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Angaben zur Veröffentlichung / Publication details:

Hoppe, Ronald H. W. 1986. "Multi-grid methods for Hamilton-Jacobi-Bellman equations." *Numerische Mathematik* 49 (2-3): 239–54. https://doi.org/10.1007/bf01389627.

Multi-Grid Methods for Hamilton-Jacobi-Bellman Equations

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Summary. In this paper we develop multi-grid algorithms for the numerical solution of Hamilton-Jacobi-Bellman equations. The proposed schemes result from a combination of standard multi-grid techniques and the iterative methods used by Lions and Mercier in [11]. A convergence result is given and the efficiency of the algorithms is illustrated by some numerical examples.

1. Introduction

We study multi-grid techniques for the numerical solution of the Hamilton-Jacobi-Bellman (HJB-) equation

$$\max_{1 \leqq \nu \leqq m} [A^\nu u(x) - f^\nu(x)] = 0, \quad x \in \Omega, \tag{1.1a}$$

$$u(x) = 0, \quad x \in \Gamma = \partial\Omega \tag{1.1b}$$

where Ω is a bounded, smooth domain in Euclidean space $\mathbb{R}^d$, $d \in \mathbb{N}$, the f^ν are given functions from $C^2(\bar{\Omega})$, and the A^ν are second order, uniformly elliptic operators of the form

$$A^\nu = -\sum_{i,j=1}^{d} a_{ij}^\nu \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i^\nu \frac{\partial}{\partial x_i} + c^\nu \tag{1.2}$$

the coefficients satisfying for all $1 \leqq \nu \leqq m$

$$\begin{aligned} &a_{ij}^\nu, b_i^\nu, c^\nu \in C^2(\bar{\Omega}), \quad a_{ij}^\nu(x) = a_{ji}^\nu(x), \quad x \in \bar{\Omega},\ 1 \leqq i,j \leqq d \\ &\gamma_1 |\xi|^2 \geqq \sum_{i,j=1}^{d} a_{ij}^\nu(x) \xi_i \xi_j \geqq \gamma_0 |\xi|^2, \quad x \in \bar{\Omega},\ \xi \in \mathbb{R}^d \\ &c^\nu(x) \geqq c_0, \quad x \in \bar{\Omega} \end{aligned} \tag{1.3}$$

with constants $\gamma_1 \geqq \gamma_0 > 0$ and $c_0 \geqq 0$.

The HJB-equation is often encountered in applications, for example in stochastic control the solution of (1.1 a), (1.1 b) characterizes the infimum of the cost function associated to an optimally controlled stochastic switching process without costs for switching (cf. e.g. [1, 5]). The existence, uniqueness and regularity of a solution to the HJB-equations have been intensively studied in the past years where the best result so far has been established recently by Evans in [4] who proved that, under the assumptions made above, (1.1 a), (1.1 b) has a unique solution $u \in C(\bar{\Omega}) \cap C^{2,\theta}(\Omega)$ for some $\theta > 0$. In some particular cases, namely when $m=2$ or $d=2$, it is even known that $u \in C^{2,\theta}(\bar{\Omega})$. Therefore, with regard to finite difference discretizations with step size h we can expect convergence of approximate solutions and at least interior estimates for the global discretization error of order $O(h^{\theta})$. Some iterative schemes combined with finite difference techniques have been developed and numerically implemented by P.L. Lions and Mercier in [11] where, roughly speaking, the nonlinear problem (1.1 a), (1.1 b) is reduced to a sequence of linear ones and the linear systems obtained by discretization are solved using standard techniques. On the other hand, it is well-known that multi-grid methods provide a powerful tool for solving linear or nonlinear elliptic problems (see e.g. the survey articles by Trottenberg, Stüben [13] and Hackbusch [8]). It is the purpose of this paper to show how to apply the iterative schemes used in [11] in connection with multi-grid techniques in order to get efficient algorithms for the numerical solution of the HJB-equation.

2. Methods of Successive Approximations for the HJB-Equation

Let $\mathbb{R}_h^d := \{x = (x_{\nu_1}, \ldots, x_{\nu_d}) \mid x_{\nu_i} = \nu_i h, \nu = (\nu_1, \ldots, \nu_d) \in \mathbb{Z}^d\}$ be a uniform partition of $\mathbb{R}^d$ with step size $h > 0$ and define $\Omega_h := \Omega \cap \mathbb{R}_h^d$ with $\Gamma_h = \partial\Omega_h$ being the set of intersection points of Γ and the grid-lines of $\mathbb{R}_h^d$. We denote by $C_0(\Omega_h)$ the vector space of grid-functions u_h on $\bar{\Omega}_h := \Omega_h \cup \Gamma_h$ with $u_h(x) = 0$, $x \in \Gamma_h$, and by $\|\cdot\|_p$, $1 \leqq p \leqq \infty$, the discrete L^p-norm on $C_0(\Omega_h)$. For each $\nu \in \{1, \ldots, m\}$ we then choose finite difference approximations A_h^{ν} of the elliptic operators A^{ν} which are of positive type, i.e. the corresponding discretization matrix is an irreducible, diagonally dominant M-matrix. This can be formally done, since it has been shown by Motzkin and Wasow [12] that each uniformly elliptic operator admits such a finite difference approximation. However, it should be noted that in practice special care must be taken of the discretization of the mixed second order derivatives in (1.2) (a construction of finite difference schemes of positive type for uniformly elliptic operators in the plane is given by Bramble and Hubbard in [2]). Denoting by f_h^{ν}, $1 \leqq \nu \leqq m$, the pointwise restriction of f^{ν} to Ω_h the discretized HJB-equation takes the form

$$\max_{1 \leqq \nu \leqq m} [A_h^{\nu} u_h(x) - f_h^{\nu}(x)] = 0, \quad x \in \Omega_h, \tag{2.1 a}$$

$$u_h(x) = 0, \quad x \in \Gamma_h. \tag{2.1 b}$$

If we define $v_0(x)$, $x\in\Omega_h$, as the index $v_0(x)\in\{1,\dots,m\}$ for which the maximum is attained in (2.1a) and if we set $C_h[u_h]u_h(x)=A_h^{v_0(x)}u_h(x)$, $f_h[u_h](x)=f_h^{v_0(x)}$, then (2.1a) can be formally written as

$$C_h[u_h]u_h(x)=f_h[u_h](x), \qquad x\in\Omega_h. \tag{2.2}$$

The following theorem gives an existence and uniqueness as well as a stability result for (2.1a), (2.1b):

Theorem 2.1. *Suppose that A_h^v are finite difference operators being consistent with A^v, $v=1,\dots,m$, such that $A_h^v-\lambda I_h$ is of positive type for some $\lambda>0$. Then there holds*

(i) *The vectorial* HJB-equation (2.1a), (2.1b) *admits a unique solution $u_h\in C_0(\Omega_h)$.*

(ii) *If $u_h\in C_0(\Omega_h)$* resp. *$v_h\in C_0(\Omega_h)$ is the unique solution of* (2.1a), (2.1b) resp. *the corresponding equation with f_h^v replaced by g_h^v, then for $1\leqq p\leqq\infty$*

$$\begin{aligned}\|u_h-v_h\|_p\leqq\max[&\|C_h^{-1}[u_h](f_h[u_h]-g_h[u_h])\|_p,\\ &\|C_h^{-1}[v_h](f_h[v_h]-g_h[v_h])\|_p].\end{aligned} \tag{2.3}$$

In particular, we have

$$\|u_h-v_h\|_p\leqq\lambda^{-1}\max_{1\leqq v\leqq m}\|f_h^v-g_h^v\|_p. \tag{2.4}$$

Proof. (i) The unique solvability of (2.1a), (2.1b) has been shown by Delebecque and Quadrat in [3; Theorem 3].

