DISCRETE APPROXIMATIONS OF COSINE OPERATOR FUNCTIONS. I*

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Abstract. In the present paper we are concerned with the approximation of cosine operator functions which appear in a natural way in the study of the Cauchy problem for second order evolution equations. We derive both qualitative and quantitative convergence theorems characterizing the convergence of cosine operator functions in terms of their infinitesimal generators, and we discuss the impact of these results with respect to the approximate solution of the corresponding Cauchy problems.

1. Introduction. Cosine operator functions, which are defined as operator-valued functions on $\mathbb{R}^+ = (0, \infty)$ satisfying d'Alembert's functional equation, play a decisive role in the solution of the Cauchy problem for second order evolution equations,

$$\frac{d^2}{dt^2}u(t) = Au(t), \qquad t \in \mathbb{R}^+, \tag{1.1a}$$

$$u(0) = u^0, \qquad \frac{d}{dt}u(0) = u_t^0, \tag{1.1b}$$

where A is supposed to be a linear operator with domain and range in a Banach space E. It is well known (cf. [6]) that the Cauchy problem under consideration is well posed in $\bar{\mathbb{R}}^+ = [0, \infty)$, i.e., solutions of (1.1a), (1.1b) exist, are unique and depend continuously on the initial data u^0 and u_t^0 if and only if the operator A is the infinitesimal generator of a strongly continuous cosine operator function. Therefore, the situation is quite similar to that in the case of the Cauchy problem for a first order evolution equation

$$\frac{d}{dt}v(t) = Lv(t), \qquad t \in \mathbb{R}^+, \tag{1.2a}$$

$$v(0) = v^0, \tag{1.2b}$$

which is well posed in $\bar{\mathbb{R}}^+$ if and only if L is the infinitesimal generator of a strongly continuous semigroup of operators. In some circumstances, imposing additional assumptions on A, a well-posed second order problem can be reduced to a well-posed first order system. For example, if $-A$ is a positive self-adjoint operator on a Hilbert space H, then (1.1a), (1.1b) can be written as a well-posed first order problem (1.2a), (1.2b) in the product space $D((-A)^{1/2}) \times H$ with $v(t) = (u(t), (d/dt)u(t))^T$, $v^0 = (u^0, v_t^0)^T$ and L given by

$$L = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}.$$

Indeed, the thus defined operator L generates a strongly continuous group of operators $\exp(tL)$, $t \in \mathbb{R}$, on $D((-A)^{1/2}) \times H$. With regard to an approximate solution of (1.1a), (1.1b) in this particular case, such a device was used in [2] to create high order one-step schemes based on rational approximations of $\exp(tL)$. A somewhat more direct approach can be found in [3] and [4], where high order two-step methods were constructed using rational approximations of the cosine function. But again the

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reduction of (1.1a), (1.1b) to a first order system was used as an essential step in the analysis of the probelm. Moreover, as it was shown recently in [12], where similar two-step methods were generated by means of Turán type quadrature formulas, the schemes used in [3], [4] can be derived easily from those in [2]. Although a formal reduction of a well-posed second order problem to a first order system as indicated above is always possible, the resulting first order problem can fail to be well posed in the general case. Therefore, it seems to be more reasonable to approach the given Cauchy problem (1.1a), (1.1b) directly. It is the main purpose of the present paper to study discrete approximations of cosine operator functions which in turn give rise to discrete schemes for the approximate solution of the associated Cauchy problem. In § 2 of this paper we will define strongly continuous as well as discrete cosine operator functions and state some basic results with special emphasis on the mutual relationship between these operator-valued functions and their generators. In order to allow rather general approximation schemes, in § 3 we introduce the concept of discrete convergence in discrete limit spaces, a theory developed by Grigorieff, Stummel et al. (cf. e.g. [9], [10], [18], [19]) which has proved to be of considerable importance in the approximate solution of operator equations. In this framework we first consider the approximation of strongly continuous cosine operator functions by means of discrete approximations of their infinitesimal generators. Therefore, this case can be viewed in a certain sense as a semidiscretization of the given problem. In applications, for example when (1.1a), (1.1b) represents a second order hyperbolic problem, it reflects discretization in the space variables. In particular we will establish necessary and sufficient conditions for discrete convergence of cosine operator functions in terms of their generators. Moreover, we will derive an a priori estimate which shows that convergence occurs with at least the order of consistency of the generators. We will also study convergence of the adjoint cosine operator functions based on the concept of discrete weak convergence with respect to the dual spaces. Then, we will be concerned with the case of discretization in time, i.e., we will characterize the convergence of a sequence of discrete cosine operator functions to a given strongly continuous cosine operator function. Since discrete cosine operator functions also define difference schemes, we can relate the rate of convergence to that of the local discretization error with respect to the approximation of the associated Cauchy problem. Finally, we will show how to combine the results in order to get qualitative convergence theorems as well as error estimates in the fully discrete case. In § 4 we will give some examples and an outlook on further work.

2. Strongly continuous and discrete cosine operator functions. In this section we define strongly continuous and discrete cosine operator functions as operator-valued functions defined on $\mathbb{R}^+$ (resp. on a discrete subset of $\bar{\mathbb{R}}^+$) satisfying d'Alembert's functional equation, and we briefly discuss their most important features. Let E be a real or complex Banach space with norm $\|\cdot\|$, and let $\mathscr{L}(E)$ be the set of linear operators with domain $D(A)$, a linear manifold of E, and range $R(A)$ in E. In particular, we denote by $\mathscr{C}(E)$ the set of all densely defined closed linear operators and by $\mathscr{B}(E)$ the Banach algebra of all bounded linear operators with norm $\|A\| = \sup_{u\neq 0} \|Au\|/\|u\|$, $A \in \mathscr{B}(E)$. The sets $\rho(A)$ (resp. $\sigma(A)$) refer to the resolvent set (resp. the spectrum) of $A \in \mathscr{L}(E)$, and for $\lambda \in \rho(A)$ the resolvent $(\lambda I - A)^{-1}$ is denoted by $R(\lambda, A)$. E^* stands for the dual space to E with norm $\|f\| = \sup\{|\langle f, u\rangle|, \|u\| \leqq 1, u \in E\}$, $f \in E^*$, where $\langle\cdot,\cdot\rangle$ refers to the dual pairing between E and E^*. For $A \in \mathscr{C}(E)$ we denote by A^* the adjoint operator to A. Finally, given a uniform partition $\Delta_k^+ = \{jk \mid j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, k \in \mathbb{R}^+\}$ of $\bar{\mathbb{R}}^+$ with step size k and a grid function $u: \Delta_k^+ \to E$, we denote by D_k^+ (resp. D_k^-) the forward (resp. backward) difference operator given by $D_k^+u(t)=u(t+k)-u(t)$, $t\in\Delta_k^+$, (resp. $D_k^-u(t)=u(t)-u(t-k)$, $t\in\Delta_k^+\backslash\{0\}$).

In the sequel we will be concerned with functions defined on $\mathbb{R}^+$ (resp. Δ_k^+) with values in $\mathscr{B}(E)$ which satisfy d'Alembert's functional equation.

A transformation $C:\mathbb{R}^+\to\mathscr{B}(E)$ (resp. $C_\Delta:\Delta_k^+\to\mathscr{B}(E)$) is called a cosine operator function (resp. discrete cosine operator function) if

$$C(t+s)+C(t-s)=2C(t)C(s),\qquad t, s\in\mathbb{R}^+,\quad t>s \text{ resp.,}\tag{2.1}$$

$$C_\Delta(t+s)+C_\Delta(t-s)=2C_\Delta(t)C_\Delta(s),\qquad t, s\in\Delta_k^+,\quad t\geqq s.\tag{2.1$'$}$$

A cosine operator function $C:\mathbb{R}^+\to\mathscr{B}(E)$ is called strongly continuous if $C(\cdot)u$ is continuous on $\mathbb{R}^+$ for each $u\in E$. A sufficient condition for a cosine operator function $C:\mathbb{R}^+\to\mathscr{B}(E)$ to be strongly continuous is that $\lim_{t\to+0}C(t)u=u$ for each $u\in E$ (cf. e.g. [17]).

Remark 2.1. (i) A strongly continuous cosine operator function C can be extended to the real line $\mathbb{R}$ simply by setting $C(0)=I$ and $C(t)=C(-t)$ for $t<0$. Then (2.1) holds for all t, $s\in\mathbb{R}$.

(ii) If $U(t)$, $t\in\mathbb{R}$, is a strongly continuous group of operators, then

$$C(t)=\tfrac{1}{2}(U(t)+U(-t)),\qquad t\in\mathbb{R}\tag{2.2}$$

defines a strongly continuous cosine operator function on $\mathbb{R}$.

(iii) If $C_\Delta:\Delta_k^+\to\mathscr{B}(E)$ is a discrete cosine operator function, then we have by (2.1)$'$ that C_Δ satisfies the three-term recurrence formula,

$$C_\Delta((j+2)k)-2C_\Delta((j+1)k)C_\Delta(k)+C_\Delta(jk)=0,\qquad j\in\mathbb{N}_0.\tag{2.3}$$

Therefore, for a given operator $T\in\mathscr{B}(E)$ we can set $C_\Delta(0)=I$, $C_\Delta(k)=T$ and define a discrete cosine operator function recursively by means of (2.3).

For a cosine operator function $C:\mathbb{R}^+\to\mathscr{B}(E)$, we define an operator $A\in\mathscr{L}(E)$ in the following way: $D(A)$ is the set of all $u\in E$ such that the limit of $t^{-2}(C(t)u-u)$ exists as $t\to+0$, and we take

$$Au:=2\lim_{t\to+0}t^{-2}(C(t)u-u),\qquad u\in D(A).\tag{2.4}$$

We refer to A as the infinitesimal generator of C.

In the case of a discrete cosine operator function $C_\Delta:\Delta_k^+\to\mathscr{B}(E)$, we proceed in an analogous manner and associate with C a generator $A_\Delta\in\mathscr{B}(E)$ according to

$$A_\Delta u:=2k^{-2}(C_\Delta(k)u-u),\qquad u\in E.\tag{2.5}$$

We will now state the most important properties of strongly continuous and discrete cosine operator functions, which will be used later on. Some of the results in the strongly continuous case are well known and therefore will be given without proof. (For proofs and further discussion see [5], [6], [7], [17].)

LEMMA 2.1. *Every strongly continuous (resp. discrete) cosine operator function C (resp. C_Δ) is quasibounded in the sense that there exist nonnegative constants M, ω (resp. M_Δ, ω_Δ) such that*

$$\|C(t)\|\leqq M\cosh(\omega t),\qquad t\in\mathbb{R}^+\quad \textit{resp.,}\tag{2.6}$$

$$\|C_\Delta(t)\|\leqq M_\Delta\cosh(\omega_\Delta t),\qquad t\in\Delta_k^+.\tag{2.6$'$}$$

Proof. See e.g., [17, Thm. 2.5].

