APPROXIMATE SOLUTION OF SYSTEMS OF EVOLUTIONARY QUASI-VARIATIONAL INEQUALITIES

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APPROXIMATE SOLUTION OF SYSTEMS OF EVOLUTIONARY QUASI-VARIATIONAL INEQUALITIES

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Dedicated to Prof. Dr. E. Mohr on occasion of his 75th birthday.

Abstract. We will construct approximations to the maximum solution of systems of evolutionary quasi--variational inequalities based on a nonlinear semigroup approach using the concept of order preserving convergence in discrete approximations of ordered Banach spaces.

1. Introduction

Let H be a separable order complete real Hilbert lattice and let V be a separable reflexive Banach space with dual V' such that $V \hookrightarrow H \hookrightarrow V'$ each space being dense and compactly embedded in the following one. We assume V to be a vector lattice with respect to the ordering induced by that on H and E to be an M-normed Banach lattice with order unit e, E a closed subspace and sublattice of H with continuous embedding $E \hookrightarrow H$. Given $m \in \mathbb{N}$ we denote by H^m, V^m and E^m the product spaces $H^m = \prod_{\nu=1}^{m} H$ etc. with canonically defined norms and orderings. For $\nu = 1,\dots,m$ let $A^\nu : V \to V'$ be linear monotone operators, let $M^\nu : H^m \to H$ be order preserving concave operators satisfying $M^\nu((V \cap E)^m) \subset V \cap E$, $M^\nu(0) \geq 0$ and let $f^\nu : [0,T] \to \operatorname{int} E^+$ be given functions with bounded variation. Then, for $u^0 \in (V \cap E^+)^m$ we are looking for a function

$u : [0,T] \to V^m$, $u = (u^1,\dots,u^m)$, satisfying $u(t) \in (V \cap E^+)^m$, $u_t(t) \in (V')^m$, $t \in (0,T)$, $u(0) = u^0$ such that for all $\nu = 1,\dots,m$ and $t \in (0,T)$

$$u^\nu(t) \le M^\nu(u(t)), \tag{1.1a}$$

$$\langle u_t^\nu(t) + A^\nu u^\nu(t), v^\nu - u^\nu(t)\rangle \ge (f^\nu(t), v^\nu - u^\nu(t))_H \tag{1.1b}$$

for all $v^\nu \le M^\nu(u(t))$,

where $\langle\cdot,\cdot\rangle$ denotes the dual pairing between V', V and $(\cdot,\cdot)_H$ is the inner product on $H \times H$.
The system (1.1a),(1,1b) constitutes a system of variational inequalities with implicitly given upper obstacles, so-called quasi-variational inequalities which typically arise in optimal switching control of stochastic processes (cf.[3]). Consider e.g. a stochastic system operating in m different regimes that can be described by the diffusion processes

$$dy_x(t) = b^\nu(y_x(t))dt + \sigma^\nu(y_x(t))dw, \quad y_x(0) = x$$

where $x \in \Omega$, Ω is a bounded domain in Euclidean space $\mathbb{R}^d$, $d \in \mathbb{N}$, and w is a standard d-dimensional Wiener process, the drift $b^\nu = (b_1^\nu,\dots,b_d^\nu)$ and the diffusion $\sigma^\nu = (\sigma_{ij}^\nu)_{ij=1}^d$, $1 \le \nu \le m$, being sufficiently smooth functions on $\mathbb{R}^d$. Then, given continuous running costs $f^\nu = f^\nu(x,t)$, $x \in \Omega$, $t \ge 0$, $1 \le \nu \le m$ and nonnegative, subadditive switching costs $k(\nu,\mu,x)$, $x \in \Omega$, $1 \le \nu$, $\mu \le m$, the control objective is to find an optimal switching control policy $v_{\nu,x,t} = (\tau_1,\nu_1;\tau_2,\nu_2,\dots)$ of random stopping times τ_i and regimes ν_i such that the total cost

$$J_{\nu,x,t} = E_{\nu,x,t}\Big[\int_t^{T\wedge\tau_x} \exp(-c(s)s) f^{\nu(s)}(y_x(s),s)\,ds$$

$$+ \sum_{i=1}^{N_{T\wedge\tau_x}} \exp(-c(\tau_i)\tau_i) k(\nu_{i-1},\nu_i,y_x(\tau_i))\Big]$$

is minimized, where $\nu(t) = \nu_i$, $\tau_i \leq t \leq \tau_{i+1}$, with given $\nu_0 = \nu$, $c(t) = c^{\nu_i}$, $\tau_i \leq t \leq \tau_{i+1}$, $c^{\nu} = c^{\nu}(x)$, $x \in \Omega$, $1 \leq \nu \leq m$, being nonnegative discount factors, τ_x is the first exit time of y_x from Ω, $N_{T\wedge\tau_x}$ the total number of switchings and $E_{\nu,x,t}$ denotes the expectation.

If $z^{\nu}(x,t) = \inf\limits_{v_{\nu,x,t}} J_{\nu,x,t}$ and $u^{\nu}(x,t) = z^{\nu}(x,T-t)$, then a formal application of the dynamic programming principle shows that $u = (u^1,\ldots,u^m)$ satisfies a system like (1.1a),(1.1b) with $H = L^2(\Omega)$, $V = H_0^1(\Omega)$ and $E = C(\bar{\Omega})$, the operators A^{ν} and M^{ν}, $1 \leq \nu \leq m$, given by

$$A^{\nu}v = -\sum_{i,j=1}^{d} a_{ij}^{\nu}(x) v_{x_i x_j} - \sum_{i=1}^{d} b_i^{\nu}(x) v_{x_i} + c^{\nu}(x)v,$$

$$M^{\nu}v = \min_{\substack{\mu=1,\ldots,m \\ \mu\neq\nu}} [k(\nu,\mu,x) + v^{\mu}(x)]$$

where $a^{\nu} = (a_{ij}^{\nu})_{ij=1}^{d}$, $a^{\nu} = \frac{1}{2}\sigma^{\nu}(\sigma^{\nu})^*$, $1 \leq \nu \leq m$.