(ii) (cf. [11; Theorem 7.1].) From (2.1a), (2.2) it follows that

$$\begin{aligned}C_h[u_h](v_h-u_h)&\leqq g_h[u_h]-f_h[u_h],\\ C_h[v_h](u_h-v_h)&\leqq f_h[v_h]-g_h[v_h].\end{aligned}$$

Using the fact that $C_h[u_h]$ and $C_h[v_h]$ are of positive type, we get

$$-C_h^{-1}[u_h](g_h[u_h]-f_h[u_h])\leqq u_h-v_h\leqq C_h^{-1}[v_h](f_h[v_h]-g_h[v_h]), \tag{2.5}$$

and hence

$$\begin{aligned}|u_h-v_h|\leqq\max[&|C_h^{-1}[u_h](f_h[u_h]-g_g[u_h])|,\\ &|C_h^{-1}[v_h](f_h[v_h]-g_h[v_h])|],\end{aligned} \tag{2.6}$$

where $|w_h|=w_h^+ + w_h^-$.

Since the discrete L^p-norms $\|\cdot\|_p$, $1\leqq p\leqq\infty$, are lattice norms, (2.3) is a direct consequence of (2.6).

Finally, $C_h[u_h]=(C_h[u_h]-\lambda I_h)+\lambda I_h$. But $C_h[u_h]-\lambda I_h$ is of positive type whence $C_h^{-1}[u_h]\leqq\lambda^{-1}I_h$. Analogously, $C_h^{-1}[v_h]\leqq\lambda^{-1}I_h$. Using this in (2.5) and arguing as before, we arrive at (2.4).

Setting $v_h=u|_{\bar\Omega_h}$ and $g_h^v=A_h^v(u|_{\bar\Omega_h})$, $1\leqq v\leqq m$, in Theorem 2.1, we get the following a-priori estimate for the discretization error:

Corollary 2.2. *Under conditions* (1.3) *let us assume that* $u \in C^{2,\theta}(\bar{\Omega})$ *and let the hypotheses of Theorem* 2.1 *be fulfilled. Then there holds*

$$\|u_h - u|_{\bar{\Omega}_h}\|_p = O(h^\theta) \quad (h \to 0). \tag{2.7}$$

We now present two iterative schemes for the numerical solution of the discretized HJB-equation which have been proposed in [11] and which will serve as the basis for our multi-grid algorithms. The first scheme requires the solution of an unilateral variational inequality at each step of the iteration:

Scheme I

Step 1. For $\hat{\nu} \in \{1, \ldots, m\}$ determine $u^0_{h,m}$ as the solution of

$$A^{\hat{\nu}}_h u^0_{h,m}(x) = f^{\hat{\nu}}_h(x), \quad x \in \Omega_h, \tag{2.8a}$$

$$u^0_{h,m}(x) = 0, \quad x \in \Gamma_h. \tag{2.8b}$$

Step 2. For $n \geqq 1$ and $1 \leqq \nu \leqq m$, given $u^{N+\nu-1}_{h,\nu-1}$, $N = (n-1)m$, where $u^N_{h,0} = u^N_{h,m}$, compute $u^{N+\nu}_{h,\nu}$ as the solution of

$$\max[A^\nu_h u^{N+\nu}_{h,\nu}(x) - f^\nu_h(x), u^{N+\nu}_{h,\nu}(x) - u^{N+\nu-1}_{h,\nu-1}(x)] = 0, \quad x \in \Omega_h \tag{2.9a}$$

$$u^{N+\nu}_{h,\nu}(x) = 0, \quad x \in \Gamma_h. \tag{2.9b}$$

Note that (2.9a), (2.9b) can be interpreted as an obstacle problem for the determination of the iterate $u^{N+\nu}_{h,\nu}$ with the obstacle given by its predecessor $u^{N+\nu-1}_{h,\nu-1}$.

The second scheme is based upon a linearization of (2.1a) by choosing locally that equation for which the maximum is attained:

Scheme II

Step 1. Determine u^0_h as the solution of (2.8a), (2.8b).

Step 2. For $n \geqq 1$, given u^{n-1}_h, set

$$\alpha^{n-1}_\nu(x) = \begin{cases} 1 & \text{if } \nu = \nu_0 \\ 0 & \text{otherwise} \end{cases} \tag{2.10}$$

where

$$\nu_0 := \min\{\nu \in \{1, \ldots, m\} \mid A^\nu_h u^{n-1}_h(x) - f^\nu_h(x) = \max_{1 \leqq \mu \leqq m} [A^\mu_h u^{n-1}_h(x) - f^\mu_h(x)]\}$$

and then compute u^n_h as the solution of

$$\sum_{\nu=1}^{m} \alpha^{n-1}_\nu(x)[A^\nu_h u^n_h(x) - f^\nu_h(x)] = 0, \quad x \in \Omega_h, \tag{2.11a}$$

$$u^n_h(x) = 0, \quad x \in \Gamma_h. \tag{2.11b}$$

Remark 2.3. In practice the problem (2.9a), (2.9b) in *Step 2* of *Scheme I* is solved by *Step 2* of *Scheme II* for the case of two operators, namely A^ν_h and I_h.

Denoting by N_h the total number of grid-points in Ω_h, *Step 2* of the second algorithm requires the solution of a linear system of the form

$$C_h^{n-1}[u_h^{n-1}]u_h^n = f_h^{n-1}[u_h^{n-1}]. \tag{2.12}$$

Here, the i-th row of the (N_h, N_h)-matrix $C_h^{n-1}[u_h^{n-1}]$ resp. the i-th component of the vector $f_h^{n-1}[u_h^{n-1}]$, $1 \leqq i \leqq N_h$, are given by the i-th row of the matrix corresponding to $A_h^{\nu_0}$ resp. the i-th component of the vector $f_h^{\nu_0}$. It is clear that $C_h^{n-1}[u_h^{n-1}]$ also is an irreducible, diagonally dominant M-matrix and hence, (2.11a), (2.11b) is uniquely solvable. Note however, that $C_h^{n-1}[u_h^{n-1}]$ is not necessarily positive definite.

