Constructive aspects in time optimal control

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Constructive Aspects in Time Optimal Control

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Abstract. Approximations of time optimal control problems are considered in the framework of discrete convergence in discrete approximations. The control systems are formulated in an abstract Banach space setting including both the case of distributed and boundary control. Controllability of the given and the approximating systems is studied in terms of the corresponding input maps and general convergence results are established for the reachable sets, optimal controls and minimum times.

1. INTRODUCTION

Given an initial state u^o in a reflexive, separable Banach space E, we consider a control system (C) evolving according to

$$(1.1) \qquad u(t) = S(t)u^o + L_t f, \quad t \geq 0$$

where $S(t) : E \to E, t \geq 0$, is a C_o -semigroup of type (M,ω) with infinitesimal generator $A : D(A) \subset E \to E$, the operator L_t is a bounded linear map from $L^\infty((0,t);V)$ in E, V being another reflexive, separable Banach space, and the input f is taken from the class of admissible controls

$$(1.2) \qquad F_t = \{f \in L^\infty([0,t];V) \mid \|f(\tau)\|_V \leq 1 \text{ a.e. in } [0,t]\}.$$

A state $u^1 \in$ E is said to be approximately controllable if there exist

$t^\circ > 0$ and an admissible control $f \in F_{t^\circ}$ transferring the system from u° to $B(u^1,\varepsilon) = \{u \in E \mid \| u-u^1 \|_E \le \varepsilon\}$, $\varepsilon > 0$, in time t°, i.e.

$$(1.3) \qquad u(0) = u^\circ, \quad u(t^\circ) \in B(u^1,\varepsilon)$$

where $u(t)$, $t \in [0,t^\circ]$, is the corresponding admissible trajectory obtained from (1.1). The smallest t° for which (1.3) holds true is called the transition time of the admissible control f and the infimum t^* of the transition times of all admissible controls is called the minimum time with respect to u°, $B(u^1,\varepsilon)$ and F. Finally, if there exists an $f^* \in F_{t^*}$ such that (1.3) is satisfied with transition time t^*, then f^* will be denoted as optimal control.

The abstract control system (C) can serve as a model for both distributed and boundary control. In fact, if $V = E$ and L_t is given by

$$(1.4) \qquad L_t^d f = \int_0^t S(t-\tau) f(\tau)\, d\tau,$$

then $u(t)$, $t \ge 0$, represents the mild solution of the evolution equation

$$(1.5) \qquad \frac{d}{dt}u(t) = Au(t) + f(t), \quad t \ge 0,$$

and we may interpret (C) as a distributed control problem. On the other hand, if L_t is given by

$$(1.6) \qquad L_t^b f = - \int_0^t AS(t-\tau) Df(\tau)\, d\tau,$$

where $S(t)$, $t \ge 0$, is additionally supposed to be analytic and D is a bounded linear map from V in E such that

$$(1.7) \qquad \| AS(t)D \| = O(t^{\theta - 1})$$

for some $0 < \theta < 1$, then (C) may be viewed as the Banach space formulation of a boundary control problem, the operator D denoting for example the Dirichlet map (cf. [19]).

Remark. Note that in view of [19;Thm.3] condition (1.7) ensures that the input map L_t^b given by (1.6) is indeed a bounded linear operator from $L^\infty([0,t];V)$ in E. In both cases it is well known (cf. e.g. [1], [5], [10]) that if u^1 is approximately controllable, then there exists an optimal control f^* which, under some additional assumptions, is uniquely determined and satisfies the bang-bang principle.
In studying the above control problems a decisive role will be played by the adjoint operators L_t^* which can be interpreted as observability maps for the corresponding dual observed systems (see e.g. [4]). The maps L_t^* can be shown to be bounded linear operators from E^* in $L^1([0,t];V^*) \subset (L^\infty([0,t];V)^*$ given by

(1.8) $$(L_t^d)^* = S^*(t - \cdot)$$

for distributed control and by

(1.9) $$(L_t^b)^* = D^*S^*(t - \cdot)A^*$$

in case of boundary control.
For notational convenience the spaces $L^\infty([0,t];V)$ respectively $L^1([0,t];V^*)$ will henceforth be denoted by W^∞ respectively W^1.

The approximate solution of time optimal control problems both in case of distributed and boundary control has been studied by various authors (cf. e.g. [3],[8],[11],[12],[13],[14],[15]). In the sequel, following the approach in [8], we will develop a unified theory based on the concept of discrete convergence in discrete approximations. For this purpose, let us assume that $(E_n)_{\mathbb{N}}$ and $(V_n)_{\mathbb{N}}$ are sequences of reflexive Banach spaces approximating E and V in a sense which will be made precise in the next section. Furthermore, let $(S_n(t))_{\mathbb{N}}$ be a sequence of C_o-semigroups $S_n(t) : E_n \to E_n$, $t \geq 0$, $n \in \mathbb{N}$, of type (M_n, ω_n) with infinitesimal generators $A_n : D(A_n) \subseteq E_n \to E_n$ and let $(L_{t,n})_{\mathbb{N}}$ be a sequense of input maps $L_{t,n} : W_n^\infty \to E_n$, $n \in \mathbb{N}$. Given initial states u_n^o and final states u_n^1

both in E_n, $n\in\mathbb{N}$, we consider control systems (C_n) given by

$$(1.10) \qquad u_n(t) = S_n(t)u_n^\circ + L_{t,n}f_n, \quad t \geq 0,$$

and we are looking for admissible controls $f_n \in F_{t,n}$ within the class

$$(1.11) \qquad F_{t,n} = \{f_n \in W_n^\infty = L^\infty([0,t];V_n) \mid \| f_n(\tau) \| \leq 1 \text{ a.e. in } [0,t]\}$$

steering the system from u_n° to $B_n(u_n^1,\varepsilon)$ in some finite time t_n°, i.e.

