A CONSTRUCTIVE APPROACH TO THE BELLMAN SEMIGROUP

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1. INTRODUCTION

IN THIS paper we are concerned with a constructive method for solving the parabolic Hamilton–Jacobi–Bellman (HJB-)equation in the following abstract setting.

Let H be a separable order complete real Hilbert lattice and let V be a separable reflexive Banach space with dual V' such that $V \hookrightarrow H \hookrightarrow V'$ each space being dense and continuously embedded in the following one. Let us further assume that V is a sublattice of H and let us denote by V^* the order dual of V. Finally, let E by an M-normed Banach lattice with order unit e such that E is a closed subspace and sublattice of H with continuous injection $E \hookrightarrow H$. Then, given linear operators $\mathscr{A}^\nu: V \to V'$, $\nu = 1, \ldots, m$, and elements $f^\nu \in \operatorname{int} E^+$, $u^0 \in V \cap E^+$, a function u: $[0, T] \to V$ is sought satisfying $u(t) \in V \cap E^+$, $u_t(t) \in V'$, $\mathscr{A}^\nu u(t) - f^\nu \in V^*$, $t \in (0, T)$, $\nu = 1, \ldots, m$, and the nonlinear evolution equation

$$u_t + \bigvee_{\nu=1}^{m} (\mathscr{A}^\nu u - f^\nu) = 0, \qquad t \in (0, T) \tag{1.1a}$$

$$u(0) = u^0. \tag{1.1b}$$

In case $H = L^2(\Omega)$, Ω being a bounded domain in Euclidean space $\mathbb{R}^d$, $V = H_0^1(\Omega)$, $V' = H^{-1}(\Omega)$, $E = C(\bar{\Omega})$ resp. $E = L^\infty(\Omega)$ and $\mathscr{A}^\nu$ being linear second order uniformly elliptic operators, (1.1a), (1.1b) reduces to the parabolic HJB-equation of dynamic programming characterizing the infimum of the cost function associated to an optimally controlled diffusion process (cf. [6, 12]). The parabolic HJB-equation has been intensively studied in recent years (cf., e.g. [11, 17, 18]) mainly applying techniques previously used in the time-independent case. A semigroup approach using accretive operator methods has been given by Pliska [22]. Under suitable assumptions on the data of the problem, the Bellman operator

$$\mathscr{B}u = \bigvee_{\nu=1}^{m} (\mathscr{A}^\nu u - f^\nu)$$

can be shown to be accretive. Hence, if additionally $R(I + \lambda\mathscr{B}) = C(\bar{\Omega})$, $\lambda \in \mathbb{R}^+$, the operator $-\mathscr{B}$ generates a nonlinear contraction semigroup $S(t)$, $t \in \bar{\mathbb{R}}^+ = [0, \infty)$, on $C(\bar{\Omega})$ which is called the Bellman semigroup. Moreover, denoting by $T^\nu(t)$, $t \in \bar{\mathbb{R}}^+$, $\nu = 1, \ldots, m$, the linear semigroups generated by $-\mathscr{B}^\nu$ where $\mathscr{B}^\nu u = \mathscr{A}^\nu u - f^\nu$, Pliska [23] (using an idea due to Evans

[10]) has also proved that

$$S(t) = \lim_{n\to\infty} \left(\bigwedge_{\nu=1}^{m} T^{\nu}(t/n)u^{0} \right)^{n} \tag{1.2}$$

the convergence being uniform in $C(\bar{\Omega})$. Using methods not based on accretiveness, that result has been established first by Nisio [21] (in the sense of pointwise convergence) and therefore, (1.2) is usually referred to as the Nisio formula.

In the case $E = L^{\infty}(\Omega)$ a generalized version of the parabolic HJB-equation involving nonlinear operators $\mathscr{A}^{\nu}$ has been studied by Catté [8], (cf. also [5]). Employing an idea due to Belbas [2, 3] she has also used a nonlinear semigroup approach by characterizing the generator implicitly via its resolvent which can be defined as the maximum solution of an associated system of variational inequalities.

As far as constructive methods are concerned, in case of the elliptic HJB-equation some numerical schemes have been investigated by Lions and Mercier [20] while in the parabolic case a Trotter-like algorithm based on the Nisio formula (1.2) has been proposed by Lions in [19].

The purpose of this paper is to give an approximate solution of the abstract HJB-equation (1.1a), (1.1b) in the framework of discrete approximations of ordered Banach spaces which will be developed in the next section. Using this concept and adopting techniques from [8] we will consider a sequence of approximating HJB-equations and we will establish the convergence of the associated Bellman semigroups as well as a discrete analogue of the Nisio formula.

2. DISCRETE APPROXIMATIONS OF ORDERED BANACH SPACES

Let X and X_n, $n \in \mathbb{N}$, be real Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_{X_n}$ respectively and let $(R^X = (R_n^X)_{\mathbb{N}}$ be a sequence of (not necessarily linear) restriction operators $R_n^X\colon X \to X_n$. Then the triple $(X, \Pi X_n, R^X)$ is called a discrete cn-approximation (approximation with convergent norms) if there holds:

$$\begin{aligned} &\text{(i)} \quad \|R_n^X(\alpha u + \beta v) - \alpha R_n^X u - \beta R_n^X v\|_{X_n} \to 0, \quad u, v \in X, \quad \alpha, \beta \in \mathbb{R}; \\ &\text{(ii)} \quad \|R_n^X u\|_{X_n} \to \|u\|_X, \quad u \in X; \\ &\text{(iii)} \quad \sup_{n\in\mathbb{N}} \|R_n^X u\|_{X_n} < \infty, \quad u \in X. \end{aligned} \tag{2.1}$$

Based upon this definition, the discrete strong convergence of a sequence $(u_n)_{\mathbb{N}'}$ of elements $u_n \in X_n$, $n \in \mathbb{N}' \subset \mathbb{N}$, to an element $u \in X$ is defined by

$$s - \lim_X u_n = u \Leftrightarrow \|u_n - R_n^X u\|_{X_n} \to 0.$$

In the following we will also require an additional condition, namely

$$\begin{aligned} &\text{(iv) if } u^{(n)} \in X, n \in \mathbb{N}' \subset \mathbb{N}, \text{ and } u \in X \text{ such that } u^{(n)} \to u \text{ in } X \\ &\qquad \text{then } s\text{-}\lim_X R_n^X u^{(n)} = u. \end{aligned} \tag{2.1}$$

The concept of discrete strong convergence in discrete cn-approximations is well-known and has proven to be a useful tool in developing constructive methods for solving operator equations (cf., e.g., [14, 25, 26]). In particular, conditions (2.1) (i), (ii) and (iii) ensure the uniqueness of the limit of a discretely strongly convergent sequence as well as the linearity of discrete strong convergence while (2.1) (iv) exhibits a certain uniformity of that convergence (for further properties see the papers cited above).

Denoting by X', X'_n, $n \in \mathbb{N}$, the duals of X, X_n, the triple $((X', X), \Pi(X'_n, X_n), (R^{X'}, R^X))$ is called a discrete dual cn-approximation if there holds (cf. [14]):

(i) $(X, \Pi X_n, R^X)$, $(X', \Pi X'_n, R^{X'})$ are discrete cn-approximations;
(ii) if $u_n \in X_n$, $f_n \in X'_n$, $n \in \mathbb{N}' \subset \mathbb{N}$, and $u \in X$, $f \in X'$ such that $s - \lim_X u_n = u$, $s - \lim_{X'} f_n = f$ then $\langle f_n, u_n \rangle \to \langle f, u \rangle$.

For sequences $(f_n)_{\mathbb{N}'}, f_n \in X'_n, n \in \mathbb{N}' \subset \mathbb{N}$, we may also introduce a discrete weak convergence as follows.

The sequence $(f_n)_{\mathbb{N}'}$ is said to converge discretely weakly to an $f \in X'$ (w-$\lim_{X'} f_n = f$) if for all sequences $(u_n)_{\mathbb{N}'}$, $u_n \in X_n$, such that s-$\lim_X u_n = u$ for some $u \in X$ there holds $\langle f_n, u_n \rangle \to \langle f, u \rangle$.

Moreover, in case of reflexive Banach spaces we can analogously define a discrete weak convergence of sequences $(u_n)_{\mathbb{N}'}$, $u_n \in X_n$, $n \in \mathbb{N}' \subset \mathbb{N}$, to an element $u \in X$. It is easy to verify the uniqueness of the limit of discretely weakly convergent sequences as well as the fact that discretely weakly convergent sequences are bounded and that discrete strong implies discrete weak convergence. If we additionally assume X to be separable we have the following approximation property, compactness result and equivalent characterization of discrete strong convergence (cf. [14, 25]).

THEOREM 2.1. Let $((X', X), (X'_n, X_n), (R^{X'}, R^X))$ be a discrete dual cn-approximation of reflexive Banach spaces X, X^n, $n \in \mathbb{N}$, which X being separable. Then the following hold.

(i) For each $u \in X$ (resp. $f \in X'$) there exists a sequence of elements $u_n \in X_n$ (resp. $f_n \in X'_n$), $n \in \mathbb{N}$, such that w-$\lim_X u_n = u$ (resp. w-$\lim_{X'} f_n = f$).

(ii) If $(u_n)_{\mathbb{N}'}$ (resp. $(f_n)_{\mathbb{N}'}$) is a bounded sequence of elements $u_n \in X_n$ (resp. $f_n \in X'_n$), $n \in \mathbb{N}' \subset \mathbb{N}$, then there exist a subsequence $\mathbb{N}'' \subset \mathbb{N}'$ and an element $u \in X$ (resp. $f \in X'$) such that w-$\lim_x u_n = u$ $(n \in \mathbb{N}'')$ (resp. w-$\lim_{X'} f_n = f (n \in \mathbb{N}''))$.

(iii) Let $u_n \in X_n$ (resp. $f_n \in X'_n$), $n \in \mathbb{N}' \subset \mathbb{N}$, and $u \in X$ (resp. $f \in X'$). Then s-$\lim_X u_n = u$ (resp. s-$\lim_{X'} f_n = f$) if and only if for all sequences of elements $f_n \in X'_n$ (resp. $u_n \in X_n$), $n \in \mathbb{N}'$, and $f \in X'$ (resp. $u \in X$) the discrete weak convergence w-$\lim_{X'} f_n = f$ (resp. w-$\lim_X u_n = u$) yields $\langle f_n, u_n \rangle \to \langle f, u \rangle$.