Convergence results for both iterative schemes have been established in [3] resp. [11]:

Theorem 2.4. *Let the assumptions of Theorem* 2.1 *be satisfied. Then, for all* $\nu \in \{1, \ldots, m\}$ *the sequence* $\{u_{h,\nu}^{N+\nu}\}$, $N=(n-1)m$, $n \in \mathbb{N}$, *generated by Scheme I, and the sequence* $\{u_h^n\}$, $n \in \mathbb{N}$, *obtained by means of Scheme II, converge monotonely decreasingly to the solution of the discretized* HJB-*equation* (2.1a), (2.1b).

3. Multi-Grid Algorithms

The proposed algorithms are a suitable combination of standard multi-grid techniques applied to the discretized HJB-equation (2.1a), (2.1b) and the iterative schemes of Sect. 2. Concerning the first scheme, we remark that multi-grid methods for variational inequalities have been considered recently by Hackbusch and Mittelmann in [9] but in a different context, namely for variational inequalities stemming from obstacle problems for minimal surfaces.

Let $\Omega_{h_k} := \Omega \cap \mathbb{R}_{h_k}^d$, $k=0,1,\ldots,l$, be grid-point sets where $\{h_k\}_{k=0}^l$ is a decreasing sequence of positive real numbers such that

$$h_k < h_{k-1} \leqq C h_k, \quad 1 \leqq k \leqq l \tag{3.1}$$

and let $A_{h_k}^\nu$ be finite difference approximations of positive type consistent with A^ν, $\nu=1,\ldots,m$. For the fine-to-coarse resp. the coarse-to-fine transfer within the multi-grid cycles we define restriction operators r_k^{k-1}: $C_0(\Omega_{h_k}) \to C_0(\Omega_{h_{k-1}})$ resp. prolongation operators p_k^{k+1}: $C_0(\Omega_{h_k}) \to C_0(\Omega_{h_{k+1}})$.

In the multi-grid algorithm we start from an approximation $u_{h_l}^0$ on the finest grid Ω_{h_l} which we choose as a supersolution of the discretized HJB-equation with respect to Ω_{h_l}. Such a startvector can be obtained by the following nested iteration technique applied to the discretized elliptic boundary-value problem (2.8a), (2.8b). For convenience the algorithm is written in quasi-ALGOL:

Nested iteration for generating a startvector (3.2)
$\tilde{u}_{h_0}^{\hat{\nu}} :=$ approximation of $(A_{h_0}^{\hat{\nu}})^{-1} f_{h_0}^{\hat{\nu}}$;
for $k := 1$ **step** 1 **until** l **do**
begin $\tilde{u}_{h_k}^{\hat{\nu}} := p_{k-1}^k \tilde{u}_{h_{k-1}}^{\hat{\nu}}$;
  for $j := 1$ **step** 1 **until** μ_1 **do** $mgm(k, \tilde{u}_{h_k}^{\hat{\nu}}, f_{h_k}^{\hat{\nu}})$
end nested iteration;

procedure $mgm(k,u,f)$; (3.3)
if $k=0$ **then** $u:=(A_{h_0}^{\hat{v}})^{-1}f$ **else**
begin integer j; **array** v,d;
for $j:=1$ **step** 1 **until** i_1 **do** $u:=G_{h_k}u$; (3.3a)
$d:=r_k^{k-1}(A_{h_k}^{\hat{v}}u-f)$;
$v:=0$;
for $j:=1$ **step** 1 **until** μ_2 **do** $mgm(k-1,v,d)$;
$u:=u-p_{k-1}^k v$;
for $j:=1$ **step** 1 **until** i_2 **do** $u:=G_{h_k}u$; (3.3b)
end.

Note that in the smoothing parts (3.3a), (3.3b) of the linear multi-grid procedure $mgm(k,u,f)$ the operation $G_{h_k}u$ denotes the application of the relaxation step with respect to a chosen smoothing method applied to the equation $A_{h_k}^{\hat{v}}u=f$ with the input u as startvector.

We now describe an algorithm (*nmgm I*) based upon the first iterative scheme of Sect. 2:

Having determined an iterate $u_{h_l,\nu-1}^{N+\nu-1}$, $1\leqq\nu\leqq m$, $N=(n-1)m$, $n\geqq 1$, on the finest grid Ω_{h_l}, we obtain a new iterate $u_{h_l,\nu}^{N+\nu}$ by performing the following multi-grid cycle on the variational inequality (2.9a), (2.9b):

We first apply a certain smoothing procedure which consists of several steps of a suitable iterative scheme, e.g. Gauss-Seidel relaxation, to the linearized version of (2.9a) obtained from $u_{h_l,\nu-1}^{N+\nu-1}$ according to *Step 2* of *Scheme II*. We thus obtain a smoothed grid-function $\tilde{u}_{h_l,\nu}^{N+\nu}$ by

$$\tilde{u}_{h_l,\nu}^{N+\nu}=G_{h_l}^{i_1}[u_{h_l,\nu-1}^{N+\nu-1}]u_{h_l,\nu-1}^{N+\nu-1} \tag{3.4}$$

where i_1 is the number of relaxation steps. Denoting the defect by

$$d_{h_l,\nu}^{N+\nu}=A_{h_l}^{\nu}\tilde{u}_{h_l,\nu}^{N+\nu}-f_{h_l}^{\nu}, \tag{3.5}$$

the defect correction equation to be solved on the coarser grid is given by

$$\max[A_{h_{l-1}}^{\nu}u_{h_{l-1},\nu}^{N+\nu}(x)-d_{h_{l-1},\nu}^{N+\nu}(x), \tag{3.6a}$$
$$u_{h_{l-1},\nu}^{N+\nu}(x)-r_l^{l-1}\tilde{u}_{h_l,\nu}^{N+\nu}(x)]=0, \qquad x\in\Omega_{h_{l-1}}$$

$$u_{h_{l-1},\nu}^{N+\nu}(x)=0, \qquad x\in\Gamma_{h_{l-1}} \tag{3.6b}$$

where

$$d_{h_{l-1},\nu}^{N+\nu}=A_{h_{l-1}}^{\nu}r_l^{l-1}\tilde{u}_{h_l,\nu}^{N+\nu}-r_l^{l-1}d_{h_l,\nu}^{N+\nu}. \tag{3.7}$$

We then compute a corrected iterate $\tilde{u}_{h_l,\nu}^{N+\nu,\mathrm{new}}$ according to

$$\tilde{u}_{h_l,\nu}^{N+\nu,\mathrm{new}}=\tilde{u}_{h_l,\nu}^{N+\nu}-p_{l-1}^{l}(r_l^{l-1}\tilde{u}_{h_l,\nu}^{N+\nu}-u_{h_{l-1},\nu}^{N+\nu}). \tag{3.8}$$

Finally, we obtain a new iterate $u_{h_l,\nu}^{N+\nu}$ by performing i_2 steps of the smoothing method applied to the linearized version of (2.9a) on Ω_{h_l} with $u_{h_l,\nu-1}^{N+\nu-1}$ replaced by $\tilde{u}_{h_l,\nu}^{N+\nu,\mathrm{new}}$ and $\tilde{u}_{h_l,\nu}^{N+\nu,\mathrm{new}}$ as startvector, i.e.