$$(1.12) \qquad u_n(0) = u_n^\circ, \qquad u_n(t_n^\circ) \in B_n(u_n^1,\varepsilon).$$

For distributed control the input maps $L_{t,n}$ are specified by

$$(1.13) \qquad L_{t,n}^d f_n = \int_0^t S_n(t-\tau)f_n(\tau)d\tau,$$

while for boundary control

$$(1.14) \qquad L_{t,n}^b f_n = -\int_0^t A_n S_n(t-\tau)D_n f_n(\tau)d\tau$$

assuming $S_n(t)$, $t \geq 0$, $n\in\mathbb{N}$, analytic and $D_n : V_n \to E_n$, $n\in\mathbb{N}$, bounded with

$$(1.15) \qquad \| A_n S_n(t) D_n \| = O(t^{\theta_n - 1}), \quad 0 < \theta_n < 1.$$

2. DISCRETE CONVERGENCE IN DISCRETE APPROXIMATIONS

We will shortly review some highlights in the theory of discrete convergence in discrete approximations which will serve as a basic tool in the subsequent sections. For details we refer to [6],[7],[16] and [17]. Given real Banach spaces E, E_n, $n\in\mathbb{N}$, and a sequence $R = (R_n)_{\mathbb{N}}$ of restriction operators $R_n : E \to E_n$, $n\in\mathbb{N}$, the triple $(E, \Pi E_n, R)$ is called a discrete approximation with convergent norms (cn-approximation) iff

(i) $$\| R_n(\alpha u + \beta v) - \alpha R_n u - \beta R_n v \|_{E_n} \to 0 \quad (n\in \mathbb{N}),$$
$$\alpha,\beta\in \mathbb{R}; \quad u,v\in E,$$

(ii) $$\| R_n u \|_{E_n} \to \| u\|_E \quad (n\in \mathbb{N}), \quad u\in E,$$

(iii) $$\sup_{n\in \mathbb{N}} \| R_n u \|_{E_n} < \infty, \quad u\in E.$$

A sequence $(u_n)_{\mathbb{N}'}$ of elements $u_n \in E_n$, $n\in \mathbb{N}' \subset \mathbb{N}$, is said to converge discretely strongly to an element $u\in E$ ($\text{s-lim}_E\, u_n = u\ (n\in \mathbb{N}')$) iff $\| u_n - R_n u \|_{E_n} \to 0\ (n\in \mathbb{N}')$.

Obviously, if $\text{s-lim}_E\, u_n = u\ (n\in \mathbb{N}')$, we also have norm-convergence, i.e. $\| u_n\|_{E_n} \to \| u\|_E\ (n\in \mathbb{N}')$, which justifies to call $(E,\Pi E_n,R)$ a cn-approximation.

A dual concept is the discrete weak convergence of functionals: A sequence $(u_n^*)_{\mathbb{N}'}$ of elements $u_n^*\in E_n^*$, $n\in \mathbb{N}' \subset \mathbb{N}$, converges discretely weakly to an element $u^*\in E^*$ ($\text{w-lim}_{E^*} u_n = u\ (n\in \mathbb{N}')$) iff for each $u \in E$ and any sequence $(u_n)_{\mathbb{N}'}$, $u_n\in E_n$, $n\in \mathbb{N}'$, we have

$$\text{s-lim}_E\, u_n = u\ (n\in \mathbb{N}') \Longleftrightarrow \langle u_n^*,u_n \rangle_{E_n^*,E_n} \to \langle u^*,u \rangle_{E^*,E} \quad (n\in \mathbb{N}'),$$

where $\langle \cdot,\cdot \rangle_{E_n^*,E_n}$ respectively $\langle .,. \rangle_{E^*,E}$ refers to the dual pairing between E_n and E_n^* resp. E and E^*.

It is easy to see that if $\text{w-lim}_{E^*} u_n^* = u^*\ (n\in \mathbb{N}')$ then $\| u^*\|_{E^*} \le \liminf \| u_n^*\|_{E_n^*}$.

If $(E^*,\Pi E_n^*,Q)$ is a cn-approximation of the dual space E^*, and the Banach spaces E,E_n, $n\in \mathbb{N}$, are reflexive ones, we may likewise define a discrete weak convergence of sequences of elements in E_n (cf. [9]) : A sequence $(u_n)_{\mathbb{N}'}$, $u_n \in E_n$, $n \in \mathbb{N}' \subset \mathbb{N}$, converges discretely weakly to $u \in E$ ($w\text{-}\lim_E u_n = u \; (n \in \mathbb{N}')$) iff for each $u^* \in E^*$ and any sequence $(u_n^*)_{\mathbb{N}'}$, $u_n^* \in E_n^*$, $n \in \mathbb{N}'$, there holds

$$s\text{-}\lim_{E^*} u_n^* = u^* \; (n \in \mathbb{N}') \Rightarrow \langle u_n^*, u_n \rangle_{E_n^*, E_n} \to \langle u^*, u \rangle_{E^*, E} \; (n \in \mathbb{N}')$$

If E is separable, we have the following equivalent characterizations of discrete strong resp. discrete weak convergence (cf. [9], [16]):