For subsets $G_n \subset X_n$, $n \in \mathbb{N}$, we introduce the following discrete limit sets:

$$s\text{-Lim}\inf_X G_n = \{u \in X \mid \exists (u_n)_{\mathbb{N}}, u_n \in X_n, n \in \mathbb{N}: s\text{-}\lim_X u_n = u\},$$

$$s\text{-Lim}\sup_X G_n = \{u \in X \mid \exists, (u_n)_{\mathbb{N}'}, u_n \in X_n, n \in \mathbb{N}' \subset \mathbb{N}: s\text{-}\lim_X u_n = u\}.$$

Obviously, in general s-Lim $\inf_X G_n \subseteq s$-Lim $\sup_X G_n$, but if equality holds we will denote that limit set by s-$\text{Lim}_X G_n$. We can analogously define the limit sets w-Lim $\inf_X G_n$, w-Lim $\sup_X G_n$ and w-$\text{Lim}_X G_n$. It is clear that s-Lim $\inf_X G_n \subseteq w$-Lim $\sup_X G_n$. If both limit sets coincide, we will denote the common limit set by $\text{Lim}_X G_n$.

Now, if $(X, \Pi X_n, R^X)$ is a discrete cn-approximation of Banach spaces and vector lattices $X, X_n, n \in \mathbb{N}$, the discrete strong convergence will be called order preserving if for all sequences

$(u_n)_{\mathbb{N}'}$, $u_n \in X_n$, $n \in \mathbb{N}' \subset \mathbb{N}$, and elements $u \in X$ there holds

$$s\text{-}\lim_X u_n = u \Rightarrow s\text{-}\lim_X u_n^+ = u^+. \tag{2.2}$$

Remark 2.2. It follows from the linearity of discrete strong convergence and the relations $u^- = u^+ - u, |u| = u^+ + u^-, u \vee v = u + (v - u)^+, u \wedge v = u - (u - v)^+$ that if (2.2) is satisfied we have discrete strong convergence of all basic lattice operations.

We can give a necessary condition for the discrete strong convergence to be order preserving in terms of the restriction operators R_n^X which in case of Banach lattices is also a sufficient one.

LEMMA 2.3. If the discrete strong convergence in a discrete cn-approximation $(X, \Pi X_n, R^X)$ is order preserving then

$$\|R_n^X u^+ - (R_n^X u)^+\|_{X_n} \to 0, \qquad u \in X. \tag{2.3}$$

If X_n, $n \in \mathbb{N}$, are Banach lattices then also the converse holds true.

Proof. Taking $u_n = R_n^X u$ in (2.2) we immediately get the necessity of (2.3). Conversely, if (2.3) is satisfied and $s\text{-}\lim_X u_n = u$ we have

$$\|u_n^+ - R_n^X u^+\|_{X_n} \leqq \|u_n^+ - (R_n^X u)^+\|_{X_n} + \|R_n^X u^+ - (R_n^X u)^+\|_{X_n}. \tag{2.4}$$

But $|u_n^+ - (R_n^X u)^+| \leqq |u_n - R_n^X u|$ whence $\|u_n^+ - (R_n^X u)^+\|_{X_n} \leqq \|u_n - R_n^X u\|_{X_n}$ where we have used the assumption that $\|\cdot\|_{X_n}$ is a lattice norm. It follows that the right-hand side in (2.4) converges to zero which implies $s\text{-}\lim_X u_n^+ = u^+$.

It is easy to show that in case of order preserving discrete strong convergence we have strong convergence of the positive cones and of order intervals.

LEMMA 2.4. Let $(X, \Pi X_n, R^X)$ be a discrete cn-approximation with order preserving discrete strong convergence. Then there holds:

(i) $s\text{-}\mathrm{Lim}_X X_n^+ = X^+$.

(ii) if $u_n \in X_n$, $v_n \in X_n$ with $u_n < v_n$, $n \in \mathbb{N}' \subset \mathbb{N}$, and $u \in X$, $v \in X$ such that $s\text{-}\lim_X u_n = u$, $s\text{-}\lim_X v_n = v$ then

$$s\text{-}\mathrm{Lim}_X [u_n, v_n] = [u, v].$$

Proof. (i) If $u \in X^+$ set $u_n = (R_n^X u)^+$. Then $u_n \in X_n^+$ and it follows from (2.3) that $s\text{-}\lim_X u_n = u$. Conversely, suppose that $u_n \in X_n^+$, $n \in \mathbb{N}' \subset \mathbb{N}$, and $u \in X$ such that $s\text{-}\lim_X u_n = u$. Then by (2.2) we get $0 = s\text{-}\lim_X u_n^- = u^-$ whence $u \in X^+$.

(ii) If $w_n \in [u_n, v_n]$, $n \in \mathbb{N}' \subset \mathbb{N}$, and $w \in X$ such that $s\text{-}\lim_X w_n = w$ then $w_n - u_n \in X_n^+$, $v_n - w_n \in X_n^+$ and we may apply part (i) of the proof to conclude $w \in [u, v]$. Conversely, if $w \in [u, v]$ choose $w_n = (R_n^X w \vee u_n) \wedge v_n$, $n \in \mathbb{N}$. Then obviously $w_n \in [u_n, v_n]$ and $s\text{-}\lim_X w_n = w$.

In the case of a discrete dual cn-approximation of reflexive Banach spaces and vector lattices

X, X_n, we can introduce the concept of order preserving discrete weak convergence (replace $s\text{-}\lim_X$ by $w\text{-}\lim_X$ in (2.2)). In view of theorem 2.1(i) it is easy to see that, if X is separable, the analogue of lemma 2.4 holds true.

Theorem 2.1(iii) indicates that there is a duality between discrete strong and discrete weak convergence. The following result shows that there is also a duality relation concerning the notation of order preserving discrete convergence.

THEOREM 2.5. Let $((X'X), \Pi(X_n', X_n), (R^{X'}, R^X))$ be a discrete dual cn-approximation of reflexive Banach spaces and vector lattices $X, X_n, n \in \mathbb{N}$, with X being separable. Then there holds:

(i) if the discrete strong (resp. weak) convergence of sequences $(u_n)_{\mathbb{N}'}$, $u_n \in X_n$, $n \in \mathbb{N}' \subset \mathbb{N}$, is order preserving then so is the discrete weak (resp. strong) convergence of sequences $(f_n)_{\mathbb{N}'}$, $f_n \in X_n^*$, $n \in \mathbb{N}'$;

(ii) if X, X_n are Banach lattices then also the converse of the statements in (i) hold true.

Proof. Let $f_n \in X_n^*$, $n \in \mathbb{N}' \subset \mathbb{N}$, and $f \in X^*$ such that $w\text{-}\lim_{X'} f_n = f$ (resp. $s\text{-}\lim_{X'} f_n = f$). According to theorem 2.1(iii) we have to show that for any sequence $(u_n)_{\mathbb{N}'}$, $u_n \in X_n$, $n \in \mathbb{N}'$, such that $s\text{-}\lim_X u_n = u$ (resp. $w\text{-}\lim_X u_n = u$) for some $u \in X$ we have $\langle f_n^+, u_n \rangle \to \langle f^+, u \rangle$. Since, by assumption, the discrete strong (resp. weak) convergence of $(u_n)_{\mathbb{N}'}$ is order preserving, we may assume $u_n \in X_n^+$ and $u \in X^+$. But

$$\langle f_n^+, u_n \rangle = \sup\{\langle f_n, v_n \rangle | 0 \leqq v_n \leqq u_n\}, \qquad u_n \in X_n^+$$

and $s\text{-}\mathrm{Lim}_X[0, u_n] = [0, u]$ (resp. $w\text{-}\mathrm{Lim}_X[0, u_n] = [0, u]$) because of lemma 2.4 (resp. its analogue) which gives the assertion. If X, X_n are Banach lattices we have $X^* = X'$, $X_n^* = X_n'$ and we can deduce the converse of the statements following the same pattern of proof.

As a by-product of the preceding results we get the following corollary.

COROLLARY 2.6. Under the hypothesis of theorem 2.5 assume the discrete strong (resp. weak) convergence of sequences $(u_n)_{\mathbb{N}'}$, $u_n \in X_n$, $n \in \mathbb{N}' \subset \mathbb{N}$, being order preserving. Then, if $u \in X$, $v \in X$ with $u < v$ (resp. $f \in X'$, $g \in X'$ with $f < g$) and if $u_n \in X_n$, $v_n \in X_n$ (resp. $f_n \in X_n'$, $g_n \in X_n'$), $n \in \mathbb{N}$, such that $s\text{-}\lim_X u_n = u$, $s\text{-}\lim_X v_n = v$ (resp. $s\text{-}\lim_{X'} f_n = f$, $s\text{-}\lim_{X'} g_n = g$) we also have $u_n < v_n$ (resp. $f_n < g_n$) for at least a final piece $\mathbb{N}_f = \{n \in \mathbb{N} | n > n_0$ for some $n_0 \in \mathbb{N}\}$.

Proof. Assume $v_n - u_n \notin \operatorname{int} E_n^+$ for a subsequence $\mathbb{N}' \subset \mathbb{N}$. Then we can find $f_n \in (X_n')^+$, $\|f_n\|_{X_n'} = 1$, $n \in \mathbb{N}'$, such that $\langle f_n, v_n - u_n \rangle \leqq 0$. Theorem 2.1(ii) gives us the existence of $\mathbb{N}'' \subset \mathbb{N}'$ and $f \in X'$ such that $w\text{-}\lim_{X'} f_n = f$ $(n \in \mathbb{N}'')$ whence $f \in (X')^+$ in view of theorem 2.5(i). On the other hand, we have $\langle f_n, v_n - u_n \rangle \to \langle f, v - u \rangle$ and thus $\langle f, v - u \rangle \leqq 0$ contradicting $v - u \in \operatorname{int} E^+$. The statement in parentheses can be proved analogously.

