Representation of fractional powers of infinitesimal generators of cosine operator functions[1]

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Abstract. By elementary means from the calculus of integral transforms we give a representation of fractional powers of the infinitesimal generator Λ of an equibounded C_0-cosine operator function C on a Banach space A. The result can be used in the theory of interpolation spaces concerning the characterization of the domains $D((-\Lambda)^\alpha)$, $0<\alpha<r$, $r\in\mathbb{N}$, as intermediate spaces of A and $D(\Lambda^r)$.

1. Introduction

Let A be a real or complex Banach space with norm $\|\cdot\|_A$, $\mathscr{C}(A)$ the class of all densely defined closed linear operators Λ with both domain and range in A and $\mathscr{B}(A)$ the Banach algebra of all bounded linear operators on A.

A transformation $C:\bar{\mathbb{R}}^+\to\mathscr{B}(A)$, $\bar{\mathbb{R}}^+:=[0,\infty)$, is called a C_0-cosine operator function if $C(0)=I$, $C(\cdot)$ satisfies d'Alembert's functional equation

$$C(t+s)+C(t-s)=2C(t)C(s), \qquad t,s\in\bar{\mathbb{R}}^+, \qquad t>s \tag{1.1}$$

and $C(\cdot)a$ is continuous on $\bar{\mathbb{R}}^+$ for each $a\in A$. $C(\cdot)$ is said to be equibounded if $\|C(t)\|\leq M$, $t\in\bar{\mathbb{R}}^+$, for some $M\in\mathbb{R}^+$. The infinitesimal generator Λ of $C(\cdot)$ is the linear operator

$$\Lambda a:=2\,\underset{t\to 0^+}{s\text{-lim}}\, t^{-2}[C(t)a-a], \qquad a\in D(\Lambda) \tag{1.2}$$

where $D(\Lambda)$ is the set of all $a\in A$ for which the strong limit in (1.2) does exist. For a systematic treatment of cosine operator theory we refer to [5], [6], [7] and [12].

In this paper we will give a representation of fractional powers $(-\Lambda)^\alpha$, $0<\alpha<r$, $r\in\mathbb{N}$, in terms of the r-th Riemann differences $[C(t)-I]^r$. For equibounded C_0-semigroups $T(t)$, $t\in\bar{\mathbb{R}}^+$, with generator $\Lambda\in\mathscr{C}(A)$ such representations have been given in [2], [13] while in case Λ generates an equibounded C_0-group $T(t)$, $t\in\mathbb{R}$, the fractional powers $(-\Lambda^2)^\alpha$ can be characterized by means

[1] This work was supported by a grant from the Deutsche Forschungsgemeinschaft (DFG).

of the central differences $[T(t/2)-T(-t/2)]^{2r}$ (cf. [14]). In interpolation theory these representations are a useful tool in the study of the spaces $D((-\Lambda)^\alpha)$ as intermediate spaces of A and $D(\Lambda^r)$ (cf. e.g. [4]). The results thus obtained can be applied not only in general approximation theory but also in the numerical solution of the Cauchy problem for first order evolution equations (cf. [3]). We remark that if $T(\cdot)$ is an equibounded C_0-group with generator $U \in \mathscr{C}(A)$ then $C(t)=[T(t)+T(-t)]/2$, $t \in \bar{\mathbb{R}}^+$, defines an equibounded C_0-cosine operator function with generator $\Lambda = U^2$ and representations of $(-\Lambda)^\alpha$ in terms of Riemann differences will follow from the results in [14] taking into account that $C(t)-I=[T(t/2)-T(-t/2)]^2/2$. However, not every C_0-cosine operator function can be related to a C_0-group in the above manner (for examples see [10; pp. 111–114]).

2. Representation results

In case $a \in D(\Lambda^k)$ and $k-1<\alpha<k$, $k \in \mathbb{N}$, the desired representation of $(-\Lambda)^\alpha$ can be derived from Balakrishnan's formula (cf. [1])

$$(-\Lambda)^\alpha a = -\pi^{-1} \sin \alpha\pi \int_0^\infty \lambda^{\alpha-k}(\lambda I-\Lambda)^{-1}\Lambda^k a\, d\lambda$$

and from the fact that $\lambda(\lambda^2 I-\Lambda)^{-1}$ is the operational Laplace transform of $C(\cdot)$ (cf. [6]) by mimicking the proof of [14; Lemma 10.1]:

LEMMA 2.1. *Let $k \in \mathbb{N}$ and $k-1<\alpha<k$. Then there holds*

$$(-\Lambda)^\alpha a = C_{\alpha,k}^{-1} \int_0^\infty t^{-2\alpha}[C(t)-I]^k a \frac{dt}{t}, \qquad a \in D(\Lambda^k) \tag{2.1}$$

where

$$C_{\alpha,k} = \int_0^\infty t^{-2\alpha}(\cos t-1)^k \frac{dt}{t}.$$

In the sequel we will make use of the fact that $C_{\alpha,k} = \int_{-\infty}^{+\infty} q_{\alpha,k}(t)\, dt$ where $q_{\alpha,k}(t)$ is the inverse cosine transform of the function $\frac{1}{2}\int_s^\infty t^{-2\alpha-1}(\cos t-1)^k\, dt$. Explicit representations of $q_{\alpha,k}(t)$ resp. $C_{\alpha,k}$ may be found in [14; pp. 114–117].