Lemma 2.1. Let $(E, \Pi E_n, R)$ and $(E^*, \Pi E_n^*, Q)$ be cn-approximations where E, E_n, $n \in \mathbb{N}$, are reflexive Banach spaces and E is separable. Then for each $u \in E$ and any sequence $(u_n)_{\mathbb{N}'}$, $u_n \in E_n$, $n \in \mathbb{N}' \subset \mathbb{N}$, we have $s\text{-}\lim_E u_n = u \; (n \in \mathbb{N}')$ [resp. $w\text{-}\lim_E u_n = u \; (n \in \mathbb{N}')$] iff each $u^* \in E^*$ and any sequence $(u_n^*)_{\mathbb{N}'}$, $u_n^* \in E_n^*$, $n \in \mathbb{N}'$, such that $w\text{-}\lim_{E^*} u_n^* = u^* \; (n \in \mathbb{N}')$ [resp. $s\text{-}\lim_{E^*} u_n^* = u^* \; (n \in \mathbb{N}')$] there holds $\langle u_n^*, u_n \rangle_{E_n^*, E_n} \to \langle u^*, u \rangle_{E^*, E} (n \in \mathbb{N}')$.

Moreover, we have the following discrete weak compactness of bounded sequences in E_n resp. E_n^* (cf. [6]):

Lemma 2.2. Under the same hypotheses as in Lemma 2.1, for any bounded sequence $(u_n)_{\mathbb{N}'}$, $u_n \in E_n$, $n \in \mathbb{N}' \subset \mathbb{N}$ [respectively $(u_n^*)_{\mathbb{N}'}$, $u_n \in E_n^*$, $n \in \mathbb{N}' \subset \mathbb{N}$] there exist a subsequence $\mathbb{N}'' \subset \mathbb{N}'$ and an element $u \in E$ [resp. $u^* \in E^*$] such that $w\text{-}\lim_E u_n = u \; (n \in \mathbb{N}'')$ and $\|u_n\|_{E_n} \to \|u\|_E \; (n \in \mathbb{N}'')$ [resp. $w\text{-}\lim_{E^*} u_n^* = u^* (n \in \mathbb{N}'')$ and $\|u_n^*\|_{E_n^*} \to \|u^*\|_{E^*} \; (n \in \mathbb{N}'')$].

We also need the notions of strong resp. weak limits of subsets, $\Omega_n \subset E_n$, $n \in \mathbb{N}' \subset \mathbb{N}$. We define

$$s\text{-}\mathrm{Lim\,sup}_E \Omega_n = \{ u \in E \mid \exists \, (u_n)_{\mathbb{N}''}, u_n \in \Omega_n, n \in \mathbb{N}'' \subset \mathbb{N}' : s\text{-}\lim_E u_n = u \; (n \in \mathbb{N}'') \} ,$$

$$s\text{-}\mathrm{Lim\,inf}_E \Omega_n = \{ u \in E \mid \exists \, (u_n)_{\mathbb{N}'}, u_n \in \Omega_n, n \in \mathbb{N}' : s\text{-}\lim_E u_n = u \; (n \in \mathbb{N}') \} .$$

Similarly, we introduce the sets w-Lim $\sup_E \Omega_n$ and w-Lim $\inf_E \Omega_n$ replacing discrete strong by discrete weak convergence in the above definitions.

Clearly, we have the following inclusions

(i) w-Lim $\inf_E \Omega_n \subseteq$ w-Lim $\sup_E \Omega_n$, s-Lim $\inf_E \Omega_n \subseteq$ s-Lim $\sup_E \Omega_n$,

(ii) s-Lim $\inf_E \Omega_n \subseteq$ w-Lim $\inf_E \Omega_n$, s-Lim $\sup_E \Omega_n \subseteq$ w-Lim $\sup_E \Omega_n$.

If in (i) equality holds, we simply write w-$\mathrm{Lim}_E\, \Omega_n$ respectively s-$\mathrm{Lim}_E\, \Omega_n$.

Moreover, if w-Lim $\sup_E \Omega_n \subseteq$ s-Lim $\inf_E \Omega_n$, all the limit sets coincide and will be denoted by $\mathrm{Lim}_E\, \Omega_n$.

We denote by $\mathcal{B}(E,F)$ the Banach algebra of bounded linear operators $B : E \to F$ and by $C(E,F)$ the class of densely defined linear operators A with domain $D(A)$ in E range $R(A)$ in F. Let us assume that $(E, \Pi E_n, R^E)$, $(E^*, \Pi E_n^*, Q^{E^*})$ and $(F, \Pi F_n, R^F)$, $(F^*, \Pi F_n^*, Q^{F^*})$ are cn-approximations satisfying the assumptions of Lemma 2.1. A sequence $(A_n)_{\mathbb{N}}$ of operators $A_n \in C(E_n, F_n)$, $n \in \mathbb{N}$, is said to converge discretely strongly [discretely weakly] to an operator $A \in C(E,F)$ $(A_n \to A\ (n \in \mathbb{N}))$[respectively $A_n \rightharpoonup A\ (n \in \mathbb{N})$] iff for each $u \in D(A)$ there is a sequence $(u_n)_{\mathbb{N}}$, $u_n \in D(A_n)$, $n \in \mathbb{N}$, such that s-$\lim_E u_n = u\ (n \in \mathbb{N})$ and s-$\lim_F A_n u_n = Au\ (n \in \mathbb{N})$ [resp. w-$\lim_E u_n = u\ (n \in \mathbb{N})$ and w-$\lim_F A_n u_n = Au\ (n \in \mathbb{N})$].

