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New solutions of reflection equation derived from type B BMW algebras

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1. Introduction

Two-dimensional integrable systems are described by solutions of the spectral parameter dependent Yang–Baxter equation (YBE). With multiplicatively written spectral parameter it reads

$$R_1(t_1)R_2(t_1t_2)R_1(t_2) = R_2(t_2)R_1(t_1t_2)R_2(t_1) \qquad \forall t_1, t_2 . \tag{1}$$

This equation lives on $V \otimes V \otimes V$ where $R \in \mathrm{End}(V \otimes V)$ acts according to its subscript either in the first and second or second and third factor.

If the system is restricted to a half plane with reflecting boundary then a second matrix is needed describing the boundary particle interaction. That is, we need a spectral parameter dependent $K(t) \in \mathrm{End}(V)$ satisfying Sklyanin's reflection equation [8]

$$R(t_1/t_2)(K(t_1) \otimes 1)R(t_1t_2)(K(t_2) \otimes 1) = (K(t_2) \otimes 1)R(t_1t_2)(K(t_1) \otimes 1)R(t_1/t_2) . \tag{2}$$

This paper presents a solution of equations (1), (2) where $R(t)$ is the usual Baxterization [5] of the R-matrix of orthogonal quantum groups. $K(t)$ is constructed algebraically from representations of a new generalization of the Birman–Wenzl algebra which is associated with the Coxeter type B braid group. It is worth noting that the type B Hecke algebra does not allow analogous Baxterization [7]. The problem of Baxterization has been treated in greater generality by Bellon *et al* [1]. However, we hope that our explicit solution may nevertheless be interesting.

2. The restricted type B Birman–Wenzl algebra

For every root system there exists an associated Weyl group (Coxeter group). For type A_n root systems it is the permutation group. For type B_n it is a semi-direct product of a permutation group with $\mathbb{Z}_2^n$. It has generators $\tau_0, \tau_1, \ldots, \tau_{n-1}$ and relations $\tau_i^2 = 1$, $|i-j| > 1 \Rightarrow \tau_i\tau_j = \tau_j\tau_i$, $i+1 = j > 0 \Rightarrow \tau_j\tau_i\tau_j = \tau_i\tau_j\tau_i$ and $\tau_0\tau_1\tau_0\tau_1 = \tau_1\tau_0\tau_1\tau_0$. Omitting the quadratic relations from the Coxeter presentations of these groups one obtains the braid

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group of the root system. tom Dieck initiated in [3] a systematic study of these braid groups. Among the quotients of the group algebra of the type B braid group there is the following restricted BMW algebra.

Definition 1. The restricted type B Birman–Wenzl algebra BB_n is defined to have invertible generators $\{Y, X_1, \dots, X_{n-1}\}$ and parameters λ, q, q_1. Using the definitions

$$\delta := q - q^{-1} \qquad x := 1 - \frac{\lambda - \lambda^{-1}}{\delta} \qquad q_0 := q^{-1} \tag{3}$$

$$e_i := 1 - \frac{X_i - X_i^{-1}}{q - q^{-1}} \qquad i = 1, \dots, n-1 \tag{4}$$

the relations are

$$X_i X_i^{-1} = X_i^{-1} X_i = 1 \tag{5}$$

$$X_i X_j = X_j X_i \qquad |i - j| > 1 \tag{6}$$

$$X_i X_j X_i = X_j X_i X_j \qquad |i - j| = 1 \tag{7}$$

$$X_i e_i = e_i X_i = \lambda e_i \tag{8}$$

$$e_i X_{i-1}^{\pm 1} e_i = \lambda^{\mp 1} e_i \tag{9}$$

$$X_1 Y X_1 Y = Y X_1 Y X_1 \tag{10}$$

$$Y^2 = q_1 Y + q_0 \tag{11}$$

$$Y X_1 Y e_1 = e_1 \tag{12}$$

$$Y X_i = X_i Y \qquad i > 1\,. \tag{13}$$

The term ‘restricted’ refers to the fact that Y satisfies a quadratic relation while the X_i satisfy cubic polynomials. The value of q_0 is enforced by (12). The algebra BB_n is studied in detail in [6].