LEMMA 2.2. *Let $k, r \in \mathbb{N}$, $0<\alpha<\min(k, r)$ and $\varepsilon, \eta>0$. Then, for each $a \in A$*

$$\int_\eta^\infty t^{-2\alpha}[C(t)-I]^k \frac{dt}{t} \int_0^\infty q_{\alpha,r}\left(\frac{s}{\varepsilon}\right) C(s) a \frac{ds}{\varepsilon}$$
$$= \int_\varepsilon^\infty t^{-2\alpha}[C(t)-I]^r \frac{dt}{t} \int_0^\infty q_{\alpha,k}\left(\frac{s}{\eta}\right) C(s) a \frac{ds}{\eta}. \tag{2.2}$$

Proof. Evaluating the first integral on the right-hand side of (2.2) by means of

$$[C(t)-I]^r=2^{-r}\left[2\sum_{j=1}^{r}(-1)^{r-j}\binom{2r}{r-j}C(jt)+(-1)^r\binom{2r}{r}I\right]$$

which can be easily derived via induction, using (1.1) and the fact that a C_0-cosine operator function can be continuously extended to the whole real line by $C(t)=C(-t)$, $t<0$, we obtain

$$\int_\varepsilon^\infty t^{-2\alpha}[C(t)-I]^r\frac{dt}{t}\int_0^\infty q_{\alpha,k}\left(\frac{s}{\eta}\right)C(s)a\frac{ds}{\eta} \tag{2.3}$$

$$=\frac{1}{2}\int_{-\infty}^{+\infty}\left[\int_0^\infty c_\varepsilon^{(\alpha,r)}(t)q_{\alpha,k}\left(\frac{s-t}{\eta}\right)dt+d_\varepsilon^{(\alpha,r)}q_{\alpha,k}\left(\frac{s}{\eta}\right)\right]C(s)a\frac{ds}{\eta}$$

where

$$c_\varepsilon^{(\alpha,r)}(t)=2^{-r+1}\sum_{j=1}^{r}(-1)^{r-j}\binom{2r}{r-j}j^{2\alpha}b_{j\varepsilon}(t)$$

$$d_\varepsilon^{(\alpha,r)}=(-1)^r2^{-r}\binom{2r}{r}(2\alpha)^{-1}\varepsilon^{-2\alpha},\qquad b_\varepsilon(t)=\begin{cases}0, & 0<t<\varepsilon\\ t^{-2\alpha-1}, & t\geq\varepsilon.\end{cases}$$

The left-hand side in (2.2) can be transformed in the same way giving an analogous formula (2.3) with ε, r mutually exchanged by η, k. Therefore, (2.2) is verified if we can show that the bracketed term in (2.3) is symmetric in the pairs (k, r) and (ε, η). This can be done by proving (2.2) in the special case $C(t)=\cos(\lambda t)$, $\lambda\in\mathbb{R}$, and $a\in\mathbb{C}$, since then the desired result follows from the uniqueness of the inverse cosine transform. ■

LEMMA 2.3. *Let $r\in\mathbb{N}$ and $0<\alpha<r$. Then, for $a\in A$ the integral $\int_0^\infty q_{\alpha,r}(t)C(t)a\,dt$ belongs to $D((-A)^\alpha)$ and*

$$(-A)^\alpha\left[\int_0^\infty q_{\alpha,r}\left(\frac{t}{\varepsilon}\right)C(t)a\frac{dt}{t}\right]=\frac{1}{2}\int_\varepsilon^\infty t^{-2\alpha}[C(t)-I]^r a\frac{dt}{t},\qquad(\varepsilon>0). \tag{2.4}$$

Proof. It is easy to show that $\int_0^\infty q_{\alpha,r}(t)C(t)a\,dt$ does exist in Bochner's sense. Moreover, we have

$$\underset{\varepsilon\to0^+}{s\text{-lim}}\int_0^\infty q_{\alpha,k}\left(\frac{t}{\varepsilon}\right)C(t)a\frac{dt}{\varepsilon}=\tfrac{1}{2}C_{\alpha,k}a,\qquad a\in A,$$

and thus it follows from (2.2) that

$$\underset{\eta\to0^+}{s\text{-lim}}\,C_{\alpha,k}^{-1}\int_\eta^\infty t^{-2\alpha}[C(t)-I]^k a\frac{dt}{t}\int_0^\infty q_{\alpha,r}\left(\frac{s}{\varepsilon}\right)C(s)a\frac{ds}{\varepsilon}=\frac{1}{2}\int_\varepsilon^\infty t^{-2\alpha}[C(t)-I]^r a\frac{dt}{t}.$$