The sequence $(A_n)_{\mathbb{N}}$ is consistent with A iff for each $u \in D(A)$ there exists a sequence $(u_n)_{\mathbb{N}}$, $u_n \in D(A_n)$, $n \in \mathbb{N}$, such that s-$\lim_E u_n = u (n \in \mathbb{N})$ and s-$\lim_F A_n u_n = Au\ (n \in \mathbb{N})$. If $B_n \in \mathcal{B}(E_n, F_n)$, $n \in \mathbb{N}$, the sequence $(B_n)_{\mathbb{N}}$ is called stable iff $\limsup_{n \in \mathbb{N}} \| B_n \| < \infty$ and inversely stable iff there exist a positive constant γ and a final piece $\mathbb{N}_1 \subset \mathbb{N}$ such that $\| B_n u_n \|_{F_n} \geq \gamma \| u_n \|_{E_n}$, $u_n \in E_n$, $n \in \mathbb{N}_1$.

Lemma 2.3. (*cf.* [16]). Let $B_n \in \mathcal{B}(E_n, F_n)$, $n \in \mathbb{N}$, *and* $B \in \mathcal{B}(E,F)$. *Then there holds:*

(i) $B_n \to B$ $(n \in \mathbb{N})$ *if and only if the sequence* $(B_n)_{\mathbb{N}}$ *is stable and consistent with* B.

(ii)
$$B_n \to B \ (n \in \mathbb{N}) \iff B_n^* \to B^* \quad (n \in \mathbb{N})$$
$$B_n \to B \ (n \in \mathbb{N}) \iff B_n^* \to B^* \quad (n \in \mathbb{N}).$$

(iii) If $(B_n)_{\mathbb{N}}$ *is inversely stable and consistent with* B, *then* B *is injective*.

(iv) *The sequence* $(B_n)_{\mathbb{N}}$ *is inversely stable, consistent with* B *and* s-$\lim\sup_F R(B_n) \subset R(B)$ *if and only if* s-$\lim_F R(B_n) = R(B)$ *and* s-$\lim_F B_n u_n = Bu$ $(n \in \mathbb{N}) \Rightarrow$ s-$\lim_E u_n = u$ $(n \in \mathbb{N})$.

Another important concept is that of a-regularity (cf.[6],[7]) : A pair $A, (A_n)_{\mathbb{N}}$ of operators $A \in C(E,F)$, $A_n \in C(E_n, F_n)$, $n \in \mathbb{N}$, is said to be a-regular iff for any bounded sequence $(u_n)_{\mathbb{N}'}$, $u_n \in D(A_n)$, $n \in \mathbb{N}' \subset \mathbb{N}$, such that s-$\lim_F A_n u_n = w$ $(n \in \mathbb{N}')$ for some $w \in F$ there exist a subsequence $\mathbb{N}'' \subset \mathbb{N}'$ and an $u \in D(A)$ such that s-$\lim_E u_n = u$ $(n \in \mathbb{N}'')$ and $w = Au$.

Obviously, a-regular operators satisfy s-$\lim\sup_F R(A_n) \subseteq R(A)$.

Moreover, we have (cf. [6],[7]):

Lemma 2.4. *Suppose* $B \in \mathcal{B}(E,F)$ *and* $B_n \in \mathcal{B}(E_n, F_n)$, $n \in \mathbb{N}$, *to be a-regular. Then there holds:*

(i) *If* B *is injective, then* $(B_n)_{\mathbb{N}}$ *is inversely stable.*

(ii) *Suppose* $(B_n)_{\mathbb{N}}$ *to be stable and let* $A \in C(E,F)$, $A_n \in C(E_n, F_n)$, $n \in \mathbb{N}$. *Then, if the pair* $A, (A_n)_{\mathbb{N}}$ *is a-regular, so is the pair* $BA, (B_n A_n)_{\mathbb{N}}$.

(iii) *If* B *is injective and the pair* $B, (B_n)_{\mathbb{N}}$ *is a-regular and consistent, then there exists a constant* $\gamma > 0$ *such that* $\|Bu\|_F \geq \gamma \|u\|_E$, $u \in E$.

We close this section with the notion of discrete compactness of operators : A sequence $(B_n)_{\mathbb{N}}$ of operators $B_n \in \mathcal{B}(E_n,F_n)$, $n\in\mathbb{N}$ is called discretely compact iff given a bounded sequence $(u_n)_{\mathbb{N}}$, $u_n \in E_n$, $n\in\mathbb{N}$; for any subsequence $\mathbb{N}'\subset\mathbb{N}$ the sequence $(B_n u_n)_{\mathbb{N}'}$ contains a discretely strongly convergent subsequence.

3. CONVERGENCE OF INPUT MAPS AND REACHABLE SETS

Throughout the following we will assume that $(E,\Pi E_n,R^E)$,$(E^*, \Pi E_n^*, Q^{E^*})$, $(V,\Pi V_n,R^V)$ and $(V^*,\Pi V_n^*,Q^{V^*})$ are cn-approximations of reflexive, separable Banach spaces E,V respectively their duals by sequences of reflexive Banach spaces E_n,V_n $n\in\mathbb{N}$, respectively their duals. Then, we canonically get cn-approximations $(W^\infty,\Pi W_n^\infty,R^{W^\infty})$ $(W^1,\Pi W_n^1,Q^{W^1})$ by setting $(R_n^{W^\infty} f(\tau)=R_n^V f(\tau)$, $(Q_n^{W^1} f^*)(\tau) = Q_n^{V^*} f^*(\tau)$, $\tau\in[0,t]$, $n\in\mathbb{N}$.

