Y_1, Y_2} = c_{\tilde{Y}_1, \tilde{Y}_2} \circ (F(f_1) \otimes F(f_2))$. To see this we introduce $v_i \in F(Y_i)$, $i = 1, 2$, as

$$v_i = \bigoplus_{A_i \in \nabla} g^{(A_i)} \otimes x^{(A_i)}, \qquad x^{(A_i)} \in F(A_i), g^{(A_i)} \in \mathrm{Mor}(A_i, Y_i).$$

Using the definitions and the shorthand $C_{A_1,A_2}(x^{(A_1)} \otimes x^{(A_2)}) = \bigoplus_{B \in \nabla} q^B_{A_1,A_2} \otimes x^B_{A_1,A_2}$ we get

$$\begin{aligned}
& c_{\tilde{Y}_1,\tilde{Y}_2} \circ \big(F(f_1) \otimes F(f_2)\big)(v_1 \otimes v_2) \\
&\quad = \bigoplus_{A_1,A_2 \in \nabla} (\Gamma \otimes \mathrm{id}) f_1 \circ g^{(A_1)} \otimes f_2 \circ g^{(A_2)} \otimes C_{A_1,A_2}(x^{(A_1)} \otimes x^{(A_2)}) \\
&\quad = \bigoplus_{A_1,A_2,B \in \nabla} \big(f_1 \circ g^{(A_1)} \otimes f_2 \circ g^{(A_2)}\big) \circ q^B_{A_1,A_2} \otimes x^B_{A_1,A_2} \\
&\quad = F(f_1 \otimes f_2) \circ \Big(\bigoplus_{A_1,A_2,B \in \nabla} \big(g^{(A_1)} \otimes g^{(A_2)}\big) \circ q^B_{A_1,A_2} \otimes x^B_{A_1,A_2} \Big) \\
&\quad = F(f_1 \otimes f_2) \circ c_{Y_1,Y_2}(v_1 \otimes v_2). \quad \blacksquare
\end{aligned}$$

Remark 1. The functorial isomorphisms $d_Y \colon F(Y)^* \to F(Y^*)$ can be displayed explicitly. First note that

$$F(Y^*) = \bigoplus_{X \in \nabla} \mathrm{Mor}(X, Y^*) \otimes F(X) = \bigoplus_{X \in \nabla} \mathrm{Mor}(\hat{X}, Y^*) \otimes F(\hat{X}).$$

We calculate d_Y^{-1} operating on one summand $u^* \otimes v \in \mathrm{Mor}(\hat{X}, Y^*) \otimes F(\hat{X}) \subset F(Y^*)$. We have to choose (actually z_X and $\tilde{d}_X$ are fixed by demanding $F(\mathrm{ev}_Y) \circ c_{Y^*,Y} \circ (d_Y \otimes \mathrm{id}) = \mathrm{ev}_{F(Y)}$) isomorphisms $z_X = \tilde{z}_X^* \in \mathrm{Mor}(X^*, \hat{X})$ and $\tilde{d}_X \colon F(X)^* \to F(\hat{X})$ for all $X \in \nabla$. Now we map

$$u^* \otimes v \mapsto u^* \circ \widetilde{z_X}^* \otimes v = \big(\widetilde{z_X} \circ u\big)^* \otimes v \in \bigoplus \mathrm{Mor}(Y, X)^{*\mathcal{C}} \otimes F(\hat{X}).$$

With the techniques of the preceding lemma this is $\lambda_{\widetilde{z_X \circ u}} \otimes v \in \mathrm{Mor}(X, Y)^{*\mathrm{Vec}}$. Finally applying $\mathrm{id} \otimes \widetilde{d_X}^{-1}$ yields $\lambda_{\tilde{z}_X \circ u} \otimes \tilde{d}_X^{-1}(v) \in \bigoplus \mathrm{Mor}(X, Y)^{\otimes \mathrm{Vec}} \otimes F(X)^* = F(Y)^*$.

Using this description of d_Y one can show $d_W \circ F(f)^* = F(f^*) \circ d_Y$ for $f \in \mathrm{Mor}(W, Y)$.

Remark 2. With arbitrary choices of the C morphisms in the proof of the theorem the constructed functor will in general not be compatible with the associativity constraints in the sense of (6). For a strict (i.e., $\Phi = \mathrm{id}$) category (6) reads

$$c_{X \otimes Y, Z} \circ (c_{X,Y} \otimes \mathrm{id}) = c_{X, Y \otimes Y} \circ (\mathrm{id} \otimes c_{Y,Z}).$$

This equation can be interpreted as a non-abelian two-cocycle condition. We will take up this point later on.

Proposition 1 reduces the problem of finding a functor to finding a dimension function. This is possible:

PROPOSITION 3. *On rational, semisimple, rigid, braided tensor categories there exist always weak dimension functions,*

$$D(1) := 1, \qquad D(X) := \dim \bigoplus_{Y,Z\in\nabla} \mathrm{Mor}(Y\otimes X, Z) = \sum_{i,j} N_{X,i}^{j}. \tag{11}$$

Other possibilities are $D(1) := 1$, $D(X) := \max_{I,J\neq 0}\sum_K N_{I,J}^{K}$, *and in the algebraic formulation of* QFT [1], $D(\rho) := \dim(\mathrm{span}\{(\rho_I\rho, \rho_J) | \rho_I, \rho_J \in \nabla\})$.