If $a \in D(\Lambda^k)$ then Lemma 2.1 gives the conclusion. Since $D(\Lambda^k)$ is dense in A and $(-\Lambda)^\alpha \in \mathscr{C}(A)$, it follows that (2.4) holds true for all $a \in A$. ■

The main result characterizing the domain of the fractional power $(-\Lambda)^\alpha$ can now be easily deduced from Lemma 2.3:

THEOREM 2.4. *Let $r \in \mathbb{N}$ and $0<\alpha<r$. An element $a \in A$ belongs to $D((-\Lambda)^\alpha)$ if and only if the strong limit*

$$s\text{-}\lim_{\varepsilon\to 0^+} C_{\alpha,r}^{-1} \int_\varepsilon^\infty t^{-2\alpha}[C(t)-I]^r a \frac{dt}{t}$$

exists in which case it is equal to $(-\Lambda)^\alpha a$.

The Phillips adjoint operator $\Lambda^{(*)}$ is the maximal restriction of Λ^* with both domain and range in $A_0^* = \mathrm{cl}\, D(\Lambda^*)$. It is well known that $\Lambda^{(*)}$ generates an equibounded C_0-cosine operator function $C^{(*)}: \bar{\mathbb{R}}^+ \to \mathscr{B}(A_0^*)$ (cf. [8], [11]). Hence, we may also define fractional powers of $-\Lambda^{(*)}$, and we obtain the following dual counterpart of Theorem 2.4:

THEOREM 2.5. *Let $r \in \mathbb{N}$ and $0<\alpha<r$. An element $a^* \in A^*$ belongs to $D((-\Lambda^{(*)})^\alpha)$ if and only if $a^* \in A_0^*$ and the weak* limit*

$$w^*\text{-}\lim_{\varepsilon\to 0+} C_{\alpha,r}^{-1} \int_\varepsilon^\infty t^{-2\alpha}[C^{(*)}(t)-I^*]^r a^* \frac{dt}{t}$$

exists in which case the limit is equal to $(-\Lambda^{(*)})^\alpha a^*$.

3. Applications

Denoting by $\omega_r(\cdot, a)$, $a \in A$, the r-th modulus of continuity of $C(\cdot)a$ (cf. [4; p. 229]) we define $A_{\alpha,r;\infty}$, $0 \leqq \alpha \leqq r$, as the set of all $a \in A$ for which

$$\|a\|_{\alpha,r;\infty} := \|a\|_A + \sup_{t\in\mathbb{R}^+} (t^{-2\alpha}\omega_r(t^r, a)) < \infty.$$

The thus defined space $A_{\alpha,r;\infty}$ is a Banach space which is norm-equivalent to the intermediate space $(A, D(\Lambda^r))_{\theta,\infty}$, $\theta = \alpha/r$ (for details see [9]). If $A = L^p(\Omega)$, $\Omega \subset \mathbb{R}^d$, $1<p<\infty$, and $S(\cdot)$ denotes the strong integral of $C(\cdot)$, then $S(t)A_{\alpha,1;\infty} \subset D((-\Lambda)^{\alpha+1/2})$, $0 \leqq \alpha \leqq 1/2$, and $(-\Lambda)^{\alpha+1/2}S(t)a$, $a \in A_{\alpha,1;\infty}$, is strongly continuous in t (cf. [7] for the special case $\alpha = 0$). In particular, $i(-\Lambda)^{1/2}$ generates a C_0-group given by $T(t)=C(t)+i(-\Lambda)^{1/2}S(t)$, $t\in\mathbb{R}$. These results can be used in the reduction of well-posed second order evolution equations to equivalent first order systems (cf. [9]).

Finally, as a simple example let us consider the case where A is the space of all complex, continuous, 2π-periodic and odd functions on $\mathbb{R}$ equipped with the sup-norm $\|\cdot\|_\infty$. Then $(C(t)a)(x)=[a(x+t)+a(x-t)]/2$, $x, t\in\mathbb{R}$, $a\in A$, defines an equibounded C_0-cosine operator function with generator $(\Lambda a)(x)=(d^2/dx^2)a(x)$. We remark that $C(\cdot)$ cannot be related to an equibounded C_0-group (cf. [10; p. 111]). In lights of the preceding results and since

$$\omega_r(t^r, a)=\sup_{|s|\leqq t}\|\Delta_s^{2r}a\|_\infty, \qquad (\Delta_s^{2r}a)(x)=\sum_{j=-r}^{+r}(-1)^{r-j}\binom{2r}{r-j}a(x+jt)$$

we can identify $D((-\Lambda)^\alpha)$, $0<\alpha<r$, with a closed subspace of the generalized Lipschitz space $\mathrm{Lip}\,(2\alpha, 2r, \infty; A)$ (cf. [4; p. 228]). Note that in case $A=L^p(\mathbb{R})$, $1<p<\infty$, we thus obtain the well-known characterization of the Besov spaces $B^{2\alpha}_{p\infty}$ by means of the $2r$-th central differences Δ_s^{2r}.

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