In the sequel we will say that C (resp. C_Δ) is of type (M, ω) (resp. $(M_\Delta, \omega_\Delta)$) if (2.6) (resp. (2.6)$'$) is fulfilled.

LEMMA 2.2. *Let C (resp. C_Δ) be a strongly continuous (resp. discrete) cosine operator function. Then we have for $u \in E$*

$$C(t_1)C(t_2)u = C(t_2)C(t_1)u, \qquad t_1, t_2 \in \mathbb{R}^+ \quad \text{resp.,} \tag{2.7}$$

$$C_\Delta(t_1)C_\Delta(t_2)u = C_\Delta(t_2)C_\Delta(t_1)u, \qquad t_1, t_2 \in \Delta_k^+. \tag{2.7$'$}$$

Proof. In view of (2.3) we can show by induction that $C_\Delta(t_m)$, $m \in \mathbb{N}$, is a polynomial of degree m in $C_\Delta(k)$, and this fact immediately implies (2.7)$'$. In the strongly continuous case, we get in the same way $C(mt)C(nt)u = C(nt)C(mt)u$, $m, n \in \mathbb{N}$, for each $t \in \mathbb{R}^+$. Therefore, (2.7) is valid for $t_1, t_2 \in \mathbb{Q}$ with $t_1 = r2^{-p}$, $t_2 = s2^{-\sigma}$ where p, q, r, s are positive integers. Since the set of rational numbers of that kind is dense in $\bar{\mathbb{R}}^+$, the continuity of $C(\cdot)u$, $u \in E$, on $\mathbb{R}^+$ gives the conclusion.

For a strongly continuous cosine operator function C, we can define $S(t) \in \mathcal{B}(E)$, $t \in \mathbb{R}^+$, by means of the (Riemann) integral

$$S(t)u = \int_0^t C(s)u\, ds, \qquad u \in E. \tag{2.8}$$

In the discrete case we associate with $C_\Delta(t)$, $t \in \Delta_k^+$, the operator $S_\Delta(t)$, given by

$$S_\Delta(t) = \frac{k}{2}I + k\sum_{j=1}^{t/k} C(jk). \tag{2.9}$$

The next result exhibits differentiability properties of strongly continuous cosine operator functions and their counterparts in the discrete case.

LEMMA 2.3. (i) *Let C be a strongly continuous cosine operator function with infinitesimal generator A. Then $A \in \mathscr{C}(E)$ and for $u \in D(A)$, $C(\cdot)$ is twice differentiable in $\mathbb{R}^+$ with*

$$\frac{d}{dt}C(t)u = AS(t)u = S(t)Au, \qquad t \in \mathbb{R}^+, \tag{2.10a}$$

$$\frac{d^2}{dt^2}C(t)u = AC(t)u = C(t)Au, \qquad t \in \mathbb{R}^+. \tag{2.10b}$$

(ii) *Let C_Δ be a discrete cosine operator function with generator A_Δ. Then we have for $u \in E$*

$$k^{-1}D_k^+C_\Delta(t)u = A_\Delta S_\Delta(t)u = S_\Delta(t)A_\Delta u, \qquad t \in \Delta_k^+, \tag{2.11a}$$

$$k^{-2}D_k^+D_k^-C_\Delta(t)u = A_\Delta C_\Delta(t)u = C_\Delta(t)A_\Delta u, \qquad t \in \Delta_k^+\backslash\{0\}. \tag{2.11b}$$

Proof. For the proof of part (i) we refer to [6, Lemma 5.4]. In order to show (2.11a), let $t_m = mk$, $m \in \mathbb{N}_0$. For $m = 0$ we have, in view of (2.5) and (2.9),

$$k^{-1}D_k^+C_\Delta(t_0)u = k^{-1}(C_\Delta(k)u - u) = \tfrac{1}{2}kA_\Delta u = S_\Delta(t_0)A_\Delta u.$$

Now let us assume that $k^{-1}D_k^+C_\Delta(t_m)u = S_\Delta(t_m)A_\Delta u$ holds for some $m \in \mathbb{N}$. For $m+1$ we then get by means of (2.3), (2.5) and (2.9):

$$\begin{aligned} k^{-1}D_k^+C_\Delta(t_{m+1})u &= 2k^{-1}C_\Delta(t_{m+1})(C_\Delta(k) - I)u + k^{-1}D_k^+C_\Delta(t_m)u \\ &= kC_\Delta(t_{m+1})A_\Delta u + S_\Delta(t_m)A_\Delta u = S_\Delta(t_{m+1})A_\Delta u. \end{aligned}$$

Similarly we get for $m \in \mathbb{N}$ and $u \in E$:

$$k^{-2}D_k^+D_k^-C_\Delta(t_m)u = k^{-2}[C_\Delta((m+1)k) - 2C_\Delta(mk)C_\Delta(k) + C_\Delta((m-1)k)]u$$
$$= 2k^{-2}C_\Delta(mk)(C_\Delta(k) - I)u = C_\Delta(t_m)A_\Delta u.$$

Since $C_\Delta(t)$, $t \in \Delta_k^+$, is a polynomial in $C_\Delta(k)$ as stated in the proof of Lemma 2.2, and since $C_\Delta(k) = I + \frac{1}{2}k^2 A_\Delta$, it follows that A_Δ commutes with $C_\Delta(t)$ as well as $S_\Delta(t)$ for each $t \in \Delta_k^+$.

If the operator A in (1.1) is the infinitesimal generator of a strongly continuous cosine operator function C, then C and S can be viewed as the propagators of the Cauchy problem (1.1a), (1.1b). Indeed, it follows easily by (2.10a), (2.10b) that, if u^0, $u_t^0 \in D(A)$, then the solution $u(t)$, $t \geqq 0$, of (1.1a), (1.1b) is given by

$$u(t) = C(t)u^0 + S(t)u_t^0. \tag{2.12}$$

On the other hand, the notion of a discrete cosine operator function C_Δ and its generator A_Δ is intimately connected with the two-step method,

$$k^{-2}D_k^+D_k^-u_k(t) = A_\Delta u_k(t), \qquad t \in \Delta_k^+\backslash\{0\}. \tag{2.13}$$

It is an immediate consequence of (2.11a), (2.11b) that for arbitrary v, $w \in E$, both $C_\Delta(t)v$ and $S_\Delta(t)w$, $t \in \Delta_k^+\backslash\{0\}$ satisfy the difference equation (2.13). Moreover, C_Δ and a slight modification $\tilde{S}_\Delta$ of S_Δ given by

$$\tilde{S}_\Delta(t)u := S_\Delta(t)u - \tfrac{1}{2}kC_\Delta(t)u, \qquad t \in \Delta_k^+, \quad u \in E \tag{2.14}$$

appear as the propagators of (2.13) in the sense that, for given starting values $u_k^0 = u_k(0)$ and $u_k^1 = u_k(k)$, the solution of (2.13) turns out to be

$$u_k(t) = C_\Delta(t)u_k^0 + \tilde{S}_\Delta(t)\tilde{u}_{k,t}^0, \qquad t \in \Delta_k^+, \tag{2.15}$$

where $\tilde{u}_{k,t}^0$ is given in terms of u_k^0 and u_k^1 by

$$\tilde{u}_{k,t}^0 = (I + \tfrac{1}{4}k^2 A_\Delta)^{-1}(k^{-1}(u_k^1 - u_k^0) - \tfrac{1}{2}kA_\Delta u_k^0). \tag{2.16}$$

In (2.16) $k \in \mathbb{R}^+$ is certainly assumed to be small enough so that the operator $I + \frac{1}{4}k^2 A_\Delta$ is invertible.

Concerning the generation of strongly continuous cosine operator functions of type (M, ω), we have the following result, which can be seen as the analogy of the Hille–Phillips–Yosida generation theorem for semigroups of operators.

THEOREM 2.1. *An operator $A \in \mathscr{C}(E)$ is the infinitesimal generator of a strongly continuous cosine function of type (M, ω) if and only if*

(i) *for each $\lambda > \omega^2$ we have $\lambda \in \rho(A)$,*
(ii) $\lambda R(\lambda, A)u \to u\,(\lambda \to \infty, \lambda > \omega^2)$, $u \in D(A)$, *and*
(iii) *for each $\lambda > \omega$ and $n \in \mathbb{N}_0$,*

$$\left\| \frac{d^n}{d\lambda^n}(\lambda R(\lambda^2, A)) \right\| \leqq \tfrac{1}{2}Mn![(\lambda + \omega)^{-(n+1)} + (\lambda - \omega)^{-(n+1)}].$$

Proof. See [17, Thms. 3.1, 3.2].

Remark 2.2. (i) The proof of the necessary part of the preceding theorem relies heavily upon the fact that for $\lambda > \omega$ the resolvent $R(\lambda^2, A)$ can be represented via the operational Laplace transform of $C(t)$, $t \in \mathbb{R}^+$:

$$\lambda R(\lambda^2, A)u = \int_0^\infty \exp(-\lambda s)C(s)u\, ds, \qquad u \in E. \tag{2.17}$$

(ii) It is an easy matter to conclude by means of conditions (i), (ii) and (iii) of Theorem 2.1 and the Hille–Phillips–Yoshida generation theorem for semigroups that if A is the infinitesimal generator of a strongly continuous cosine operator function C of type (M, ω), then A also generates a strongly continuous semigroup $\exp(tA)$ of type (M, ω^2), i.e., $\|\exp(tA)\| \leqq M \exp(\omega^2 t)$, $t \in \mathbb{R}^+$. However, the converse is not necessarily true: For example, it is well known that the Laplacian generates a strongly continuous semigroup on $L^p(\mathbb{R}^m)$, $1 \leqq p \leqq \infty$, $m \in \mathbb{N}$, but fails to generate a strongly continuous cosine operator function on these spaces unless either $p = 2$ or $m = 1$ (cf. [15]).

In most cases it is not possible to give an explicit representation of a strongly continuous cosine operator function C in terms of the infinitesimal generator A. However, if $A \in \mathcal{B}(E)$, then C admits the representation

$$C(t) = \sum_{\nu=0}^{\infty} \frac{t^{2\nu} A^{\nu}}{(2\nu)!}. \tag{2.18}$$

Moreover, C converges uniformly to the identity on E as $t \to +0$. The converse is also true, i.e., if $C(t) \to I \, (t \to +0)$ with respect to the operator topology on $\mathcal{B}(E)$, then there exists a unique operator $A \in \mathcal{B}(E)$ such that (2.18) holds (cf. [13]). It follows from this remark that every discrete cosine operator function gives rise to a strongly continuous cosine operator function of the form (2.18) by means of its generator $A_\Delta \in \mathcal{B}(E)$.