$$u_{h_l,\nu}^{N+\nu}=G_{h_l}^{i_2}[\tilde{u}_{h_l,\nu}^{N+\nu,\mathrm{new}}]\tilde{u}_{h_l,\nu}^{N+\nu,\mathrm{new}}. \tag{3.9}$$

So far we have described one cycle of a two-grid iteration. In the multi-grid case with $l \geqq 2$, the solution of the defect correction equations on Ω_{h_k}, $k<l$, is successively replaced by the technique described above, using $r_{k+1}^k \tilde{u}_{h_{k+1},\nu}^{N+\nu}$ as startvector, until the coarsest grid Ω_{h_0} is reached. This means that in the fine-to-coarse part of the cycle the smoothing is applied to the linear equation

$$C_{h_k}^{\nu} u_{h_k,\nu}^{N+\nu}(x) = \tilde{d}_{h_k,\nu}^{N+\nu}(x), \qquad x \in \Omega_{h_k}, \tag{3.10a}$$

$$u_{h_k,\nu}^{N+\nu}(x) = 0, \qquad x \in \Gamma_{h_k} \tag{3.10b}$$

where

$$C_{h_k}^{\nu} = A_{h_k}^{\nu}, \qquad \tilde{d}_{h_k,\nu}^{N+\nu}(x) = d_{h_k,\nu}^{N+\nu}(x), \qquad \text{if } (r_{k+1}^k d_{h_{k+1},\nu}^{N+\nu})(x) > 0,$$

and

$$C_{h_k}^{\nu} = I_{h_k}, \qquad \tilde{d}_{h_k,\nu}^{N+\nu}(x) = r_{k+1}^k \tilde{u}_{h_{k+1},\nu}^{N+\nu}(x), \qquad \text{if } (r_{k+1}^k d_{h_{k+1},\nu}^{N+\nu})(x) \leqq 0.$$

The correction problem on Ω_{h_0} can be solved by means of the second iterative scheme with $r_1^0 \tilde{u}_{h_1,\nu}^{N+\nu}$ as startvector. If the chosen relaxation method is convergent, it can also be used as an iterative solver on level $k=0$.

Moreover, we may use nested iteration which means that several cycles are performed at each level $k \in \{1, \ldots, l\}$.

Finally, the multi-grid iteration will be terminated if for all $\nu \in \{1, \ldots, m\}$ the difference $\|u_{h_l,\nu}^{N+\nu} - u_{h_l,\nu}^{N-m+\nu}\|_\infty$ between two successive iterates is less than a given bound $\varepsilon > 0$. Hence, the complete multi-grid scheme (MGS I) takes the form:

Scheme MGS I (3.11)

$u_{h_l,m}^0 := \tilde{u}_{h_l}^{\hat{\nu}}$ startvector obtained by nested iteration;
$n := 0$;
start $n := n+1$
$N := (n-1) * m$;
for $\nu := 1$ **step** 1 **until** m **do**
begin $f := f_{h_l}^{\nu}$;
  if $\nu = 1$ **then** $u := u_{h_l,m}^N$ **else** $u := u_{h_l,\nu-1}^{N+\nu-1}$;
  for $j := 1$ **step** 1 **until** κ_1 **do** $nmgm\ I(\nu, l, u, f)$;
  $u_{h_l,\nu}^{N+\nu} := u$;
end;
if $n = 1$ **go to start else**

for $\nu := 1$ **step** 1 **until** m **do** $d_\nu := u_{h_l,\nu}^{N-m+\nu} - u_{h_l,\nu}^{N+\nu}$;

if $\min_{1 \leqq \nu \leqq m} \|d_\nu\|_\infty > \varepsilon$ **then go to start else**
go to end MGS I;
end MGS I.
procedure $nmgm\ I(\nu, k, u, f)$ (3.12)
if $k = 0$ **then** $u :=$ solution of defect correction (3.12a)
  equation **else**
begin integer j; **array** v_1, v_2, d;

for $j:=1$ **step** 1 **until** i_1 **do** $u:=G_{h_k}u$; (3.12b)
$v_1:=r_k^{k-1}u$; $v_2:=v_1$;
$d:=A_{h_{k-1}}^{v}v_1-r_k^{k-1}(A_{h_k}^{v}u-f)$;
for $j:=1$ **step** 1 **until** κ_2 **do** *nmgm* $I(v,k-1,v_2,d)$;
$u:=u-p_{k-1}^{k}(v_1-v_2)$; (3.12c)
for $j:=1$ **step** 1 **until** i_2 **do** $u:=G_{h_k}u$; (3.12d)
end *nmgm I*.

Note that the iteration processes G_{h_k} in the fine-to-coarse part (3.12b) resp. the coarse-to-fine part (3.12d) are not the same at each occurrence but are determined by the linearized versions of (2.9a), (2.9b) (in case $k=l$) resp. of the defect correction equations (in case $k<l$).

If we use the second iterative scheme of Sect. 2 instead of the first one, we obtain a multi-grid algorithm (*nmgm II*) where, given an approximate solution $u_{h_l}^{n-1}$ on Ω_{h_l}, in the fine-to-coarse part of the cycle we compute $\tilde{u}_{h_k}^{n}$, $1\leqq k\leqq l$, as in *nmgm I* (cf. (3.4), (3.12b)). Note that now G_{h_k} denotes the same relaxation method used in *nmgm I* but this time applied to the linearized (according to (2.11) resp. (2.12)) versions of the vectorial HJB-equations

$$\max_{1\leqq v\leqq m}[A_{h_k}^{v}u_{h_k}^{n}(x)-\tilde{f}_{h_k,v}^{n}(x)]=0,\qquad x\in\Omega_{h_k}, \tag{3.13a}$$

$$u_{h_k}^{n}(x)=0,\qquad x\in\Gamma_{h_k} \tag{3.13b}$$

where the $\tilde{f}_{h_k,v}^{n}$, $v=1,\ldots,m$ are recursively given by

$$\begin{aligned}&\tilde{f}_{h_l,v}^{n}=f_{h_l}^{v}\\&\tilde{f}_{h_k,v}^{n}=A_{h_k}^{v}r_{k+1}^{k}\tilde{u}_{h_{k+1}}^{n}-r_{k+1}^{k}(A_{h_{k+1}}^{v}\tilde{u}_{h_{k+1}}^{n}-\tilde{f}_{h_{k+1},v}^{n}),\qquad k<l\end{aligned} \tag{3.14}$$

and $u_{h_l}^{n-1}$ resp. $r_{k+1}^{k}\tilde{u}_{h_{k+1}}^{n}$, $k<l$, are taken as startvectors. In the coarse-to-fine part of the cycle we then determine new iterates $u_{h_k}^{n}$ in a similar way as in *nmgm I*, i.e. we first compute

$$\tilde{u}_{h_k}^{n,\mathrm{new}}=\tilde{u}_{h_k}^{n}-p_{k-1}^{k}(r_k^{k-1}\tilde{u}_{h_k}^{n}-u_{h_{k-1}}^{n}), \tag{3.15}$$

with $u_{h_{k-1}}^{n}$ denoting the new iterate on the $(k-1)$-th level, and we then perform i_2 smoothing steps applied to the linearized version of (3.13) with $\tilde{u}_{h_k}^{n,\mathrm{new}}$ as startvector.

