

New solutions of reflection equation derived from type B BMW algebras

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1. Introduction

Two-dimensional integrable systems are described by solutions of the spectral parameter dependent Yang–Baxter equation (YBE). With multiplicatively written spectral parameter it reads

$$R_1(t_1)R_2(t_1t_2)R_1(t_2) = R_2(t_2)R_1(t_1t_2)R_2(t_1) \quad \forall t_1, t_2. \quad (1)$$

This equation lives on $V \otimes V \otimes V$ where $R \in \text{End}(V \otimes V)$ acts according to its subscript either in the first and second or second and third factor.

If the system is restricted to a half plane with reflecting boundary then a second matrix is needed describing the boundary particle interaction. That is, we need a spectral parameter dependent $K(t) \in \text{End}(V)$ satisfying Sklyanin’s reflection equation [8]

$$R(t_1/t_2)(K(t_1) \otimes 1)R(t_1t_2)(K(t_2) \otimes 1) = (K(t_2) \otimes 1)R(t_1t_2)(K(t_1) \otimes 1)R(t_1/t_2). \quad (2)$$

This paper presents a solution of equations (1), (2) where $R(t)$ is the usual Baxterization [5] of the R -matrix of orthogonal quantum groups. $K(t)$ is constructed algebraically from representations of a new generalization of the Birman–Wenzl algebra which is associated with the Coxeter type B braid group. It is worth noting that the type B Hecke algebra does not allow analogous Baxterization [7]. The problem of Baxterization has been treated in greater generality by Bellon *et al* [1]. However, we hope that our explicit solution may nevertheless be interesting.

2. The restricted type B Birman–Wenzl algebra

For every root system there exists an associated Weyl group (Coxeter group). For type A_n root systems it is the permutation group. For type B_n it is a semi-direct product of a permutation group with \mathbb{Z}_2^n . It has generators $\tau_0, \tau_1, \dots, \tau_{n-1}$ and relations $\tau_i^2 = 1, |i - j| > 1 \Rightarrow \tau_i \tau_j = \tau_j \tau_i, i + 1 = j > 0 \Rightarrow \tau_j \tau_i \tau_j = \tau_i \tau_j \tau_i$ and $\tau_0 \tau_1 \tau_0 \tau_1 = \tau_1 \tau_0 \tau_1 \tau_0$. Omitting the quadratic relations from the Coxeter presentations of these groups one obtains the braid

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group of the root system. tom Dieck initiated in [3] a systematic study of these braid groups. Among the quotients of the group algebra of the type B braid group there is the following restricted BMW algebra.

Definition 1. The restricted type B Birman–Wenzl algebra BB_n is defined to have invertible generators $\{Y, X_1, \dots, X_{n-1}\}$ and parameters λ, q, q_1 . Using the definitions

$$\delta := q - q^{-1} \quad x := 1 - \frac{\lambda - \lambda^{-1}}{\delta} \quad q_0 := q^{-1} \quad (3)$$

$$e_i := 1 - \frac{X_i - X_i^{-1}}{q - q^{-1}} \quad i = 1, \dots, n-1 \quad (4)$$

the relations are

$$X_i X_i^{-1} = X_i^{-1} X_i = 1 \quad (5)$$

$$X_i X_j = X_j X_i \quad |i - j| > 1 \quad (6)$$

$$X_i X_j X_i = X_j X_i X_j \quad |i - j| = 1 \quad (7)$$

$$X_i e_i = e_i X_i = \lambda e_i \quad (8)$$

$$e_i X_{i-1}^{\pm 1} e_i = \lambda^{\mp 1} e_i \quad (9)$$

$$X_1 Y X_1 Y = Y X_1 Y X_1 \quad (10)$$

$$Y^2 = q_1 Y + q_0 \quad (11)$$

$$Y X_1 Y e_1 = e_1 \quad (12)$$

$$Y X_i = X_i Y \quad i > 1. \quad (13)$$

The term ‘restricted’ refers to the fact that Y satisfies a quadratic relation while the X_i satisfy cubic polynomials. The value of q_0 is enforced by (12). The algebra BB_n is studied in detail in [6].