3. DISCRETE APPROXIMATION OF THE HJB-EQUATION

With regard to the situation considered in the introduction let us assume that $(H, \Pi H_n, R^H)$ is a discrete cn-approximation of separable order complete real Hilbert lattices with order preserving discrete strong convergence and let $((V', V), \Pi(V_n', V_n), (R^{V'}, R^V))$ be a discrete dual cn-approximation of reflexive, separable real Banach spaces such that $V \hookrightarrow H \hookrightarrow V'$, $V_n \hookrightarrow H_n \hookrightarrow V'_n$, each space being dense and compactly embedded in the following one. Moreover, we assume the embeddings $V_n \hookrightarrow H_n$ being discretely compact in the sense that given a bounded sequence $(u_n)_{\mathbb{N}}$, $u_n \in V_n$, $n \in \mathbb{N}$, for each subsequence $(u_n)_{\mathbb{N}'}$, $\mathbb{N}' \subset \subset \mathbb{N}$, there exist another subsequence $\mathbb{N}'' \subset \mathbb{N}'$ and an $u \in H$ such that $s\text{-}\lim_H u_n = u$ $(n \in \mathbb{N}'')$. In particular, by the discrete compactness of the embeddings we have that $w\text{-}\lim_V u_n = u$ implies $s\text{-}\lim_H u_n = u$. We further suppose that V, V_n are sublattices of H, H_n, the positive cones V^+, V_n^+ having nonvoid interior, the norms on V_n satisfying

$$\|u_n^+\|_{V_n} \leqq \|u_n\|_{V_n}, \qquad u_n \in V_n \tag{3.1}$$

and the restriction operators R_n^V fulfilling (2.3). Finally, let $(E, \Pi E_n, R^E)$ be a discrete cn-approximation of M-normed Banach lattices E, E_n with order units e, e_n (the order relations induced by those on H, H_n) such that the discrete strong convergence is order preserving with $s\text{-}\lim_E e_n = e$. We suppose E, E_n being continuously embedded in H, H_n with discrete continuous embeddings $E_n \hookrightarrow H_n$, i.e. $s\text{-}\lim_E u_n = u$ gives $s\text{-}\lim_H u_n = u$, and we assume $R_n^V(V \cap E) \subset V_n \cap E_n$, $R_n^E(V \cap E) \subset V_n \cap E_n$, $n \in \mathbb{N}$, $V \cap E^+$ and $V_n \cap E_n^+$ having nonvoid interior. We further suppose $(u_n - e_n)^+ \in V_n$ for each $u_n \in V_n$, the mapping $u_n \to (u_n - e_n)^-$ being bounded on bounded sets of V_n. Since the norms $\|\cdot\|_{V_n}$ are not assumed to be lattice norms we cannot apply lemma 2.2 to deduce that the discrete strong convergence in $(V, \Pi V_n, R^V)$ is order preserving. Nevertheless, we have:

THEOREM 3.1. Let V, H and V_n, H_n, $n \in \mathbb{N}$, be given as above. Then the discrete weak convergence of sequences $(u_n)_{\mathbb{N}'}$, $u_n \in V_n$, $n \in \mathbb{N}' \subset \mathbb{N}$, is order preserving and there holds $\mathrm{Lim}_V V_n^+ = V^+$.

Proof. Suppose that $w\text{-}\lim_V u_n = u$ for some $u \in V$. Then $(u_n)_{\mathbb{N}'}$ is bounded and so is $(u_n^+)_{\mathbb{N}'}$ because of (3.1). Hence, by theorem 2.1(ii) there exists $w \in V$ such that $w\text{-}\lim_V u_n^+ = w$ for at least a subsequence $\mathbb{N}'' \subset \mathbb{N}'$. But then the discrete compactness of the embeddings $V_n \hookrightarrow H_n$ implies $s\text{-}\lim_H u_n^+ = w$. On the other hand, we also have $s\text{-}\lim_H u_n = u$ and thus $s\text{-}\lim_H u_n^+ = u^+$ whence $w = u^+$. It follows from the above that $w\text{-}\mathrm{Lim\,sup}_V V_n^+ \subset V^+$. Since $V^+ \subset s\text{-}\mathrm{Lim\,inf}_V V_n^+$ by (2.3), we get $\mathrm{Lim}_V V_n^+ = V^+$.

The following result will be needed later on.

LEMMA 3.2. For $u \in V$ and $u_n \in V_n$, $n \in \mathbb{N}$, let $K(u) = \{v \in V \mid v \leqq u\}$, $K_n(u_n) = \{v_n \in V_n \mid v_n \leqq u_n\}$ and suppose that $w\text{-}\lim_V u_n = u$. Then $\mathrm{Lim}_V K_n(u_n) = K(u)$.

Proof. The preceding theorem immediately gives $w\text{-}\mathrm{Lim\,sup}_V K_n(u_n) \subset K(u)$. In order to show $K(u) \subset s\text{-}\mathrm{Lim\,inf}_V K_n(u_n)$ let us first assume that $v \in \mathrm{int}\, K(u)$. Setting $v_n = R_n^V v$ we have $s\text{-}\lim_V v_n = v$ and thus $s\text{-}\lim_H v_n = v$. Since also $s\text{-}\lim_H u_n = u$, corollary 2.6 tells us that $v_n < u_n$ for at least a final piece $\mathbb{N}_f \subset \mathbb{N}$. If v is not an interior point of $K(u)$ we can find a sequence $v^{(m)} \in \mathrm{int}\, K(u)$, $m \in \mathbb{N}$, such that $v^{(m)} \to v$ in V. For each $m \in \mathbb{N}$ there exists $(v_n^{(m)})_{\mathbb{N}}$, $v_n^{(m)} \in K_n(u_n)$ such that $s\text{-}\lim_V v_n^{(m)} = v^{(m)}$ and the uniformity of discrete strong convergence implies $s\text{-}\lim_V v_n^{(n)} = v$.

We will now construct a discrete approximation of the abstract HJB-equation (1.1a), (1.1b).

For this purpose let $\mathscr{A}_n^\nu: V_n \to V_n'$, $n \in \mathbb{N}$, $\nu = 1, \ldots, m$, be linear monotone operators satisfying

$$\langle \mathscr{A}_n^\nu u_n, (u_n - ce_n)^+ \rangle \geqq 0, \qquad u_n \in V_n \tag{3.2}$$

for all $c \in \mathbb{R}^+$ and being coercive in the sense that there exists $\gamma_n \in \mathbb{R}^+$ such that for all $\nu = 1, \ldots, m$ and all $u^0 \in V_n$ there holds

$$\|u^{(m)}\|_{V_n}^{-1}(\langle \mathscr{A}_n^\nu u^{(m)}, u^{(m)} - u^0 \rangle + \gamma_n \|u^{(m)}\|_{H_n}^2) \to \infty \tag{3.3}$$

for all sequences $(u^{(m)})_{\mathbb{N}}$, $u^{(m)} \in V_n$, $m \in \mathbb{N}$, with $\|u^{(m)}\|_{V_n} \to \infty$ $(m \in \mathbb{N})$. We further assume that the sequences $(\mathscr{A}_n^\nu)_{\mathbb{N}}$ are stable, i.e.

$$\sup_{n \in \mathbb{N}} \|\mathscr{A}_n^\nu\| < \infty, \qquad \nu = 1, \ldots, m \tag{3.4}$$

and discretely uniformly coercive in the sense that there is a $\gamma \in \mathbb{R}^+$ such that for all $\nu = 1, \ldots, m$, and all bounded sequences $(u_n^0)_{\mathbb{N}}$, $u_n^0 \in V_n$, $n \in \mathbb{N}$, the analogue of (3.3) holds true for all sequences $(u_n)_{\mathbb{N}}$, $u_n \in V_n$, $n \in \mathbb{N}$, with $\|u_n\|_{V_n} \to \infty$.

Then, given elements $f_n^\nu \in \operatorname{int} E_n^+$, $\nu = 1, \ldots, m$, and $u_n^0 \in V_n \cap E_n^+$, we are looking for functions $u_n: [0, T] \to V_n$ satisfying $u_n(t) \in V_n \cap E_n^-$, $(u_n)_t(t) \in V_n'$, $\mathscr{A}_n^\nu u_n(t) - f_n^\nu \in V_n^*$, $t \in (0, T)$, $\nu = 1, \ldots, m$, and the HJB-equations

$$(u_n)_t + \bigvee_{\nu=1}^{m} (\mathscr{A}_n^\nu u_n - f_n^\nu) = 0, \qquad t \in (0, T) \tag{3.5a}$$

$$u_n(0) = u_n^0. \tag{3.5b}$$

In order to guarantee that (3.5a), (3.5b) defines an approximation of (1.1a), (1.1b) we will assume

$$s\text{-}\lim_E f_n^\nu = f^\nu, \qquad \nu = 1, \ldots, m, \qquad s\text{-}\lim_E u_n^0 = u^0. \tag{3.6}$$

Moreover, denoting by A^ν, A_n^ν the restrictions of the graphs of $\mathscr{A}^\nu$, $\mathscr{A}_n^\nu$ to E resp. E_n (i.e. $u \in D(A^\nu)$ and $w = A^\nu u$ if $u \in V \cap E$ and $w \in E$), we assume that for each $\nu = 1, \ldots, m$ the operators $\mathscr{A}^\nu$, $\mathscr{A}_n^\nu$ as well as $A^\nu|_{V \cap E^-}$, $A_n^\nu|_{V_n \cap E_n^-}$ are consistent in the following sense.