Proof.

$$\begin{aligned} D(X)D(Y) &= \Big(\sum_{s,r} N_{X,S}^{r}\Big)\Big(\sum_{S,R} N_{Y,S}^{R}\Big) = \sum_{s,r,S,R} N_{X,s}^{r} N_{Y,S}^{R} \\ &\geq \sum_{K,N,M} N_{X,N}^{K} N_{Y,K}^{M} = \sum_{K,N,M} N_{X,Y}^{K} N_{K,N}^{M} = D(X\otimes Y). \quad \blacksquare \end{aligned}$$

2.5. *Weak and Ultra Weak Quasi Hopf Algebras*

The structure of most rational semisimple tensor categories does not allow non-weak dimension functions [5, 16]. This results from the fact that ordinary quantum groups at roots of unity have indecomposable representations of zero (quantum) dimension d. They arise in the tensor product decomposition of simple representations and spoil many of the intended applications, e.g., the interpretation of ordinary quantum groups as gauge symmetry algebras is impossible.

To discard the indecomposable representations one has to allow that the coproduct of unity, $\Delta(1)$, is not $1\otimes 1$, but a projector on the fully decomposable part. This is the idea of Mack and Schomerus encoded in the definition of *weak quasi Hopf algebras* as modifications of Drinfeld's quasi Hopf algebras. As those they are unital algebras H together with a comultiplication $\Delta: H \to H\otimes H$, counit $\epsilon: H \to \mathbb{K}$, and antipode $S: H \to H$. The coproduct is commutative up to conjugation by $R \in H\otimes H$ and associative up to conjugation by $\phi \in H\otimes H\otimes H$, that is, for all $h\in H$ one has $\phi((\mathrm{id}\otimes\Delta)\circ\Delta(h)) = ((\Delta\otimes\mathrm{id})\circ\Delta(h))\phi$.

$$\phi^{-1}\phi = (\mathrm{id}\otimes\Delta)\Delta(1) \tag{12}$$

$$\phi\phi^{-1} = (\Delta\otimes\mathrm{id})\Delta(1) \tag{13}$$

$$RR^{-1} = \Delta'(1), \qquad \Delta' := \tau\circ\Delta \tag{14}$$

$$R^{-1}R = \Delta(1) \tag{15}$$

$$(\mathrm{id}\otimes\mathrm{id}\otimes\epsilon)(\phi) = (\mathrm{id}\otimes\epsilon\otimes\mathrm{id})(\phi) = (\epsilon\otimes\mathrm{id}\otimes\mathrm{id})(\phi) = \Delta(1). \tag{16}$$

For the sake of completeness we also recall Drinfeld's form of the antipode axiom for quasi Hopf algebras. It states the existence of two invertible elements $\alpha, \beta \in H$ such that the following relations hold:

$$\epsilon(a)\alpha = \sum_i S\big(a_i^{(1)}\big)\alpha a_i^{(2)} \qquad \forall a \in H \tag{17}$$

$$\epsilon(a)\beta = \sum_i a_i^{(1)}\beta S\big(a_i^{(2)}\big) \qquad \forall a \in H \tag{18}$$

with

$$\Delta(a) = \sum_i a_i^{(1)} \otimes a_i^{(2)}.$$

Is there some kind of algebra generalizing the ((weak) quasi) quantum groups and observable algebras of algebraic quantum field theory? We believe that ultra weak quasi quantum groups as introduced in [2] may provide an answer.

DEFINITION 2 (Ultra Weak Quasi Hopf Algebra). Let A denote an unital algebra. An *A ultra weak quasi Hopf algebra H* is an A-bialgebra H (left and right multiplication are denoted by μ_l: $A \otimes H \to H$, μ_r: $H \otimes A \to H$) and algebra morphisms η: $A \to H$, ϵ: $H \to A$ such that all axioms of a weak quasi Hopf algebra are fulfilled with the exception of unit/counit properties which are replaced by

$$\begin{aligned} &\mu_l(\epsilon \otimes \mathrm{id})\Delta = \mu_r(\mathrm{id} \otimes \epsilon)\Delta = \mathrm{id}_H, \qquad m(\mathrm{id} \otimes \eta) = \mu_r, \\ &m(\eta \otimes \mathrm{id}) = \mu_l. \end{aligned} \tag{19}$$

3. RECONSTRUCTION THEOREMS

Historically the first reconstruction theorem was the famous Tannaka–Krein theorem: Given a symmetric tensor category and a faithful tensor functor to Vec there is a group with the given category as a representation category. Majid proved reconstruction theorems for quasi-triangular Hopf algebras and quasi Hopf algebras. A reconstruction theorem for weak quasi Hopf algebras was suggested by Kerler without a proof.

The *forgetful functor V*: $\mathrm{Rep}(H) \to \mathrm{Vec}$ assigns to each representation the underlying vector space.

We start in Lemma 4 by reviewing Majid's reconstruction theorem for quasi Hopf algebras. The starting point for his construction is the set

$\mathrm{Nat}(F, F)$ of natural transformations of F.

$$H := H(\mathcal{C}, F) := \mathrm{Nat}(F, F)$$
$$= \{h\colon \mathrm{Obj}(\mathcal{C}) \to \mathrm{End}_{\mathrm{Vec}} | h_X \in \mathrm{End}(F(X)),$$
$$F(f) \circ h_X = h_Y \circ F(f)\ \forall X, Y \in \mathrm{Obj}(\mathcal{C})\ \forall f \in \mathrm{Mor}(X, Y)\}.$$

LEMMA 4. *H is a quasitriangular (quasi) Hopf algebra if F is a (quasi) tensor functor.*

Proof. H is a vector space by pointwise addition. The multiplication is also defined pointwise: $(hg)_X := h_X \circ g_X$ $X \in \mathrm{Obj}(\mathcal{C})$, $h, g \in H$. The unit is $X \mapsto 1_X = \mathrm{id}_{F(X)}$. (The ultra weak case is handled in Lemma 15.)