When $H = L^2(\Omega)$, $V = H_0^1(\Omega)$, $E = L^{\infty}(\Omega)$, $A = -\Delta$ and $M : L^{\infty}(\Omega)^+ \to L^{\infty}(\Omega)^+$ a T-contraction, a single quasi-variational inequality has been studied by Barthêlemy and Cattê in [1] associating with this inequalitiy a nonlinear semigroup in $L^{\infty}(\Omega)$ whose generator is given implicitly via its resolvent which is shown to be the maximum solution of an appropriately defined elliptic quasi--variational inequality. Moreover, in case of vanishing switching costs when (1.1a),(1.1b) reduces to the parabolic Hamilton-

-Jacobi-Bellman equation the same techniques have been used by Bénilan and Catté in [2],[4], while, adopting their ideas, in [9] an approximation to the Bellman semigroup associated with an abstract Hamilton-Jacobi-Bellman equation has been given within the framework of order preserving discrete convergence in discrete approximations of ordered Banach spaces. In the subsequent sections we will use the same concept to generate approximations to the maximum solution of the system (1.1a), (1.1b).

2. Order preserving discrete convergence

In this section we will give a short introduction to the concept of order preserving discrete convergence in discrete approximations of ordered Banach spaces. Since most of the material is taken from [9], proofs will be omitted.

Let X, X_n, $n \in \mathbb{N}$, be real Banach spaces with norms $\|\cdot\|_X, \|\cdot\|_{X_n}$ and let $R^X = (R_n^X)_{\mathbb{N}}$ be a sequence of restriction operators $R_n^X : X \to X_n$ satisfying

$$\text{(i)} \quad \|R_n^X(\alpha u+\beta v)-\alpha R_n^X u-\beta R_n^X v\|_{X_n} \to 0, \quad u,v \in X, \quad \alpha,\beta \in \mathbb{R},$$

$$\text{(ii)} \quad \|R_n^X u\|_{X_n} \to \|u\|_X, \quad u \in X, \tag{2.1}$$

$$\text{(iii)} \quad \sup_{n\in\mathbb{N}} \|R_n^X u\|_{X_n} < \infty, \quad u \in X.$$

Then the triple $(X, \Pi X_n, R^X)$ is called a discrete cn-approximation (approximation with convergent norms) and the discrete strong convergence of a sequence $(u_n)_{\mathbb{N}'}$, $u_n \in X_n$, $n \in \mathbb{N}' \subset \mathbb{N}$, to an $u \in X$ is defined by

$$\text{s-lim}_X\, u_n = u \quad \Leftrightarrow \quad \|u_n - R_n^X u\|_{X_n} \to 0.$$

We will require an additional condition, namely

(iv) If $u^{(n)} \in X$, $n \in \mathbb{N}' \subset \mathbb{N}$, and $u \in X$ such that $u^{(n)} \to u$ in X then $\text{s-lim}_X\, R_n^X u^{(n)} = u$, (2.1)

which expresses a certain uniformity of discrete strong convergence (for a detailed discussion of that convergence, its properties and its importance in constructive functional analysis see the basic papers by Grigorieff and Stummel [7],[13],[14]). If we denote by X', X_n', $n \in \mathbb{N}$, the duals of X, X_n, then the triple $((X',X), \Pi(X_n',X_n), (R^{X'}, R^X))$ is said to be a discrete dual cn-approximation if there holds (cf.[11]):

(i) $(X, \Pi X_n, R^X), (X', \Pi X_n', R^{X'})$ are discrete cn-approximations,

(2.2)

(ii) if $u_n \in X_n$, $f_n \in X_n'$, $n \in \mathbb{N}' \subset \mathbb{N}$, and $u \in X$, $f \in X'$ are such that $\text{s-lim}_X\, u_n = u$, $\text{s-lim}_{X'} f_n = f$, then $\langle f_n, u_n \rangle \to \langle f, u \rangle$.

For sequences $(f_n)_{\mathbb{N}'}$, $f_n \in X_n'$, $n \in \mathbb{N}' \subset \mathbb{N}$, we can define a discrete weak convergence to an element $f \in X'$ as follows:

$$\text{w-lim}_{X'}\, f_n = f \quad \Leftrightarrow \quad \langle f_n, u_n \rangle \to \langle f, u \rangle \text{ for all } u \in X \text{ and all sequences } (u_n)_{\mathbb{N}'},\ u_n \in X_n,\ n \in \mathbb{N}', \text{ such that } \text{s-lim}_X\, u_n = u.$$

It is obvious that in case of discrete dual cn-approximations involving reflexive Banach spaces we may likewise define a discrete weak convergence of sequences $(u_n)_{\mathbb{N}'}$, $u_n \in X_n$, $n \in \mathbb{N}' \subset \mathbb{N}$, to an element $u \in X$.

For subsets $G_n \subset X_n$, $n \in \mathbb{N}$, we introduce the following limit sets

$$\text{s-Lim}\inf_X G_n = \{u \in X \mid \exists (u_n)_{\mathbb{N}}, u_n \in X_n, n \in \mathbb{N} : \text{s-lim}_X u_n = u\},$$

$$\text{s-Lim}\sup_X G_n = \{u \in X \mid \exists (u_n)_{\mathbb{N}'}, u_n \in X_n, n \in \mathbb{N}' \subset \mathbb{N} : \text{s-lim}_X u_n = u\}.$$

Obviously, $\text{s-Lim}\sup_X G_n \subseteq \text{s-Lim}\inf_X G_n$. If equality holds, the common limit set will be denoted by $\text{s-Lim}_X G_n$. In discrete dual cn-approximations of reflexive Banach spaces we can analogously define the sets $\text{w-Lim}\sup_X G_n$, $\text{w-Lim}\inf_X G_n$ and $\text{w-Lim}_X G_n$. It is clear that in general $\text{s-Lim}\inf_X G_n \subseteq \text{w-Lim}\sup_X G_n$. If both sets coincide we will denote the common limit by $\text{Lim}_X G_n$.
So far we have not taken care of a possible lattice structure on X resp. X_n, $n \in \mathbb{N}$. Now, if $(X, \Pi X_n, R^X)$ is a discrete cn-approximation of Banach spaces and vector lattices X, X_n, $n \in \mathbb{N}$, the discrete strong convergence will be called order preserving if for all $u \in X$ and all sequences $(u_n)_{\mathbb{N}'}$, $u_n \in X_n$, $n \in \mathbb{N}' \subset \mathbb{N}$, there holds

$$\text{s-lim}_X u_n = u \Rightarrow \text{s-lim}_X u_n^+ = u^+. \tag{2.3}$$

It follows immediately from the linearity of discrete strong convergence that, if (2.3) is satisfied, then we have discrete strong convergence of all basic lattice operations.
A necessary condition for order preserving discrete strong convergence can be given in terms of the restriction operators R_n^X (cf.[9; Lemma 2.3]):

<u>LEMMA 2.1</u>. If the discrete strong convergence in a discrete cn-approximation $(X, \Pi X_n, R^X)$ is order preserving then

$$\| R_n^X u^+ - (R_n^X u)^+ \|_{X_n} \to 0, \quad u \in X. \tag{2.4}$$

If X_n, $n \in \mathbb{N}$, are Banach lattices then also the converse holds true.