It should be noted that throughout this paper we are working with generic parameters. For non-generic values one would have to introduce the e_i as generators in their own right and take care of poles.

It is obvious that $X_1, \dots, X_{n-1}$ generate a standard Birman–Wenzl algebra [9] (which is of type A).

Lemma 1.

$$e_i^2 = x e_i \tag{14}$$

$$X_i^{-1} = X_i - \delta + \delta e_i \tag{15}$$

$$X_i^2 = 1 + \delta X_i - \delta\lambda e_i \tag{16}$$

$$0 = (X_i - \lambda)(X_i + q^{-1})(X_i - q) \tag{17}$$

$$e_i e_j = e_j e_i \qquad |i - j| > 1 \tag{18}$$

$$Y^{-1} = q_0^{-1} Y - q_1 q_0^{-1} \tag{19}$$

$$0 = [X_1 Y X_1 Y, \{Y, e_1, X_1\}] \tag{20}$$

$$e_1 Y X_1 Y = e_1 \tag{21}$$

$$e_1 Y e_1 = x q_1 (1 - q_0 \lambda)^{-1} e_1\,. \tag{22}$$

The proofs are straightforward with the possible exception of the last equation:

$$e_1Ye_1 = e_1YYX_1Ye_1 = q_1e_1YX_1Ye_1 + q_0e_1X_1Ye_1 = q_1xe_1 + q_0\lambda e_1Ye_1$$

$$\Rightarrow (1-q_0\lambda)e_1Ye_1 = q_1xe_1 \Rightarrow e_1Ye_1 = q_1x(1-q_0\lambda)^{-1}e_1\,.$$

3. Solution of the reflection equation

Solutions of the Yang–Baxter equation can be obtained from the standard (type A) Birman–Wenzl algebra by the following Baxterization procedure [2]:

$$R_i(t) = -\delta t(t+q\lambda^{-1}) + (t-1)(t+q\lambda^{-1})X_i + \delta t(t-1)e_i\,. \tag{23}$$

To also find a solution of (2) we make the ansatz

$$K(t) = f_0(t) + f_1(t)Y\,. \tag{24}$$

Using the relations of the previous section (equations (12) and (21) are multiplied with Y^{-1} and then used) it is then a tedious but straightforward computation to reduce (2) to

$$\begin{aligned}\mathrm{LHS(2)} - \mathrm{RHS(2)} = {} & (1-q^2)(t_1f_0(t_2)f_1(t_1) - t_1t_2^2f_0(t_2)f_1(t_1) - t_2f_0(t_1)f_1(t_2)\\ & +t_1^2t_2f_0(t_1)f_1(t_2) + q_1t_1^2t_2f_1(t_1)f_1(t_2) - q_1t_1t_2^2f_1(t_1)f_1(t_2))\\ & \big(-(\lambda q^3t_1Ye_1) + \lambda q^3t_2Ye_1 + \lambda^2t_1^2t_2Ye_1 + \lambda qt_1^2t_2Ye_1 - \lambda^2q^2t_1^2t_2Ye_1\\ & -\lambda q^3t_1t_2^2Ye_1 - \lambda q^2t_1YX_1 - q^3t_2YX_1 - \lambda^2qt_1^2t_2YX_1 - \lambda q^2t_1t_2^2YX_1\\ & +\lambda q^3t_1e_1Y - \lambda q^3t_2e_1Y - \lambda^2t_1^2t_2e_1Y - \lambda qt_1^2t_2e_1Y + \lambda^2q^2t_1^2t_2e_1Y\\ & +\lambda q^3t_1t_2^2e_1Y + \lambda q^2t_1X_1Y + q^3t_2X_1Y + \lambda^2qt_1^2t_2X_1Y + \lambda q^2t_1t_2^2X_1Y\big)\,.\end{aligned}$$