In Vec the following relation holds: $\mathrm{End}(F(X)) \otimes \mathrm{End}(F(Y)) \cong \mathrm{End}(F(X) \otimes F(Y))$ so that $H \otimes H$ is given by functions in two variables X, Y (i.e., we understand the tensor product algebraically), which map to $\mathrm{End}(F(X) \otimes F(Y))$. The coproduct $\Delta\colon H \to H \otimes H$ is defined by

$$\Delta(h)_{X,Y} := c^{-1}_{X,Y} \circ h_{X\otimes Y} \circ c_{X,Y}. \tag{20}$$

This is compatible with multiplication:

$$(\Delta(h)\Delta(g))_{X,Y} = \Delta(h)_{X,Y} \circ \Delta(g)_{X,Y}$$
$$= c^{-1}_{X,Y} \circ h_{X\otimes Y} \circ c_{X,Y} \circ c^{-1}_{X,Y} \circ g_{X\otimes Y} \circ c_{X,Y}$$
$$= c^{-1}_{X,Y} \circ h_{X\otimes Y} \circ g_{X\otimes Y} \circ c_{X,Y} = \Delta(hg)_{X,Y}.$$

The counit is $\epsilon\colon H \to \mathbb{K}$, $\epsilon(h) := h_1$,

$$((\mathrm{id} \otimes \epsilon)\Delta(h))_X = \Delta(h)_{X,1} = c^{-1}_{X,1} \circ h_{X\otimes 1} \circ c_{X,1} = h_{X\otimes 1} = h_X.$$

The associator $\phi \in H \otimes H \otimes H$ is given by

$$\phi_{X,Y,Z} := (c^{-1}_{X,Y} \otimes \mathrm{id}) \circ c^{-1}_{X\otimes Y,Z} \circ F(\Phi_{X,Y,Z}) \circ c_{X,Y\otimes Z} \circ (\mathrm{id} \otimes c_{Y,Z}). \tag{21}$$

For tensor functors this is trivial because of (6). For quasi tensor functors it is invertible,

$$(\phi(1 \otimes \Delta)\Delta(h))_{X,Y,Z}$$
$$= \phi_{X,Y,Z} \circ (c_{X,Y\otimes Z} \circ (\mathrm{id} \otimes c_{Y,Z}))^{-1} \circ h_{X\otimes(Y\otimes Z)} \circ c_{X,Y\otimes Z} \circ (\mathrm{id} \otimes c_{Y,Z})$$
$$= (c^{-1}_{X,Y} \otimes \mathrm{id}) \circ c^{-1}_{X\otimes Y,Z} \circ F(\Phi_{X,Y,Z}) \circ h_{X\otimes(Y\otimes Z)} \circ c_{X,Y\otimes Z} \circ (\mathrm{id} \otimes c_{Y,Z})$$

$$((\Delta \otimes 1)\Delta(h)\phi)_{X,Y,Z}$$
$$= (c^{-1}_{X,Y} \otimes \mathrm{id}) \circ c^{-1}_{X\otimes Y,Z} \circ h_{(X\otimes Y)\otimes Z} \circ c_{X\otimes Y,Z} \circ (c_{X,Y} \otimes \mathrm{id}) \circ \phi_{X,Y,Z}$$
$$= (c^{-1}_{X,Y} \otimes 1) c^{-1}_{X\otimes Y,Z} h_{(X\otimes Y)\otimes Z} F(\Phi_{X,Y,Z}) c_{X,Y\otimes Z} (1 \otimes c_{Y,Z}).$$

Both expressions are the same because of naturality: "$F(\Phi)h = hF(\Phi)$". This shows quasi coassociativity. For tensor functors this reduces to coassociativity and for weak quasi tensor functors ϕ remains quasi invertible.

For the proof of $(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\phi) \cdot (\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\phi) = (1 \otimes \phi)(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\phi)(\phi \otimes 1)$, we refer to Majid's original work [12].

F is a functor between rigid braided tensor categories. There are isomorphisms $d_X\colon F(X)^* \cong F(X^*)$ and $d_X^*\colon F(X^*)^* \cong F(X)$. They are used in the definition of the antipode:

$$(Sh)_X := d_X^* \circ (h_{X^*})^* \circ d_X^{-1}. \tag{22}$$

The proof of the antipode identity will be given in Lemma 12.

H is quasitriangular by means of $R \in H \otimes H$:

$$R_{X,Y} := \Psi^{\mathrm{Vec}\,-1}_{F(X),F(Y)} \circ c^{-1}_{Y,X} \circ F(\Psi_{X,Y}) \circ c_{X,Y}. \tag{23}$$

R relates the coproduct and the opposite coproduct:

$$\begin{aligned}
&(R\Delta(h)R^{-1})_{X,Y}\\
&\quad= \Psi^{\mathrm{Vec}\,-1}_{F(X),F(Y)} \circ c^{-1}_{Y,X} \circ F(\Psi_{X,Y}) \circ c_{X,Y} \circ c^{-1}_{X,Y} \circ h_{X\otimes Y} \circ c_{X,Y}\\
&\qquad \times c^{-1}_{X,Y} \circ F(\Psi_{X,Y})^{-1} \circ c_{Y,X} \circ \Psi^{\mathrm{Vec}}_{F(X),F(Y)}\\
&\quad= \Psi^{\mathrm{Vec}\,-1}_{F(X),F(Y)} \circ c^{-1}_{Y,X} \circ F(\Psi_{X,Y}) \circ h_{X\otimes Y}\\
&\qquad \circ F(\Psi_{X,Y})^{-1} \circ c_{Y,X} \circ \Psi^{\mathrm{Vec}}_{F(X),F(Y)}\\
&\quad= \Psi^{\mathrm{Vec}\,-1}_{F(X),F(Y)} \circ c^{-1}_{Y,X} \circ h_{Y\otimes X} \circ c_{Y,X} \circ \Psi^{\mathrm{Vec}}_{F(X),F(Y)} = \Delta'(h)_{X,Y}.
\end{aligned}$$

For the proof of the other two quasitriangularity equations we refer once more to [11, 2]. ∎

LEMMA 5. *If F is a weak quasi tensor functor then H is a weak quasi Hopf algebra.*