We will begin with some basic controllability results. The control systems (C) abd (C_n) are exactly controllable iff the input maps L_t and $L_{t,n}$, $n\in\mathbb{N}$, $t > 0$, are surjective, i.e. $R(L_t) = E$ and $R(L_{t,n}) = E_n$, $n\in\mathbb{N}$, and approximately controllable iff $\mathrm{cl}\, R(L_t) = E$ and $\mathrm{cl}\, R(L_{t,n}) = E_n$, $n\in\mathbb{N}$. A necessary and sufficient condition for exact controllability is the existence of positive constants $\gamma(t)$ and $\gamma_n(t)$, $n\in\mathbb{N}$, such that

$$\text{(3.1a)} \qquad \| L_t^* u^* \|_{W^1} \geq \gamma(t) \| u^* \|_{E^*}, \quad u^* \in E^*$$

$$\text{(3.1b)} \qquad \| L_{t,n}^* u_n^* \|_{W_n^1} \geq \gamma_n(t) \| u_n^* \|_{E_n^*}, \quad u_n^* \in E_n^*,$$

while approximate controllability holds iff $N(L_t^*) = \{0\}$ and $N(L_{t,n}^*) = \{0\}$, $N(L_t^*)$ and $N(L_{t,n}^*)$ denoting the null spaces of L_t^* and $L_{t,n}^*$ respectively.

Clearly, (3.1a) resp. (3.1b) holds true if and only if $B_E(0,\gamma(t)) \subseteq L_t B_{W^\infty}(0,1)$ resp. $B_{E_n}(0,\gamma_n(t)) \subseteq L_{t,n} B_{W_n^\infty}(0,1)$. Due to this fact, the control systems (C_n) are said to be asymptotically uniformly exactly controllable if there exist $\gamma_o(t) > 0$ and a final piece $\mathbb{N}_1 \subset \mathbb{N}$ such that (3.1b) is satisfied for all $n \in \mathbb{N}_1$ with $\gamma_n(t)$ replaced by $\gamma_o(t)$. Consequently, we have the following obvious criterion for asymptotic uniform exact controllability:

Lemma 3.1. The control systems (C_n) are asympotically uniformly exactly controllable if and only if the sequence $(L^*_{t,n})_{\mathbb{N}}$ is inversely stable.

Moreover, in view of Lemma 2.3(iii) and Lemma 2.4(i),(iii) we get the following relationship between approximate controllability of (C) and asymptotic uniform exact controllability of (C_n):

Theorem 3.2.

(i) If (C) is approximately controllable and the pair $L^*_t, (L^*_{t,n})_{\mathbb{N}}$ is a-regular, then (C_n) is asymptotically uniformly exactly controllable.

(ii) Conversely, if for (C_n) asymptotic uniform exact controllability holds true and $(L^*_{t,n})$ is consistent with L^*_t, then (C) is approximately controllable. If, additionally, the pair $L^*_t, (L^*_{t,n})$ is a-regular, then (C) is exactly controllable.

In finite dimensional spaces the notions of exact and approximate controllability coincide while in the infinite dimensional case it is well known that many control systems are not exactly but only approximately controllable (cf.[18]). So, with regard to applications, where (C_n) is usually obtained from (C) by finite difference or finite element techniques, the standard situation will be that (C) is approximately controllable while (C_n) is asymptotically uniformly exactly controllable.

The results of Theorem 3.2 require a detailed study of the input maps and their adjoints. As a first step in this direction we will establish convergence criteria for the semigroups $S_n(t)$, $n \in \mathbb{N}$, and their adjoints. For this purpose let us make the following assumptions: $S(t)$ and $S_n(t)$, $n \in \mathbb{N}$, $t \geq 0$, are C_0-semigroups of type (M,ω) and (M_n,ω_n) with infinitesimal generators $A \in C(E,E)$ and $A_n \in C(E_n,E_n)$ such that

(A_1) $\bar{M} = \limsup M_n < \infty$, $\bar{\omega} = \limsup \omega_n < \infty$,

(A_2) the pair $(\lambda I - A),(\lambda I_n - A_n)_{\mathbb{N}}$, $\lambda > \max(\omega,\bar{\omega})$, is a-regular and consistent,

(A_3) the pair $(\lambda I^* - A^*),(\lambda I_n^* - A_n^*)_{\mathbb{N}}$, $\lambda > \max(\omega,\bar{\omega})$ is a-regular and consistent.

Theorem 3.3.
<u>Suppose that assumptions</u> $(A_1),(A_2)$ [<u>respectively</u> $(A_1),(A_3)$] <u>hold true. Then</u> $S_n(t) \to S(t)$ $(n \in \mathbb{N})$ [<u>respectively</u> $S_n^*(t) \to S^*(t)$ $(n \in \mathbb{N})$] <u>uniformly on finite subintervals of</u> $[0,\infty)$.

Proof. The a-regularity of $(\lambda I - A),(\lambda I_n - A_n))_{\mathbb{N}}$, $\lambda > \max(\omega,\bar{\omega})$, yields s-lim $\sup_E R(\lambda I_n - A_n) \subseteq R(\lambda I - A)$ while (A_1) implies the inverse stability of the sequence $(\lambda I_n - A_n)_{\mathbb{N}}$. Together with the consistency of $(\lambda I - A),(\lambda I_n - A_n)_{\mathbb{N}}$, Lemma 2.3 (iv) gives discrete strong convergence of the resolvents, i.e. $(\lambda I_n - A_n)^{-1} \to (\lambda I - A)^{-1}$ $(n \in \mathbb{N})$. Then, by standard arguments, one can easily deduce uniform discrete strong convergence of the semigroups $S_n(t) \to S(t)$ $(n \in \mathbb{N})$. To establish uniform discrete strong convergence of the adjoint semigroups, exactly the same arguments as before do apply. #

An immediate consequence of the preceding result is :

Corollary 3.4. <u>Under the hypotheses of Theorem</u> 3.3 <u>we have</u> $L_{t,n}^d \to L_t^d$

$(n\in\mathbb{N})$ <u>[resp.</u> $(L^d_{t,n})^* \to (L^d_t)^*$ $(n\in\mathbb{N})$] <u>uniformly on bounded subintervals of</u> $[0,\infty)$.