For discrete cn-approximations satisfying (2.4) we have the following approximation property for the positive cone X^+ resp. closedness result for the positive cones $(X_n')^+$ which generate the order duals X_n^* (cf.[9; Lemma 2.4]):

<u>LEMMA 2.2</u>. Let $(X,\Pi X_n,R^X)$ be a discrete cn-approximation with $(R_n^X)_{\mathbb{N}}$ satisfying (2.4). Then there holds

(i) $X^+ \subseteq \text{s-Lim}\inf_X X_n^+$,

(ii) $\text{w-Lim}\sup_{X'} (X_n')^+ \subset (X')^+$.

Under the stronger assumption of order preserving discrete strong convergence we can prove strong convergence of positive cones and order intervals (cf.[9; Lemma 2.4]):

<u>LEMMA 2.3</u>. Let $(X,\Pi X_n,R^X)$ be a discrete cn-approximation with order preserving discrete strong convergence. Then there holds

(i) $\text{s-Lim}_X X_n^+ = X^+$,

(ii) If $u,v \in X$, $u \leq v$ and $u_n,v_n \in X_n$, $n \in \mathbb{N}' \subset \mathbb{N}$, such that $\text{s-lim}_X u_n = u$, $\text{s-lim}_X v_n = v$, then

$$\text{s-Lim}_X [u_n,v_n] = [u,v].$$

Using the concept of order preserving discrete convergence developed so far, in the remaining part of this section we will construct suitable discrete approximations of the spaces involved in the definition of the system (1.1a),(1.1b) of quasi-variational inequalities. We will assume that $(H,\Pi H_n,R^H)$ is a discrete cn--approximation of separable order complete real Hilbert lattices with order preserving discrete strong convergence and that

$((V',V),\Pi(V_n',V_n),(R^{V'},R^V))$ is a discrete dual cn-approximation of reflexive, separable real Banach spaces V,V_n, $n\in\mathbb{N}$, with $(R_n^V)_{\mathbb{N}}$ satisfying (2.1)(iv), such that $V\hookrightarrow H\hookrightarrow V'$, $V_n\hookrightarrow H_n\hookrightarrow V_n'$, the embeddings $V_n\hookrightarrow H_n$ being discretely compact. This means that, given a bounded sequence $(u_n)_{\mathbb{N}}$, $u_n\in V_n$, $n\in\mathbb{N}$, for each subsequence $\mathbb{N}'\subset\mathbb{N}$ there exist another subsequence $\mathbb{N}''\subset\mathbb{N}'$ and an $u\in H$ such that $\text{s-lim}_H\, u_n = u$. In particular, as a consequence of that discrete compactness we have

$$\text{w-lim}_V\, u_n = u \;\Rightarrow\; \text{s-lim}_H\, u_n = u\ . \tag{2.5}$$

We further suppose V,V_n to be vector lattices with respect to the orderings induced by those on H,H_n. We assume the restriction operators R_n^V to be order preserving and to satisfy (2.4), the positive cones V^+,V_n^+ having nonvoid interior and the norms $\|\cdot\|_{V_n}$ being such that

$$\|u_n^+\|_{V_n} \le \|u_n\|_{V_n}\ ,\qquad u_n\in V_n\ . \tag{2.6}$$

Moreover, we suppose $E\subset H$ and $E_n\subset H_n$, $n\in\mathbb{N}$, to be M-normed Banach lattices with order units e resp. e_n the order relations induced by those on H,H_n, such that $(E,\Pi E_n,R^E)$ is a discrete cn-approximation with order preserving discrete strong convergence and $\text{s-lim}_E\, e_n = e$, $(R_n^E)_{\mathbb{N}}$ fulfilling (2.1)(iv). We assume that E resp. E_n is continuously embedded in H resp. H_n, the embeddings $E\hookrightarrow H$, $E_n\hookrightarrow H_n$, $n\in\mathbb{N}$, being discretely strongly convergent, i.e. $\text{s-lim}_E\, u_n = u$ implies $\text{s-lim}_H\, u_n = u$, and we further assume $R_n^V(V\cap E)\subset V_n\cap E_n$, $R_n^E(V\cap E)\subset V_n\cap E_n$, $n\in\mathbb{N}$, $V\cap E^+$ and $V_n\cap E_n^+$ having nonvoid interior. Finally, we require $(u_n-e_n)^+\in V_n$ for each $u_n\in V_n$, the mappings $u_n\to(u_n-e_n)^+$ being bounded on bounded

sets of V_n.
Although, in general we cannot expect order preserving discrete strong convergence in $(V,\Pi V_n,R^V)$, since $\|\cdot\|_{V_n}$ are not lattice norms, we have (cf.[9;Thm.3.1]):

THEOREM 2.4. Let $(H,\Pi H_n,R^H)$ and $((V',V),\Pi(V_n',V_n),(R^{V'},R^V))$ be given as above. Then the discrete weak convergence of sequences $(u_n)_{\mathbb{N}'}$, $u_n \in V_n$, $n\in\mathbb{N}'\subset\mathbb{N}$, is order preserving and $\mathrm{Lim}_V\, V_n^+ = V^+$.