Proof. The additional axioms (the statements already proven remain true!) are easily verified using $cc^{-1} = 1$, $c^{-1}c \neq 1$: For (16) we calculate

$$\begin{aligned}
&(\mathrm{id} \otimes \mathrm{id} \otimes \epsilon)(\phi)_{X,Y}\\
&\quad= (c^{-1}_{X,Y} \otimes \mathrm{id}) \circ c^{-1}_{X\otimes Y,1} \circ F(\Phi_{X,Y,1}) \circ c_{X,Y} \circ (\mathrm{id} \otimes c_{Y,1})\\
&\quad= c^{-1}_{X,Y} \circ c_{X,Y} = \Delta(1)_{X,Y}.
\end{aligned}$$

And for (15),

$$
\begin{aligned}
&(R^{-1}R)_{X,Y}\\
&\quad = c_{X,Y}^{-1} \circ F\big(\Psi_{X,Y}^{-1}\big) \circ c_{Y,X} \circ \Psi^{\mathrm{Vec}} \circ \Psi^{\mathrm{Vec}-1} \circ c_{Y,X}^{-1} \circ F(\Psi_{X,Y}) \circ c_{X,Y}\\
&\quad = c_{X,Y}^{-1} \circ c_{X,Y} = \Delta(1)_{X,Y}.
\end{aligned}
$$

Similarly one gets (13):

$$
\begin{aligned}
&\phi_{X,Y,Z} \circ \phi_{X,Y,Z}^{-1}\\
&\quad = \big(c_{X,Y}^{-1} \otimes \mathrm{id}\big) \circ c_{X\otimes Y,Z}^{-1} \circ F(\Phi_{X,Y,Z}) \circ c_{X,Y\otimes Z} \circ (\mathrm{id} \otimes c_{Y,Z})\\
&\qquad \times \big(\mathrm{id} \otimes c_{Y,Z}^{-1}\big) \circ c_{X,Y\otimes Z}^{-1} \circ F(\Phi_{X,Y,Z})^{-1} \circ c_{X\otimes Y,Z} \circ (c_{X,Y} \otimes \mathrm{id})\\
&\quad = \big(c_{X,Y}^{-1} \otimes \mathrm{id}\big) \circ c_{X\otimes Y,Z}^{-1} \circ c_{X\otimes Y,Z} \circ (c_{X,Y} \otimes \mathrm{id})\\
&\quad = \big(c_{X,Y}^{-1} \otimes \mathrm{id}\big)\Delta(1)_{X\otimes Y,Z} \circ (c_{X,Y} \otimes \mathrm{id})\\
&\quad = ((\Delta \otimes \mathrm{id})\Delta(1))_{X,Y,Z}.
\end{aligned}
$$

Equation (12) is proven in the same way, just as (14). ∎

LEMMA 6. *The vector spaces $F(X)$ are representation spaces of H. The functor $G\colon \mathcal{C} \to \mathrm{Rep}(H)$ is a full tensor functor.*

Proof. The representations are $\varrho_X(h).v := h_X(v)$ $h \in H$, $v \in F(X)$. This induces a functor $G\colon \mathcal{C} \to \mathrm{Rep}(H)$. Morphisms $f \in \mathrm{Mor}(X,Y)$ are mapped to intertwiners $G(f) = F(f)$: $G(f) \circ \varrho_X(h) = F(f) \circ h_X = h_Y \circ F(f) = \varrho_Y(h) \circ G(f)$. G is a tensor functor:

$$
\begin{aligned}
(G(X) \otimes G(Y))(h) &= (\varrho_X \otimes \varrho_Y)(\Delta(h)) = \Delta(h)_{X,Y}\\
&= c_{X,Y}^{-1} \circ h_{X\otimes Y} \circ c_{X,Y} = c_{X,Y}^{-1} \circ G(X \otimes Y) \circ c_{X,Y}.
\end{aligned}
$$

Here the $c_{X,Y}$ are as maps of vector spaces the same as the $c_{X,Y}$ of the functor F, but because the tensor product on the left hand side of this equation is in the representation category of H they are restricted to the representation subspace and are therefore isomorphisms. The definitions of R and ϕ are precisely the statements that G is compatible with associativity and braid isomorphisms.

G is full, because every morphism T in $\mathrm{Rep}(H)$ $(T\varrho_Y = \varrho_X T)$ is a constraint that can only exist if it is of the form $T = F(f)$. ∎

LEMMA 7. *Let $X, Y \in \mathrm{Obj}(\mathcal{C})$ and $h \in H$. If X and Y are isomorphic then h_X is determined uniquely by h_Y. If $\mathcal{C}$ is semisimple then $h \in H$ is determined by its values on ∇ where it may take arbitrary values.*

Proof. If $\phi \in \mathrm{Mor}(X, Y)$ is iso then the naturality condition can be expressed as $h_Y = F(\phi) \circ h_X \circ F(\phi^{-1})$.

Let h be defined on ∇. Since we assume $\mathcal{C}$ to be semisimple, every object is isomorphic to a direct sum of objects in ∇. By the above remark h is therefore uniquely defined on all objects if it is uniquely defined on direct sums. Consider $\oplus_{i \in I} X_i$, $X_i \in \nabla$. We have morphisms $p_j \in \mathrm{Mor}(\oplus_i X_i, X_j)$ and $q_j \in \mathrm{Mor}(X_j, \oplus_i X_i)$ such that $\mathrm{id}_{\oplus X_i} = \sum_j q_j \circ p_j$. Naturality implies $F(p_j) \circ h_{\oplus X_i} = h_{X_j} \circ F(p_j)$ and hence we have

$$\begin{aligned} h_{\oplus X_i} &= F\Big(\sum_j q_j \circ p_j\Big) \circ h_{\oplus X_i} \\ &= \sum_j F(q_j) \circ F(p_j) \circ h_{\oplus X_i} \\ &= \sum_j F(q_j) \circ h_{X_j} \circ F(p_j). \end{aligned}$$