The complete multi-grid scheme (MGS II) reads as follows:

Scheme MGS II

$u_{h_l}^{0}:=\tilde{u}_{h_l}^{\bar{v}}$ startvector obtained by nested iteration; (3.16)
$n:=0$;
start $n:=n+1$;
for $v:=1$ **step** 1 **until** m **do** $f_v:=f_{h_l}^{v}$;
$u:=u_{h_l}^{n-1}$;
for $j:=1$ **step** 1 **until** κ_1 **do** *nmgm II*$(l,u,f_1,\ldots,f_m)$;
$u_{h_l}^{n}:=u$;
$d:=u_{h_l}^{n-1}-u_{h_l}^{n}$
if $\|d\|_\infty>\varepsilon$ **then go to start else go to end** MGS II;
end MGS II.

procedure $nmgm\ II(k,u,f_1,\ldots,f_m)$ (3.17)
if $k=0$ **then** $u:=$solution of defect correction
equation **else**
begin integer j; **array** $v_1,v_2,d_1,\ldots,d_m$;
for $j:=1$ **step** 1 **until** i_1 **do** $u:=G_{h_k}u$;
$v_1:=r_k^{k-1}u$; $v_2:=v_1$;
for $v:=1$ **step** 1 **until** m **do**
$d_v:=A^v_{h_{k-1}}v_1-r_k^{k-1}(A^v_{h_k}u-f_v)$;
for $j:=1$ **step** 1 **until** κ_2 **do** $nmgm\ II(k-1,v_2,d_1,\ldots,d_m)$;
$u:=u-p^k_{k-1}(v_1-v_2)$;
for $j:=1$ **step** 1 **until** i_2 **do** $u:=G_{h_k}u$;
end $nmgm\ II$.

Convergence results for multi-grid methods involving finite difference approximations for elliptic boundary-value problems have been given by Hackbusch in [6]. The main requirements are an approximation and a smoothing property, the former describing the approximability of the difference equation on level k by the correction process involving the defect equation on level $k-1$ and the latter reflecting the effect of the relaxation process. Both properties can be stated in terms of suitably chosen discrete norms: Denoting by $|\cdot|_s$ the discrete analogues of the Sobolew norms and by $|\cdot|_{s,r}$ the corresponding operator norms as in [6], we require that for all $1\leqq v\leqq m$ and $k\geqq 1$

$$r_k^{k-1}A^v_{h_k}p^k_{k-1}=A^v_{h_{k-1}}+\delta^v_{h_{k-1}} \tag{3.18a}$$

with

$$|\delta^v_{h_{k-1}}|_{2,-2}<Ch^2_{k-1}. \tag{3.18b}$$

If we choose p^k_{k-1} based on interpolation of order 2 as in [6] and r_k^{k-1} as the associated full weighted restriction, (3.18a), (3.18b) is easy to fulfill in view of the assumed smoothness of the operator coefficients. Moreover, with that choice of prolongation and restriction we have

$$C^{-1}|u_{h_{k-1}}|_0\leqq|p^k_{k-1}u_{h_{k-1}}|_0\leqq C|u_{h_{k-1}}|_0,\quad u_{h_{k-1}}\in C_0(\Omega_{h_{k-1}}), \tag{3.19a}$$

$$|r_k^{k-1}|_{s,s}\leqq C,\quad s=-2,0;\quad |I_{h_k}-p^k_{k-1}r_k^{k-1}|_{2,0}\leqq Ch^2_{k-1}. \tag{3.19b}$$

Finally, we require stability of the difference operators $C_{h_k}[\cdot]$, $k\geqq 0$ according to

$$|C^{-1}_{h_k}[\cdot]|_{s,s+2}\leqq C,\quad s=-2,0;\quad |C_{h_k}[\cdot]|_{0,-2}\leqq C. \tag{3.20}$$

While by construction of $C_{h_k}[\cdot]$ the third estimate in (3.20) can be easily established, if we require the same for all difference operators $A^v_{h_k}$, $1\leqq v\leqq m$, i.e.

$$|A^v_{h_k}|_{0,-2}\leqq C, \tag{3.21}$$

the first two estimates can be deduced as follows: The operator $C_{h_k}[\cdot]$ can be viewed as a consistent difference approximation of an uniformly elliptic differential operator C with piecewise smooth coefficients for which the corresponding stability results hold true with respect to the continuous-case Sobolew norms (cf. [10]). Then, taking into account the $|\cdot|_{0,0}$-stability of $C_{h_k}[\cdot]$ guaranteed by Theorem 2.1, we may conclude following the ideas in [7].

Since the smoothing iterations are applied to the linearized versions of the discrete HJB-equations, denoting by $S_{h_k}[\cdot]$ the iteration operator of the relaxation process $G_{h_k}[\cdot]$ we have to require that

$$|C_{h_k}[\cdot]S_{h_k}^i[\cdot]|_{0,0} \leqq C(i)h_k^{-2}, \quad 1 \leqq i \leqq i_{\max}(h_k) \tag{3.22}$$

with $C(i) \to 0$ $(i \to \infty)$ and $i_{\max}(h) \to \infty$ $(h \to 0)$.

In [6] the smoothing property is deduced for Gauss-Seidel iteration as relaxation process under the assumptions that the principal part of the difference operator is positive definite and that the difference operator admits a 2-cyclic splitting. While the latter condition will be satisfied for both MGS I and MGS II, the principal part of $C_{h_k}[\cdot]$ is indeed positive definite for MGS I, but for MGS II we cannot expect that unless we are dealing with operators A^ν having a common principal part, i.e. $a_{ij}^\nu = a_{ij}$ for all $1 \leqq \nu \leqq m$. Nevertheless, in many cases we can assume (3.22) for MGS II by the following reasoning: Let us denote by $\Omega_{h_k}^1$ the set of all grid points $x \in \Omega_{h_k}$ such that $C_{h_k}[\cdot]$ corresponds to $A_{h_k}^{\nu_0(x)}$ for all neighbors of x, i.e.