3. Discrete strong convergence of maximum solutions

We will now construct approximating systems of quasi-variational inequalities. To do so let $A_n^\nu : V_n \to V_n'$, $n\in\mathbb{N}$, $1\le\nu\le m$, be linear monotone operators satisfying

$$\langle A_n^\nu u_n,(u_n-ce_n)^+\rangle \ge 0, \quad u_n\in V_n \tag{3.1}$$

for all $c\in\mathbb{R}^+$. We assume A_n^ν to be uniformly coercive in the sense that there exists $\gamma\in\mathbb{R}^+$ such that for all $\lambda\in\mathbb{R}^+$

$$\langle A_n^\nu u_n,u_n\rangle + \lambda\|u_n\|_{H_n}^2 \ge \gamma|u_n|_{V_n}^2, \quad u_n\in V_n, \quad 1\le\nu\le m, \tag{3.2}$$

and we require stability of the sequences $(A_n^\nu)_{\mathbb{N}}$, i.e.

$$\sup_{n\in\mathbb{N}}\|A_n^\nu\| < \infty, \qquad 1\le\nu\le m. \tag{3.3}$$

Moreover, we suppose M_n^ν, $n\in\mathbb{N}$, $1\le\nu\le m$, to be order preserving concave operators $M_n^\nu : H_n^m\to H_n$ satisfying $M_n^\nu((V_n\cap E_n)^m)\subset V_n\cap E_n$, $M_n^\nu(0)\ge 0$ and

$$M_n^\nu(u_n+re_n^m) = M_n^\nu(u_n)+re_n, \quad u_n\in (V_n\cap E_n)^m \tag{3.4}$$

for all $r\in\mathbb{R}^+$, where $E_n^m=(e_n,\dots,e_n)$ is the order unit in E_n^m.

Then, given functions $f_n^\nu : [0,T] \to E_n^+$, $n \in \mathbb{N}$, $1 \le \nu \le m$, with bounded variation and initial data $u_n^o \in (V_n \cap E_n^+)^m$, $n \in \mathbb{N}$, functions $u_n : [0,T] \to V_n^m$, $u_n = (u_n^1,\dots,u_n^m)$ are sought satisfying $u_n(t) \in (V_n \cap E_n^+)^m$, $(u_n)_t(t) \in (V_n')^m$, $t \in (0,T)$, $u_n(0) = u_n^o$ and the system of quasi-variational inequalities

$$u_n^\nu(t) \le M_n^\nu(u_n(t)) \tag{3.5a}$$

$$\langle (u_n^\nu)_t(t) + A_n^\nu u_n^\nu(t), v_n^\nu - u_n^\nu(t) \rangle \ge (f_n^\nu(t), v_n^\nu - u_n^\nu(t))_{H_n} \tag{3.5b}$$

for all $v_n^\nu \le M_n^\nu(u_n(t))$, $t \in (0,T)$, $1 \le \nu \le m$.

Clearly, we want the system (3.5a),(3.5b) to approximate (1.1a), (1.1b). To assure this we require discrete convergence of the data, i.e.

$$\text{s-lim}_E\, f_n^\nu(t) = f^\nu(t) \quad \text{uniformly in} \quad t \in [0,T], \quad 1 \le \nu \le m \tag{3.6a}$$

$$\text{s-lim}_{E^m}\, u_n^o = u^o. \tag{3.6b}$$

Concerning the operators A^ν, A_n^ν, $1 \le \nu \le m$, we assume consistency of the pairs $A^\nu, (A_n^\nu)_{\mathbb{N}}$ and $A^\nu|_{D(A^\nu)}$, $(A_n^\nu|_{D(A_n^\nu)})_{\mathbb{N}}$ where $D(A^\nu) = \{u \in V \cap E^+ \mid A^\nu u \in V \cap E\}$, $D(A_n^\nu)$ defined correspondingly: For each $u \in V$ (resp. $u \in D(A^\nu)$) there exists a sequence $(u_n)_{\mathbb{N}}$, $u_n \in V_n$, $n \in \mathbb{N}$ (resp. $u_n \in D(A_n^\nu)$, $n \in \mathbb{N}$) such that $\text{s-lim}_V\, u_n = u$ and $\text{s-lim}_{V'}\, A_n^\nu u_n = A^\nu u$ (resp. $\text{s-lim}_E\, u_n = u$ and $\text{s-lim}_E\, A_n^\nu u_n = A^\nu u$). In particular, it follows from our assumptions and [13;1.2(6)] that $(A_n^\nu)_{\mathbb{N}}$ converges discretely strongly to A^ν with respect to $((V',V),\Pi(V'_n,V_n),(R^{V'},R^V))$, i.e. for $u \in V$ and sequences of elements $u_n \in V_n$, $n \in \mathbb{N}$, such that $\text{s-lim}_V\, u_n = u$ we have $\text{s-lim}_{V'}\, A_n^\nu u_n = A^\nu u$. Moreover, it follows from [11] that the pairs $A^\nu, (A_n^\nu)_N$ are a-pseudomonotone in the following sense: For any sequence $(u_n)_{\mathbb{N}'}$, $u_n \in V_n$, $n \in \mathbb{N}' \subset \mathbb{N}$, and $u \in V$ with $\text{w-lim}_V\, u_n = u$ and all sequences $(w_n)_{\mathbb{N}'}$, $w_n \in V_n$, $n \in \mathbb{N}'$, with $\text{s-lim}_V\, w_n = u$ such that

$$\limsup_{n \in \mathbb{N}'} \langle A_n^\nu u_n, u_n - w_n \rangle \leq 0$$

there holds

$$\langle A^\nu u, u-v \rangle \leq \liminf_{n \in \mathbb{N}'} \langle A_n^\nu u_n, u_n - v_n \rangle$$

for all $v \in V$ and all sequences $(v_n)_{\mathbb{N}'}$, $v_n \in V_n$, $n \in \mathbb{N}'$, with $\text{s-lim}_V\, v_n = v$.

We further require the pairs $(A_n^\nu)_{\mathbb{N}}$ to converge discretely weakly to A^ν while the pairs $M^\nu, (M_n^\nu)_{\mathbb{N}}$, $1 \leq \nu \leq m$, are supposed to be both discretely weakly and strongly convergent with respect to $((V',V), \Pi(V_n',V_n), (R^{V'}, R^V)$ as well as discretely strongly convergent in $(E, \Pi E_n, R^E)$ satisfying additionally

$$M_n^\nu(R_n^{V^m} u) \geq R_n^V M^\nu(u), \qquad 1 \leq \nu \leq m \tag{3.7}$$

where $R_n^{V^m} u = (R_n^V u^1, \ldots, R_n^V u^m)$.