On different objects in ∇ the function h may take arbitrary values because there are no morphisms (and hence no naturality constraints) between inequivalent irreducible objects in an abelian category. ∎

LEMMA 8. *G is surjective in the sense that it hits every class of irreps of H.*

Proof. We use Lemma 7. It shows that H is a direct sum of full matrix algebras $M_n(\mathbb{K})$. Each of them has only one irrep. And so H has no other irreducible representations, because all representations have to reflect commutativity of the summands and must therefore annihilate all summands but one. Therefore H has no more irreducible representation classes than $\mathcal{C}$ has irreducible object classes. ∎

LEMMA 9. *Faithfulness of F implies that inequivalent objects yield inequivalent representations.*

Proof. Assume X, Y to be inequivalent objects which are mapped to equivalent representations, i.e., $F(X) = F(Y)$, $\forall h \in H$, $h_X = \varphi \circ h_Y \circ \varphi^{-1}$ with an isomorphism φ: $F(X) \to F(Y) = F(X)$. So the value of h on X is determined uniquely by its value on Y. This can be done by naturality only if $\exists f \in \mathrm{Mor}(X, Y) \exists g \in \mathrm{Mor}(Y, X)$ such that $F(f) = \varphi$, $F(g) = \varphi^{-1}$. But then (by faithfulness) f and g are iso ($\mathrm{id}_{F(Y)} = F(f)F(g) = F(fg)$; because of faithfulness only id_Y is mapped to $\mathrm{id}_{F(Y)}$ and hence $f = g^{-1}$) contracting our hypothesis. ∎

LEMMA 10. *$F(\phi^i(\substack{Z \\ XY})) \circ c$ form a basis of morphisms in* $\mathrm{Mor}(F(X) \otimes F(Y), F(Z))$ *if F is faithful.*

Proof. They are linearly independent: Assume $0 = \sum_i \alpha_i F(\phi^i(\binom{Z}{XY})) \circ c$. By surjectivity of c and linearity of F this implies $0 = F\,(\sum_i \alpha_i \phi^i(\binom{Z}{XY}))$ and faithfulness of F yields a contradiction. Further they span the whole space since G is full. ∎

LEMMA 11. *If F is faithful then* $\mathcal{C}$ *and* Rep(H) *have the same structural constants and are therefore equivalent as braided tensor categories.*

Proof. Describe $\mathcal{C}$ as in Subsection 2.2. According to this presentation we have for $X, Y, Z \in \nabla$ matrices Ω that satisfy $\phi^i(\binom{Z}{XY}) \circ \Psi_{Y,X} = \sum_j \Omega_{i,j} \phi^j(\binom{Z}{YX})$. We apply F, multiply c from the right, introduce $1 = cc^{-1}$, and use linearity of F to get $F(\phi^i(\binom{Z}{XY})) \circ c \circ c^{-1} \circ F(\Psi_{Y,X}) \circ c = \sum_j \Omega_{i,j} F(\phi^j(\binom{Z}{YX})) \circ c$. Taking Lemma 10 into account and observing that $c^{-1} \circ F(\Psi) \circ c$ is (by (2) and (23)) nothing than $\Psi^{\mathrm{Rep}(H)}$ this shows that $\mathcal{C}$ and Rep(H) have the same structure constants. ∎

LEMMA 12. Rep(H) *is rigid if F is faithful. The antipode* (22) *satisfies* (17).

Proof. Rep(H) is rigid. The dual representation of ϱ_X is given by $(\varrho_X)^*(h) := (\varrho_X(S(h)))^*$ acting on $F(X)^*$. Note that $*$ on the left hand side of this definition is the duality in Rep while on the right hand side it is the duality in Vec.

Evaluation and coevaluation are given by

$$\mathrm{ev}^{\mathrm{Rep}}_{\varrho X} = F(\mathrm{ev}_X) \circ c_{X^*,X} \circ (d_X \otimes \mathrm{id})$$
$$\mathrm{coev}^{\mathrm{Rep}}_{\varrho X} = \left(\mathrm{id} \otimes d_X^{-1}\right) \circ c^{-1}_{X,X^*} \circ F(\mathrm{coev}_X).$$

We verify the intertwining property for $\mathrm{ev}^{\mathrm{Rep}}$ (the proofs for the coevaluation are identical up to duality symmetry and are not displayed):

$$\begin{aligned}
&\mathrm{ev}^{\mathrm{Rep}}_{\varrho X} \circ (\varrho_X^* \otimes \varrho_X)(h) \\
&\quad = F(\mathrm{ev}_X) \circ c_{X^*,X} \circ (d_X \otimes \mathrm{id}) \\
&\qquad \circ \left((d_X^* \otimes \mathrm{id}) \circ \left(c^{-1}_{X^*,X} \circ h_{X^* \otimes X} \circ c_{X^*,X}\right)^{* \otimes \mathrm{id}} \circ \left(d_X^{*-1} \otimes \mathrm{id}\right)\right)^{* \otimes \mathrm{id}} \\
&\quad = F(\mathrm{ev}_X) \circ c_{X^*,X} \circ (d_X \otimes \mathrm{id}) \circ \left(d_X^{-1} \otimes \mathrm{id}\right) \circ c^{-1}_{X^*,X} \\
&\qquad \circ h_{X^* \otimes X} \circ c_{X^*,X} \circ (d_X \otimes \mathrm{id}) \\
&\quad = F(\mathrm{ev}_X) \circ h_{X^* \otimes X} \circ c_{X^*,X} \circ (d_X \otimes \mathrm{id}) \\
&\quad = h_1 \circ F(\mathrm{ev}_X) \circ c_{X^*,X} \circ (d_X \otimes \mathrm{id}) = \varrho_1(h) \circ \mathrm{ev}^{\mathrm{Rep}}_{\varrho X}.
\end{aligned}$$