$$\Omega_{h_k}^1 = \{x \in \Omega_{h_k} | \nu_0(y) = \nu_0(x),\ y = x \pm h_k e_\mu \in \Omega_{h_k},\ 1 \leqq \mu \leqq d\},$$

where the e_μ's are the d unit vectors in $\mathbb{R}^d$. Then the set $\Omega_{h_k}^2 = \Omega_{h_k} \setminus \Omega_{h_k}^1$ describes the discrete internal free boundary. For sufficiently small h_k the number $N_{h_k}^{(2)} = \operatorname{card} \Omega_{h_k}^2$ is small compared to $N_{h_k}^{(1)} = N_{h_k} - N_{h_k}^{(2)}$ so that $C_{h_k}[\cdot]$ can be interpreted as a perturbation of a positive definite operator, the perturbation being so small that (3.22) can still be verified. Furthermore, it has to be noticed that the solution u_{h_k} of the discrete HJB-equation on level k is not a fixed point of the iteration process $G_{h_k}[w_{h_k}]$, $w_{h_k} \neq u_{h_k}$. Nevertheless, we may assume

$$u_{h_k} = G_{h_k}^{i_j}[w_{h_k}^j](u_{h_k} + \eta_{h_k}^j), \tag{3.23a}$$

$$|\eta_{h_k}^j|_0 \leqq C|w_{h_k}^j - u_{h_k}|_0, \tag{3.23b}$$

where in MGS I $w_{h_k}^1 = u_{h_k,\nu-1}^{N+\nu-1}$, $w_{h_k}^2 = \tilde{u}_{h_k,\nu}^{N+\nu,\text{new}}$ and in MGS II $w_{h_k}^1 = u_{h_k}^{n-1}$, $w_{h_k}^2 = \tilde{u}_{h_k}^{n,\text{new}}$.

Finally, observing $r_k^{k-1}(A_{h_k}^\nu u_{h_k} - d_{h_k}^\nu) \leqq 0$, $1 \leqq \nu \leqq m$, it turns out that in MGS I $r_k^{k-1}u_{h_k}$ is the solution of the corresponding defect equation on level $k-1$ and thus a fixed point of the iteration but not in MGS II where $r_k^{k-1}u_{h_k}$ only is a subsolution. Therefore, in MGS II we have to choose another restriction operator $\tilde{r}_k^{k-1}$ in order to guarantee that

$$\max_{1 \leqq \nu \leqq m} [r_k^{k-1}(A_{h_k}^\nu u_{h_k} - d_{h_k}^\nu)] = 0. \tag{3.24}$$

Denoting by $\hat{r}_k^{k-1}$ the operator of pointwise restriction to the coarser grid, this can be achieved if we take

$$\tilde{r}_k^{k-1} = \begin{cases} r_k^{k-1}, & \text{if } x \in \Omega_{h_k}^1 \\ \hat{r}_k^{k-1}, & \text{if } x \in \Omega_{h_k}^2 \end{cases}. \tag{3.25}$$

Since the modification of r_k^{k-1} given by (3.25) is confined to the internal free boundary, in case $N_{h_k}^{(2)} \ll N_{h_k}^{(1)}$ assumptions (3.18) and (3.19b) can still be expect-

ed to hold true. After these prerequisites we are now in a position to prove convergence for the multi-grid schemes MGS I and MGS II:

Theorem 3.1. *In addition to the hypotheses of Theorem* 2.1 *let us suppose that the assumptions* (3.1), (3.18)–(3.23) *are satisfied with* $\tilde{r}_k^{k-1}$ *instead of* r_k^{k-1} *in case of* MGS II. *Further, let* $\{u_{h_l,\nu}^{N+\nu}\}$, $N=(n-1)m$, $n\in\mathbb{N}$, $1\leqq\nu\leqq m$, resp. $\{u_{h_l}^n\}$, $n\in\mathbb{N}$, *be the sequences generated by* MGS I resp. MGS II *with respect to a startvector* $u_{h_l,m}^0\in C_0(\Omega_{h_l})$ resp. $u_{h_l}^0\in C_0(\Omega_{h_l})$. *Then, for both schemes we can find* $i_{\min}>1$ *and* $h_{\max}>0$ *such that for* $\kappa_2\geqq 2$ *and* $i_{\min}\leqq i=i_1+i_2\leqq i(h_l)$, $h_l\leqq h_{\max}$, *there holds: For each* $\varepsilon>0$ *there exists* $n_0(\varepsilon)\geqq 1$ *such that for* $n\geqq n_0(\varepsilon)$

$$|u_{h_l,\nu}^{N+\nu}-u_{h_l}|_0<\varepsilon,\qquad 1\leqq\nu\leqq m, \tag{3.26a}$$

$$|u_{h_l}^n-u_{h_l}|_0<\varepsilon. \tag{3.26b}$$

Proof. Since the idea of proof is essentially the same for both schemes, we will only give the proof for MGS II. Moreover, we may restrict ourselves to the case of a two-grid scheme and $i_2=0$, because the proof for $i_2\geqq 1$ only requires minor modifications and the convergence in the multi-grid case can be easily deduced from that of the two-grid method.

Denoting by $u_{h_{l-1}}^n$ the solution of the defect equation on level $l-1$ within the n-th iteration step, we get

$$u_{h_l}^n-u_{h_l}=(I_{h_l}-p_{l-1}^l\tilde{r}_l^{l-1})(\tilde{u}_{h_l}^n-u_{h_l})+p_{l-1}^l u_{h_{l-1}}^n-p_{l-1}^l\tilde{r}_l^{l-1}u_{h_l}. \tag{3.27}$$

Using (3.19b), (3.20), (3.22) and (3.23), for the first term on the right-hand side of (3.27) we find

$$\begin{aligned}|(I_{h_l}-p_{l-1}^l\tilde{r}_l^{l-1})(\tilde{u}_{h_l}^n-u_{h_l})|_0&\leqq|I_{h_l}-p_{l-1}^l\tilde{r}_l^{l-1}|_{2,0}|C_{h_l}^{-1}[u_{h_l}^{n-1}]|_{0,2}\\&\quad\cdot|C_{h_l}[u_{h_l}^{n-1}]S_{h_l}^{i_1}[u_{h_l}^{n-1}]|_{0,0}[|u_{h_l}^{n-1}-u_{h_l}|_0+|\eta_{h_l}^1|_0]\\&\leqq CC(i_1)|u_{h_l}^{n-1}-u_{h_l}|_0.\end{aligned}$$

Concerning the second term, we observe that $\tilde{r}_l^{l-1}u_{h_l}$ is the solution of

$$\max_{1\leqq\nu\leqq m}[A_{h_{l-1}}^\nu u_{h_{l-1}}-A_{h_{l-1}}^\nu\tilde{r}_l^{l-1}u_{h_l}+\tilde{r}_l^{l-1}(A_{h_l}^\nu u_{h_l}-f_{h_l}^\nu)]=0.$$

Hence, taking advantage of (2.3) and (3.18)–(3.23) we get

$$\begin{aligned}|p_{l-1}^l(u_{h_{l-1}}^n-\tilde{r}_l^{l-1}u_{h_l})|_0&\leqq\max(|C_{h_{l-1}}^{-1}[u_{h_{l-1}}^n]|_{-2,0},|C_{h_{l-1}}^{-1}[\tilde{r}_l^{l-1}u_{h_l}]|_{-2,0})\\&\quad\cdot\{\max_{1\leqq\nu\leqq m}|\delta_{h_{l-1}}^\nu|_{2,-2}+|\tilde{r}_l^{l-1}|_{-2,-2}\max_{1\leqq\nu\leqq m}|A_{h_l}^\nu|_{0,-2}|I_{h_l}-p_{l-1}^l\tilde{r}_l^{l-1}|_{2,0}\}\\&\quad\cdot|C_{h_l}^{-1}[u_{h_l}^{n-1}]|_{0,2}|C_{h_l}[u_{h_l}^{n-1}]S_{h_l}^{i_1}[u_{h_l}^{n-1}]|_{0,0}[|u_{h_l}^{n-1}-u_{h_l}|_0+|\eta_{h_l}^1|_0]\\&\leqq CC(i_1)|u_{h_l}^{n-1}-u_{h_l}|_0.\end{aligned}$$

Consequently,

$$|u_{h_l}^n-u_{h_l}|_0\leqq CC(i_1)|u_{h_l}^{n-1}-u_{h_l}|_0,$$

from which the conclusion of the theorem can be drawn immediately.