The main result of this paper is to prove the existence of nonlinear contraction semigroups $S(t) : (E^+)^m \to (E^+)^m$, $S_n(t) : (E_n^+)^m \to (E_n^+)^m$, $n \in \mathbb{N}$, $t \in \bar{\mathbb{R}}^+ = [0,\infty)$ which define in a generalized sense the maximal solutions to (1.1a),(1.1b) resp. (3.5a),(3.5b) and to establish discrete strong convergence $S_n(t) \to S(t)$ uniformly in $t \in [0,T]$. For this purpose let $g^\nu \in \text{int}\, E^+$, $g_n^\nu \in E_n^+$, $n \in \mathbb{N}$, $1 \leq \nu \leq m$, $\lambda \in \mathbb{R}^+$ and consider the following system of stationary quasi-variational inequalities

$$u^\nu \in V \cap E^+, \quad u^\nu \le M^\nu(u) \tag{3.8a}$$

$$\langle A^\nu u^\nu, v^\nu - u^\nu\rangle \ge \lambda^{-1}(g^\nu - u^\nu, v^\nu - u^\nu)_H, \quad v^\nu \le M^\nu(u) \tag{3.8b}$$

resp. its discrete counterparts

$$u_n^\nu \in V_n \cap E_n^+, \quad u_n^\nu \le M_n^\nu(u_n) \tag{3.9a}$$

$$\langle A_n^\nu u_n^\nu, v_n^\nu - u_n^\nu\rangle \ge \lambda^{-1}(g_n^\nu - u_n^\nu, v_n^\nu - u_n^\nu)_{H_n}, \quad v_n^\nu \le M_n^\nu(u_n). \tag{3.9b}$$

We associate with (3.8a),(3.8b) resp. (3.9a),(3.9b) the selection maps $S : (V \cap E^+)^m \to 2^{(V \cap E^+)^m}$ resp. $S_n : (V_n \cap E_n^+)^m \to 2^{(V_n \cap E_n^+)^m}$ which assign to $u \in (V \cap E^+)^m$ resp. $u_n \in (V_n \cap E_n^+)^m$ the set of solutions $S(u)$ resp. $S_n(u_n)$ of the system of variational inequalities

$$w^\nu \in V \cap E, \quad w^\nu \le M^\nu(u) \tag{3.10a}$$

$$\langle A^\nu w^\nu, v^\nu - u^\nu\rangle \ge \lambda^{-1}(g^\nu - w^\nu, v^\nu - w^\nu)_H, \quad v^\nu \le M^\nu(u) \tag{3.10b}$$

resp. its discrete counterparts. Clearly, $u \in (V \cap E^+)^m$ resp. $u_n \in (V_n \cap E_n^+)^m$ is a solution to (3.8a),(3.8b) resp. (3.9a), (3.9b) iff $u \in S(u)$ resp. $u_n \in S_n(u_n)$.
It follows from the assumptions that (3.10a),(3.10b) and the discrete counterparts are uniquely solvable, i.e. S and S_n are single-valued.
The first step will be to prove that there exist maximal solutions to (3.8a),(3.8b) resp. (3.9a),(3.9b). Based on our assumptions on A_n^ν there holds (cf.[1; Lemme 1])

<u>LEMMA 3.1</u>. For each $\nu = 1,\dots,m$ the operators $A_n^\nu|_{D(A_n^\nu)}$, $n \in \mathbb{N}$, are m-T-accretive.

As an immediate consequence of Theorem 3.1 we have that the equations

$$(I_n+\lambda A_n^\nu)u_n^\nu = g_n^\nu, \quad \nu = 1,\dots,m \tag{3.11}$$

admit unique solutions $\bar{u}_n^\nu \in V_n \cap E_n^+$, $n \in \mathbb{N}$. It follows from the comparison theorem for variational inequalities that $\bar{u}_n = (\bar{u}_n^1,\dots,\bar{u}_n^m)$ is a supersolution to (3.9a),(3.9b). On the other hand, it is obvious that $\underline{u}_n = (0,\dots,0)$ is a subsolution. Since the selection map S_n is increasing, the Knaster-Kantorovich-Birkhoff fixed point theorem (cf.[12],[15]) implies that the solution set of (3.9a),(3.9b) is nonvoid and possesses a minimum and a maximum element.

Next, we define the closed, convex sets

$$\begin{aligned} K_n(\lambda;g_n) = \{u_n \in (V_n \cap E_n^+)^m \mid & \langle A_n^\nu u_n^\nu, v_n^\nu - u_n^\nu\rangle \geq \\ & \lambda^{-1}(g_n^\nu - u_n^\nu, v_n^\nu - u_n^\nu)_{H_n}, \; v_n^\nu \in V_n, \\ & v_n^\nu \leq u_n^\nu, \; u_n^\nu \leq M_n^\nu(u_n), \; 1 \leq \nu \leq m\} \end{aligned} \tag{3.12}$$

which can be considered as the sets of subsolutions to (3.9a), (3.9b). To be more precise, we have

<u>THEOREM 3.2</u>. The maximum solution $u_n \in (V_n \cap E_n^+)^m$ of (3.9a), (3.9b) is the maximum element $J_n^\lambda(g_n)$ of the set $K_n(\lambda;g_n)$, $n \in \mathbb{N}$.

<u>Proof</u>. Choosing $v_n \leq u_n$ in (3.9b) it is clear that $u_n \in K_n(\lambda;g_n)$. On the other hand, if $w_n \in K_n(\lambda;g_n)$ then we may take $v_n^\nu = u_n^\nu - (w_n^\nu - u_n^\nu - ce_n)^+$, $1 \leq \nu \leq m$, $c \in \mathbb{R}^+$ in (3.12) and we get

$$(w_n^\nu - u_n^\nu, (w_n^\nu - u_n^\nu - ce_n)^+)_{H_n} \leq 0$$

whence $w_n \leq u_n$, since $c \in \mathbb{R}^+$ can be chosen arbitrarily.