For each $u \in V$ (resp. $u \in V \cap E^-$) there exists a sequence $(u_n)_{\mathbb{N}}$ of elements $u_n \in V_n$ (resp. $u_n \in V_n \cap E_n^+$) such that $s\text{-}\lim_V u_n = u$ and $s\text{-}\lim_{V'} \mathscr{A}_n^\nu u_n = \mathscr{A}^\nu u$ (resp. $s\text{-}\lim_E u_n = u$ and $s\text{-}\lim_E A_n^\nu u_n = A^\nu u$). The consistency of $\mathscr{A}^\nu$, $\mathscr{A}_n^\nu$ and the stability of $(\mathscr{A}_n^\nu)_{\mathbb{N}}$ implies $\mathscr{A}_n^\nu \to \mathscr{A}^\nu$, i.e. for any sequence $(u_n)_{\mathbb{N}'}$, $u_n \in V_n$, $n \in \mathbb{N}' \subset \mathbb{N}$, and $u \in V$ such that $s\text{-}\lim_V u_n = u$ we have $s\text{-}\lim_{V'} \mathscr{A}_n^\nu u_n = \mathscr{A}^\nu u$ (cf. [25]). From that convergence we may also deduce the boundedness of $\mathscr{A}^\nu$. Moreover, it follows from [16] that the pair $\mathscr{A}^\nu, \mathscr{A}_n^\nu$ is a-pseudomonotone in the following sense:

for any sequence $(u_n)_{\mathbb{N}'}$, $u_n \in V_n$, $n \in \mathbb{N}' \subset \mathbb{N}$, and $u \in V$ with $w\text{-}\lim_V u_n = u$ such that

$$\limsup_{n \in \mathbb{N}'} \langle \mathscr{A}_n^\nu u_n, u_n - w_n \rangle \leqq 0$$

for all sequences $(w_n)_{\mathbb{N}'}$, $w_n \in V_n$, with $s\text{-}\lim_V w_n = u$ there holds

$$\langle \mathscr{A}^\nu u, u - v \rangle \leqq \liminf_{n \in \mathbb{N}'} \langle \mathscr{A}_n^\nu u_n, u_n - v_n \rangle$$

for all $v \in V$ and all sequences $(v_n)_{\mathbb{N}'}$, $v_n \in V_n$, with $s\text{-}\lim_V v_n = v$. Using these results we will now show that we can assign to (1.1a), (1.1b) and (3.5a), (3.5b) nonlinear contraction semigroups $S(t): E^+ \to E^+$, $t \in \bar{\mathbb{R}}^-$, resp. $S_n(t): E_n^+ \to E_n^+$, $t \in \bar{\mathbb{R}}^-$, $n \in \mathbb{N}$, which are in a generalized sense the solution operators of the HJB-equations and therefore will be called Bellman semigroups. Moreover, we will prove convergence of these semigroups, i.e. $s\text{-}\lim_E S_n(t) u_n^0 = S(t) u^0$, $t \in \bar{\mathbb{R}}^+$. To this end, given elements $g \in E^+$, $g_n \in E_n^+$, $n \in \mathbb{N}$, and $\lambda \in \mathbb{R}^+$ we consider the time-independent equations

$$u + \lambda(A^\nu u - f^\nu) = g, \qquad u \in V \cap E^+, \qquad \nu = 1, \dots, m \tag{3.7}$$

and their discrete counterparts

$$u_n + \lambda(A_n^\nu u_n - f_n^\nu) = g_n, \qquad u_n \in V_n \cap E_n^+, \qquad \nu = 1, \dots, m. \tag{3.8}$$

THEOREM 3.3. For each $\nu = 1, \dots, m$ the operators $B^\nu = A^\nu - f^\nu$ and $B_n^\nu = A_n^\nu - f_n^\nu$ are m-T-accretive. Moreover, if $s\text{-}\lim_E g_n = g$ then $s\text{-}\lim_E (I_n + \lambda B_n^\nu)^{-1} g_n = (I + \lambda B^\nu)^{-1} g$.

Proof. Using the assumptions on $\mathscr{A}_n^\nu$ it follows from [1, lemma 1] that the operators A_n^ν are m-T-accretive and then so are the operators B_n^ν. In particular, if $u_n^\nu \in V_n \cap E_n^+$ is the unique solution of (3.8) we have

$$0 \leqq u_n^\nu \leqq (\|f_n^\nu\|_{E_n} + \|g_n\|_{E_n}) e_n. \tag{3.9}$$

To prove $R((I + \lambda B^\nu)|_{V \cap E^-}) = E^-$ we note that due to (3.4) and the consistency of $A^\nu|_{V \cap E^+}, A_n^\nu|_{V_n \cap E_n^-}$ we have

$$R((I + \lambda B^\nu)|_{V \cap E^-}) \subset s\text{-Lim inf}_E R((I_n + \lambda B_n^\nu)|_{V_n \cap E_n^-}) = s\text{-Lim inf}_E E_n^- = E^-.$$

Conversely, let $g \in E^+$. Then there exists a sequence $(g_n)_\mathbb{N}$, $g_n \in E_n^+$, $n \in \mathbb{N}$, such that $s\text{-}\lim_E g_n = g$. Let us denote by u_n^ν the corresponding solutions of (3.8). Using the discrete uniform coerciveness of $(\mathscr{A}_n^\nu)_\mathbb{N}$ it is easy to show that $(u_n^\nu)_\mathbb{N}$ is bounded, and hence, there exists an $u^\nu \in V$ such that $w\text{-}\lim_V u_n^\nu = u^\nu$ for at least a subsequence $\mathbb{N}' \subset \mathbb{N}$. Now, let $w_n \in V_n$, $n \in \mathbb{N}'$, such that $s\text{-}\lim_V w_n = u^\nu$. Since $s\text{-}\lim_H u_n^\nu = u^\nu$ it follows that $\lim_{n \in \mathbb{N}'} \langle \mathscr{A}_n^\nu u_n^\nu, u_n^\nu - w_n \rangle = 0$. Then in view of the a-pseudomonotonicity of $\mathscr{A}^\nu$, $\mathscr{A}_n^\nu$ we get

$$\langle \mathscr{A}^\nu u^\nu, u^\nu - v \rangle \leqq \liminf_{n \in \mathbb{N}'} \langle \mathscr{A}_n^\nu u_n^\nu, u_n^\nu - v_n \rangle = (f^\nu + \lambda^{-1}(g - u), u - v)_H$$

for each $v \in V$ and any sequence $(v_n)_{\mathbb{N}'}$, $v_n \in V_n$, with $s\text{-}\lim_V v_n = v$ whence $u^\nu + \lambda \mathscr{B}^\nu u^\nu = g$. Using again $s\text{-}\lim_H u_n^\nu = u^\nu$ and the discrete strong convergence of order intervals in $(H, \Pi H_n, R^H)$ (cf. lemma 2.3(ii)), it follows from (3.9) that $0 \leqq u^\nu \leqq (\|f^\nu\|_E + \|g\|_E) e$. Hence $u^\nu \in V \cap E^+$, $\mathscr{B}^\nu u^\nu \in E$ and thus $g \in R((I + \lambda B^\nu)|_{V \cap E^-})$. Taking into account the consistency of $A^\nu|_{V \cap E^+}$, $A^\nu{}_n|_{V_n \cap E^-{}_n}$ and the m-T-accretiveness of B_n^ν, we are now in a position to apply [25, theorem 1.3(3)] which gives us the unique solvability of (3.7), the discrete strong convergence of the resolvents $(I_n + \lambda B_n^\nu)^{-1} \to (I + \lambda B^\nu)^{-1}$ and the T-accretiveness of B^ν.

As an immediate consequence of the preceding results we have:

COROLLARY 3.4. The operators $-B^\nu$, $-B_n^\nu$ generate strongly continuous linear contraction semigroups $T^\nu(t): E^+ \to E^+$, $t \in \bar{\mathbb{R}}^+$, and $T_n(t)$: $E_n^+ \to E_n^+$, $t \in \bar{\mathbb{R}}^+$, such that $s\text{-}\lim_E T_n^\nu(t) u_n = T^\nu(t) u$, $t \in \bar{\mathbb{R}}^+$, for any sequence $(u_n)_\mathbb{N}$, $u_n \in E_n^+$ and any $u \in E^+$ with s--$\lim_E u_n = u$, the convergence being uniform on bounded subintervals of $\bar{\mathbb{R}}^+$.

With equations (3.7), (3.8) we associate the closed, convex sets

$$K^\nu(\lambda;g) = \{u \in V \cap E^+ | \langle \mathcal{B}^\nu u, u-v\rangle \leqq \lambda^{-1}(g-u, u-v)_H, v \in V, v \leqq u\} \tag{3.10}$$

$$\begin{aligned} K_n^\nu(\lambda;g_n) = \{u_n \in V_n \cap E_n^+ | \langle \mathcal{B}_n^\nu u_n, u_n - v_n\rangle \\ \leqq \lambda^{-1}(g_n - u_n, u_n - v_n)_{H_n}, v_n \in V_n, v_n \leqq u_n\}. \end{aligned} \tag{3.11}$$

It is easy to show that $K_n^\nu(\lambda;g_n)$ is the set of positive subsolutions of (3.8).

LEMMA 3.5. Let $u_n^\nu \in V_n \cap E_n^+$ be the unique solution of (3.8). Then u_n^ν is the maximum element of the set $K_n^\nu(\lambda;g_n)$.

Proof. Obviously, $u_n^\nu \in K_n^\nu(\lambda;g_n)$. Then, if $w_n \in K_n^\nu(\lambda;g_n)$ and $c \in \mathbb{R}^+$ we have

$$\langle \mathcal{A}_n^\nu u_n^\nu - f_n^\nu, (w_n - u_n^\nu - ce_n)^+\rangle = \lambda^{-1}(g_n - u_n^\nu, (w_n - u_n^\nu - ce_n)^+)_{H_n} \tag{3.12}$$

$$\langle \mathcal{A}_n^\nu u_n^\nu - f_n^\nu, (w_n - u_n^\nu - ce_n)^+\rangle \leqq \lambda^{-1}(g_n - w_n, (w_n - u_n^\nu - ce_n)^+)_{H_n} \tag{3.13}$$

where we have chosen $v_n = u_n^\nu - (w_n - u_n^\nu - ce_n)^-$ in (3.11). Subtracting (3.12) from (3.13) and using (3.2) we get

$$(w_n - u_n^\nu, (w_n - u_n^\nu - ce_n)^+)_{H_n} \leqq 0. \tag{3.14}$$

Since (3.14) holds true for all $c \in \mathbb{R}^+$ it follows that $w_n \leqq u_n^\nu$.