In order to get convergence results for the input maps $L^b_t, L^b_{t,n}$ of the abstract boundary control systems let us state another set of assumptions:

(B_1) The sequence $(D_n)_{\mathbb{N}}$ is stable and

$$0 < \liminf \theta_n < \limsup \theta_n < 1,$$

(B_2) The pair $D,(D_n)_{\mathbb{N}}$ is consistent,

(B_3) The pair $D^*,(D^*_n)_{\mathbb{N}}$ is consistent,

(B_4) $A_n \to A$ $(n\in\mathbb{N})$,

(B_5) The pair $A^*,(A^*_n)_{\mathbb{N}}$ is consistent

(B_6) The pairs $D^*,(D^*_n)_{\mathbb{N}}$, $(L^d_t)^*,((L^d_{t,n})^*)_{\mathbb{N}}$ and $A^*,(A_n)^*$ are a-regular,

(B_7) The sequence $(D_n)_{\mathbb{N}}$ is discretely compact,

(B_8) The pair $D,(D_n)_{\mathbb{N}}$ is a-regular.

Theorem 3.5.

<u>Under conditions</u> $(A_1),(B_1)$ <u>there holds:</u>

(i) <u>If assumptions</u> $(A_2),(B_2),(B_4)$ <u>[resp.</u> $(A_3),(B_3),(B_5)$] <u>are satisfied then</u> $L^b_{t,n} \to L^b_t$ $(n\in\mathbb{N})$ <u>[resp.</u> $(L^b_{t,n})^* \to (L^b_t)^*$ $(n\in\mathbb{N})$] <u>uniformly on bounded subintervals of</u> $[0,\infty)$.

(ii) <u>If assumption</u> (B_6) <u>holds true, then the pair</u> $(L^b_t)^*,((L^b_{t,n})^*)_{\mathbb{N}}$ <u>is a-regular.</u>

(iii) Under conditions (A_2),(B_4) and (B_7) the sequence $(L^b_{t,n})_{\mathbb{N}}$ is discretely compact.

Proof. (i) In view of (B_1),(B_2), Lemma 2.2 (i) gives $D_n \to D(n\in\mathbb{N})$.

Since $S_n(t) \to S(t)$ $(n\in\mathbb{N})$ by means of Theorem 3.3 and $A_n \to A(n\in\mathbb{N})$ because of (B_4), we immediately get $A_nS_n(t-\cdot)D_n \to AS(t-\cdot)D(n\in\mathbb{N})$ which in turn yields $L^b_{t,n} \to L^b_t$ $(n\in\mathbb{N})$.

In order to prove convergence of the adjoints, we claim that $((L^b_{t,n})^*)_{\mathbb{N}}$ is consistent with $(L^b_t)^*$. In fact, by (B_5) for each $u^*\in E^*\subseteq \subseteq D(A^*)$ there exists $(u^*_n)_{\mathbb{N}}$, $U^*_n\in D(A^*_n)$, $n\in\mathbb{N}$, such that $s\text{-}\lim_{E^*} u^*_n = u^*(n\in\mathbb{N})$ and $s\text{-}\lim_{E^*} A^*_n u^*_n = A^*u^*$ $(n\in\mathbb{N})$. But $S^*_n(t-\cdot) \to S^*(t-\cdot)$ $(n\in\mathbb{N})$ and $D^*_n \to D^*(n\in\mathbb{N})$ because of Theorem 3.3 and Lemma 2.2 (i) whence $s\text{-}\lim_{V^*} D^*_n S^*_n(t-\cdot)A^*_n u^*_n = D^*S^*(t-\cdot)A^*u^*$ $(n\in\mathbb{N})$ and thus $s\text{-}\lim_{W^1}(L^b_{t,n})^* u^*_n = (L^b_t)^* u^*$ $(n\in\mathbb{N})$. On the other hand, (B_1) implies the stability of $((L^b_{t,n})^*)_{\mathbb{N}}$ and consequentely, the assertion follows again from Lemma 2.2 (i).

Obviously, in both cases the discrete convergence is uniform on bounded subintervals of $[0,\infty)$.

(ii) The a-regularity of the adjoint input maps follows by applying Lemma (2.4) (ii) twice.

(iii) If $(f_n)_{\mathbb{N}}$, $f_n\in W^\infty_n$, $n\in\mathbb{N}$, is a bounded sequence, then $(f_n(\tau))_{\mathbb{N}}$ is bounded for almost all $\tau\in[0,t]$ and hence, by (B_7) for any $\mathbb{N}'\subset\mathbb{N}$ there exist $\mathbb{N}''\subset\mathbb{N}'$ and $v(\tau)\in E$ such that $s\text{-}\lim_E D_nf_n(\tau) = v(\tau)$ $(n\in\mathbb{N}'')$. Since $S_n(t) \to S(t)$ $(n\in\mathbb{N})$ and because of (B_4), we arrive at $s\text{-}\lim_E A_nS_n(t-\tau)D_nf_n(\tau) = AS(t-\tau)v(\tau)$ $(n\in\mathbb{N}'')$ whence $s\text{-}\lim_E L^b_{t,n} f_n = w$ $(n\in\mathbb{N}'')$ where

$$w = \int_0^t AS(t-\tau)v(\tau)d\tau .$$ #

We now consider the reachable sets

(3.2a) $$R_t = \{u \in E \mid u = S(t)u^o + L_t f,\ f \in F_t\},$$

(3.2b) $$R_{t,n} = \{u_n \in E_n \mid u_n = S_n(t)u_n^o + L_{t,n} f_n,\ f_n \in F_{t,n}\}.$$

If $u^o = 0$, we will write R_t^o, and if $L_t = L_t^d$ respectively $L_t = L_t^b$, the corresponding sets will be denoted by $R_t^d, R_t^{d,o}$ respectively $R_t^b, R_t^{b,o}$.