Recalling that Theorem 2.4 states monotone convergence of the sequences generated by *Scheme I* resp. *Scheme II* it is an interesting question whether that monotonicity can be preserved by the multi-grid schemes MGS I resp. MGS II. Concerning MGS I the answer is affirmative, because the multi-grid process directly applies to equations (2.9a), (2.9b). However, for MGS II this argument cannot be used, since the multi-grid scheme does provide approximations to the full nonlinear HJB-equations and not to its linearized versions. Nevertheless, for MGS II we do have monotonicity as long as $u_{h_l}^{n-1}$ is a supersolution in the sense that

$$\max_{1 \leq \nu \leq m} [A_{h_l}^{\nu} u_{h_l}^{n-1} - f_{h_l}^{\nu}] \geqq 0. \tag{3.28}$$

Provided the chosen relaxation method is a monotonely convergent iterative scheme (as it is Gauss-Seidel iteration applied to difference equations of positive type), it follows from (3.28) that $\tilde{u}_{h_l}^{n} \leqq u_{h_l}^{n-1}$. Then $\tilde{r}_l^{l-1} \tilde{u}_{h_l}^{n}$ is a supersolution of the defect correction equation whence $\tilde{u}_{h_l}^{n,\text{new}} \leqq \tilde{u}_{h_l}^{n}$. If (3.28) also holds true for $\tilde{u}_{h_l}^{n,\text{new}}$, by the same argument as before we achieve at $u_{h_l}^{n} \leqq \tilde{u}_{h_l}^{n,\text{new}}$ and thus $u_{h_l}^{n} \leqq u_{h_l}^{n-1}$.

4. Numerical Results

As a first example we have considered the case of two uniformly elliptic operators A^1, A^2 with constant coefficients

$$A_1 = -\frac{\partial^2}{\partial x^2} - 0.5 \frac{\partial^2}{\partial x \partial y} - \frac{\partial^2}{\partial y^2}, \qquad A^2 = -0.5 \frac{\partial^2}{\partial x^2} - 0.1 \frac{\partial^2}{\partial x \partial y} - \frac{\partial^2}{\partial y^2} \tag{4.1}$$

and functions f^1, f^2 given by

$$f^1 = f^2 = \max(A^1 u_1, A^2 u_1), \qquad u_1 = x(1-x)y(1-y) \tag{4.2}$$

so that u_1 turns out to be the exact solution of the corresponding HJB-equation (1.1a), (1.1b) with respect to $\Omega = \{(x, y) | 0 < x, y < 1\}$. This simple example has been treated by Lions and Mercier in [11] using the first and the second iterative scheme of Sect. 2 based upon a discretization of the second order derivatives by

$$\begin{aligned} &\frac{\partial^2}{\partial x^2} \approx h^{-2} D_{h,x}^{+} D_{h,x}^{-}, \qquad \frac{\partial^2}{\partial y^2} \approx h^{-2} D_{h,y}^{+} D_{h,y}^{-} \\ &\frac{\partial^2}{\partial x \partial y} \approx \frac{1}{2} h^{-2} [D_{h,x}^{+} D_{h,y}^{+} + D_{h,x}^{-} D_{h,y}^{-}] \end{aligned} \tag{4.3}$$

where $D_{h,x}^{\pm}$, $D_{h,y}^{\pm}$ denote the forward resp. backward difference in x resp. y. Since $a_{12}^{\nu} \leqq \frac{1}{2} \min(a_{11}^{\nu}, a_{22}^{\nu})$, $\nu = 1, 2$, the discretizations (4.3) result in a difference scheme of positive type. The linear systems obtained in *Step 2* of both schemes have been solved in [11] by the conjugate gradient method in case of *Scheme I* and by Gaussian elimination in case of *Scheme II*. The termination criterion for the iteration was $\varepsilon = 10^{-5}$.

For the implementation of the multi-grid schemes MGS I and MGS II we have taken the same discretizations (4.3) with respect to the grid size sequences

$$\begin{aligned} h_0 &= \tfrac{1}{2}, \quad h_1 = \tfrac{1}{4}, \quad h_2 = \tfrac{1}{8} \\ h_0 &= \tfrac{1}{4}, \quad h_1 = \tfrac{1}{8}, \quad h_2 = \tfrac{1}{16} \end{aligned} \tag{4.4}$$

and we have chosen $\hat{\nu}=1$, $\mu_1=\mu_2=2$ in (3.2) and $\kappa_1=\kappa_2=2$ in (3.11) resp. (3.16). As smoothing procedure we have used Gauss-Seidel relaxation with relaxation parameter $\omega=1$. The number of relaxation steps has been chosen to be $i_1=i_2=2$ in *mgm*, *nmgm* I and *nmgm* II (cf. (3.3), (3.12) resp. (3.17)), and the defect correction equation on the coarsest grid has been approximated by $i_3=5$ Gauss-Seidel iterations applied to the linearized equation obtained by means of *Step 2* of *Scheme II*. Finally, the coarse-to-fine prolongation has been implemented on the basis of bilinear interpolation and the fine-to-coarse transfer as the corresponding weighted restriction. The startvector on the coarsest grid has been taken as the restricted exact solution and the accuracy bound was again $\varepsilon = 10^{-5}$.

Table 1a contains the results obtained by MGS I, MGS II (with all the parameters as specified above) at the center of the grid-point sets, i.e. $x=y=0.5$ where the exact solution is $u_I=0.0625$. Table 1b shows the corresponding results for a two-grid scheme ($h_0=\frac{1}{5}$, $h_1=\frac{1}{10}$; all other parameters as before) as well as the comparable results obtained in [11].

We remark that for *Scheme II* resp. MGS II the total number of iterations is much smaller than for *Scheme I* resp. MGS I. This is mainly due to the fact that for *Scheme I* resp. MGS I in each iteration cycle (corresponding to *Step 2* of *Scheme I*) the obstacle is active only once so that the iterates are the same.