In view of $\mathcal{A}^\nu 0 = 0$, $\mathcal{A}_n^\nu 0 = 0$ we have $0 \in K^\nu(\lambda;g)$, $0 \in K_n^\nu(\lambda;g_n)$. Moreover, due to the continuity of $\mathcal{A}^\nu$, $\mathcal{A}_n^\nu$ and since f^ν, f_n^ν are assumed to be interior points of E^+ resp. E_n^+, it follows that $K^\nu(\lambda;g)$ and $K_n^\nu(\lambda;g_n)$ have nonvoid interior. We can then show:

THEOREM 3.6. Let $K^\nu(\lambda;g)$, $K_n^\nu(\lambda;g_n)$ be the sets given by (3.10), (3.11). Then, if s-$\lim_E g_n = g$ there holds

$$\operatorname{Lim}_V K_n^\nu(\lambda;g_n) = K^\nu(\lambda;g).$$

Proof. First, let us assume that $u \in \operatorname{int} K^\nu(\lambda;g)$. In view of $R_n^\nu(V\cap E) \subset V_n \cap E_n$ and (2.3) there exists a sequence $(u_n)_{\mathbb{N}}$, $u_n \in V_n \cap E_n^+$, such that s-$\lim_V u_n = u$. Setting $z_n = u_n + \lambda \mathcal{B}_n^\nu u_n$ the convergence $\mathcal{B}_n^\nu \to \mathcal{B}^\nu$ gives s-$\lim_{V'}$, $z_n = u + \lambda\mathcal{B}^\nu u$ and consequently, corollary 2.6 implies $z_n < g_n$, i.e. $u_n \in K_n^\nu(\lambda;g_n)$, for at least a final piece $\mathbb{N}_f \subset \mathbb{N}$. On the other hand, if $u \in K^\nu(\lambda;g)$ is not an interior point there exists a sequence $(u^{(m)})_{\mathbb{N}}$, $u^{(m)} \in \operatorname{int} K^\nu(\lambda;g)$, $m \in \mathbb{N}$, such that $u^{(m)} \to u$ in V, and we may proceed as in the proof of lemma 3.2. Conversely, let $u_n \in K_n^\nu(\lambda;g_n)$, $n \in \mathbb{N}' \subset \mathbb{N}$, and $u \in V$ such that w-$\lim_V u_n = u$. In order to prove $u \in K^\nu(\lambda;g)$ we remark that, given $v \in V$, $v \leqq u$, by lemma 3.2 we can find sequences $(w_n)_{\mathbb{N}'}$, $(v_n)_{\mathbb{N}'}$, $w_n \leqq u_n$, $v_n \leqq u_n$, $n \in \mathbb{N}'$, such that s-$\lim_V w_n = u$ and s-$\lim_V v_n = v$. Then we have

$$\limsup_{n\in\mathbb{N}'} \langle \mathcal{A}_n^\nu u_n, u_n - w_n\rangle \leqq 0$$

and hence, the a-pseudomonotonicity of $\mathcal{A}^\nu$, $\mathcal{A}_n^\nu$ implies

$$\langle \mathcal{A}^\nu u, u - v\rangle \leqq \liminf_{n\in\mathbb{N}'} \langle \mathcal{A}_n^\nu u_n, u_n - v_n\rangle.$$

On the other hand

$$\limsup_{n\in\mathbb{N}'} \langle \mathcal{A}_n^\nu u_n, u_n - v_n\rangle \leqq (f^\nu + \lambda^{-1}(g-u), u-v)_H$$

whence $\langle \mathcal{B}^\nu u, u-v\rangle \leqq \lambda^{-1}(g-u,u-v)_H$. Finally, by lemma 3.4 $0 \leqq u_n \leqq (\|f_n^\nu\|_{E_n} + \|g_n\|_{E_n})e_n$, and as in the proof of theorem 3.3 this gives us $0 \leqq u \leqq (\|f^\nu\|_E + \|g\|_E)e$, i.e. $u \in V \cap E^+$ and thus $u \in K^\nu(\lambda; g)$.

If $(u_n^\nu)_{\mathbb{N}}$ is a sequence of elements $u_n^\nu \in K_n^\nu(\lambda; g_n)$ we may again use the discrete uniform coerciveness of $(\mathcal{A}_n^\nu)_{\mathbb{N}}$ to deduce that $(u_n^\nu)_{\mathbb{N}}$ is bounded and hence, there is an $u^\nu \in V$ such that $w\text{-lim}_V u_n^\nu = u^\nu$ for at least a subsequence $\mathbb{N}' \subset \mathbb{N}$. Theorem 3.6 tells us then that $u^\lambda \in K^\nu(\lambda; g)$. Denoting by $J^{\lambda,\nu}(g), J_n^{\lambda,\nu}(g_n)$ the maximum elements of $K^\nu(\lambda; g), K_n^\nu(\lambda; g_n)$ let us consider the case $u_n^\nu = J_n^{\lambda,\nu}(g_n)$. Since the discrete weak convergence is order preserving, we must have $u^\nu = J^{\lambda,\nu}(g)$. On the other hand, by lemma 3.5 we already know that $J_n^{\lambda,\nu}(g_n) = (I_n + \lambda B_n^\nu)^{-1} g_n$ while theorem 3.3 says $s\text{-lim}_E (I_n + \lambda B_n^\nu)^{-1} g_n = (I + \lambda B^\nu)^{-1} g$. Since then $s\text{-lim}_H J_n^{\lambda,\nu}(g_n) = J^{\lambda,\nu}(g)$ and $s\text{-lim}_H J_n^{\lambda,\nu}(g_n) = (I + \lambda B^\nu)^{-1} g$, it follows that $J^{\lambda,\nu}(g) = (I + \lambda B^\nu)^{-1} g$, i.e. we may also consider $K^\nu(\lambda; g)$ as the set of positive subsolutions of (3.7). We now introduce the sets

$$K(\lambda; g) = \bigcap_{\nu=1}^m K^\nu(\lambda; g), \qquad K_n(\lambda; g_n) = \bigcap_{\nu=1}^m K_n^\nu(\lambda; g_n).$$

Obviously, these sets are closed and convex. Since $K^\nu(\lambda; g)$ and $K_n^\nu(\lambda; g_n)$ have zero as a common element and have nonvoid interior, the same holds true for $K(\lambda; g)$ and $K_n(\lambda; g_n)$. Hence, we may use the preceding theorem to show:

COROLLARY 3.7. Under the hypotheses of theorem 3.6 there holds

$$\text{Lim}_V K_n(\lambda; g_n) = K(\lambda; g).$$

Denoting by $J^\lambda(g)$ and $J_n^\lambda(g_n)$ the maximum elements of $K(\lambda; g)$ resp. $K_n(\lambda; g_n)$ it follows by the same arguments as above that $w\text{-Lim}_V J_n^\lambda(g_n) = J^\lambda(g)$ at least for a subsequence $\mathbb{N}' \subset \mathbb{N}$. Moreover, if $u_n \in D(A_n^\nu)$, $\nu = 1, \ldots, m$, is a solution of

$$u_n + \lambda \bigvee_{\nu=1}^m (A_n^\nu u_n - f_n^\nu) = g_n \tag{3.15}$$

we can prove as in lemma 3.5 that $K_n(\lambda; g_n)$ is the set of positive subsolutions of (3.15) with $u_n = J_n^\lambda(g_n)$ which shows that $K_n(\lambda; g_n)$ is closely related to the HJB-equations (3.5a), (3.5b). By combining techniques used in [1, 8, 20] we will now show:

THEOREM 3.8. Let J^λ resp. J_n^λ be the operators which assign to $g \in E^+$ resp. $g_n \in E_n^+$ the maximum elements of $K(\lambda; g)$ resp. $K_n(\lambda; g_n)$. Then $(J^\lambda)_{\lambda\in\mathbb{R}^+}$ resp. $(J_n^\lambda)_{\lambda\in\mathbb{R}^+}$ are families of T-contractive resolvent operators $J^\lambda : E^+ \to E^+$ resp. $J_n^\lambda : E_n^+ \to E_n^+$. Moreover, if $s\text{-lim}_E g_n = g$ then $s\text{-lim}_E J_n^\lambda(g_n) = J^\lambda(g)$.

Proof. Using a well-known penalization technique, for each $n \in \mathbb{N}$ we will first approximate $J_n^\lambda(g_n)$ by a family of T-contractive resolvent operators acting on $E^m = \prod_{\nu=1}^m E$ and then we will identify the discrete strong limit of the sequence of these families as $J^{\lambda}(g)$. For this purpose we define an operator $B\colon (V\cap E)^m \to E^m$ by $u\in D(B)$ and $w=Bu$ iff $u^\nu\in D(B^\nu)$ and $w^\nu = B^\nu u^\nu \in E$, $\nu=1,\dots,m$. Since the operators B^ν are m-T-accretive, the same holds true for B. We further introduce penalization operators $P^\varepsilon\colon E^m\to E^m$, $\varepsilon\in\mathbb{R}^+$, by $P^\varepsilon u = (\varepsilon^{-1}(u^\nu - u^{\nu+1})^+)_{1\leqq\nu\leqq m}$ where $u^{m+1}=u^1$. The operators P^ε can be shown to be T-accretive and Lipschitzian with Lipschitz constant $2\varepsilon^{-1}$ (cf. [8, lemma 2]). Consequently, as the sum of an m-T-accretive and a T-accretive Lipschitzian operator $C^\varepsilon = B + P^\varepsilon$ is m-T-accretive. Finally, for each $\nu=1,\dots,m$ we assume $(g^{\nu,\varepsilon})_{\varepsilon\in\mathbb{R}^+}$ to be a monotone decreasing family of elements $g^{\nu,\varepsilon}\in E^+$ such that $g^{\nu,\varepsilon}\to g$ $(\varepsilon\to 0)$ in E. Then, for each $n\in\mathbb{N}$ we can define operators B_n, P_n^ε and C_n^ε in the same way. Moreover, since $E^+\subset s\text{-}\mathrm{Lim\,sup}_E E_n^+$ and due to the fact that the discrete strong convergence in $(E,\Pi E_n, R^E)$ is order preserving, we can construct monotone decreasing families $(g_n^{\nu,\varepsilon})_{\varepsilon\in\mathbb{R}^+}$ of elements $g_n^{\nu,\varepsilon}\in E_n^+$ such that $g_n^{\nu,\varepsilon}\to g_n$ $(\varepsilon\to 0)$ in E_n and $s\text{-}\lim_E g_n^{\nu,\varepsilon} = g^{\nu,\varepsilon}$ for each $\varepsilon\in\mathbb{R}^+$. Denoting by $J_n^{\lambda,\varepsilon}$ the resolvent of C_n^ε and setting $u_n^\varepsilon = J_n^{\lambda,\varepsilon}(g_n^\varepsilon)$ where $g_n^\varepsilon = (g_n^{\nu,\varepsilon})_{1\leqq\nu\leqq m}$, it follows from the coerciveness assumption (3.3) and the pseudomonotonicity of the operators $\mathcal{A}_n^\nu$ that there exists an element $u_n = (u_n^\nu)_{1\leqq\nu\leqq m}$, $u_n^\nu\in K_n(\lambda; g_n)$, $\nu=1,\dots,m$, such that $u_n^{\nu,\varepsilon}\rightharpoonup u_n^\nu$ $(\varepsilon\to 0)$ in V_n (cf. [8, proposition 1]). In order to show $u_n^\nu = J_n^\lambda(g_n)$ for all $\nu=1,\dots,m$ we first prove $w_n\leqq u_n^{\nu,\varepsilon}$ for an arbitrarily given $w_n\in K_n(\lambda; g_n)$. To do this, let $(u_n^{\nu,k})_{k\in\mathbb{N}_0}$, $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, be recursively defined by (cf. [20])

$$u_n^{\nu,0} + \lambda B_n^\nu u_n^{\nu,0} = g_n^{\nu,\varepsilon} \tag{3.16a}$$

$$u_n^{\nu,k} + \lambda(B_n^\nu u_n^{\nu,k} + \varepsilon^{-1}(u_n^{\nu,k} - u_n^{\nu+1,k-1})^+) = g_n^{\nu,\varepsilon}, \qquad k\in\mathbb{N}. \tag{3.16b}$$