Because Rep is in general not strict (even if $\mathcal{C}$ is strict ($\Phi^{\mathcal{C}} = \mathrm{id}$) which we will assume) we have to insert an associator into the fundamental ev/coev property (1):

$$
\begin{aligned}
&\left(\mathrm{id} \otimes \mathrm{ev}^{\mathrm{Rep}}_{\varrho X}\right) \circ \Phi^{\mathrm{Rep}-1} \circ \left(\mathrm{coev}^{\mathrm{Rep}}_{\varrho X} \otimes \mathrm{id}\right) \\
&\quad = \left(\mathrm{id} \otimes \mathrm{ev}^{\mathrm{Rep}}_{\varrho X}\right) \circ (\varrho_X \otimes \varrho^*_X \otimes \varrho_X)(\phi^{-1}) \circ \left(\mathrm{coev}^{\mathrm{Rep}}_{\varrho X} \otimes \mathrm{id}\right) \\
&\quad = \left(\mathrm{id} \otimes F(\mathrm{ev}_X)\right) \circ \left(\mathrm{id} \otimes c_{X^*,X}\right) \circ \left(\mathrm{id} \otimes d_X \otimes \mathrm{id}\right) \\
&\qquad \circ \left(\mathrm{id} \otimes d^{-1}_X \otimes \mathrm{id}\right) \circ \left(\mathrm{id} \otimes c^{-1}_{X^*,X}\right) \circ c^{-1}_{X, X^* \otimes X} \circ F(\Phi^{\mathcal{C}-1}) \\
&\qquad \circ c_{X \otimes X^*, X} \circ \left(c_{X, X^*} \otimes \mathrm{id}\right) \\
&\qquad \circ \left(\mathrm{id} \otimes d_X \otimes \mathrm{id}\right) \circ \left(\mathrm{id} \otimes d^{-1}_X \otimes \mathrm{id}\right) \circ \left(c^{-1}_{X,X^*} \otimes \mathrm{id}\right) \\
&\qquad \circ \left(F(\mathrm{coev}_X) \otimes \mathrm{id}\right) \\
&\quad = \left(\mathrm{id} \otimes F(\mathrm{ev}_X)\right) \circ \left(c^{-1}_{X, X^* \otimes X}\right) \circ c_{X \otimes X^*, X} \circ \left(F(\mathrm{coev}_X) \otimes \mathrm{id}\right) \\
&\quad = c^{-1}_{X,1} \circ \left(\mathrm{id} \otimes F(\mathrm{ev}_X)\right) \circ \left(F(\mathrm{coev}_X) \otimes \mathrm{id}\right) \circ c_{1,X} = \mathrm{id}.
\end{aligned}
$$

The antipode identities involve elements $\alpha, \beta \in H(\mathcal{C}, F)$,

$$
\alpha_X := \left(\mathrm{id} \otimes \mathrm{ev}^{\mathrm{Rep}}_{\varrho X}\right) \circ \left(\mathrm{coev}^{\mathrm{Vec}}_{F(X)} \otimes \mathrm{id}\right) : F(X) \to F(X) \tag{24}
$$

$$
\beta_X := \left(\mathrm{id} \otimes \mathrm{ev}^{\mathrm{Vec}}_{F(X)}\right) \circ \left(\mathrm{coev}^{\mathrm{Rep}}_{\varrho X} \otimes \mathrm{id}\right) : F(X) \to F(X). \tag{25}
$$

This gives well defined elements in H: $F(f) \circ \alpha_X = \alpha_Y \circ F(f)$ holds because d, c are functorial.

Applying $\mathrm{ev}^{\mathrm{Vec}} \otimes \mathrm{id}$ yields $\mathrm{ev}^{\mathrm{Rep}}_{\varrho X} = \mathrm{ev}^{\mathrm{Vec}} \circ (\mathrm{id} \otimes \alpha_X)$. Obviously $\alpha = \beta^{-1}$ if $\mathrm{Rep}(H)$ is strict. The proof of the antipode identity is given in the following calculation.

$$
\begin{aligned}
&(\epsilon(h)\alpha)_X \\
&\quad = \left(\mathrm{id} \otimes h_1\right) \circ \left(\mathrm{id} \otimes F(\mathrm{ev}_X)\right) \circ \left(\mathrm{id} \otimes c_{X^*,X}\right) \circ \left(\mathrm{id} \otimes d_X \otimes \mathrm{id}\right) \\
&\qquad \circ \left(\mathrm{coev}^{\mathrm{Vec}}_{F(X)} \otimes \mathrm{id}\right) \\
&\quad = \left(\mathrm{id} \otimes F(\mathrm{ev}_X)\right) \circ \left(\mathrm{id} \otimes h_{X^* \otimes X}\right) \circ \left(\mathrm{id} \otimes c_{X^*,X}\right) \\
&\qquad \otimes \left(d^*_X \otimes \mathrm{id} \otimes \mathrm{id}\right) \circ \left(\mathrm{coev}^{\mathrm{Vec}} \otimes \mathrm{id}\right) \\
&\quad = \left(d^*_X \otimes \mathrm{id} \otimes \mathrm{id}\right) \circ \left(\mathrm{id} \otimes \left(F(\mathrm{ev}_X) \circ c_{X^*,X}\right)\right)
\end{aligned}
$$

$$\circ\Big(\mathrm{id}\otimes\big(d_X\circ d_X^{-1}\big)\otimes\mathrm{id}\Big)$$

$$\circ\sum_i\Big(\mathrm{id}\otimes h_i^{(1)}\otimes h_i^{(2)}\Big)\circ\big(\mathrm{coev}^{\mathrm{Vec}}\otimes\mathrm{id}\big)$$

$$=\sum_i d_X^*\circ h_i^{(1)*}\circ d_X^{-1*}\circ\big(\mathrm{id}\otimes F(\mathrm{ev}_X)\circ c_{X^*,X}\big)\circ\big(\mathrm{id}\otimes d_X\otimes\mathrm{id}\big)$$

$$\circ\Big(\mathrm{coev}^{\mathrm{Vec}}\otimes h_i^{(2)}\Big)$$

$$=\Big(\sum_i S\big(h_i^{(1)}\big)\alpha h_i^{(2)}\Big)_X.$$

The second antipode axiom (18) involving β is established similarly. ∎

LEMMA 13. *If F is isometric then H is involutive and the representations are unitary*: $\varrho(h^\dagger)=\varrho(h)^\dagger$.