We conclude this section with some remarks about cosine operator functions adjoint to a given strongly continuous cosine operator function C with infinitesimal generator A. It is plain that the adjoint operator A^* also generates a cosine operator function $C^*: \mathbb{R}^+ \to \mathcal{B}(E^*)$ with $C^*(t) = C(t)^*$, $t \in \mathbb{R}^+$. But since in general $\operatorname{cl} D(A^*)$ is a proper subspace of E^*, the cosine operator function C^* fails to be strongly continuous. However, it can be shown that the maximal restriction $A^{(*)}$ of A^* with domain and range in $\operatorname{cl} D(A^*)$ generates a strongly continuous cosine operator function which is of the same type as C. More precisely we have:

LEMMA 2.4. *Let $A \in \mathcal{C}(E)$ be the infinitesimal generator of a strongly continuous cosine operator function C of type (M, ω). Then the operator $A^{(*)}$ with $D(A^{(*)}) = \{f \in E^* \mid f \in D(A^*), A^* f \in \operatorname{cl} D(A^*)\}$ generates a strongly continuous cosine operator function $C^{(*)}: \mathbb{R}^+ \to \mathcal{B}(\operatorname{cl} D(A^*))$ of the same type (M, ω) with $C^{(*)} = C^*|_{\operatorname{cl} D(A^*)}$.*

Proof. It follows from [11, Thm. 14.3.3] that $\rho(A^{(*)}) = \rho(A)$ and $R(\lambda, A^{(*)}) = R(\lambda, A^*)|_{\operatorname{cl} D(a^*)}$ for each $\lambda \in \rho(A^{(*)})$. Therefore, for each $\lambda > \omega^2$ we have $R(\lambda, A^{(*)}) \in \mathcal{B}(\operatorname{cl} D(A^*))$. Moreover, since $\lambda R(\lambda, A^*) f \to f \, (\lambda \to \infty, \lambda > \omega^2)$, $f \in \operatorname{cl} D(A^*)$, we also have $\lambda R(\lambda, A^{(*)}) f \to f \, (\lambda \to \infty, \lambda > \omega^2)$, $f \in \operatorname{cl} D(A^*)$, which implies in particular that $D(A^{(*)}) = R(\lambda, A^{(*)})(\operatorname{cl} D(A^*))$ is dense in $\operatorname{cl} D(A^*)$. Finally, for $\lambda > \omega$ and $f \in \operatorname{cl} D(A^*)$ we get

$$\begin{aligned}\left\| \frac{d^n}{d\lambda^n}(\lambda R(\lambda^2, A^{(*)}))f \right\| &= \sup \left\{ \left| \left\langle \frac{d^n}{d\lambda^n}(\lambda R(\lambda^2, A^{(*)}))f, u \right\rangle \right| \,\middle|\, u \in E, \|u\| \leqq 1 \right\} \\ &= \sup \left\{ \left| \left\langle f, \frac{d^n}{d\lambda^n}(\lambda R(\lambda^2, A))u \right\rangle \right| \,\middle|\, u \in E, \|u\| \leqq 1 \right\} \\ &\leqq \tfrac{1}{2} M n! [(\lambda + \omega)^{-(n+1)} + (\lambda - \omega)^{-(n+1)}] \|f\|, \qquad n \in \mathbb{N}_0.\end{aligned}$$

Hence in view of Theorem 2.1, the operator $A^{(*)}$ generates a strongly continuous cosine operator function $C^{(*)}: \mathbb{R}^+ \to \mathcal{B}(\operatorname{cl} D(A^*))$ of type (M, ω). Moreover, we have for each $u \in E$

$$\langle \lambda R(\lambda^2, A^{(*)})f, u\rangle = \int_0^\infty \exp(-\lambda s)\langle f, C(s)u\rangle\, ds$$
$$= \int_0^\infty \exp(-\lambda s)\langle C^*(s)f, u\rangle\, ds,$$

and therefore

$$\lambda R(\lambda^2, A^{(*)})f = \int_0^\infty \exp(-\lambda s)C^*(s)f\, ds, \qquad f \in \mathrm{cl}\, D(A^*), \quad \lambda > \omega, \tag{2.19}$$

which implies by (2.17) that $C^{(*)} = C^*|_{\mathrm{cl}\, D(A^*)}$.

3. Discrete approximations of cosine operator functions. There are two major attempts in the approximation of a strongly continuous cosine operator function C. The first one consists in approximating the infinitesimal generator A by a sequence $(A_n)_{\mathbb{N}}$ of operators which may also generate cosine operator functions C_n, and investigating the impact of that approximation on the approximation of C by C_n. In applications when C and A are related to a Cauchy problem (1.1a), (1.1b), representing for example a second order hyperbolic initial boundary value problem, this kind of approximation usually will result from a semidiscretization of the problem with respect to the space variables. The second approach is to approximate C by a sequence $(C_{\Delta,n})_{\mathbb{N}}$ of discrete cosine operator functions defined on uniform partitions $\Delta_{k_n}^+$ of the positive real half-axis with step sizes k_n converging to zero as $n \to \infty$. With reference again to the Cauchy problem (1.1a), (1.1b), this kind of approximation can emerge from a discretization in time by means of a two-step difference method like that given by (2.13).

In order to achieve maximum generality in the investigation of the mutual relationship between the approximation of C by C_n on the one hand and the approximation of their infinitesimal generators A, A_n on the other hand, we allow C_n (resp. A_n) to act on Banach spaces E_n, $n \in \mathbb{N}$, which are not necessarily subspaces of E. Therefore, with regard to applications, this approach covers approximation schemes based on perturbation of the domain, penalty techniques or nonconforming finite element methods. A convenient framework which enables us to handle the approximation process in this way is delivered by the theory of discrete convergence in discrete limit spaces, as developed by Aubin [1], Grigorieff [9], [10] and Stummel [18], [19]. The basic notion throughout the sequel will be that of a discrete approximation of a Banach space E by a sequence of Banach spaces E_n, $n \in \mathbb{N}$.

Let E, E_n, $n \in \mathbb{N}$, be Banach spaces over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) and let $R = (R_n)_{\mathbb{N}}$ be a sequence of (not necessarily linear) operators $R_n : E \to E_n$, $n \in \mathbb{N}$, satisfying

(3.1)

(i) $\|R_n(\alpha_1 u_1 + \alpha_2 u_2) - \alpha_1 R_n u_1 - \alpha_2 R_n u_2\| \to 0 \qquad (n \in \mathbb{N}),$

(ii) $\|R_n u\| \to \|u\| \qquad (n \in \mathbb{N}), \quad u \in E,$

(iii) $\sup_{n \in \mathbb{N}} \|R_n u\| < \infty, \qquad u \in E.$

Then the triple $(E, \Pi E_n, R)$ is called a discrete approximation.

In the sequel we will sometimes impose an additional condition on $(E, \Pi E_n, R)$:

(3.1) (iv) If $u \in E$ and $u^{(n)} \in E$, $n \in \mathbb{N}$, such that $u^{(n)} \to u\,(n \in \mathbb{N})$, in E, then

$$\|R_n u^{(n)} - R_n u\| \to 0 \qquad (n \in \mathbb{N}).$$

For a given discrete approximation $(E, \Pi E_n, R)$ the discrete convergence of a sequence $(u_n)_{\mathbb{N}'}$ of elements $u_n \in E_n$, $n \in \mathbb{N}' \subset \mathbb{N}$ to an element $u \in E$ is defined by

$$u_n \to u(n \in \mathbb{N}') \leftrightarrow \|u_n - R_n u\| \to 0 \ (n \in \mathbb{N}'). \tag{3.2}$$

Remark 3.1. Conditions (i), (ii) and (iii) in the above definition of a discrete approximation ensure the linearity of discrete convergence, the uniqueness of limits of discrete convergent sequences and the continuity of norms with respect to discrete convergence. The additional condition (iv) exhibits a certain uniformity of discrete convergence.

Let us now consider operators $A \in \mathscr{C}(E)$ and $A_n \in \mathscr{C}(E_n)$, $n \in \mathbb{N}$, where $(E, \Pi E_n, R)$ is assumed to be a discrete approximation. Then the sequence $(A_n)_{\mathbb{N}}$ is said to converge discretely to A ($A_n \to A(n \in \mathbb{N})$) if for each $u \in D(A)$ and any sequence $(u_n)_{\mathbb{N}}$, $u_n \in D(A_n)$, $n \in \mathbb{N}$, we have

$$u_n \to u(n \in \mathbb{N}) \Rightarrow A_n u_n \to Au(n \in \mathbb{N}). \tag{3.3}$$

A notion closely related to the discrete convergence of operators is that of consistency. The pair A, $(A_n)_{\mathbb{N}}$ is called consistent if for each $u \in D(A)$ there exists a sequence $(u_n)_{\mathbb{N}}$, $u_n \in D(A_n)$, $n \in \mathbb{N}$, such that $u_n \to u(n \in \mathbb{N})$ and $A_n u_n \to Au(n \in \mathbb{N})$. Finally, for bounded linear operators A_n, $n \in \mathbb{N}$, the sequence $(A_n)_{\mathbb{N}}$ is said to be stable if it is uniformly bounded, i.e., if there exists a constant $K \geqq 0$ such that $\|A_n\| \leqq K$, $n \in \mathbb{N}$. In the special case $A \in \mathscr{B}(E)$, $A_n \in \mathscr{B}(E_n)$, $n \in \mathbb{N}$, a basic result states that the discrete convergence $A_n \to A(n \in \mathbb{N})$ is equivalent to the consistency of the pair A, $(A_n)_{\mathbb{N}}$ and the stability of the sequence $(A_n)_{\mathbb{N}}$ (cf. e.g. [18, Thm. 1.2(6)]).

We now focus our attention to the case where $A \in \mathscr{C}(E)$ and $A_n \in \mathscr{C}(E_n)$, $n \in \mathbb{N}$, are infinitesimal generators of strongly continuous cosine operator functions $C : \mathbb{R}^+ \to \mathscr{B}(E)$ (resp. $C_n : \mathbb{R}^+ \to \mathscr{B}(E_n)$). Our first result gives a complete characterization of the discrete convergence $C_n(t) \to C(t)(n \in \mathbb{N})$, $t \in \mathbb{R}^+$, by stating necessary and sufficient conditions in terms of the generators A, A_n.