Note that equations (3.16a), (3.16b) are uniquely solvable, since the operators B_n^ν resp. $B_n^\nu + \varepsilon^{-1}(\cdot - z_n)$, $z_n\in V_n\cap E_n^+$, are both m-T-accretive. Moreover, it is easily shown that $(u_n^{\nu,k})_{k\in\mathbb{N}_0}$ converges monotonely decreasingly to $u_n^{\nu,\varepsilon}$. Now, choosing $c\in\mathbb{R}^+$ arbitrarily and $v_n = w_n \wedge (u_n^{\nu,k} - ce_n)$ in (3.8) we have

$$\langle \mathcal{B}_n^\nu w_n, (w_n - u_n^{\nu,k} - ce_n)^+\rangle \leqq \lambda^{-1}(g_n - w_n, (w_n - u_n^{\nu,k} - ce)^+)_{H_n}.$$

On the other hand, in view of $g_n^{\nu,\varepsilon}\geqq g_n$

$$\langle \mathcal{B}_n^\nu u_n^{\nu,k}, (w_n - u_n^{\nu,k} - ce_n)^+\rangle \geqq \lambda^{-1}(g_n - u_n^{\nu,k} - b_n^k, (w_n - u_n^{\nu,k} - ce_n)^+)_{H_n}$$

where $b_n^0 = 0$, $b_n^k = \varepsilon^{-1}(u_n^{\nu,k} - u_n^{\nu+1,k-1})^+$, $k\in\mathbb{N}$. Using (3.2) we get

$$(w_n - u_n^{\nu,k}, (w_n - u_n^{\nu,k} - ce_n)^+)_{H_n} \leqq (b_n^k, (w_n - u_n^{\nu,k} - ce_n)^+)_{H_n}. \tag{3.17}$$

Since (3.17) holds true for all $c\in\mathbb{R}^+$ and $k\in\mathbb{N}_0$, we may deduce by induction on k that $w_n\leqq u_n^{\nu,k}$, $k\in\mathbb{N}_0$, which gives $w_n\leqq u_n^{\nu,\varepsilon}$ and thus also $w_n\leqq u_n^\nu$. Finally, using this result and the monotonicity of $\mathcal{A}_n^\nu$ we get the desired result $u_n^\nu = J_n^\lambda(g_n)$, $\nu=1,\dots,m$. Since obviously $P_n^\varepsilon\to P^\varepsilon$ for each $\varepsilon\in\mathbb{R}^+$ with respect to the discrete strong convergence in $(E,\Pi E_n, R^E)$, the consistency of $A^\nu|_{V\cap E^+}, A_n^\nu|_{V_n\cap E_n^+}$ implies that of $C^\varepsilon|_{(V\cap E^+)^m}$, $C_n^\varepsilon|_{(V_n\cap E_n^+)^m}$, and we may again apply [22, theorem 1.3(3)] to deduce that $s\text{-}\lim_E u_n^{\nu,\varepsilon} = u^{\nu,\varepsilon}$ where $u^{\nu,\varepsilon} = J^{\lambda,\varepsilon}(g) = (I+\lambda C^\varepsilon)^{-1} g$. But the discrete strong convergence is order preserving and thus $(u^{\nu,\varepsilon})_{\varepsilon\in\mathbb{R}^+}$ is a monotone decreasing family of elements $u^{\nu,\varepsilon}\in E^+$. Since E is an M-normed Banach lattice with order unit, there exists $u^\nu\in E^+$ such that $u^{\nu,\varepsilon}\to u^\nu(\varepsilon\to 0)$ in E. Then, choosing a null sequence $(\varepsilon_n)_{\mathbb{N}}$ of positive real numbers, the uniformity of discrete strong convergence implies $s\text{-}\lim_E u_n^{\nu,\varepsilon_n} = u^\nu$ and hence, we also have $s\text{-}\lim_H u_n^{\nu,\varepsilon_n} = u^\nu$. On the other hand, we already know that $u_n^{\nu,\varepsilon} \rightharpoonup J_n^\lambda(g_n)$ $(\varepsilon \to 0)$ in V_n and $w\text{-lim}_V J_n^\lambda(g_n) = J^\lambda(g)$. Consequently, $s\text{-lim}_H u_n^{\nu,\varepsilon_n} = J^\lambda(g)$ and thus $u^\nu = J^\lambda(g)$, $\nu = 1, \ldots, m$.

Defining operators C and C_n by their graphs according to

$$C = \bigcup_{\lambda \in \mathbb{R}^+} \{(J^\lambda(g), \lambda^{-1}(g - J^\lambda(g))) \,|\, g \in E^+\}$$

$$C_n = \bigcup_{\lambda \in \mathbb{R}^+} \{(J_n^\lambda(g_n), \lambda^{-1}(g_n - J_n^\lambda(g_n))) \,|\, g_n \in E_n^+\}$$

it follows from above that both C and C_n are accretive operators with $\mathrm{cl}D(C) = R(I + \lambda C) = E^+$, $\mathrm{cl}D(C_n) = R(I_n + \lambda C_n) = E_n^+$ and thus, $-C$ and $-C_n$ generate nonlinear contraction semigroups $S(t)$: $E^+ \to E^+$, $t \in \bar{\mathbb{R}}^+$, resp. $S_n(t): E_n^+ \to E_n^+$, $t \in \bar{\mathbb{R}}^+$, in the Crandall–Liggett sense (cf. [9, theorem III]). Since theorem 3.8 exhibits the discrete strong convergence of the generator resolvents, we get

COROLLARY 3.9. Suppose that $u \in E^+$, $u_n \in E_n^+$, $n \in N$, such that $s\text{-lim}_E u_n = u$. Then $s\text{-lim}_E S_n(t)u_n = S(t)u$, $t \in \bar{\mathbb{R}}^+$, the convergence being uniform on bounded subintervals of $\bar{\mathbb{R}}^+$.

It follows from [8, theorem 2] that $S_n(t)u_n^0$ can be interpreted as an integral solution of the HJB-equation (3.5a), (3.5b) in the sense of Bénilan [4]. Hence, due to this and the preceding corollary we will refer to $S(t)$ resp. $S_n(t)$ as Bellman semigroups associated to the HJB-equations (1.1a), (1.1b) resp. (3.5a), (3.5b).

Finally, let us consider the operators $T_n(t)$: $E_n^+ \to E_n^+$, $t \in \mathbb{R}^+$, given by

$$T_n(t)u_n = \bigwedge_{\nu=1}^{m} T_n^\nu(t)u_n, \qquad u_n \in E_n^+.$$

We will prove the following analogue of the Nisio formula (1.2):

THEOREM 3.10. Suppose that (3.15) is solvable for each $g_n \in E_n^+$ and let $u \in E^+$, $u_n \in E_n^+$, $n \in N$, such that $s\text{-lim}_E u_n = u$. Then for any sequence $(k_n)_{\mathbb{N}}$ of positive integers such that $k_n \to \infty$ we have

$$s\text{-lim}_E (T_n(t/k_n)u_n)^{k_n} = S(t)u.$$

Proof. Since we already know that $s\text{-lim}_E S_n(t)u_n = S(t)u$, $t \in \bar{\mathbb{R}}^+$, we have only to verify $\lim_{k\to\infty} (T_n(t/k)u_n)^k = S_n(t)u_n$, $t \in \bar{\mathbb{R}}^+$, which is exactly the nonlinear Chernoff formula (cf. [7, corollary 4.3]). Since $T_n(t)$ is contractive, it only remains to be shown that $\lim_{t\to 0+} t^{-1}(T_n(t)u_n - u_n) = -C_n u_n$ for each $u_n \in D(C_n)$. For this purpose let $u_n = J_n^\lambda(g_n)$ for some $\lambda \in \mathbb{R}^+$ and $g_n \in E_n^+$. In view of $T_n(t)J_n^\lambda(g_n) \geqq S_n(t)J_n^\lambda(g_n)$, $t \in \bar{\mathbb{R}}^+$, we have

$$\liminf_{t\to 0+} t^{-1}(T_n(t)J_n^\lambda(g_n) - J_n^\lambda(g_n)) \geqq \liminf_{t\to 0+} t^{-1}(S_n(t)J_n^\lambda(g_n) - J_n^\lambda(g_n)) = -C_n J_n^\lambda(g_n).$$

On the other hand, from the discussion of (3.15) we know that $J_n^\lambda(g_n) \in D(B_n^\nu)$, $\nu = 1, \ldots, m$, and $\bigvee_{\nu=1}^{m} B_n^\nu J_n^\lambda(g_n) = \lambda^{-1}(g_n - J_n^\lambda(g_n)) = C_n J_n^\lambda(g_n)$. Now, by definition $T_n(t)J_n^\lambda(g_n) \leqq$

$T_n^\nu(t)J_n^\lambda(g_n)$, $t \in \bar{\mathbb{R}}^+$, $\nu = 1, \ldots, m$, and thus

$$\limsup_{t\to 0+} t^{-1}(T_n(t)J_n^\lambda(g_n) - J_n^\lambda(g_n)) \leqq \limsup_{t\to 0+} t^{-1}(T_n^\nu(t)J_n^\lambda(g_n) - J_n^\lambda(g_n)) = -B_n^\nu J_n^\lambda(g_n).$$

whence

$$\limsup_{t\to 0+} t^{-1}(T_n(t)J_n^\lambda(g_n) - J_n^\lambda(g_n)) \leqq \bigwedge_{\nu=1}^{m} (-B_n^\nu J_n^\lambda(g_n)).$$

But

$$\bigwedge_{\nu=1}^{m} (-B_n^\nu J_n^\lambda(g_n)) = -\bigvee_{\nu=1}^{m} B_n^\nu J_n^\lambda(g_n) = -C_n J_n^\lambda(g_n)$$

which gives the assertion.