The following results establish convergence of the reachable sets both in case of distributed and boundary control (the terms in brackets always will refer to the boundary control systems, i.e. $R_t = R_t^b$ etc.):

Theorem 3.6.

Under assumption (A_1) we have for each $t > 0$:

(i) If condition (A_3) is satisfied [respectively conditions $(A_3),(B_1)$, $(B_3),(B_5)$], then the sequence $(R_{t,n}^o)_{\mathbb{N}}$ is discretely weakly compact and there holds

$$\text{w-Lim sup}_E\, R_{t,n}^o \subseteq R_t^o.$$

Moreover, if $\text{w-lim}_E\, u_n^o = u^o$ $(n \in \mathbb{N})$, then we also have

$$\text{w-Lim sup}_E\, R_{t,n} \subseteq R_t .$$

(ii) If conditions $(A_2),(B_4),(B_7),(B_8)$ hold true, then the sequence $(R_{t,n}^o)_{\mathbb{N}}$ is discretely compact and

$$\text{s-Lim sup}_E\, R_{t,n}^{b,o} \subseteq R_t^{b,o} .$$

(iii) Suppose that condition (A_2) is fulfilled [respectively conditions $(A_2),(B_2),(B_4)$]. Then there holds

$$R_t^o \subseteq \text{s-Lim inf}_E\, R_{t,n}^o .$$

If additionally $\text{s-lim}_E u_n^0 = u^0$ $(n\in\mathbb{N})$, then also

$$R_t \subseteq \text{s-Lim inf}_E R_{t,n} \quad .$$

(iv) If assumptions $(A_2),(A_3)$ [respectively $(A_2),(A_3),(B_1)-(B_5)$] are met, then

$$\text{Lim}_E R_{t,n}^0 = R_t^0 \quad .$$

Furthermore, if $\text{s-lim}_E u_n^0 = u^0$ $(n\in\mathbb{N})$, then also

$$\text{Lim}_E R_{t,n} = R_t \quad .$$

Proof. Assertions (i),(iii) and (iv) will only be shown in case of boundary control, because the corresponding proofs for distributed control follow the same pattern:

(i) Let $(u_n)_{\mathbb{N}}$ be a bounded sequence of reachable states $u_n \in R_{t,n}^{b,0}$, $n\in\mathbb{N}$, and let $\mathbb{N}'\subset\mathbb{N}$. Then there exist $f_n \in F_{t,n}$, $n\in\mathbb{N}'$, such that $u_n = L_{t,n}^b f_n$. After a correction on sets of measure zero, we may assume that for each $\tau\in[0,t]$ the sequence $(f_n(\tau))_{\mathbb{N}}$ is bounded and hence, by Lemma 2.2 there exist a subsequence $\mathbb{N}''\subset\mathbb{N}'$ and $f(\tau)\in V$ such that $\text{w-lim}_V f_n(\tau) = f(\tau)$ $(n\in\mathbb{N}'')$ and $\|f(\tau)\|_V \le \liminf \|f_n(\tau)\|_{V_n} \le 1$, i.e. $f\in F_t$. Now, let $u_n^* \in E_n^*$, $n\in\mathbb{N}''$, and $u^*\in E^*$ such that $\text{s-lim}_{E^*} u_n^* = u^*$ $(n\in\mathbb{N}'')$. Then we have

$$\langle u_n^*, L_{t,n}^b f_n \rangle_{E_n^*,E_n} = \langle (L_{t,n}^b)^* u_n^*, f_n \rangle_{W_n^1, W_n^\infty} =$$

$$= \int_0^t \langle D_n^* S_n^*(t-\tau) A_n^* u_n^*, f_n(\tau) \rangle_{V_n^*,V_n} \, d\tau .$$

Since $\text{s-lim}_{V^*} D_n^* S_n^*(t-\tau) A_n^* u_n^* = D^* S^*(t-\tau) A^* u^*$ $(n\in\mathbb{N}'')$ because of Theorem 3.5(i), Lemma 2.1 implies that the integrand converges pointwise to $< D^*S^*(t-\tau)A^*u^*, f(\tau)>_{V^*,V}$. Moreover, the integrand is uniformly bounded and consequently, the integral converges to

$$\int_0^t < D^* S^*(t-\tau)A^* u^*, f(\tau) >_{V^*,V} d\tau =$$

$$= \langle (L_t^b)^* u^*, f \rangle_{W^1,W^\infty} = \langle u^*, L_t^b f \rangle_{E^*,E} .$$

We have thus shown $\text{w-lim}_E\, u_n = u \quad (n \in \mathbb{N}'')$ where $u = L_t^b f$, $f \in F_t$.

Let us now assume that $u \in \text{w-lim sup}_E R_{t,n}^{b,0}$, i.e. there exists a sequence $(u_n)_{\mathbb{N}'}$, $u_n \in R_{t,n}^{b,0}$, $n \in \mathbb{N}' \subset \mathbb{N}$, such that $\text{w-lim}_E\, u_n = u$ $(n \in \mathbb{N}')$. Because discretely weakly convergent sequences are bounded, we may use the above result to conclude that there are a subsequence $\mathbb{N}'' \subset \mathbb{N}'$ and an element $f \in F_t$ such that $\text{w-lim}_E\, u_n = L_t^b f$ $(n \in \mathbb{N}'')$. Since limits of discretely weakly convergent sequences are unique, we get $u = L_t^b f$, i.e. $u \in R_t^{b,0}$.