THEOREM 3.1. *Let $(E, \Pi E_n, R)$ be a discrete approximation and let $A \in \mathscr{C}(E)$, $A_n \in \mathscr{C}(E_n)$, $n \in \mathbb{N}$, be infinitesimal generators of strongly continuous cosine operator functions $C : \mathbb{R}^+ \to \mathscr{B}(E)$, $C_n : \mathbb{R}^+ \to \mathscr{B}(E_n)$ of type (M_0, ω_0) (resp. (M_n, ω_n)). Suppose that $\bar{\omega} = \limsup_{n\to\infty} \omega_n < \infty$ and $\bar{M} = \limsup_{n\to\infty} M_n < \infty$. Then the consistency of the pair $(A, (A_n)_{\mathbb{N}}$ is a necessary and sufficient condition for the discrete convergence $C_n(t) \to C(t)$ $(n \in \mathbb{N})$ to hold uniformly on finite intervals of $\mathbb{R}^+$.*

Proof. We will first show that consistency of the generators is sufficient to ensure discrete convergence. Since A_n, $n \in \mathbb{N}$, is the infinitesimal generator of a strongly continuous cosine operator function C_n of type (M_n, ω_n), A_n also generates a strongly continuous semigroup of type (M_n, ω_n^2) (cf. Remark 2.2 (ii)). Therefore, if $\lambda > \max(\omega_0^2, \bar{\omega}^2)$, the generation theorem implies that $\lambda \in \rho(A_n)$, $n \in \mathbb{N}$, and

$$\|R(\lambda, A_n)\| \leqq M(\lambda - \bar{\omega}^2)^{-1}, \qquad n \in \mathbb{N}. \tag{3.4}$$

Since also $\lambda \in \rho(A)$, it follows by [19, Thm. 2.1(5)] that

$$R(\lambda, A_n) \to R(\lambda, A) \qquad (n \in \mathbb{N}). \tag{3.5}$$

Let now $u \in E$ and $v := R(\lambda, A)u$, $w := R(\lambda, A)v$. Then, for $t \in \mathbb{R}^+$ we have

$$\begin{aligned} &\|C_n(t)R_n w - R_n C(t)w\| \\ &\leqq \|C_n(t)[R_n R(\lambda, A)v - R(\lambda, A_n)R_n v]\| + \|R(\lambda, A_n)R_n C(t)v - R_n R(\lambda, A)C(t)v\| \\ &\quad + \|R(\lambda, A_n)[C_n(t)R_n v - R_n C(t)v]\|. \end{aligned} \tag{3.6}$$

By the uniform boundedness of $(C_n(t))_{\mathbb{N}}$ and because of (3.5) the first two terms on the right-hand side of (3.6) tend to zero as $n\to\infty$. To prove the convergence of the remaining term we first notice that by (2.10a) we get for $0<s<t$:

$$\frac{d}{ds}R(\lambda, A_n)C_n(t-s)R_nC(s)v = C_n(t-s)R(\lambda, A_n)R_n(\lambda R(\lambda, A)-I)S(s)u$$
$$-S_n(t-s)(\lambda R(\lambda, A_n)-I)R_nR(\lambda, A)C(s)u.$$

Integrating with respect to s on $(0, t)$, we obtain by partial integration

$$R(\lambda, A_n)[C_n(t)R_nv - R_nC(t)v] = \int_0^t S_n(t-s)[R(\lambda, A_n)R_n - R_nR(\lambda, A)]C(s)u\, ds.$$

From (2.8) it follows that

$$\|S_n(t-s)\| \leqq \bar{M}(t-s)\cosh(\bar{\omega}(t-s)), \qquad 0<s<t, \tag{3.7}$$

which gives us

$$\begin{aligned}&\|R(\lambda, A_n)[C_n(t)R_nv - R_nC(t)v]\|\\ &\qquad \leqq \bar{M}t\cosh(\bar{\omega}t)\int_0^t \|[R(\lambda, A_n)R_n - R_nR(\lambda, A)]C(s)u\|\, ds.\end{aligned} \tag{3.8}$$

Now (3.5) shows that the integrand in (3.8) converges pointwise to zero as $n\to\infty$, while (3.1) (iii) and (3.4) guarantee that the integrand is bounded independently of $n\in\mathbb{N}$. Therefore, the right-hand side in (3.8) tends to zero as $n\to\infty$. We have thus shown the consistency of $C(t)$, $(C_n(t))_{\mathbb{N}}$, $t\in\mathbb{R}^+$, on the dense subset $R(R(\lambda, A)^2)\subset E$. Since $(C_n(t))_{\mathbb{N}}$ is uniformly bounded in $n\in\mathbb{N}$, it follows by [18, Thm. 1.2(6)] that $C_n(t)\to C(t)(n\in\mathbb{N})$. From a review of the preceding steps of proof, it is an easy matter to conclude that the convergence is uniform on finite intervals of $\mathbb{R}^+$.

Conversely, assume that $C_n(t)\to C(t)(n\in\mathbb{N})$, $t\in\mathbb{R}^+$, and let $u\in E$, $u_n\in E_n$, $n\in\mathbb{N}$, such that $u_n\to u(n\in\mathbb{N})$. Then, for $\lambda>\max(\omega_0^2, \bar{\omega}^2)$ we have by (2.17)

$$\|R(\lambda, A_n)u_n - R_nR(\lambda, A)u\| \leqq \int_0^\infty \lambda^{-1/2}\exp(-\lambda^{1/2}t)\|C_n(t)u_n - R_nC(t)u\|\, dt. \tag{3.9}$$

Since the integrand in (3.9) converges pointwise and is uniformly bounded, the right-hand side in (3.9) tends to zero as $n\to\infty$, and therefore we have $R(\lambda, A_n)\to R(\lambda, A)(n\in\mathbb{N})$. Finally, for $u\in D(A)$ there exists a $v\in E$ such that $u=R(\lambda, A)v$. If we set $u_n = R(\lambda, A_n)R_nv$, then $u_n\in D(A_n)$ and $u_n\to u(n\in\mathbb{N})$. Moreover, we get

$$A_nu_n = (\lambda R(\lambda, A_n)-I)R_nv \to (\lambda R(\lambda, A)-I)v = Au \qquad (n\in\mathbb{N}),$$

which proves the consistency of the pair A, $(A_n)_{\mathbb{N}}$.

Remark 3.2. In view of (2.8), the discrete convergence of the propagators $S_n(t)$, $n\in\mathbb{N}$, to $S(t)$, $t\in\mathbb{R}^+$, can be characterized in exactly the same way.

In applications, one is particularly interested in a priori estimates for the global discretization error $C_n(t)u_n - R_nC(t)u$ (resp. $S_n(t)u_n - R_nS(t)u$), $n\in\mathbb{N}$. For this purpose let us assume that $(E, \Pi E_n, R)$ is a discrete approximation with $R=(R_n)_{\mathbb{N}}$, $R_n(D(A))\subset D(A_n)$, $n\in\mathbb{N}$, and for $t\in\mathbb{R}^+$, $v_1, v_2\in D(A)$ let us then define

$$\tau_n(t; v_1, v_2) := \tau_n^{(1)}(t; v_1) + \tau_n^{(2)}(t; v_2), \qquad n\in\mathbb{N}, \tag{3.10}$$

where

$$\tau_n^{(1)}(t; v_1) := R_nAC(t)v_1 - A_nR_nC(t)v_1, \tag{3.11a}$$

$$\tau_n^{(2)}(t; v_2) := R_nAS(t)v_2 - A_nR_nS(t)v_2. \tag{3.11b}$$

We can interpret $\tau_n(t; v_1, v_2)$ as the local discretization error with respect to the approximation of the Cauchy problem (1.1a), (1.1b) by a sequence of similar Cauchy problems in E_n, $n \in \mathbb{N}$:

$$\frac{d^2}{dt^2}u_n(t) = A_nu_n(t), \qquad t \in \mathbb{R}^+, \tag{3.12a}$$

$$u_n(0) = u_n^0, \qquad \frac{d}{dt}u_n(0) = u_{n,t}^0. \tag{3.12b}$$

Indeed, if $u(t)$ is the solution of (1.1a), (1.1b) for given u^0, $u_t^0 \in D(A)$, then we get according to (2.12)

$$\begin{aligned}\tau_n(t; u^0, u_t^0) &= \tau_n^{(1)}(t; u^0) + \tau_n^{(2)}(t; u_t^0)\\ &= (R_nA - A_nR_n)[C(t)u^0 + S(t)u_t^0] = \frac{d^2}{dt^2}R_nu(t) - A_nR_nu(t).\end{aligned}$$

We shall show now that, provided u_n converges to u with at least the order of consistency, the order of discrete convergence $C_n(t)u_n \to C(t)u\,(n \in \mathbb{N})$ (resp. $S_n(t)u_n \to S(t)u$ $(n \in \mathbb{N})$) is the same as that of consistency.

THEOREM 3.2. *Let* $(E, \Pi E_n, R)$ *be a discrete approximation with* $R = (R_n)_{\mathbb{N}}$, $R_n(D(A)) \subset D(A_n)$, $n \in \mathbb{N}$, *and suppose that the assumptions of Theorem* 3.1 *are satisfied. Then there exist positive constants* $K_\nu = K_\nu(\bar{M}, \bar{\omega}, t)$, $\nu = 1, 2$, *such that for* $u \in D(A)$, $u_n \in E_n$, $n \in \mathbb{N}$, *and* $t \in \mathbb{R}^+$

$$\|C_n(t)u_n - R_nC(t)u\| \leqq K_1[\|u_n - R_nu\| + \max_{0<s\leqq t}\|\tau_n^{(1)}(s; u^0)\|], \tag{3.13}$$

$$\|S_n(t)u_n - R_nS(t)u\| \leqq K_2[\|u_n - R_nu\| + \max_{0<s\leqq t}\|\tau_n^{(2)}(s; u_t^0)\|]. \tag{3.14}$$

Proof. Starting from the inequality

$$\|C_n(t)u_n - R_nC(t)u\| \leqq \|C_n(t)(u_n - R_nu)\| + \|C_n(t)R_nu - R_nC(t)u\|, \tag{3.15}$$

we can handle the second term on the right-hand side of (3.15) in almost the same way as in Theorem 3.1 to obtain

$$\|C_n(t)R_nu - R_nC(t)u\| \leqq \int_0^t \|S_n(t-s)\|\|A_nR_nC(s)u - R_nAC(s)u\|\,ds. \tag{3.16}$$

If we insert (3.16) into (3.15), we get the asserted a priori estimate (3.13) by means of (3.7) and (3.11a). The other estimate (3.14) can be proven in an analogous manner, since with regard to (2.8) we have

$$\|S_n(t)R_nu - R_nS(t)u\| \leqq \int_0^t \int_0^s \|C_n(s-\tau)\|\|A_nR_nS(\tau)u - R_nAS(\tau)u\|\,d\tau ds.$$

The preceding inequalities (3.13), (3.14) instantly provide us with an a priori estimate for the global discretization error when approximating the Cauchy problem (1.1a), (1.1b) by (3.11a), (3.11b).