4. APPROXIMATE SOLUTION OF THE PARABOLIC HJB-EQUATION

Since the approach as described in the preceding sections is a constructive one, it enables us to develop various schemes to the approximate solution of the HJB-equation (1.1a), (1.1b). As an example we consider the parabolic HJB-equation of dynamic programming

$$u_t + \bigvee_{\nu=1}^{m} (A^\nu u - f^\nu) = 0 \text{ in } Q := \Omega \times (0, T) \tag{4.1a}$$

$$u = 0 \text{ on } \Gamma \times (0, T), \qquad \Gamma = \partial\Omega \tag{4.1b}$$

$$u(0) = u^0 \text{ in } \Omega \tag{4.1c}$$

where Ω is a bounded domain in Euclidean space $\mathbb{R}^d$ and the operators A^ν, $1 \leqq \nu \leqq m$, are linear second order elliptic operators given by

$$A^\nu = -\sum_{i,j=1}^{d} a_{ij}^\nu(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i^\nu(x) \frac{\partial}{\partial x_i} + c^\nu(x) \tag{4.2}$$

with coefficients satisfying

$$a_{ij}^\nu, \frac{\partial}{\partial x_i} a_{ij}^\nu, b_i^\nu, c^\nu \in L^\infty(\Omega), \qquad 1 \leqq i, j \leqq d.$$

To put this problem into the setting of Section 1 we choose $H = L^2(\Omega)$, $V = W_0^{1,2}(\Omega)$ and $E = L^\infty(\Omega)$, the spaces $L^2(\Omega)$ and $L^\infty(\Omega)$ being equipped with the canonical ordering and the standard norms such that in particular $L^\infty(\Omega)$ appears as an M-normed Banach lattice with order unit e given by $e(x) = 1$, $x \in \Omega$ (cf. [24]). The operators $\mathcal{A}^\nu: W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega)$ are defined via

$$\langle \mathcal{A}^\nu u, v \rangle = a^\nu(u, v), \qquad u, v \in W_0^{1,2}(\Omega) \tag{4.3}$$

the bilinear forms $a^\nu(\cdot, \cdot)$ being given by

$$a^\nu(u, v) = \sum_{i,j=1}^{d} \int_\Omega a_{ij}^\nu(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} \mathrm{d}x + \sum_{i=1}^{d} \int_\Omega b_i^{\nu,*}(x) \frac{\partial u}{\partial x_i} v \, \mathrm{d}x$$
$$+ \int_\Omega c^\nu(x) uv \, \mathrm{d}x, \qquad u, v \in W_0^{1,2}(\Omega) \tag{4.4}$$

$$b_i^{\nu,*} = b_i^\nu(x) + \sum_{j=1}^{d} \frac{\partial a_{ij}^\nu(x)}{\partial x_i}.$$

Let us now suppose that $(\Omega_n)_{\mathbb{N}}$ is a uniformly bounded sequence of domains $\Omega_n \subset \mathbb{R}^d$, $n \in \mathbb{N}$, such that

(i) for any compact subset $S \subset \Omega$ we have

$$M_{1,2}(S \setminus \Omega_n) \to 0 \; (n \in \mathbb{N}) \tag{4.5}$$

where $M_{1,2}(S \setminus \Omega_n)$ denotes the (1,2)-capacity of the set $S \setminus \Omega_n$,

(ii) $\operatorname{meas}(\Omega_n \setminus \Omega) \to 0$ $(n \in \mathbb{N})$,

(iii) the subset

$$\Gamma^* = \bigcap_{j=0}^{\infty} \operatorname{cl}\left[\bigcup_{n \in \mathbb{N}_j} (\Omega_n \cap \Gamma)\right], \qquad \mathbb{N}_j = \{n \in \mathbb{N} \mid n > j\}$$

satisfies the segment property.

Moreover, let us assume that there are functions $a_{ij}^{\nu,n}$, $b_i^{\nu,n}$ and $c^{\nu,n}$, $1 \leqq i, j \leqq d$, $1 \leqq \nu \leqq m$, $n \in \mathbb{N}$, satisfying

$$\text{(i) } \Psi_n \in L^\infty(\Omega_n), \qquad \|\Psi_n\|_{L^\infty(\Omega_n)} \leqq C, \tag{4.6}$$

$$\text{(ii) } \|\tilde{\Psi}_n - \tilde{\Psi}\|_{L^\infty(\mathbb{R}^d)} \to 0 \qquad (n \in \mathbb{N})$$

where $\tilde{\Psi}$, $\tilde{\Psi}_n$ denote the extensions of Ψ, Ψ_n by zero to all of $\mathbb{R}^d$ and Ψ, Ψ_n are given by

$$\Psi = a_{ij}^\nu, \frac{\partial}{\partial x_i} a_{ij}^\nu, b_i^\nu, c^\nu$$

and

$$\Psi_n = a_{ij}^{\nu,n}, \frac{\partial}{\partial x_i} a_{ij}^{\nu,n}, b_i^{\nu,n}, c^{\nu,n}$$

respectively,

(iii) $\sum_{i,j=1}^{d} a_{ij}^{\nu,n}(x)\, \xi_i \xi_j \geqq \gamma |\xi|^2$, $\quad x \in \Omega_n$, $\xi \in \mathbb{R}^d$ $(\gamma > 0)$,

(iv) $c^{\nu,n}(x) \geqq c_0 \geqq 0$, $x \in \Omega_n$.

We choose $H_n = L^2(\Omega_n)$, $V_n = W_0^{1,2}(\Omega_n)$ and $E_n = L^\infty(\Omega_n)$, and we define operators $\mathcal{A}_n^\nu : W_0^{1,2}(\Omega_n) \to W^{-1,2}(\Omega_n)$ as in (4.3), (4.4) with $a_{ij}^\nu, b_i^\nu, c^\nu$ replaced by $a_{ij}^{\nu,n}, b_i^{\nu,n}, c^{\nu,n}$. For functions $u \in L^p(\Omega)$, $1 \leqq p \leqq \infty$, denoting again by $\tilde{u}$ the extension to $L^p(\mathbb{R}^d)$ via $\tilde{u} = 0$ in $\mathbb{R}^d \setminus \Omega$, we introduce "restriction" operators $R_n : L^p(\Omega) \to L^p(\Omega_n)$ by $R_n u = \tilde{u}|_{\Omega_n}$, $n \in \mathbb{N}$. It is immediately clear that $(L^2(\Omega), \Pi L^2(\Omega_n), R)$ and $(L^\infty(\Omega), \Pi L^\infty(\Omega_n), R)$, $R = (R_n)_{\mathbb{N}}$, are discrete cn-approximations satisfying (2.1)(i)–(iv) whereas under assumption (4.5)(i) the same holds true for $(W_0^{1,2}(\Omega), \Pi W_0^{1,2}(\Omega_n), R)$ as has been shown in [13]. It is also evident that the discrete strong convergence in $(L^2(\Omega), \Pi L^2(\Omega_n), R)$ is order preserving in the sense of (2.2), that

$$\|R_n u^+ - (R_n u)^+\|_{W^{1,2}(\Omega_n)} = \|\widetilde{(u^+)} - (\tilde{u})^+\|_{W^{1,2}(R^d)} \to 0 \qquad (n \in \mathbb{N}) \tag{4.7}$$

(cf. (2.3)), $\|u_n^+\|_{W^{1,2}(\Omega_n)} \leqq \|u_n\|_{W^{1,2}(\Omega_n)}$ (cf. (3.1)) and $(u_n - e_n)^+ \in W_0^{1,2}(\Omega_n)$ with $\|(u_n - e_n)^+\|_{W^{1,2}(\Omega_n)} \leqq C\|u_n\|_{W^{1,2}(\Omega_n)}$. Moreover, under conditions (4.5)(i), (ii), (iii) the sequence of embeddings $W^{1,2}(\Omega_n) \hookrightarrow L^2(\Omega_n)$, $n \in \mathbb{N}$, is discretely compact (cf. [13]). In view of (4.6)(i) the bilinear forms $a^{\nu,n}(\cdot,\cdot)$ are uniformly bounded from which we may deduce the stability of the sequence $(\mathcal{A}_n^\nu)_{\mathbb{N}}$, and because of (4.6)(iii), (iv) it is easy to establish the existence of constants $\kappa_1 > \kappa_0 \geqq 0$ such that for all $1 \leqq \nu \leqq m$ and $n \in \mathbb{N}$

$$a_n^\nu(u_n, u_n) \geqq \kappa_1 \|u_n\|^2_{W^{1,2}(\Omega_n)} - \kappa_0 \|u_n\|^2_{L^2(\Omega_n)}, \qquad u_n \in W_0^{1,2}(\Omega_n) \tag{4.8}$$

which yields the discrete uniform coerciveness of $(\mathcal{A}_n)_{\mathbb{N}}$. Moreover, for any $c \in \mathbb{R}^+$ it follows from (4.6)(iv) and (4.8) that

$$\begin{aligned}\langle \mathcal{A}_n^\nu u_n, (u_n - ce_n)^+\rangle &= \langle \mathcal{A}_n^\nu(u_n - ce_n), (u_n - ce_n)^+\rangle \\ &+ c\langle c^{\nu,n} e_n, (u_n - ce_n)^+\rangle \geqq 0, \qquad n \in \mathbb{N}.\end{aligned}$$