We will now show that the system (3.7a),(3.7b) possesses a maximum solution which is the maximum element $J^{\lambda}(g)$ of the set $K(\lambda;g)$ defined in the same way as the sets $K_n(\lambda;g_n)$. For this purpose we first prove

<u>THEOREM 3.3</u>. Suppose that $\text{s-lim}_E\, g_n^{\nu} = g^{\nu}$, $1 \le \nu \le m$, and $u \in (V \cap E^+)^m$, $u_n \in (V_n \cap E_n^+)^m$, $n \in \mathbb{N}$, such that $\text{s-lim}_V\, u_n^{\nu} = u^{\nu}$, $1 \le \nu \le m$. Then the sequence $(S_n(u_n))_{\mathbb{N}}$ is discretely compact and $\text{s-lim}_{V^m}\, S_n(u_n) =$
$= S(u)$.

<u>Proof</u>. In view of (3.2) it follows that $(w_n)_{\mathbb{N}}$, $w_n = S_n(u_n)$, $n \in \mathbb{N}$, is bounded and hence, by [11;1.(7)] there exist a $w \in V$ and a subsequence $\mathbb{N}' \subset \mathbb{N}$ such that $\text{w-lim}_V\, w_n = w$ $(n \in \mathbb{N}')$. Let $K(u) =$
$= \{v \in V^m \mid v^{\nu} \le M^{\nu}(u),\ 1 \le \nu \le m\}$ and $K_n(u_n)$ defined analogously. Since $\text{s-lim}_V\, M_n^{\nu}(u_n) = M^{\nu}(u)$, $1 \le \nu \le m$, we have $\text{Lim}_{V^m}\, K_n(u_n) = K(u)$ and thus $w \in K(u)$ because of $w_n \in K_n(u_n)$. On the other hand, for each $v \in K(u)$ there exists $(v_n)_{\mathbb{N}}$, $v_n \in K_n(u_n)$, $n \in \mathbb{N}$, with $\text{s-lim}_{V^m}\, v_n = v$. Moreover, there exists $(z_n)_{\mathbb{N}}$, $z_n \in K_n(u_n)$, $n \in \mathbb{N}$, such that $\text{s-lim}_V\, z_n^{\nu} = w^{\nu}$, $1 \le \nu \le m$. Then

$$\lim_{n\in\mathbb{N}'} \langle A_n^{\nu} w_n^{\nu}, w_n^{\nu} - z_n^{\nu} \rangle = 0,$$

and the a-pseudomonotonicity of $A^{\nu}, (A_n^{\nu})_{\mathbb{N}}$ gives

$$\begin{aligned} \langle A^{\nu} w^{\nu}, w^{\nu} - v^{\nu} \rangle &\le \liminf_{n\in\mathbb{N}'} \langle A_n^{\nu} w_n^{\nu}, w_n^{\nu} - v_n^{\nu} \rangle \\ &\le \lambda^{-1} (g^{\nu} - w^{\nu}, w^{\nu} - v^{\nu})_H, \quad v \in K(u). \end{aligned} \tag{3.13}$$

Again, by the comparison theorem for variational inequalities $\bar{u}_n$ is a supersolution to (3.10a),(3.10b) and hence, using the T-contractiveness of $(I_n + \lambda A_n^{\nu})^{-1}$ we conclude that $0 \le w_n^{\nu} \le$
$\le \|g_n^{\nu}\|_{E_n} e_n$, $1 \le \nu \le m$. Taking advantage of Lemma 2.3(ii) we then get $0 \le w^{\nu} \le \|g^{\nu}\|_E\, e$ whence $w^{\nu} \in V \cap E^+$. Since $w \in K(u)$ and w satisfies

(3.13) we must have $w = S(u)$. Finally, it follows from (3.2) that

$$\lambda\gamma\|w_n^\nu - z_n^\nu\|_{V_n}^2 \leq \langle (I_n+\lambda A_n^\nu)(w_n^\nu - z_n^\nu), w_n^\nu - z_n^\nu \rangle,$$

the right-hand side converging to zero as $n \to \infty$ which yields $\text{s-lim}_V\, w_n^\nu = w^\nu$, $1 \leq \nu \leq m$.

<u>THEOREM 3.4</u>. If $\text{s-lim}_E\, g_n^\nu = g^\nu$, $1 \leq \nu \leq m$, then

$$\text{Lim}_{V^m}\, K_n(\lambda;g_n) = K(\lambda;g).$$

<u>Proof</u>. Let $u_n \in K_n(\lambda;g_n)$, $n \in \mathbb{N}' \subset \mathbb{N}$, and $u \in V^m$ such that $\text{w-lim}_V\, u_n^\nu = u^\nu$, $1 \leq \nu \leq m$. Then $\text{w-lim}_{V'}\, (I_n+\lambda A_n^\nu)u_n^\nu = (I+\lambda A^\nu)u^\nu$, $\text{w-lim}_V\, M_n^\nu(u_n) = M^\nu(u)$ and it follows from Lemma 2.2(ii) and Theorem 2.4 that $(I+\lambda A^\nu)u^\nu \leq g^\nu$, $u^\nu \leq M^\nu(u)$, $1 \leq \nu \leq m$. As in the proof of Theorem 3.3 we deduce that $u \in (V \cap E^+)^m$ whence $u \in K(\lambda;g)$. We have thus shown $\text{w-Lim sup}_{V^m}\, K_n(\lambda;g_n) \subset K(\lambda;g)$. To prove $K(\lambda;g) \subset \text{s-Lim inf}_{V^m}\, K_n(\lambda;g_n)$ we remark that $K(\lambda;g) = K^1(\lambda;g) \cap K^2(M^\nu)$ and $K_n(\lambda;g_n) = K_n^1(\lambda;g_n) \cap K_n^2(M_n^\nu)$ where $K^1(\lambda;g) = \{u \in (V \cap E^+)^m \mid (I+\lambda A^\nu)u^\nu \leq g^\nu, 1 \leq \nu \leq m\}$, $K^2(M^\nu) = \{u \in (V \cap E^+)^m \mid u^\nu \leq M^\nu(u),\ 1 \leq \nu \leq m\}$ and $K_n^1(\lambda;g_n)$, $K_n^2(M_n^\nu)$ defined correspondingly. Now, if $u \in K^1(\lambda;g)$ set $h_n^\nu = R_n^E((I+\lambda A^\nu)u^\nu) \wedge g_n^\nu$, $n \in \mathbb{N}$, $1 \leq \nu \leq m$. There exists $u_n \in (V_n \cap E_n^+)^m$ such that $(I_n+\lambda A_n^\nu)u_n^\nu = h_n^\nu$, $1 \leq \nu \leq m$. But $\text{s-lim}_{V'}\, h_n^\nu = g^\nu$ and thus it follows from [13;1.3(2)] that $\text{s-lim}_V\, u_n^\nu = u^\nu$. On the other hand, taking advantage of (3.7) and the fact that R_n^V is order preserving, we have $K^2(M^\nu) \subset \text{s-Lim inf}_{V^m}\, K_n^2(M_n^\nu)$. Since $K(\lambda;g)$ has nonvoid interior, it is then easy to draw the conclusion.