It should be noted that throughout this paper we are working with generic parameters. For non-generic values one would have to introduce the e_i as generators in their own right and take care of poles.

It is obvious that X_1, \dots, X_{n-1} generate a standard Birman–Wenzl algebra [9] (which is of type A).

Lemma 1.

$$e_i^2 = x e_i \quad (14)$$

$$X_i^{-1} = X_i - \delta + \delta e_i \quad (15)$$

$$X_i^2 = 1 + \delta X_i - \delta \lambda e_i \quad (16)$$

$$0 = (X_i - \lambda)(X_i + q^{-1})(X_i - q) \quad (17)$$

$$e_i e_j = e_j e_i \quad |i - j| > 1 \quad (18)$$

$$Y^{-1} = q_0^{-1} Y - q_1 q_0^{-1} \quad (19)$$

$$0 = [X_1 Y X_1 Y, \{Y, e_1, X_1\}] \quad (20)$$

$$e_1 Y X_1 Y = e_1 \quad (21)$$

$$e_1 Y e_1 = x q_1 (1 - q_0 \lambda)^{-1} e_1. \quad (22)$$

The proofs are straightforward with the possible exception of the last equation:

$$\begin{aligned} e_1 Y e_1 &= e_1 Y Y X_1 Y e_1 = q_1 e_1 Y X_1 Y e_1 + q_0 e_1 X_1 Y e_1 = q_1 x e_1 + q_0 \lambda e_1 Y e_1 \\ &\Rightarrow (1 - q_0 \lambda) e_1 Y e_1 = q_1 x e_1 \Rightarrow e_1 Y e_1 = q_1 x (1 - q_0 \lambda)^{-1} e_1 . \end{aligned}$$

3. Solution of the reflection equation

Solutions of the Yang–Baxter equation can be obtained from the standard (type A) Birman–Wenzl algebra by the following Baxterization procedure [2]:

$$R_i(t) = -\delta t(t + q\lambda^{-1}) + (t - 1)(t + q\lambda^{-1})X_i + \delta t(t - 1)e_i . \quad (23)$$

To also find a solution of (2) we make the ansatz

$$K(t) = f_0(t) + f_1(t)Y . \quad (24)$$

Using the relations of the previous section (equations (12) and (21) are multiplied with Y^{-1} and then used) it is then a tedious but straightforward computation to reduce (2) to

$$\begin{aligned} \text{LHS}(2) - \text{RHS}(2) &= (1 - q^2)(t_1 f_0(t_2) f_1(t_1) - t_1 t_2^2 f_0(t_2) f_1(t_1) - t_2 f_0(t_1) f_1(t_2)) \\ &\quad + t_1^2 t_2 f_0(t_1) f_1(t_2) + q_1 t_1^2 t_2 f_1(t_1) f_1(t_2) - q_1 t_1 t_2^2 f_1(t_1) f_1(t_2) \\ &\quad (-\lambda q^3 t_1 Y e_1) + \lambda q^3 t_2 Y e_1 + \lambda^2 t_1^2 t_2 Y e_1 + \lambda q t_1^2 t_2 Y e_1 - \lambda^2 q^2 t_1^2 t_2 Y e_1 \\ &\quad - \lambda q^3 t_1 t_2^2 Y e_1 - \lambda q^2 t_1 Y X_1 - q^3 t_2 Y X_1 - \lambda^2 q t_1^2 t_2 Y X_1 - \lambda q^2 t_1 t_2^2 Y X_1 \\ &\quad + \lambda q^3 t_1 e_1 Y - \lambda q^3 t_2 e_1 Y - \lambda^2 t_1^2 t_2 e_1 Y - \lambda q t_1^2 t_2 e_1 Y + \lambda^2 q^2 t_1^2 t_2 e_1 Y \\ &\quad + \lambda q^3 t_1 t_2^2 e_1 Y + \lambda q^2 t_1 X_1 Y + q^3 t_2 X_1 Y + \lambda^2 q t_1^2 t_2 X_1 Y + \lambda q^2 t_1 t_2^2 X_1 Y . \end{aligned}$$