Proof. The involution is given by $(h^\dagger)_X:=(h_X)^\dagger$. Applying $F\circ\dagger=\dagger\circ F$ to the naturality condition implies that H is closed under this operation. Multiplicativity carries over from vector space endomorphisms. $\Delta(h^\dagger)=\Delta(h)^\dagger$ follows easily from the fact that $c^\dagger=c^{-1}$: $\Delta(h)^\dagger_{X,Y}=(c^{-1}_{X,Y}\circ h_{X\otimes Y}\circ c_{X,Y})^\dagger=c^{-1}_{X,Y}\circ h^\dagger_{X\otimes Y}\circ c_{X,Y}$.

The proof of the compatibility of the involution $\dagger$ and the antipode S uses the fact that in the category of finite dimensional Hilbert spaces the duality map is given by $V^*=V$ and ev/coev is given by the scalar product. Using this one calculates

$$\big(S(h)^\dagger\big)_X=\big(S(h)_X\big)^\dagger=\big(d_X^*\circ h^*_{X^*}\circ d_X^{*-1}\big)^\dagger$$

$$=d_X^{*-1\dagger}\circ h^{*\dagger}_{X^*}\circ d_X^{*\dagger}=d_X^*\circ h^{\dagger *}_{X^*}\circ d_X^{*-1}=S(h^\dagger).\quad ∎$$

LEMMA 14. *If C is a ribbon category then H is a ribbon Hopf algebra in the sense of* [17].

Proof. The ribbon element $v\in H$ is defined by $v_X:=F(\sigma(X))$. It is central because σ is functorial. Further one calculates $\epsilon(v)=F(\sigma(1))=1$ and

$$S(v)_X=d_X^*\circ(v_{X^*})^*\circ d_X^{*-1}=d_X^*\circ F(\sigma(X^*))^*\circ d_X^{*-1}$$

$$=d_X^*\circ F(\sigma(X)^*)^*\circ d_X^{*-1}=F(\sigma(X))=v_X$$

$$
\begin{aligned}
\Delta(v)_{X,Y} &= c_{X,Y}^{-1} \circ F(\sigma(X \otimes Y)) \circ c_{X,Y} \\
&= c_{X,Y}^{-1} \circ F\big(\Psi_{X,Y}^{-1}\big) \circ c_{Y,X} \circ c_{Y,X}^{-1} \circ F\big(\Psi_{Y,X}^{-1}\big) \circ c_{X,Y} \circ c_{X,Y}^{-1} \\
&\quad \circ F(\sigma(X) \otimes \sigma(Y)) \circ c_{X,Y} \\
&= \big((R_{2,1}R)^{-1}\big)_{X,Y} \circ (F(\sigma(X)) \otimes F(\sigma(Y))) \\
&= \big((R_{2,1}R)^{-1}(v \otimes v)\big)_{X,Y}. \quad \blacksquare
\end{aligned}
$$

LEMMA 15. *If F is an ultra weak quasi tensor functor then H is an* End(F(1)) *ultra weak quasi Hopf algebra.*

Proof. The bimodule actions are defined to be

$$
\begin{aligned}
\mu_l(a \otimes h)_X &:= c_{1,X} \circ (a \otimes h) \circ c_{1,X}^{-1}, \\
\mu_r(h \otimes a)_X &:= c_{X,1} \circ (h \otimes a) \circ c_{X,1}^{-1}, \; a \in \mathrm{End}(F(1)).
\end{aligned} \tag{26}
$$

The definition of ϵ doesn't have to be changed but the unit is now defined more generally to be

$$
\eta(a) := \mu_l(a \otimes 1) = \mu_r(1 \otimes a). \tag{27}
$$

The counit property is fulfilled:

$$
(\mu_l(\epsilon \otimes \mathrm{id})\Delta(h))_X = c_{1,X} \circ c_{1,X}^{-1} \circ h_{1 \otimes X} \circ c_{1,X} \circ c_{1,X}^{-1} = h_X. \quad \blacksquare
$$

Collecting results together we have:

THEOREM 16 (Generalized Majid's Reconstruction Theorem). *Let $\mathcal{C}$ be a rigid braided tensor category and F: $\mathcal{C} \to$ Vec a weak quasi tensor functor. Then the set $H =$ Nat(F, F) carries the structure of a weak quasi Hopf algebra and there is a functor G: $\mathcal{C} \to$ Rep(H) such that $\mathcal{C} \xrightarrow{G}$ Rep(H) $\xrightarrow{V}$ Vec composes to F. G maps inequivalent objects to inequivalent representations if F is faithful. G is full. G is faithful iff F is faithful. Hence in the case of a faithful functor and a semisimple category, $\mathcal{C}$ and* Rep(H) *are equivalent braided tensor categories.* Rep(H) *is rigid if F is faithful and it is C^* if $\mathcal{C}$ is so. The structure matrices (see subsection* 2.2) *of $\mathcal{C}$ and* Rep(H) *coincide. A ribbon structure on $\mathcal{C}$ induces a ribbon structure on H* [17]. *The structure of H is determined by F:*