Denoting by $J^\lambda(g)$ the maximum element of $K(\lambda;g)$, in view of Theorem 2.4 it follows from the preceding result that $\text{w-lim}_{V^m}\, J_n^\lambda(g_n) = J^\lambda(g)$ at least for a subsequence $\mathbb{N}' \subset \mathbb{N}$. Moreover, there exists $(v_n)_{\mathbb{N}}$, $v_n \in K_n(\lambda;g_n)$, $n \in \mathbb{N}$, such that s-$\lim_{V^m} v_n =$ $= J^\lambda(g)$ and we may again use (3.2) to deduce that then s-$\lim_{V^m} J_n^\lambda(g_n) = J^\lambda(g)$. Now, by Theorem 3.2 we have $S_n(J_n^\lambda(g_n)) =$ $= J_n^\lambda(g_n)$ while Theorem 3.3 tells us s-$\lim_{V^m} S_n(J_n^\lambda(g_n)) = S(J^\lambda(g))$ whence $J^\lambda(g) = S(J^\lambda(g))$. Oviously, any solution to (3.8a),(3.8b) is an element of $K(\lambda;g)$ and thus we have shown

<u>COROLLARY 3.5</u>. The maximum element $J^\lambda(g)$ of $K(\lambda;g)$ is the maximum solution of the system of quasi-variational inequalities (3.8a),(3.8b).

Now, for $\lambda \in \mathbb{R}^+$ let us denote by J^λ and J_n^λ, $n \in \mathbb{N}$, the operators which assign to $g \in (\text{int } E^+)^m$ resp. $g_n \in (E_n^+)^m$ the maximum elements $J^\lambda(g)$ of $K(\lambda;g)$ resp. $J_n^\lambda(g_n)$ of $K_n(\lambda;g_n)$. Our first result is

<u>THEOREM 3.6</u>. For each $n \in \mathbb{N}$ $(J_n^\lambda)_{\lambda\in\mathbb{R}^+}$ is a family of T-contractive resolvent operators $J_n^\lambda = (E_n^+)^m \to (E_n^+)^m$.

<u>Proof</u>. Using standard penalization techniques for quasi-variational inequalities we will show that J_n^λ is the limit of a sequence of T-contractive resolvent operators $(J_n^{\lambda,\varepsilon})_{\varepsilon\in\mathbb{R}^+}$. We define $A_n : (V \cap E)^m \to E^m$ by $u_n \in D(A_n)$ iff $u_n^\nu \in D(A_n^\nu)$, $1 \leq \nu \leq m$. A_n is m-T-accretive, since so are the operators A_n^ν. We further define penalization operators $P_n^\varepsilon : E^m \to E^m$, $\varepsilon\in\mathbb{R}^+$, by $(P_n^\varepsilon u_n)^\nu = \varepsilon^{-1}(u_n^\nu - M_n^\nu(u_n^+))^+$, $1 \leq \nu \leq m$. Since the operators M_n^ν commute with translations by a positive constant (cf.(3.4)) and are order preserving, it is easily shown that they are T-contractive (cf. [6; Prop. 2]) and then we may use [1; Lemme 2] to deduce that the penalization operators P_n^ε are T-accretive and Lipschitzian with Lipschitz constant $2\varepsilon^{-1}$. As a sum of an m-T-accretive and a T-accretive Lipschitzian

operator the operators $C_n^\varepsilon = A_n + P_n^\varepsilon$ are m-T-accretive and hence, the resolvent operators $J_n^{\lambda,\varepsilon} = (I_n + \lambda C_n^\varepsilon)^{-1}$ are T-contractive. Now, let $(g_n^{\nu,\varepsilon})_{\varepsilon\in\mathbb{R}^+}$, $1 \le \nu \le m$, be monotonely decreasing families of element $g_n^{\nu,\varepsilon} \in E_n^+$ such that $g_n^{\nu,\varepsilon} \to g_n^\nu$ $(\varepsilon \to 0)$ in E_n. Setting $u_n^\varepsilon = J_n^{\lambda,\varepsilon}(g_n^\varepsilon)$ and using the coerciveness and pseudomonotonicity of the operators A_n^ν, it follows that there exists an element $u_n \in K_n(\lambda; g_n)$ such that $u_n^{\nu,\varepsilon} \rightharpoonup u_n^\nu$ $(\varepsilon \to 0)$, $1 \le \nu \le m$, in V_n (cf. [1; Prop.1]). To prove $u_n = J_n^\lambda(g_n)$ we recursively define $u_n^{\nu,k}$, $k = 0,1,\ldots,$ by

$$u_n^{\nu,o} + \lambda A_n^\nu u_n^{\nu,o} = g_n^{\nu,\varepsilon}$$

$$u_n^{\nu,k} + \lambda[A_n^\nu u_n^{\nu,k} + \varepsilon^{-1}(u_n^{\nu,k} - M_n^\nu((u_n^{k-1})^+))^+] = g_n^{\nu,\varepsilon}, \qquad k \ge 1.$$

It can be shown by a slight modification of the proof in [9; Thm.3.8] that $(u_n^{\nu,k})_{k\in\mathbb{N}}$ converges monotonely decreasingly to $u_n^{\nu,\varepsilon}$ and that for any $w_n \in K_n(\lambda; g_n)$ we have $w_n^\nu \le u_n^{\nu,k}$. Consequently, $w_n^\nu \le u_n^{\nu,\varepsilon}$ yielding $w_n^\nu \le u_n^\nu$ and thus $u_n = J_n^\lambda(g_n)$.