To make this vanish we take the second factor which contains all the occurrences of f_0, f_1 and divide it by $f_0(t_2) f_1(t_1)$:

$$0 = t_1 - t_1 t_2^2 + (q_1 t_2 t_1^2 - q_1 t_2^2 t_1) f_1(t_2) f_0(t_2)^{-1} + (t_2 t_1^2 - t_2) f_0(t_1) f_1(t_1)^{-1} f_0(t_2)^{-1} f_1(t_2) .$$

Introducing $F(t) := f_0(t) f_1(t)^{-1}$ and multiplying with $F(t_2)$ we obtain

$$(t_1 F(t_2) - t_2^2 t_1 (q_1 + F(t_2))) - (t_2 F(t_1) - t_1^2 t_2 (q_1 + F(t_1))) = 0 .$$

We require $0 = t_1 F(t_2) - t_2^2 t_1 (q_1 + F(t_2))$ and find $F(t) = t^2 q_1 (1 - t^2)^{-1}$.

Proposition 2. $K(t) = (t^2 q_1 (1 - t^2)^{-1} + Y) f_1(t)$ is (for all f_1) a solution of the reflection equation (2).

4. Tensor representations

In [4] tom Dieck found representations of BB_n acting on n -fold tensor products of representation spaces of orthogonal quantum groups. Following Wenzl he used the R -matrix of the quantum group $U_q(so_N)$, $N = 2m + 1, m \in \mathbb{N}$. We denote its N -dimensional defining representation by $V = \{v_i \mid i \in I\}$. The index set is $I = \{-N + 2, -N + 4, \dots, -3, -1, 0, 1, 3, \dots, N - 2\}$. Denote by $f_{i,j}$ the matrix with a single entry of 1 at position i, j . Then the R -matrix reads

$$\begin{aligned} R &= \sum_{i \neq 0} (q f_{i,i} \otimes f_{i,i} + q^{-1} f_{i,-i} \otimes f_{-i,i}) + f_{0,0} \otimes f_{0,0} + \sum_{i \neq j, -j} f_{i,j} \otimes f_{j,i} \\ &\quad + (q - q^{-1}) \sum_{i < j} f_{i,i} \otimes f_{j,j} - \sum_{j < -i} q^{\frac{i+j}{2}} f_{i,j} \otimes f_{-i,-j} . \end{aligned} \quad (25)$$

From $E = 1 - (R - R^{-1})/\delta$ one obtains

$$E = \sum_{i,j} q^{i+j/2} f_{i,j} \otimes f_{-i,-j}. \quad (26)$$

$E^2 = xE$ with $x = \sum_i q^i$ and thus $\lambda = q^{1-N}$.

The following matrix was found by tom Dieck:

$$F = -f_{0,0} + q^{-1/2} \sum_{i \neq 0} f_{-i,i} + (q^{-1} - 1) \sum_{i > 0} f_{i,i}. \quad (27)$$

It is shown in [4] that it fulfills $F^2 = (q^{-1} - 1)F + q^{-1}$ and $(F \otimes 1)B(F \otimes 1)B = B(F \otimes 1)B(F \otimes 1)$ as well as $E = E(F \otimes 1)B(F \otimes 1)$. Hence a representation of BB_n with parameters $q_1 = (q^{-1} - 1)$, $\lambda = q^{1-N}$ on the n fold tensor product is given by $\phi : B^*B_n \rightarrow \text{End}(V^{\otimes n})$, $Y \mapsto F \otimes 1 \cdots \otimes 1$, $X_i \mapsto 1 \otimes \cdots \otimes 1 \otimes B \otimes 1 \cdots \otimes 1$.

Combining this with the results of the previous section we obtain the matrix solution of the reflection equation.

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