COROLLARY 3.1. *For u^0, $u_t^0 \in D(A)$ and u_n^0, $u_{n,t}^0 \in D(A_n)$, $n \in \mathbb{N}$, let $u(t)$ (resp. $u_n(t)$), $t \in \mathbb{R}^+$, be the solutions of the Cauchy problems* (1.1a), (1.1b) (*resp.* (3.11a), (3.11b)) *and suppose that the assumptions of Theorem* 3.1 *hold. Then there exists a positive constant $K = K(\bar{M}, \bar{\omega}, t)$ such that*

$$\|u_n(t) - R_n u(t)\| \leqq K[\|u_n^0 - R_n u^0\| + \|u_{n,t}^0 - R_n u_t^0\| + \max_{0<s\leqq t} \|\tau_n^{(1)}(s; u^0)\| + \max_{0<s\leqq t} \|\tau_n^{(2)}(s; u_t^0)\|]. \tag{3.17}$$

It is natural to ask to what extent the preceding results can be used concerning the approximation of the adjoint cosine operator function $C^*|_{\mathrm{cl}\, D(A^*)}$. For that purpose we introduce the concept of discrete weak convergence of sequences $(f_n)_{\mathbb{N}}$ of bounded linear functionals $f_n \in E_n^*$, $n \in \mathbb{N}$ (cf. [18]): A sequence $(f_n)_{\mathbb{N}}$, $f_n \in E_n^*$, $n \in \mathbb{N}$, converges discretely weakly to $f \in E^*(f_n \rightharpoonup f(n \in \mathbb{N}))$ if for each $u \in E$ and any discrete convergent sequence $(u_n)_{\mathbb{N}}$, $u_n \in E_n$, $n \in \mathbb{N}$, we have

$$u_n \to u(n \in \mathbb{N}) \Rightarrow \langle f_n, u_n \rangle \to \langle f, u \rangle (n \in \mathbb{N}). \tag{3.18}$$

Remark 3.3. (i) If $(E, \Pi E_n, R)$ is a discrete approximation of a separable Banach space E, then the discrete convergence in $(E, \Pi E_n, R)$ can be completely characterized by means of the discrete weak convergence of bounded linear functionals (cf. [18, Thm. 2.1(6)]): Let $u \in E$, $u_n \in E_n$, $n \in \mathbb{N}$. Then $u_n \to u(n \in \mathbb{N})$ if and only if for each $f \in E^*$ and any discrete weak convergent sequence $(f_n)_{\mathbb{N}}$, $f_n \in E_n^*$, $n \in \mathbb{N}$, we have

$$f_n \rightharpoonup f(n \in \mathbb{N}) \Rightarrow \langle f_n, u_n \rangle \to \langle f, u \rangle (n \in \mathbb{N}). \tag{3.19}$$

(ii) Moreover, if E is separable, for each $f \in E^*$ there exists a sequence $(f_n)_{\mathbb{N}}$, $f_n \in E_n^*$, $n \in \mathbb{N}$, such that $f_n \rightharpoonup f(n \in \mathbb{N})$ and $\|f_n\| \to \|f\|$ $(n \in \mathbb{N})$ (cf. [9, Thm. 1(12)]).

Based on the definition (3.18) of discrete weak convergence of bounded linear functionals, a sequence $(A'_n)_{\mathbb{N}}$ of operators $A'_n \in \mathscr{L}(E_n^*)$, $n \in \mathbb{N}$, is said to converge discrete weakly to an operator $A' \in \mathscr{L}(E^*)$ $(A'_n \to A'(n \in \mathbb{N}))$ if for each $f \in D(A')$ and any discrete weakly convergent sequence $(f_n)_{\mathbb{N}}$, $f_n \in D(A'_n)$, $n \in \mathbb{N}$, we have

$$f_n \rightharpoonup f(n \in \mathbb{N}) \Rightarrow A'_n f_n \rightharpoonup A'f \ (n \in \mathbb{N}). \tag{3.20}$$

In the special case $A' = A^*$ and $A'_n = A_n^*$, $n \in \mathbb{N}$, where $A \in \mathscr{B}(E)$, $A_n \in \mathscr{B}(E_n)$ and $(E, \Pi E_n, R)$ is a discrete approximation, it is not difficult to show (cf. [18, Thm. 2.2(3)]) that

$$A_n \to A(n \in \mathbb{N}) \Rightarrow A_n^* \rightharpoonup A^* \ (n \in \mathbb{N}). \tag{3.21}$$

This leads to the following result:

THEOREM 3.3. *Let $(E, \Pi E_n, R)$ be a discrete approximation and suppose that the operators $A \in \mathscr{C}(E)$, $A_n \in \mathscr{C}(E_n)$, $n \in \mathbb{N}$, satisfy the assumptions of Theorem* 3.1. *Then, if the pair A, $(A_n)_{\mathbb{N}}$ is consistent, we get $C_n^*(t)|_{\mathrm{cl}D(A_n^*)} \rightharpoonup C^*(t)|_{\mathrm{cl}D(A^*)}$, $t \in \mathbb{R}^+$, while for separable E we also have the converse.*

Proof. By Theorem 3.1, the consistency of the pair A, $(A_n)_{\mathbb{N}}$ implies that $C_n(t) \to C(t)(n \in \mathbb{N})$, $t \in \mathbb{R}^+$, which gives the discrete weak convergence of the adjoint cosine operator functions by means of (3.21).

Conversely, let us assume that $C_n^*(t)|_{\mathrm{cl}\, D(A_n^*)} \rightharpoonup C^*(t)|_{\mathrm{cl}\, D(A^*)}$ $(n \in \mathbb{N})$, $t \in \mathbb{R}^+$. By (2.19) we then have $R(\lambda, A_n^{(*)}) \rightharpoonup R(\lambda, A^{(*)})$ $(n \in \mathbb{N})$, $\lambda > \max(\omega_0^2, \bar{\omega}^2)$. According to (3.18) and (3.20), this means that if $u \in E$, $u_n \in E_n$, $n \in \mathbb{N}$, such that $u_n \to u$ $(n \in \mathbb{N})$, for each $f \in \mathrm{cl}\, D(A^*)$ and any sequence $(f_n)_{\mathbb{N}}$, $f_n \in \mathrm{cl}\, D(A_n^*)$, $n \in \mathbb{N}$, with $f_n \rightharpoonup f$ $(n \in \mathbb{N})$, we get

$$\langle R(\lambda, A_n^*) f_n, u_n \rangle \to \langle R(\lambda, A^*) f, u \rangle \qquad (n \in \mathbb{N}),$$

and thus

$$\langle f_n, R(\lambda, A_n)u_n\rangle \to \langle f, R(\lambda, A)u\rangle \qquad (n \in \mathbb{N}). \tag{3.22}$$

Since $D(A^*)$ (resp. $D(A_n^*)$, $n \in \mathbb{N}$), is weakly* dense in E^* (resp. E_n^*), it follows that (3.22) also holds for each $f \in E^*$ and any sequence $(f_n)_\mathbb{N}$, $f_n \in E_n^*$, $n \in \mathbb{N}$, with $f_n \rightharpoonup f$ $(n \in \mathbb{N})$. But E is assumed to be separable, and therefore with regard to Remark 3.3(i) we conclude that $R(\lambda, A_n)u_n \to R(\lambda, A)u$ $(n \in \mathbb{N})$. This yields $R(\lambda, A_n) \to R(\lambda, A)$ $(n \in \mathbb{N})$, because the pair u, $(u_n)_\mathbb{N}$ with $u \in E$, $u_n \in E_n$ and $u_n \to u$ $(n \in \mathbb{N})$ was arbitrarily chosen. Finally, as in the proof of Theorem 3.1, the convergence of the resolvents gives us the consistency of A, $(A_n)_\mathbb{N}$, to complete our proof.

We now turn our attention to the approximation of a strongly continuous cosine operator function $C : \mathbb{R}^+ \to \mathcal{B}(E)$ by a sequence $(C_{\Delta,n})_\mathbb{N}$ of discrete cosine operator functions $C_{\Delta,n} : \Delta_{k_n}^+ \to \mathcal{B}(E_n)$, $n \in \mathbb{N}$, where $(k_n)_\mathbb{N}$ is a null sequence of positive real numbers and $(E, \Pi E_n, R)$ is assumed to be a discrete approximation. The sequence $(C_{\Delta,n})_\mathbb{N}$ is said to converge to $C(C_{\Delta,n} \to C(n \in \mathbb{N}))$ if for each $u \in E$ and any sequence $(u_n)_\mathbb{N}$, $u_n \in E_n$, $n \in \mathbb{N}$, such that $u_n \to u$ $(n \in \mathbb{N})$,

$$\max_{t_n \in \Delta_{k_n}^+} \|C_{\Delta,n}(t_n)u_n - R_n C(t_n)u\| \to 0 \qquad (n \in \mathbb{N}). \tag{3.23}$$

Remark 3.4. Note that (3.23) is satisfied if and only if for each $t \in \bar{\mathbb{R}}^+$ and any sequence $(t_n)_\mathbb{N}$, $t_n \in \Delta_{k_n}^+$, with $t_n \to t$ $(n \in \mathbb{N})$ we have $C_{\Delta,n}(t_n) \to C(t)$ $(n \in \mathbb{N})$ in the sense of discrete convergence of operators with respect to the discrete approximation $(E, \Pi E_n, R)$.

Our first result characterizes the convergence $C_{\Delta,n} \to C(n \in \mathbb{N})$ in mostly the same way as in the time-continuous case, and therefore can be seen as the time-discrete counterpart of Theorem 3.1.

THEOREM 3.4. *Let $C : \mathbb{R}^+ \to \mathcal{B}(E)$ be a strongly continuous cosine operator function of type (M_0, ω_0) with infinitesimal generator $A \in \mathcal{C}(E)$ and let $C_{\Delta,n} : \Delta_{k_n}^+ \to \mathcal{B}(E_n)$, $n \in \mathbb{N}$, be discrete cosine operator functions with generators $A_{\Delta,n} \in \mathcal{B}(E_n)$. Assume that $(E, \Pi E_n, R)$ is a discrete approximation satisfying* (3.1) (i)–(iv) *and that the sequence $(C_{\Delta,n})_\mathbb{N}$ is uniformly quasibounded in the sense of* (2.6)′ *where $(k_n)_\mathbb{N}$ is a null sequence of positive real numbers. Then, in order that $C_{\Delta,n} \to C$ $(n \in \mathbb{N})$, the consistency of the pair A, $(A_{\Delta,n})_\mathbb{N}$ is a sufficient as well as a necessary condition.*

Proof. First we will prove that the indicated condition is a sufficient one. Due to the uniform quasiboundedness of $(C_{\Delta,n})_\mathbb{N}$, it is an easy matter to show by means of the first Neumann series that there are positive constants $\bar{M}$, $\bar{\omega}$ such that $R(\lambda, A_{\Delta,n})$ exists for $\lambda > \bar{\omega}^2$ and

$$\|R(\lambda, A_{\Delta,n})\| \leqq \bar{M}(\lambda - \bar{\omega}^2)^{-1}, \qquad n \in \mathbb{N}. \tag{3.24}$$

But for $\lambda > \omega_0^2$ we also have $\lambda \in \rho(A)$, and therefore the consistency of the pair A, $(A_{\Delta,n})_\mathbb{N}$ again gives us

$$R(\lambda, A_{\Delta,n}) \to R(\lambda, A)(n \in \mathbb{N}), \qquad \lambda > \max(\omega_0^2, \bar{\omega}^2). \tag{3.25}$$