F is a tensor functor

$\Rightarrow$ *H is a quasitriangular Hopf algebra*

F is a quasi tensor functor

$\Rightarrow$ *H is a quasitriangular quasi Hopf algebra*

$$\begin{aligned} & F \textit{ is a weak quasi tensor functor} \\ \Rightarrow\ & H \textit{ is a quasitr. weak quasi Hopf algebra} \\ & F \textit{ is an ultra weak quasi tensor functor} \\ \Rightarrow\ & H \textit{ is a quasitr. ultra weak quasi Hopf algebra.} \end{aligned}$$

Combining this with the construction of weak quasi tensor functors we conclude:

COROLLARY 17. *Every rational semisimple rigid braided tensor category is the representation category of some weak quasi Hopf algebra.*

3.1. *Questions of Non-Uniqueness*

The reconstruction of H from a given category $\mathcal{C}$ presented in this paper is not unique. It can be checked that in some typical examples there is an infinite number of weak dimension functions. But there is even more freedom because of the choice of epimorphisms C in the proof of Proposition 1.

Remark 3. Let $F, \tilde{F}\colon \mathcal{C} \to \mathrm{Vec}$ denote two faithful (weak) quasi tensor functors constructed as in Proposition 1 by the same dimension function. Then the reconstructed (weak) quasi Hopf algebras H and $\tilde{H}$ are equal up to twice equivalence (in the sense of Drinfeld).

Proof. F and $\tilde{F}$ differ only by different choices of C. However, because they share the same dimension function there is a family of isomorphisms φ such that $\tilde{c}_{X,Y} = \varphi_{X,Y} \circ c_{X,Y}$. H and $\tilde{H}$ are then equal as algebras. Their coalgebra structure however differs,

$$\begin{aligned} \tilde{\Delta}(h)_{X,Y} &= \tilde{c}_{X,Y}^{-1} \circ h_{X\otimes Y} \circ \tilde{c}_{X,Y} \\ &= c_{X,Y}^{-1} \circ \varphi_{X,Y}^{-1} \circ c_{X,Y} \circ c_{X,Y}^{-1} \circ h_{X\otimes Y} \circ c_{X,Y} \circ c_{X,Y}^{-1} \circ \varphi_{X,Y} \circ c_{X,Y} \\ &= T_{X,Y} \circ \Delta(h)_{X,Y} \circ T_{X,Y}^{-1} \\ &= \left(T\Delta(h)T^{-1}\right)_{X,Y}. \end{aligned}$$

Here we have inserted $1 = cc^{-1}$ twice and introduced the twist element $T \in H \otimes H$, defined by $T_{X,Y} := c_{X,Y}^{-1} \circ \varphi_{X,Y}^{-1} \circ c_{X,Y}$. (In the weak case T is not invertible, but one has $TT^{-1} = T^{-1}T = \Delta(1)$.) Note that T really is an element of $H \otimes H$ because the dependence of $T_{X,Y}$ on X, Y obeys the naturality condition as is easily seen from the definition of F and c.

Of course the R element gets twisted alike:

$$\begin{aligned}
\tilde{R}_{X,Y} &= \Psi^{\mathrm{Vec}\,-1}_{F(Y),F(X)} \circ \tilde{c}^{-1}_{Y,X} \circ F(\Psi_{X,Y}) \circ \tilde{c}_{X,Y} \\
&= \Psi^{\mathrm{Vec}\,-1}_{F(Y),F(X)} \circ c^{-1}_{Y,X} \circ \varphi^{-1}_{Y,X} \circ F(\Psi_{X,Y}) \circ \varphi_{X,Y} \circ c_{X,Y} \\
&= \Psi^{\mathrm{Vec}\,-1}_{F(Y),F(X)} \circ c^{-1}_{Y,X} \circ \varphi^{-1}_{Y,X} \circ c_{Y,X} \circ \Psi^{\mathrm{Vec}}_{F(X),F(Y)} \circ \Psi^{\mathrm{Vec}\,-1}_{F(X),F(Y)} \circ c^{-1}_{Y,X} \\
&\quad \circ F(\Psi_{X,Y}) \circ c_{X,Y} \circ c^{-1}_{X,Y} \circ \varphi_{X,Y} \circ c_{X,Y} \\
&= \Psi^{\mathrm{Vec}\,-1}_{F(X),F(Y)} \circ T_{Y,X} \circ \Psi^{\mathrm{Vec}}_{F(X),F(Y)} \circ R_{X,Y} \circ t^{-1}_{X,Y} \\
&= \left(T_{2,1} R T^{-1}_{1,2}\right)_{X,Y}.
\end{aligned}$$

A similar calculation shows that

$$\tilde{\phi} = T^{-1}_{1,2}(\Delta \otimes \mathrm{id})(T^{-1})\,\phi(\mathrm{id} \otimes \Delta)(T)T_{2,3}. \quad \blacksquare$$

The results of the previous remark can be nicely interpreted in the language of non-abelian cohomology [9] where the n-cochains are given by the invertible elements in $H^{\otimes n}$ and the coboundary operator is defined to be

$$\delta(\gamma) := \prod_{i=1,3,\dots} \Delta_i(\gamma) \quad |p_{i=0,2,\dots} \quad \Delta_i(\gamma)^{-1} \in H^{\otimes n+1}, \qquad \gamma \in H^{\otimes n}$$

with $\Delta_0(\gamma) := 1 \otimes \gamma$, $\Delta_{n+1}(\gamma) := \gamma \otimes 1$, $\Delta_i(\gamma) := (\mathrm{id} \otimes \cdots \otimes \Delta \otimes \cdots \otimes \mathrm{id})(\gamma)$. Then the pentagon identity for $\phi \in H^{\otimes 3}$ is the statement that ϕ is a 3-cocycle, $\delta(\phi) = 1 \otimes 1 \otimes 1 \otimes 1$. Returning to the previous remark we see that $\tilde{\phi}$ can be made trivial iff ϕ is a coboundary $\phi = \delta(T)$.

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