Taking (4.6)(i), (ii) into account, for any sequence $(v_n)_{\mathbb{N}}$, $v_n \in V_n = W_0^{1,2}(\Omega_n)$ and any $v \in W_0^{1,2}(\Omega)$ such that $w\text{-}\lim_V v_n = v$ $(n \in \mathbb{N})$ we have

$$\langle \mathcal{A}_n^\nu R_n u, v_n\rangle = a_n^\nu(R_n u, v_n) \to a^\nu(u, v) = \langle \mathcal{A}^\nu u, v\rangle \qquad (n \in \mathbb{N})$$

thus establishing the consistency of the pair $\mathcal{A}^\nu$, $(\mathcal{A}_n^\nu)_{\mathbb{N}}$. Also, (4.6)(i), (ii) immediately give the consistency of $\mathcal{A}^\nu|_W, (\mathcal{A}_n^\nu|_{W_n})_{\mathbb{N}}$, $W = W_0^{1,2}(\Omega) \cap L^\infty(\Omega)^+$, $W_n = W_0^{1,2}(\Omega_n) \cap L^\infty(\Omega_n)^+$, since it is sufficient to prove consistency on a dense subspace and thus we may take $u \in C_0^\infty(\Omega) \cap L^\infty(\Omega)^+$ and $u_n = R_n u$. So far we have verified all the hypotheses made in Section 3. What remains to be clarified is how we can actually solve the approximating HJB-equations

$$(u_n)_t + \bigvee_{\nu=1}^{m} (\mathcal{A}_n^\nu u_n - f_n^\nu) = 0 \text{ in } Q_n := \Omega_n \times (0, T) \tag{4.9a}$$

$$u_n = 0 \text{ on } \Gamma_n \times (0, T), \qquad \Gamma_n = \partial\Omega_n \tag{4.9b}$$

$$u_n(0) = u_n^0 \text{ in } \Omega_n. \tag{4.9c}$$

For a discretization in space we may use either finite element methods or finite difference techniques. In the finite element case, for each $n \in \mathbb{N}$ we may choose a sequence $(S_{n,m})_{m \in \mathbb{N}}$ of approximating subspaces of $W_0^{1,2}(\Omega_n)$, e.g. piecewise linear elements with respect to given triangulations of Ω_n, and we may consider the operators $\mathcal{A}_{n,m}^\nu = \mathcal{A}_n^\nu|_{S_{n,m}}$ which share all the properties of $\mathcal{A}_n^\nu$ such that again the results of Section 3 do apply.

In the finite difference case let us for simplicity assume that $\Omega_n = \text{Lim}\ \Omega_{n,m}$ where $\Omega_{n,m}$, $m \in \mathbb{N}$, are grid-point sets

$$\Omega_{n,m} := \{x_\nu = (x_{\nu_1}, \ldots, x_{\nu_d}) \mid \nu = (\nu_1, \ldots, \nu_d) \in \Lambda_m^{(n)} \subset \mathbb{Z}^d\}$$

with uniform step size $h_m^{(n)} > 0$ such that $h_m^{(n)} \to 0$ as $m \to \infty$. We denote by $C(\Omega_{n,m})$ the vector space of grid functions $u_{n,m}$ on $\Omega_{n,m}$ and by $C_0(\Omega_{n,m})$ the subspace of all $u_{n,m} \in C(\Omega_{n,m})$ with $u_{n,m}(x) = 0$, $x \in \Gamma_{n,m} = \partial\Omega_{n,m}$. We take $H_{n,m} = (C(\Omega_{n,m}), \|\cdot\|_{L^2(\Omega_{n,m})})$, $E_{n,m} = (C(\Omega_{n,m}), \|\cdot\|_{L^\infty(\Omega_{n,m})})$, $\|\cdot\|_{L^2(\Omega_{n,m})}$ resp. $\|\cdot\|_{L^\infty(\Omega_{n,m})}$ denoting the discrete L^2-resp. L^∞-norm on $\Omega_{n,m}$, and $V_{n,m} = (C_0(\Omega_{n,m}), \|\cdot\|_{W^{1,2}(\Omega_{n,m})})$ where $\|\cdot\|_{W^{1,2}(\Omega_{n,m})}$ is the discrete Sobolev norm

$$\|u_{n,m}\|_{W^{1,2}(\Omega_{n,m})} = \left(\sum_{\nu \in \Lambda_m^{(n)}} h_m^{(n)} u_{n,m}^2(x_\nu) + \sum_{\nu \in \tilde{\Lambda}_m^{(n)}} h_m^{(n)} (D_{h_m^{(n)}}^+ u_{n,m}(x_\nu))^2\right)^{1/2},$$

$D_{h_m^{(n)}}^+$ denoting the forward difference quotient and $\tilde{\Lambda}_m^{(n)}$ the set of all $\nu \in \Lambda_m^{(n)}$ such that $x_\nu + h_m^{(n)} e_j^d \in \Omega_{n,m} \cap \Gamma_{n,m}$, $j \in \{1, \ldots, d\}$, e_j^d denoting the jth unit vector in $\mathbb{R}^d$. We introduce restriction operators $R_m^{(n)}$ by

$$(R_m^{(n)}u_n)(x) = (h_m^{(n)})^{-d}\int_{I_m^{(n)}(x)} u_n(y)\,\mathrm{d}y, \qquad x\in\Omega_{n,m}$$

where $I_m^{(n)}(x)=\{y\in\mathbb{R}^d \mid \max_{1\leq j\leq d}|y_j-x_j|\leq h_m^{(n)}/2\}$, and we define operators $\mathscr{A}_{n,m}^{\nu}$ by

$$\langle \mathscr{A}_{n,m}^{\nu}u_{n,m}, v_{n,m}\rangle = a_{n,m}^{\nu}(u_{n,m}, v_{n,m})$$

$$a_{n,m}^{\nu}(u_{n,m}, v_{n,m}) = \sum_{i,j=1}^{d}\sum_{\nu\in\bar{\Lambda}_m^{(n)}} h_m^{(n)}a_{i,j}^{\nu,n}(x_\nu)(D_{h_m^{(n)}}^{+} u_{n,m}(x_\nu))(D_{h_m^{(n)}}^{+} v_{n,m}(x_\nu))$$

$$+\sum_{i=1}^{d}\sum_{\nu\in\bar{\Lambda}_m^{(n)}} h_m^{(n)}b_i^{\nu,n,*}(D_{h_m^{(n)}}^{+}u_{n,m}(x_\nu))v_{n,m}(x_\nu)+\sum_{\nu\in\Lambda_m^{(n)}} h_m^{(n)}c^{\nu,n}(x_\nu)u_{n,m}(x_\nu)v_{n,m}(x_\nu).$$

It can be shown in a similar way as above that the discrete cn-approximations $(L^2(\Omega_n), \Pi H_{n,m}, R^{(n)})$, $(L^\infty(\Omega_n), \Pi E_{n,m}, R^{(n)})$, $(W_0^{1,2}(\Omega_n), \Pi V_{n,m}, R^{(n)})$, $R^{(n)}=(R_m^{(n)})_{m\in\mathbb{N}}$, and the operator sequence $(\mathscr{A}_{n,m})_{m\in\mathbb{N}}$ satisfy all the hypotheses of Section 3.

Finally, we discretize in time by using the backward Euler scheme with respect to a uniform partition $t_j=jk_n$, $j=0,1,\dots,l(n)$, $k_n=T/l(n)$ of the time interval $[0,T]$, i.e. we consider

$$u_{n,m}(t_{j+1})+k_n\sum_{\nu=1}^{m}(\mathscr{A}_{n,m}^{\nu}u_{n,m}^{\nu}(t_{j+1})-f_{n,m}^{\nu})=u_{n,m}(t_j)\text{ in }\Omega_{n,m} \tag{4.10a}$$

$$u_{n,m}(t_j)=0\text{ on }\Gamma_{n,m}=\partial\Omega_{m,n},\qquad j=0,1,\dots,l(n) \tag{4.10b}$$

where $u_{n,m}(0)=u_{n,m}^0$.

Let us denote by $S(\cdot)$ the nonlinear semigroup generated by the operator-$\mathscr{B}$ where $\mathscr{B}=\bigvee_{\nu=1}^{m}(\mathscr{A}^\nu-f^\nu)$ and let $\mathscr{B}_{n,m}=\bigvee_{\nu=1}^{m}(\mathscr{A}_{n,m}^\nu-f_{n,m}^\nu)$. If f^ν, f_n^ν, $f_{n,m}^\nu$ belong to $\operatorname{int}L^\infty(\Omega)^+$, $\operatorname{int}L^\infty(\Omega_n)^+$, $\operatorname{int}E_{n,m}^+$ and u^0, u_n^0, $u_{n,m}^0$ are elements of $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)^+$, $W_0^{1,2}(\Omega_n)\cap L^\infty(\Omega_n)^+$, $V_{n,m}\cap E_{n,m}^+$ respectively, such that $s\text{-}\lim_{L^\infty(\Omega)}f_n^\nu=f^\nu(n\in\mathbb{N})$, $s-\lim_{L^\infty(\Omega_n)}f_{n,m}^\nu=f_n^\nu(m\in\mathbb{N})$ and $s\text{-}\lim_{L^\infty(\Omega)}u_n^0=u^0$ $(n\in\mathbb{N})$, $s\text{-}\lim_{L^\infty(\Omega_n)}u_{n,m}^0=u_n^0$ $(m\in\mathbb{N})$, corollary 3.9 tells us that $s\text{-}\lim_{L^\infty(\Omega)}u_{n,n}(T)=s\text{-}\lim_{L^\infty(\Omega)}(I+k_n\mathscr{B}_{n,n})^{-l(n)}u_{n,n}^0=S(T)u^0$ $(n\in\mathbb{N})$.

In order to solve the nonlinear boundary-value problems (4.10a), (4.10b) we may use the algorithms proposed by Lions and Mercier in [20] or the recently developed multi-grid techniques from [15].

An alternative way is to use the Trotter–Kato like formula given by theorem 3.10 which means that instead of solving a nonlinear equation per time-step we solve the m linear problems

$$u_{n,m}^\nu(t_{j+1})+k_n(\mathscr{A}_{n,m}^\nu u_{n,m}^\nu(t_{j+1})-f_{n,m}^\nu)=\bigvee_{\nu=1}^{m}u_{n,m}^\nu(t_j)\text{ in }\Omega_{n,m} \tag{4.11a}$$

$$u_{n,m}^\nu(t_j)=0\text{ on }\Gamma_{n,m}=\partial\Omega_{n,m},\qquad j=0,1,\dots,l(n) \tag{4.11b}$$

which are uniquely solvable by means of theorem 3.3. Under the same assumptions concerning f^ν, f_n^ν, $f_{n,m}^\nu$ and u^0, u_n^0, $u_{n,m}^0$ as above, theorem 3.10 asserts that $s\text{-}\lim_{L^\infty(\Omega)}u_{n,n}(T)=S(T)u^0$ $(n\in\mathbb{N})$.

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