Morphisms from Uv(sl2) to the rotation algebra

Reinhard Oldenburg

Angaben zur Veröffentlichung / Publication details:

Oldenburg, Reinhard. 1997. "Morphisms from Uv(sl2) to the rotation algebra." *Reports on Mathematical Physics* 39 (1): 135–36. https://doi.org/10.1016/s0034-4877(97)81477-2.

MORPHISMS FROM $U_v(sl_2)$ TO THE ROTATION ALGEBRA $\mathcal{A}_\theta$

REINHARD HÄRING-OLDENBURG

Mathematisches Institut, Bunsenstr. 3-5, 37073 Göttingen, Germany
(e-mail: haering@cfgauss.uni-math.gwdg.de)

There exists a family of morphisms from the quantum group of sl_2 to the rotation C^*-algebra. This allows to construct the irreducible representations of the quantum group in a new way and implements a quantum group symmetry in quantum Hall systems.

It is the purpose of this short note to display an interesting morphism from the quantum group of the Lie algebra sl_2 to the rotation algebra.

The quantum group $U_v(sl_2)$ [2] is the associative algebra generated by E, F, K, K^{-1} with relations

$$KEK^{-1} = v^2E, \qquad KFK^{-1} = v^{-2}F, \qquad [E,F] = \delta^{-1}(K - K^{-1}). \tag{1}$$

Here v is an invertible parameter in the ground ring and $\delta := v - v^{-1}$.

The rotation algebra A has invertible generators V, U and a single relation using a parameter λ which is a complex number of norm 1

$$VU = \lambda UV. \tag{2}$$

If U, V are taken to be unitary, then this algebra can be closed to become a C^*-algebra which is usually denoted $\mathcal{A}_\theta$ with $\lambda = e^{2\pi i\theta}$. In this paper we do not need this C^*-structure.

There is a morphism ϕ: $U_v(sl_2) \to A$ defined by

$$\begin{gathered} K \mapsto U^2, \qquad E \mapsto \delta^{-1}(U - U^{-1})V, \\ F \mapsto \delta^{-1}(U^{-1} - U)V^{-1}, \qquad \lambda = v^{-1}. \end{gathered} \tag{3}$$

It is a simple calculation to establish that this really defines a morphism of algebras.

The algebra A has irreducible finite-dimensional modules iff λ is a root of unity. This is because a single eigenvector a of U with eigenvalue μ gives rise to the U eigenvectors V^ia. The set of eigenvalues $\{\lambda^{-i}\mu \mid i \in \mathbb{Z}\}$ is infinite iff λ is not a root of unity. If λ is an m-th root of unity, then there is an m-dimensional A module

[3] where U acts as the diagonal matrix with entries $1, \lambda, \lambda^2, \ldots, \lambda^{m-1}$ and V is the permutation matrix that shifts basis vectors in a cyclic manner $e_i \mapsto e_{i-1}$.

ϕ can be blown up to a whole family of morphisms $\phi_n, n \in \mathbb{N}$, given by

$$K \mapsto U^2, \quad E \mapsto \delta^{-1}(U - U^{-1})V^n, \\ F \mapsto \delta^{-1}(U^{-1} - U)V^{-n}, \quad \lambda^n = v^{-1}. \tag{4}$$

Again the verification is straightforward.

If V allows roots, then one may also allow rational $n \in \mathbb{Q}$. Different values of l (v fixed) give then rise to $U_v(sl_2)$ modules of any dimension induced by the irreducible A modules given above. It can easily be seen that they include all the irreducible $U_v(sl_2)$ modules.

The algebra A arises in quantum Hall systems [1, p. 359] as the algebra of the magnetic translation operators which commute with the Hamiltonian. Hence quantum Hall systems possess a quantum group symmetry implemented by ϕ or any of the ϕ_n.

REFERENCES

[1] A. Connes: *Noncommutative Geometry*, Academic Press, New York 1994.

[2] G. Lusztig: *Introduction to Quantum Groups*, Birkhäuser, Boston 1993.

[3] N. E. Wegge-Olsen: *K-Theory and C^*-Algebras*, Oxford University Press, Oxford 1993.