Comparing the total number of iterations for MGS I and *Scheme I*, it turns out that MGS I basically inherits the convergence behaviour of *Scheme I* which is not surprising, since in MGS I the multi-grid technique applies to the equations which have to be solved within the iteration process of *Scheme I*. Concerning MGS II and *Scheme II*, MGS II seems to be superior, since the correction process involves the full nonlinear HJB-equations while in *Scheme II* linearized HJB-equations are solved within each iteration step. Indeed, the total number of iterations for MGS II is essentially the same for both the three-grid schemes with $h_{\min}=\frac{1}{8}$ resp. $h_{\min}=\frac{1}{16}$ and the two-grid scheme with $h_{\min}=\frac{1}{10}$ thus reflecting a characteristic feature of multi-grid methods.

In the second example we have taken two second order uniformly elliptic operators with variable coefficients and with lower order terms

$$\begin{aligned} A_1 &= -(x+6)^2 \frac{\partial^2}{\partial x^2} - (x+6)(y+2) \frac{\partial^2}{\partial x \partial y} - (y+2)^2 \frac{\partial^2}{\partial y^2} \\ &\quad + [0.5(x+6)-4] \frac{\partial}{\partial x} + 0.5(y+2) \frac{\partial}{\partial y} + 1, \\ A^2 &= -(x+6)^2 \frac{\partial^2}{\partial x^2} - 0.8(x+6)(y+2) \frac{\partial^2}{\partial x \partial y} - 0.75(y+2)^2 \frac{\partial^2}{\partial y^2} \\ &\quad + [(x+6)-2] \frac{\partial}{\partial x} + (y+2) \frac{\partial}{\partial y} + 4. \end{aligned} \tag{4.5}$$

Table 1a

n	MGS I		MGS II	
	$h_{\min}=\frac{1}{8}$	$h_{\min}=\frac{1}{16}$	$h_{\min}=\frac{1}{8}$	$h_{\min}=\frac{1}{16}$
1	0.065551	0.064771	0.064798	0.064120
2	0.065551	0.064771	0.064796	0.064118
3	0.064742	0.063981		
4	0.064742	0.063981		
5	0.064553	0.063803		
6	0.064553	0.063803		
7	0.064478	0.063745		
8	0.064478	0.063745		
9	0.064468	0.063718		
10	0.064468	0.063718		
11	0.064461	0.063705		
12	0.064461	0.063705		
13		0.063698		
14		0.063698		

Table 1b

n	$h_{\min}=\frac{1}{10}$		Lions-M. ($h=\frac{1}{10}$)	
	MGS I	MGS II	Scheme I	Scheme II
1	0.065179	0.064173	0.065222	0.083143
2	0.065179	0.064153	0.065222	0.064099
3	0.064345	0.064153	0.064384	0.064099
4	0.064345		0.064384	
5	0.064199		0.064248	
6	0.064199		0.064248	
7	0.064156		0.064171	
8	0.064156		0.064171	
9	0.064140		0.064171	
10	0.064140		0.064171	
11	0.064132			
12	0.064132			
13	0.064128			
14	0.064128			

Again, the functions f^1, f^2 have been chosen according to

$$f^1 = f^2 = \max(A^1 u_{\mathrm{II}}, A^2 u_{\mathrm{II}}), \qquad u_{\mathrm{II}} = \sin \pi x \sin \pi y \tag{4.6}$$

which means that u_{II} is the exact solution of the HJB-equation (1.1 a), (1.1 b) on $\Omega = \{(x, y) | 0 < x, y < 1\}$.

The discretization has been performed with respect to the same grid size sequences as in the first example (cf. (4.4)), and the difference operators $A^{\nu}_{h_k}$, $0 \leqq k \leqq 2$, $\nu = 1, 2$, have been constructed by the Bramble-Hubbard technique as outlined in [2; Chap. III] with the only difference that we have used one-sided difference approximations for the first order derivatives to avoid any step size restriction. The prolongation and restriction operators as well as the smoothing

Table 2

n	MGS I				MGS II			
	$h_{\min}=\frac{1}{8}$		$h_{\min}=\frac{1}{16}$		$h_{\min}=\frac{1}{8}$		$h_{\min}=\frac{1}{16}$	
	$\kappa_1=2$	$\kappa_1=5$	$\kappa_1=2$	$\kappa_1=5$	$\kappa_1=2$	$\kappa_1=5$	$\kappa_1=2$	$\kappa_1=5$
1	0.74898	0.74898	0.73068	0.73068	0.72157	0.72011	0.71391	0.71319
2	0.74834	0.74824	0.72621	0.72610	0.72017	0.72009	0.71329	0.71305
3	0.72808	0.72750	0.71765	0.71744	0.72010	0.72009	0.71313	0.71304
4	0.72808	0.72750	0.71765	0.71744	0.72009		0.71306	0.71304
5	0.72364	0.72327	0.71541	0.71519	0.72009		0.71305	
6	0.72364	0.72327	0.71541	0.71519			0.71304	
7	0.72166	0.72143	0.71432	0.71419				
8	0.72166	0.72143	0.71432	0.71419				
9	0.72078	0.72069	0.71385	0.71372				
10	0.72078	0.72069	0.71385	0.71372				
11	0.72039	0.72036	0.71356	0.71348				
12	0.72039	0.72036	0.71356	0.71348				
13	0.72023	0.72021	0.71339	0.71333				
14	0.72023	0.72021	0.71339	0.71333				
15	0.72015	0.72015	0.71328	0.71322				
16	0.72015	0.72015	0.71328	0.71322				
17	0.72012	0.72012	0.71320	0.71316				
18	0.72012	0.72012	0.71320	0.71316				
19	0.72010	0.72010	0.71314	0.71311				
20	0.72010	0.72010	0.71314	0.71311				
21	0.72010	0.72010	0.71311	0.71308				
22	0.72010	0.72010	0.71311	0.71308				
23			0.71308	0.71306				
24			0.71308	0.71306				
25			0.71307	0.71305				
26			0.71307	0.71305				
27			0.71306	0.71304				
28			0.71306	0.71304				

procedure, the iteration for solving the defect equation on the coarsest grid and the termination criterion for the multi-grid iteration have been chosen as in the first example. Moreover, the nested iteration for the determination of a first iterate on the finest grid has been implemented in the same way, i.e. with $\mu_1=\mu_2=2$ and $i_1=i_2=2$ while the startvector on the coarsest grid has been taken as the slightly perturbed pointwise restricted exact solution $\tilde{u}_{II}=u_{II}+0.1$.

Table 2 above shows the values of the iterates at the grid point $x=0.5$, $y=0.25$ where the exact solution is given by $u_{II}=0.70711$. We remark that both MGS I and MGS II exhibit the same behaviour as in Example I. In particular, we have tested the effect of increasing the number κ_1 of multi-grid iterations *nmgm I* resp. *nmgm* II by choosing $\kappa_1=2$ and $\kappa_1=5$ while keeping the *W*-cycle structure of the multi-grid schemes, i.e. $\kappa_2=2$ in each case. Since in MGS I an increase in κ_1 only improves the approximate solution of the variational inequalities occurring within each iteration step, the number of MGS I iterations basically remains unchanged. In MGS II however, an increase in κ_1 directly improves the approximation of the full nonlinear HJB-equation so that the number of MGS II iterations can be diminished.

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