Furthermore, for $u \in E$, $v := R(\lambda, A)u$, $w := R(\lambda, A)v$, $t \in [0, b)$, $b \in \mathbb{R}^+$, and $t_n \in \Delta_{k_n}^+$, $t_n = m_n k_n$, $m_n \in \mathbb{N}$, with $t_n \to t(n \in \mathbb{N})$, we have again

$$\begin{aligned} &\|C_{\Delta,n}(t_n)R_n w - R_n C(t_n)w\| \\ &\quad \leqq \|C_{\Delta,n}(t_n)[R(\lambda, A_{\Delta,n})R_n v - R_n R(\lambda, A)v]\| \\ &\qquad + \|R(\lambda, A_{\Delta,n})R_n C(t_n)v - R_n R(\lambda, A)C(t_n)v\| \\ &\qquad + \|R(\lambda, A_{\Delta,n})[C_{\Delta,n}(t_n)R_n v - R_n C(t_n)v]\|. \end{aligned} \tag{3.26}$$

In view of (3.25) and the strong continuity of C, the second term on the right-hand side of (3.26) converges to zero as $n \to \infty$, and so does the first term, since the sequence $(C_{\Delta,n}(t_n))_{\mathbb{N}}$ is uniformly bounded. If we define

$$A^{(k_n)} := 2k_n^{-2}(C(k_n)-I), \qquad n \in \mathbb{N}, \tag{3.27}$$

$$S^{(k_n)}(t) := \tfrac{1}{2}k_n I + k_n \sum_{\nu=1}^{[t/k_n]} C(\nu k_n), \qquad t \in \mathbb{R}^+, \tag{3.28}$$

then, by means of (2.11a), the third term can be rewritten in the following form:

$$\begin{aligned} R(\lambda, A_{\Delta,n})&[C_{\Delta,n}(t_n)R_n v - R_n C(t_n)v] \\ &= R(\lambda, A_{\Delta,n}) \sum_{j=0}^{m_n-1} [D_{k_n}^+ C_{\Delta,n}((m_n-j-1)k_n)R_n C(jk_n)v \\ &\qquad\qquad - C_{\Delta,n}((m_n-j-1)k_n)D_{k_n}^+ R_n C(jk_n)v] \\ &= k_n \sum_{j=0}^{m_n-1} A_{\Delta,n}R(\lambda, A_{\Delta,n})S_{\Delta,n}((m_n-j-1)k_n)R_n C(jk_n)R(\lambda, A)u \\ &\quad - k_n \sum_{j=0}^{m_n-1} R(\lambda, A_{\Delta,n})C_{\Delta,n}((m_n-j-1)k_n)R_n S^{(k_n)}(jk_n)AR(\lambda, A)u \\ &\quad - k_n \sum_{j=0}^{m_n-1} R(\lambda, A_{\Delta,n})C_{\Delta,n}((m_n-j-1)k_n)R_n S^{(k_n)}(jk_n) \\ &\qquad\qquad \cdot (A^{(k_n)}-A)R(\lambda, A)u. \end{aligned} \tag{3.29}$$

Using (2.9) and (3.28), we see that

$$\begin{aligned} \sum_{j=0}^{m_n-1} &R(\lambda, A_{\Delta,n})C_{\Delta,n}((m_n-j-1)k_n)R_n S^{(k_n)}(jk_n)AR(\lambda, A)u \\ &= \sum_{j=0}^{m_n-1} R(\lambda, A_{\Delta,n})S_{\Delta,n}((m_n-j-1)k_n)R_n C(jk_n)AR(\lambda, A)u \\ &\quad + \tfrac{1}{2}k_n \sum_{j=1}^{m_n-1} R(\lambda, A_{\Delta,n})(R_n C(jk_n) - C_{\Delta,n}(jk_n)R_n)AR(\lambda, A)u. \end{aligned} \tag{3.30}$$

If we insert (3.30) into (3.29) and take advantage of (3.25) and the uniform boundedness of $C_{\Delta,n}(jk_n)$ and $S_{\Delta,n}(jk_n)$, $0 \leqq j \leqq m_n$, $n \in \mathbb{N}$, we achieve the estimate

$$\begin{aligned} \|R(\lambda, A_{\Delta,n})&[C_{\Delta,n}(t_n)R_n v - R_n C(t_n)v]\| \\ &\leqq K\Bigg[k_n \sum_{j=0}^{m_n-1} \|(R(\lambda, A_{\Delta,n})R_n - R_n R(\lambda, A))C(jk_n)u\| \\ &\quad + \max_{t\in[0,b]} \|R_n S^{(k_n)}(t)(A^{(k_n)}-A)R(\lambda, A)u\| \\ &\quad + k_n\Big(\max_{t\in[0,b]} \|R_n C(t)AR(\lambda, A)u\| + \|R_n AR(\lambda, A)u\|\Big)\Bigg]. \end{aligned} \tag{3.31}$$

The first term in brackets can be interpreted as the lower Darboux sum of the integral

$$\int_0^{t_n} \|(R(\lambda, A_{\Delta,n})R_n - R_n R(\lambda, A))C(s)u\|\, ds,$$

which can be shown to converge to zero as $n \to \infty$ by arguing exactly in the same way as in the proof of Theorem 3.1. Since $R(\lambda, A)u \in D(A)$, it follows by (2.4) and (3.27) that $A^{(k_n)}R(\lambda, A)u \to AR(\lambda, A)u\,(n \in \mathbb{N})$. Moreover, $S^{(k_n)}(t) \to S(t)(n \in \mathbb{N})$, $t \in [0, b]$, in view of (3.28). With regard to (3.1) (iii), (iv), we then conclude also that the remaining terms on the right-hand side in (3.31) tend to zero as $n \to \infty$. Altogether, we get $C_{\Delta,n}(t_n)R_n u \to C(t)u\,(u \in \mathbb{N})$, $u \in R^2(\lambda, A)(E)$. But $R^2(\lambda, A)(E)$ is dense in E, and $(C_{\Delta,n})_{\mathbb{N}}$ is uniformly quasibounded, which gives us $C_{\Delta,n} \to C|_{[0,b)}(n \in \mathbb{N})$. Finally, using (2.1), (2.1)′ we can show by induction that the asserted convergence holds with respect to any interval $[0, mb)$, $m \in \mathbb{N}$.

In order to prove the converse, let us assume that $u \in E$, $(u_n)_{\mathbb{N}}$, $u_n \in E_n$, $n \in \mathbb{N}$, with $u_n \to u\,(n \in \mathbb{N})$ and let us define

$$Z_{\Delta,n}(\lambda)u_n := \tfrac{1}{2}k_n u_n + k_n \sum_{\nu=1}^{\infty} \exp(-\nu\lambda k_n) C_{\Delta,n}(\nu k_n)u_n, \quad \lambda > \max(\omega_0^2, \bar{\omega}^2).$$

It follows by a simple calculation that

$$\begin{aligned} A_{\Delta,n} Z_{\Delta,n}(\lambda)u_n &= k_n^{-1}(C_{\Delta,n}(k_n) - I)u_n \\ &\quad + 2k_n^{-1} \sum_{\nu=1}^{\infty} \exp(-\nu\lambda k_n) C_{\Delta,n}(\nu k_n)(C_{\Delta,n}(k_n) - I)u_n \\ &= k_n^{-1}(\exp(-\lambda k_n) - 1)u_n \\ &\quad + k_n^{-1} \sum_{\nu=1}^{\infty} [\exp(-(\nu+1)k_n) - 2\exp(-\nu\lambda k_n) \\ &\qquad\qquad + \exp(-(\nu-1)\lambda k_n)] C_{\Delta,n}(\nu k_n)u_n \\ &= (2k_n)^{-1}[\exp(-\lambda k_n) - \exp(+\lambda k_n)]u_n \\ &\quad + k_n^{-2}[\exp(-\lambda k_n) - 2 + \exp(+\lambda k_n)] Z_{\Delta,n}(\lambda)u_n \\ &= (-\lambda + O(k_n^2))u_n + (\lambda^2 + O(k_n^2)) Z_{\Delta,n}(\lambda)u_n. \end{aligned}$$

Therefore, we get

$$(3.32) \quad (\lambda I - A_{\Delta,n})\lambda^{-1/2} Z_{\Delta,n}(\lambda^{-1/2})u_n = (1 + O(k_n^2))u_n + O(k_n^2) Z_{\Delta,n}(\lambda^{1/2})u_n.$$

The right-hand side in (3.32) apparently converges to u as $n \to \infty$. Since $C_{\Delta,n} \to C\,(n \in \mathbb{N})$, we have

$$\lambda^{-1/2} Z_{\Delta,n}(\lambda^{1/2})u_n \to \lambda^{-1/2} \int_0^\infty \exp(-\lambda^{1/2}s) C(s)u\, ds = R(\lambda, A)u \qquad (n \in \mathbb{N}),$$

and thus we come to the conclusion that $R(\lambda, A_{\Delta,n}) \to R(\lambda, A)(n \in \mathbb{N})$, which again gives us the consistency of the pair A, $(A_{\Delta,n})_{\mathbb{N}}$.

In analogy to the time-continuous case, we define for $t \in \Delta_{k_n}^+$ and $v_1, v_2 \in E$

$$(3.33) \qquad \tau_{\Delta,n}(t; v_1, v_2) := \tau_{\Delta,n}^{(1)}(t; v_1) + \tau_{\Delta,n}^{(2)}(t; v_2), \qquad n \in \mathbb{N},$$

where

$$(3.34a) \qquad \tau_{\Delta,n}^{(1)}(t; v_1) := (R_n A^{(k_n)} - A_{\Delta,n} R_n) C(t) v_1,$$

$$(3.34b) \qquad \tau_{\Delta,n}^{(2)}(t; v_2) := (R_n A^{(k_n)} - A_{\Delta,n} R_n) S(t) v_2.$$

Again, $\tau_{\Delta,n}(t; v_1, v_2)$ appears as the local discretization error with respect to the time-discrete approximation of the Cauchy problem (1.1a), (1.1b) by a sequence of two-step schemes of the form (2.13): If $u(t) = C(t)u^0 + S(t)u_t^0$, $t \in \mathbb{R}^+$, is the solution of (1.1a), (1.1b), it follows from (2.11a), (2.11b), (3.27) and (3.33) that

$$\tau_{\Delta,n}(t; u^0, u_t^0) = k_n^{-2} D_{k_n}^+ D_{k_n}^- u(t) - A_{\Delta,n} u(t), \qquad t \in \Delta_{k_n}^+ \backslash \{0\}.$$

Next we will derive a priori estimates for $C_{\Delta,n}(t_n)R_n u - R_n C(t_n)u$ (resp. $\tilde{S}_{\Delta,n}(t_n)u_n - R_n S(t_n)u$), $t_n \in \Delta_{k_n}^+$, $u \in D(A)$, where

$$u_n := (I + \tfrac{1}{4}k_n^2 A_{\Delta,n})^{-1} R_n k_n^{-1} S(k_n) u, \tag{3.35}$$

and k_n is assumed to be small enough that the operator $I + \frac{1}{4}k_n^2 A_{\Delta,n}$ is boundedly invertible.