We are now in a position to draw the corresponding conclusions for the family $(J^\lambda)_{\lambda\in\mathbb{R}^+}$ and to establish strong convergence of J_n^λ to J^λ:

<u>THEOREM 3.7</u>. If $\text{s-lim}_E\, g_n^\nu = g^\nu$, $1 \le \nu \le m$, then $\text{s-lim}_{E^m}\, J_n^\lambda(g_n) = J^\lambda(g)$ and $(J^\lambda)_{\lambda\in\mathbb{R}^+}$ is a family of T-contractive resolvent operators $J^\lambda : (E^+)^m \to (E^+)^m$.

<u>Proof</u>. Let $C^\varepsilon = A + P^\varepsilon$ where the operators $A : (V \cap E)^m \to E^m$ and $P^\varepsilon : E^m \to E^m$, $\varepsilon \in \mathbb{R}^+$, are defined as in the proof of Theorem 3.6. Moreover, let $(g^\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ be a monotonely decreasing family of elements $g^\varepsilon \in (E^+)^m$ with $g^{\nu,\varepsilon} \to g^\nu$ $(\varepsilon \to 0)$, $1 \le \nu \le m$, in E and

analogously choose $g_n^{\varepsilon} \in (E_n^+)^m$, $n \in \mathbb{N}$, such that $g_n^{\nu,\varepsilon} \to g_n^{\nu}$ $(\varepsilon \to 0)$ in E_n and $\text{s-lim}_E\, g_n^{\nu,\varepsilon} = g^{\nu,\varepsilon}$, $1 \leq \nu \leq m$. Due to $M_n^{\nu} \to M^{\nu}$ and the fact that the discrete strong convergence in $(E, \Pi E_n, R^E)$ is order preserving we have $P_n^{\varepsilon} \to P^{\varepsilon}$. Consequently, the pair $C^{\varepsilon}|_{D(A^{\nu})^m}, (C_n^{\varepsilon}|_{D(A_n^{\nu})^m})_{\mathbb{N}}$ is consistent and since $C^{\varepsilon}|_{D(A^{\nu})^m}$ can be easily shown to be m-T-accretive, we may apply [13;1.3(3)] to deduce that $\text{s-lim}_{E^m} J_n^{\lambda,\varepsilon}(g_n^{\varepsilon}) = u^{\varepsilon}$ where $u^{\varepsilon} = J^{\lambda,\varepsilon}(g^{\varepsilon}) = (I+\lambda C^{\varepsilon})^{-1} g^{\varepsilon}$. Then $(u^{\varepsilon})_{\varepsilon \in \mathbb{R}^+}$ is a monotonely decreasing family of elements in an M-normed Banach lattice with order unit and hence, there exists $u \in (E^+)^m$ such that $u^{\varepsilon} \to u$ $(\varepsilon \to 0)$ in E^m. Using the uniformity of discrete strong convergence, for a suitably chosen null sequence $(\varepsilon_n)_{\mathbb{N}}$, $\varepsilon_n \in \mathbb{R}^+$, $n \in \mathbb{N}$, we have $\text{s-lim}_{E^m} u_n^{\varepsilon_n} = u$, $u_n^{\varepsilon_n} = J_n^{\lambda,\varepsilon_n}(g_n^{\varepsilon_n})$, while Theorem 3.4 yields $\text{w-lim}_{V^m} u_n^{\varepsilon_n} = J^{\lambda}(g)$ whence $u = J^{\lambda}(g)$.

It follows from Theorems 3.6 and 3.7 that the operators C and C_n, $n \in \mathbb{N}$, defined by their graphs

$$C = \bigcup_{\lambda \in \mathbb{R}^+} \{J^{\lambda}(g), \lambda^{-1}(g - J^{\lambda}(g)) \mid g \in (E^+)^m\},$$

$$C_n = \bigcup_{\lambda \in \mathbb{R}^+} \{J_n^{\lambda}(g_n), \lambda^{-1}(g_n - J_n^{\lambda}(g_n)) \mid g_n \in (E_n^+)^m\}$$

are accretive and $\text{cl}\, D(C) \subset R(I+\lambda C) = E^+$, $\text{cl}\, D(C_n) \subset R(I_n + \lambda C_n) = E_n^+$, $\lambda \in \mathbb{R}^+$. Then the Crandall-Liggett generation theorem [5; Thm.III] implies that $-C$ and $-C_n$ generate nonlinear contraction semigroups $S(t) : (E^+)^m \to (E^+)^m$ resp. $S_n(t) : (E_n^+)^m \to (E_n^+)^m$, $t \in \mathbb{R}^+$, which are the solution operators to the evolution equations

$$u_t + C \ni f, \quad u(0) = u^0 \in \text{cl}\, D(C) \tag{3.14a}$$

$$(u_n)_t + C_n \ni f_n, \quad u_n(0) = u_n^0 \in \text{cl}\, D(C_n). \tag{3.14b}$$

Following [1; Thm.2] it can be shown that the solution to

(3.14a) resp. (3.14b) represents the maximum integral solution to (1.1a),(1.1b) resp. (3.5a),(3.5b). Since Theorem 3.7 assures discrete strong convergence of the generator resolvents, we have $S_n(t) \to S(t)$ uniformly on bounded subintervals of $\bar{\mathbb{R}}^+$ and thus discrete strong convergence of the maximum integral solutions for appropriately chosen initial data.

A final remark should be due to the solution of the approximating systems (3.5a),(3.5b). Discretizing in time by the backward Euler scheme, at any time-step we have to solve a stationary system of the form (3.9a),(3.9b). Starting from the supersolution $u_n^{(o)} = \bar{u}_n$ we can use Bensoussan-Goursat-Lions' iterative scheme $u_n^{(i)} = S_n(u_n^{(i-1)})$, $i \geq 1$ (cf.[3],[8]) to generate a sequence $(u_n^{(i)})_{i\in\mathbb{N}}$ of iterates converging monotonely decreasingly to the maximum solution of (3.9a),(3.9b). Thus, for each $i \in \mathbb{N}$ we must solve variational inequalities of type

$$\max[(I_n+\lambda A_n^\nu)(u_n^{(i)})^\nu-g_n^\nu,(u_n^{(i)})^\nu-M_n^\nu(u_n^{(i-1)})] = 0, \quad 1 \leq \nu \leq m$$

which in the concrete situation of stochastic switching control and approximation by finite difference schemes of positive type can be efficiently done using the multigrid techniques developed in [10].

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