To make this vanish we take the second factor which contains all the occurrences of f_0, f_1 and divide it by $f_0(t_2)f_1(t_1)$:

$$0 = t_1 - t_1t_2^2 + (q_1t_2t_1^2 - q_1t_2^2t_1)f_1(t_2)f_0(t_2)^{-1} + (t_2t_1^2 - t_2)f_0(t_1)f_1(t_1)^{-1}f_0(t_2)^{-1}f_1(t_2)\,.$$

Introducing $F(t) := f_0(t)f_1(t)^{-1}$ and multiplying with $F(t_2)$ we obtain

$$(t_1F(t_2) - t_2^2t_1(q_1 + F(t_2))) - (t_2F(t_1) - t_1^2t_2(q_1 + F(t_1))) = 0\,.$$

We require $0 = t_1F(t_2) - t_2^2t_1(q_1 + F(t_2))$ and find $F(t) = t^2q_1(1-t^2)^{-1}$.

Proposition 2. $K(t) = (t^2q_1(1-t^2)^{-1} + Y)f_1(t)$ is (for all f_1) a solution of the reflection equation (2).

4. Tensor representations

In [4] tom Dieck found representations of BB_n acting on n-fold tensor products of representation spaces of orthogonal quantum groups. Following Wenzl he used the R-matrix of the quantum group $U_q(so_N)$, $N = 2m+1, m \in \mathbb{N}$. We denote its N-dimensional defining representation by $V = \{v_i \mid i \in I\}$. The index set is $I = \{-N+2, -N+4, \ldots, -3, -1, 0, 1, 3, \ldots, N-2\}$. Denote by $f_{i,j}$ the matrix with a single entry of 1 at position i, j. Then the R-matrix reads

$$\begin{aligned}R = {} & \sum_{i\neq 0}(qf_{i,i}\otimes f_{i,i} + q^{-1}f_{i,-i}\otimes f_{-i,i}) + f_{0,0}\otimes f_{0,0} + \sum_{i\neq j,-j} f_{i,j}\otimes f_{j,i}\\ & +(q-q^{-1})\left(\sum_{i<j} f_{i,i}\otimes f_{j,j} - \sum_{j<-i} q^{\frac{i+j}{2}} f_{i,j}\otimes f_{-i,-j}\right)\,.\end{aligned} \tag{25}$$

From $E = 1 - (R - R^{-1})/\delta$ one obtains

$$E = \sum_{i,j} q^{i+j/2} f_{i,j} \otimes f_{-i,-j} \,. \tag{26}$$

$E^2 = xE$ with $x = \sum_i q^i$ and thus $\lambda = q^{1-N}$.

The following matrix was found by tom Dieck:

$$F = -f_{0,0} + q^{-1/2} \sum_{i \neq 0} f_{-i,i} + (q^{-1} - 1) \sum_{i>0} f_{i,i} \,. \tag{27}$$

It is shown in [4] that it fulfills $F^2 = (q^{-1} - 1)F + q^{-1}$ and $(F \otimes 1)B(F \otimes 1)B = B(F \otimes 1)B(F \otimes 1)$ as well as $E = E(F \otimes 1)B(F \otimes 1)$. Hence a representation of BB_n with parameters $q_1 = (q^{-1} - 1), \lambda = q^{1-N}$ on the n fold tensor product is given by $\phi : \mathrm{B}^*\mathrm{B}_n \to \mathrm{End}(V^{\otimes n}), Y \mapsto F \otimes 1 \cdots \otimes 1,\ X_i \mapsto 1 \otimes \cdots \otimes 1 \otimes B \otimes 1 \cdots \otimes 1$.

Combining this with the results of the previous section we obtain the matrix solution of the reflection equation.

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