THEOREM 3.5. *Let $C(t)$, $t \in \mathbb{R}^+$ and $C_{\Delta,n}(t)$, $t \in \Delta_{k_n}^+$, $n \in \mathbb{N}$, be given as in Theorem 3.4. Then there exist a positive integer n_0 and positive constants $K_\nu = K_\nu(\bar{M}, \bar{\omega}, b)$, $b \in \mathbb{R}^+$, $\nu = 1, 2$, such that for $u \in D(A)$, $t_n = m_n k_n \leqq b$, $m_n \in \mathbb{N}$,*

$$\|C_{\Delta,n}(t_n)R_n u - R_n C(t_n)u\| \leqq K_1 \max_{0 \leqq j \leqq m_n - 1} \|\tau_{\Delta,n}^{(1)}(t_j; u)\|, \qquad n \geqq 1, \tag{3.36}$$

$$\|\tilde{S}_{\Delta,n}(t_n)u_n - R_n S(t)u\| \leqq K_2 \max_{1 \leqq j \leqq m_n - 1} \|\tau_{\Delta,n}^{(2)}(t_j; u)\|, \qquad n \geqq n_0, \tag{3.37}$$

where u_n, $n \geqq n_0$, is given by (3.35).

Proof. For notational convenience we set $\alpha_j^{(n)} = 1$, $1 \leqq j \leqq m_n - 1$, $\alpha_0^{(n)} = \alpha_{m_n}^{(n)} = \frac{1}{2}$ and $m_n^0 := [m_n/2]$, $m_n^1 := [(m_n+1)/2]$, $j_0^{(n)} := [2j/m_n]$, $j_1^{(n)} := [2j/(m_n+1)]$, $n \in \mathbb{N}$. Then, using (2.1) and (2.3) we can show by induction that for even m_n

$$\begin{aligned} &C_{\Delta,n}(t_n)R_n u - R_n C(t_n)u \\ &= 2k_n \sum_{j=0}^{m_n^0 - 1} \Big[\alpha_{2j}^{(n)} k_n \sum_{\nu=1}^{m_n^0 - j} C_{\Delta,n}((2\nu - 1)k_n)(A_{\Delta,n}R_n - R_n A^{(k_n)})C(2jk_n)u \\ &\qquad + \alpha_{2j+1}^{(n)} k_n \sum_{\nu=0}^{m_n^0 - j - 1} \alpha_{2\nu}^{(n)} C_{\Delta,n}(2\nu k_n)(A_{\Delta,n}R_n - R_n A^{(k_n)})C((2j+1)k_n)u \Big], \end{aligned} \tag{3.38}$$

while for odd m_n

$$\begin{aligned} &C_{\Delta,n}(t_n)R_n u - R_n C(t_n)u \\ &= 2k_n \sum_{j=0}^{m_n^1 - 1} \Big[\alpha_{2j}^{(n)} k_n \sum_{\nu=0}^{m_n^1 - j - 1} \alpha_{2\nu}^{(n)} C_{\Delta,n}(2\nu k_n)(A_{\Delta,n}R_n - R_n A^{(k_n)})C(2jk_n)u \Big] \\ &\quad + 2k_n \sum_{j=0}^{m_n^1 - 2} \Big[\alpha_{2j+1}^{(n)} k_n \sum_{\nu=1}^{m_n^1 - j - 1} C_{\Delta,n}((2\nu - 1)k_n) \\ &\qquad \cdot (A_{\Delta,n}R_n - R_n A^{(k_n)})C((2j+1)k_n)u \Big]. \end{aligned} \tag{3.39}$$

Due to the fact that $(C_{\Delta,n})_{\mathbb{N}}$ is uniformly quasibounded, the first error estimate (3.36) is a direct consequence of (3.38) (resp. (3.39)).

The uniform quasiboundedness of $(C_{\Delta,n})_{\mathbb{N}}$ also ensures the existence of an integer $n_0 \in \mathbb{N}$ such that the inverse operators $(I + \frac{1}{4}k_n^2 A_{\Delta,n})^{-1}$ exist and are uniformly bounded for $n \geqq n_0$. If we take this into account and make use of (2.1), (2.3), (2.5), (3.27), (3.28) and (3.39), we get by elementary computations

$$\begin{aligned} &\tilde{S}_{\Delta,n}(t_n)u_n - R_n S(t_n)u \\ &\quad = (I + \tfrac{1}{4}k_n^2 A_{\Delta,n})^{-1} \Big[\sum_{j=0}^{m_n} \alpha_j^{(n)} C_{\Delta,n}(jk_n) \int_0^{k_n} R_n C(s)u\, ds - (I + \tfrac{1}{4}k_n^2 A_{\Delta,n}) R_n S(t_n)u \Big] \\ &\quad = (I + \tfrac{1}{4}k_n^2 A_{\Delta,n})^{-1} \Big[\sum_{j=1}^{m_n} \alpha_j^{(n)} (C_{\Delta,n}(jk_n)R_n - R_n C(jk_n)) \end{aligned}$$

$$(3.40) \qquad \cdot \int_0^{k_n} C(s)u\,ds - \tfrac{1}{4}k_n^2(A_{\Delta,n}R_n - R_nA^{(k_n)})S(t_n)u\Big]$$

$$= (I + \tfrac{1}{4}k_n^2 A_{\Delta,n})^{-1}$$

$$\cdot \Big\{ k_n \sum_{j=1}^{m_n^0} \alpha_{2j}^{(n)} \Big[k_n \sum_{\nu=0}^{j-1} (C_{\Delta,n}(2(j-\nu)-1)k_n)(A_{\Delta,n}R_n - R_nA^{(k_n)})S((2\nu+1)k_n)u$$

$$+ k_n \sum_{\nu=1}^{j-j_0^{(n)}} \alpha_{2(j-\nu)}^{(n)} C_{\Delta,n}(2(j-\nu)k_n)(A_{\Delta,n}R_n - R_nA^{(k_n)})S(2\nu k_n)u \Big]$$

$$+ k_n \sum_{j=1}^{m_n^1} \alpha_{2j-1}^{(n)} \Big[k_n \sum_{\nu=0}^{j-j_1^{(n)}-1} \alpha_{2(j-\nu-1)}^{(n)} C_{\Delta,n}(2(j-\nu-1)k_n)$$

$$\cdot (A_{\Delta,n}R_n - R_nA^{(k_n)})S((2\nu+1)k_n)u$$

$$+ k_n \sum_{\nu=1}^{j-1} C_{\Delta,n}((2(j-\nu)-1)k_n)(A_{\Delta,n}R_n - R_nA^{(k_n)})S(2\nu k_n)u \Big]\Big\}.$$

The right-hand side in (3.40) can be bounded in a straightforward manner yielding the second error estimate (3.37).

As a by-product of the preceding estimates, we obtain an a priori estimate for the global discretization error with respect to the time discretization of (1.1a), (1.1b) by (2.13).

COROLLARY 3.2. *Under the hypotheses of Theorem* 3.4, *let* $u(t)$, $t \in \mathbb{R}^+$, *be the solution of the Cauchy problem* (1.1a), (1.1b) *with* u^0, $u_t^0 \in D(A)$, *and let* $u_{k_n}(t)$, $t \in \Delta_{k_n}^+$, *satisfy* (2.13) *where* $u_{k_n}^0 = R_n u^0$ *and* $u_{k_n}^1 = R_n u(k_n)$, $n \in \mathbb{N}$. *Then there exist an integer* $n_0 \in \mathbb{N}$ *and a positive constant* $K = K(\bar{M}, \bar{\omega}, b)$, $b \in \mathbb{R}^+$, *such that for* $t_n \in \Delta_{k_n}^+$, $t_n = m_n k_n \leqq b$, $m_n \in \mathbb{N}$, $n \geqq n_0$,

$$(3.41) \quad \|u_{k_n}(t_n) - R_n u(t_n)\| \leqq K\big[\max_{0 \leqq j \leqq m_n - 1} \|\tau_{\Delta,n}^{(1)}(t_j; u^0)\| + \max_{1 \leqq j \leqq m_n - 1} \|\tau_{\Delta,n}^{(2)}(t_j; u_t^0)\|\big].$$

Proof. Since $u_{k_n}^0 = R_n u^0$ and $u_{k_n}^1 = R_n u(k_n)$, it follows from (2.14) and (2.15) that the solution $u_{k_n}(t_n)$, $t_n \in \Delta_{k_n}^+$, $n \geqq n_0$, of the two-step scheme (2.13) can be represented in terms of u^0 and u_t^0:

$$(3.42) \qquad u_{k_n}(t_n) = C_{\Delta,n}(t_n)R_n u^0 + \tilde{S}_{\Delta,n}(t_n)(I + \tfrac{1}{4}k_n^2 A_{\Delta,n})^{-1} R_n k_n^{-1} S(k_n) u_t^0.$$

Subtracting $R_n u(t_n) = R_n C(t_n)u^0 + R_n S(t_n)u_t^0$ from (3.42) and using (3.36), (3.37) gives the conclusion.

Remark 3.5. It is obvious that the error estimate (3.41) remains valid if we replace $u_{k_n}^0 = R_n u^0$ and $u_{k_n}^1 = R_n u(k_n)$ by any starting values which approximate u^0 (resp. $u(k_n)$) with at least the order of consistency.

4. Examples and concluding remarks. As an example we consider the initial boundary value problem

$$(4.1a) \qquad \frac{\partial^2}{\partial t^2}u(t,x) = p(x)\frac{\partial^2}{\partial x^2}u(t,x) + q(x)\frac{\partial}{\partial x}u(t,x) + r(x)u(x,t), \qquad t \in \mathbb{R}^+, \quad x \in I = (0,1),$$

$$(4.1b) \qquad u(t,0) = u(t,1) = 0, \qquad t \in \mathbb{R}^+,$$

$$(4.1c) \qquad u(0,x) = u^0(x), \qquad \frac{\partial}{\partial t}u(0,x) = u_t^0(x), \qquad x \in I,$$

where p, q and r are assumed to be continuous functions on $\bar{I}$ with $0<p_0\leqq p(x)\leqq p_1$, $x\in\bar{I}$, and $|p(x_1)-p(x_2)|\leqq p_2|x_1-x_2|$, $x_1, x_2\in\bar{I}$. It is well known that (4.1a)–(4.1c) is well posed in $C(\bar{I})$, and therefore the operator A given by the right-hand side in (4.1a) with $D(A)=\{u\in C^2(I)\,|\,u(0)=u(1)=0\}$ is the infinitesimal generator of a strongly continuous cosine operator function.

In order to discretize (4.1a)–(4.1c) with respect to the space variable, let $I_h :=\{x_i=ih\,|\,i=0,\cdots,M_h, M_h=h^{-1}, h\in\mathbb{R}^+\}$ be a uniform partition of I with step size h and let $I_h^0 := I_h\backslash\{0, 1\}$ be the set of inner grid points. By $C(I_h)$ we denote the Banach space of all grid functions defined on I_h with norm $\|u_h\| := \max_{x\in I_h}|u_h(x)|$. If $\mathbb{H}_0$ is a null sequence of positive real numbers and $R=(R_h)_{\mathbb{H}_0}$, R_h, $h\in\mathbb{H}_0$, denoting the operator of pointwise restriction with respect to I_h, then $(C(\bar{I}), \Pi C(I_h), R)$ defines a discrete approximation in the sense of (3.1). The operator A will be approximated by difference operators A_h, $h\in\mathbb{H}_0$, of the form

$$A_hu_h(x) := \sum_{\nu=-1}^{+1} a_{\nu,h}(x)E^\nu u_h(x), \qquad x\in I_h^0, \tag{4.2}$$

with $D(A_h)=\{u_h\in C(I_h)\,|\,u_h(0)=u_h(1)=0\}$ where $a_{\nu,h}\in C(I_h)$ and E^ν refers to the shift operator $E^\nu u_h(x) := u_h(x+\nu h)$, $x\in I_h^0$, $\nu=-1, 0, +1$. With the operators A_h, $h\in\mathbb{H}_0$, given in this way, the semidiscrete approach to (4.1a)–(4.1c) reads

$$\frac{d^2}{dt^2}u_h(t,x)=A_hu_h(t,x), \qquad t\in\mathbb{R}^+, \quad x\in I_h^0, \tag{4.3a}$$

$$u_h(0,x)=u^0(x), \qquad \frac{d}{dt}u_h(0,x)=u_t^0(x), \qquad x\in I_h. \tag{4.3b}$$

It can be shown easily that the operators A_h, $h\in\mathbb{H}_0$, are consistent with A if and only if there exist grid functions p_h, q_h and r_h such that

$$\max_{x\in I_h}[|(p_h-p)(x)|+|(q_h-q)(x)|+|(r_h-r)(x)|]\to 0 \qquad (h\in\mathbb{H}_0) \tag{4.4}$$

and A_h, $h\in\mathbb{H}_0$, admits the representation

$$A_hu_h=p_hh^{-2}D_h^+D_h^-u_h+\tfrac{1}{2}q_hh^{-1}(D_h^+ +D_h^-)u_h+r_hu_h. \tag{4.5}$$

To simplify matters we shall assume henceforth that p_h (resp. r_h, q_h) is the restriction of p (resp. r, h) to I_h. Then, for $u\in C^4(I)$, the order of consistency is $O(h^2)$. Moreover, if conditions (4.4) and (4.5) are met, then A_h is the infinitesimal generator of a strongly continuous cosine operator function, at least for sufficiently small $h\in\mathbb{H}_0$, and according to Theorem 3.2, the discretization error $u_h(t,x)-u(t,x)$, $t\in\mathbb{R}^+$, $x\in I_h$, is of order $O(h^2)$.

Now let $\mathbb{H}_1$ be another null sequence of positive real numbers, and for $k\in\mathbb{H}_1$ let Δ_k^+ be a uniform partition of $\mathbb{R}^+$ with step size k. Then, the most obvious choice of a fully discrete approximation to (4.1a)–(4.1c) is the two-step scheme (2.13) with $A_\Delta=A_h$, $h\in\mathbb{H}_0$. It is well known that the sequence of discrete cosine operator functions $C_{k,h}$, $k\in\mathbb{H}_1$, $h\in\mathbb{H}_0$, generated by A_h, is uniformly quasibounded if the mesh ratio k/h is bounded by $p_1^{-1/2}$. This method is a special case of more general approximation schemes which can be governed in the following way: Let $A_h^{(p)}$ denote the principal part of the difference operator (4.5), i.e., $A_h^{(p)}u_h := p_hh^{-2}D_h^+D_h^-u_h$, and define $A_{k,h}^{(\alpha)}$, $\alpha>0$, by

$$A_{k,h}^{(\alpha)}u_h := (I-\alpha k^2A_h^{(p)})^{-1}A_hu_h, \qquad k\in\mathbb{H}_1, \quad h\in\mathbb{H}_0.$$

Note that $(I-\alpha k^2 A_h^{(p)})^{-1}$ is well defined, since the spectrum of $A_h^{(p)}$ is confined to the negative real axis. If we take $A_\Delta = A_{k,h}^{(\alpha)}$ in (2.13), then the resulting two-step scheme corresponds to the von Neuman difference approximation to (4.1a)–(4.1c). The sequence of discrete cosine operator functions $C_{k,h}^{(\alpha)}$ generated by $A_{k,h}^{(\alpha)}$ can be shown to be uniformly quasibounded without restriction to the mesh ratio k/h if $\alpha \geqq \frac{1}{4}$, while in case $\alpha < \frac{1}{4}$, the mesh ratio has to be restricted to $k/h \leqq [(1-4\alpha)p_1]^{-1/2}$ (cf. [8], [14]). If $u \in C^4(\mathbb{R}^+ \times I)$, then the operators $A_{k,h}^{(\alpha)}$ are consistent with A of order $O(k^2+h^2)$, and therefore, due to Theorem 3.5, the global discretization error $u_{k,h}(t,x)-u(t,x)$, $t \in \Delta_k^+$, $x \in I_h$, is of the same order, provided the above stability requirements are fulfilled.

Remark 4.1. Appropriate difference equations for the approximation of the initial conditions (4.1c) are formulated in [14].

The above problem can be treated in almost the same way, if we choose $E = L^p(I)$, $1 \leqq p < \infty$, and $E_n = E_h = C(I_h)$, $h \in \mathbb{H}_0$, normed by the discrete analogue of the L^p-norm. It should be pointed out that under the above assumptions Theorem 3.3 provides us with information about the approximation of the adjoint cosine operator function generated by the operator A^* given by $A^*u = (d^2/dx^2)(pu) - (d/dx)(qu) + ru$.

In order to approximate (4.1a)–(4.1c) by finite element methods instead of finite difference schemes, let us assume that $p(x) = 1$, $x \in \bar{I}$, which is not a severe restriction of generality (a nonconstant p would contribute to the lower order term of the bilinear form below, which causes no problem). Then $u(t,\cdot) \in W_0^{1,2}(I)$, $t \in \mathbb{R}^+$, is called a weak solution of (4.1a)–(4.1b) if

$$\text{(4.6a)} \qquad \left(\frac{\partial^2 u}{\partial t^2}, v\right)_0 = (Tu, v) := -\left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}\right)_0 + \left(q\frac{\partial u}{\partial x} + ru, v\right)_0, \qquad v \in W_0^{1,2}(I),$$

$$\text{(4.6b)} \qquad (u(0), v)_0 = (u^0, v)_0, \left(\frac{\partial}{\partial t}u(0), v\right)_0 = (u_t^0, v)_0, \qquad v \in W_0^{1,2}(I),$$

where $(\cdot,\cdot)_0$ denotes the usual scalar product in $L^2(I)$. If $q \in C^1(\bar{I})$, $r \in C(\bar{I})$ with $\frac{1}{2}(d/dx)q(x) \geqq r(x)$, $x \in \bar{I}$, then the operator T, given by the right-hand side of (4.6a), is the infinitesimal generator of a strongly continuous cosine operator function of type $(1, 0)$. For a semidiscretization in space let

$$\mathcal{M}_h := \{v_h \in C(\bar{I}) | v_h|_{[x_{i-1},x_i]} \in P_r[x_{i-1}, x_i], \quad i = 1, \cdots, M_h; v_h(0) = v_h(1) = 0\},$$

when $P_r[x_{i-1}, x_i]$ denotes the set of polynomials of degree not greater than r on $[x_{i-1}, x_i]$. If $R = (R_h)_{\mathbb{H}_0}$ is the sequence of projection operators onto $\mathcal{M}_h$, then $(L^2(I), \Pi\mathcal{M}_h, R)$ defines a discrete approximation in the sense of (3.1). The corresponding Galerkin operators T_h, $h \in \mathbb{H}_0$, which are given by the right-hand side of (4.6a) with $u, v \in W_0^{1,2}(I)$ replaced by $u_h, v_h \in \mathcal{M}_h$, also generate strongly continuous cosine operator functions C_h of type $(1, 0)$. Due to the approximation properties of the subspaces $\mathcal{M}_h \subset W_0^{1,2}(I)$, $h \in \mathbb{H}_0$, the local discretization errors $\tau_h^{(1)}(t; u^0)$ and $\tau_h^{(2)}(t; u_t^0)$ are of order $O(h^r)$ if $u^0 \in W_0^{r+1,2}(I)$, $u_t^0 \in W_0^{r,2}(I)$, and it follows from Theorem 3.2 that then the propagators C_h and S_h of the semidiscrete Galerkin equations approximate C(resp. S) of the same order. A high order fully discrete scheme is given by (2.13) with $A_\Delta = T_{k,h}^{(\gamma)}$, where

$$T_{k,h}^{(\gamma)} := (I - \gamma^2 k^2 T_h)^{-s} \sum_{\nu=1}^{s} \alpha_\nu(\gamma) k^{2\nu} T_h^\nu, \qquad s \in \mathbb{N},$$

$$\alpha_\nu(\gamma) := 2 \sum_{\mu=0}^{\nu-1} \binom{s}{\mu} \frac{(-1)^\mu \gamma^{2\mu}}{(\nu-2\mu)!}, \qquad \gamma \in \mathbb{R}^+.$$

It can be shown that there exists a positive constant $\gamma_{r,s}$ such that for all $\gamma \geqq \gamma_{r,s}$ the sequence of discrete cosine operator functions $C_{k,h}^{(\gamma)}$ generated by $T_{k,h}^{(\gamma)}$ is uniformly quasibounded without restriction of the mesh ratio k/h (cf. [4], [12]). Since the operators $T_{k,h}^{(\gamma)}$ are consistent with T of order $O(k^s + h^r)$, Theorem 3.5 then tells us that the solutions $u_{k,h}(t, x)$ of the two-step scheme (2.13) approximate the weak solution of (4.1a)–(4.1c) by exactly the same order.

The approximation of (4.1a)–(4.1c) by either finite difference schemes or finite element methods based on the approximation of the corresponding cosine operator function C can easily be extended to higher dimensions if E is chosen to be $L^2(\Omega)$, $\Omega \subset \mathbb{R}^m$, $m \geqq 2$. However, this is not possible if $E = C(\Omega)$ or $E = L^p(\Omega)$, $p \neq 2$, even if $q = r \equiv 0$, since then C is not strongly continuous (cf. Remark 2.2 (ii)). Nevertheless, in these cases the operators $C(t)$, $t \in \mathbb{R}^+$, can still be shown to be bounded operators from $W^{s,p}(\Omega)$ in $L^p(\Omega)$ if $s > m|\frac{1}{2} - 1/p|$ (cf. [16]). Such constellations will be investigated in a forthcoming paper.

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