Moreover, since $S_n^*(t) \to S^*(t)$ $(n \in \mathbb{N})$, $t \geq 0$, because of Theorem 3.3, Lemma 2.1(ii) tells us $S_n(t) \rightharpoonup S(t)$ $(n \in \mathbb{N})$. So, if $\text{w-lim}_E u_n^0 = u^0$ $(n \in \mathbb{N})$ we get $\text{w-lim}_E S_n(t) u_n^0 = S(t) u^0$ $(n \in \mathbb{N})$. Combined with that what we have shown before, we have $\text{w-Lim sup}_E R_{t,n}^b \subset R_t^b$.

(ii) Again, taking $(u_n)_{\mathbb{N}'}$, $u_n \in R_{t,n}^{b,0}$, $n \in \mathbb{N}' \subset \mathbb{N}$, as a bounded sequence of reachable states $u_n = L_{t,n}^b f_n$, $f_n \in F_{t,n}$, $n \in \mathbb{N}'$, we may assume $(f_n(\tau))_{\mathbb{N}'}$ to be bounded for each $\tau \in [0,t]$. Assumption (B_7) implies the existence of a subsequence $\mathbb{N}'' \subset \mathbb{N}'$ and an element $v(\tau) \in E$ such that $\text{s-lim}_E D_n f_n(\tau) = v(\tau)$ $(n \in \mathbb{N}'')$. Then, by (B_8) we deduce the existence of another subsequence $\mathbb{N}''' \subset \mathbb{N}''$ and an element $f(\tau) \in V$ such that $\text{s-lim}_V f_n(\tau) = f(\tau)$ $(n \in \mathbb{N}''')$ and $v(\tau) = Df(\tau)$. Since $\| f_n(\tau) \|_{V_n} \leq 1$, we also have

$\|f(\tau)\|_V \le 1$, i.e. $f \in F_t$. Moreover, $S_n(t) \to S(t)$ $(n \in \mathbb{N})$, $t \ge 0$, by Theorem 3.3 and $A_n \to A$ $(n \in \mathbb{N})$ because of (B_4) whence s-$\lim_E A_n S_n(t-\tau) D_n f_n(\tau) = AS(t-\tau)Df(\tau)$ $(n \in \mathbb{N}''')$ and thus also s-$\lim_E L^b_{t,n} f_n = L^b_t f$ $(n \in \mathbb{N}''')$. If $u \in$ s-Lim $\sup_E R^{b,0}_{t,n}$, the above arguments and the uniqueness of discrete strong limits imply the existence of an $f \in F_t$ such that $u = L^b_t f$, i.e. $u \in R^{b,0}_t$.

(iii) If $u \in R^{b,0}_t$ there exists $f \in F_t$ such that $u = L^b_t f$. Assuming $\|f(\tau)\|_V \le 1$ for all $\tau \in [0,t]$, by (B_2) for each $\tau \in [0,t]$ there exists a sequence $(f_n(\tau))_{\mathbb{N}}$, $f_n(\tau) \in V_n$, $n \in \mathbb{N}$, such that s-$\lim_V f_n(\tau) = f(\tau)$ $(n \in \mathbb{N})$ and s-$\lim_E D_n f_n(\tau) = Df(\tau)$ $(n \in \mathbb{N})$. Note that we also have norm convergence, i.e. $\|f_n(\tau)\|_{V_n} \to \|f(\tau)\|_V$ $(n \in \mathbb{N})$. So, if $\|f(\tau)\|_V < 1$, for a final piece $\mathbb{N}_1 \subset \mathbb{N}$ we also have $\|f_n(\tau)\|_{V_n} < 1$, $n \in \mathbb{N}_1$. We set $g_n(\tau) = \|f_n(\tau)\|^{-1}_{V_n} f_n(\tau)$, $n \in \mathbb{N} \setminus \mathbb{N}_1$, and $g_n(\tau) = f_n(\tau)$, $n \in \mathbb{N}_1$. If $\|f(\tau)\|_V = 1$, we define $g_n(\tau) = f_n(\tau)$, if $\|f_n(\tau)\|_{V_n} \le 1$, and $g_n(\tau) = \|f_n(\tau)\|^{-1}_{V_n} f_n(\tau)$ otherwise. Consequently, in any case $\|g_n(\tau)\|_{V_n} \le 1$ and s-$\lim_V g_n(\tau) = f(\tau)$ $(n \in \mathbb{N})$ as well as s-$\lim_E D_n g_n(\tau) = Df(\tau)$ $(n \in \mathbb{N})$. But $S_n(t) \to S(t)$ $(n \in \mathbb{N})$, $t \ge 0$, by Theorem 3.3 and $A_n \to A$ $(n \to \mathbb{N})$ in view of (B_4) which yields s-$\lim_E A_n S_n(t-\tau) D_n g_n(\tau) = AS(t-\tau)Df(\tau)$ $(n \in \mathbb{N})$ and also s-$\lim_E L^b_{t,n} g_n = L^b_t f$ $(n \in \mathbb{N})$. Since $g_n \in F_{t,n}$, we have $L^b_{t,n} g_n \in R^{b,0}_{t,n}$ which gives the assertion.

If s-$\lim_E u^0_n = u^0$ $(n \in \mathbb{N})$, we get s-$\lim_E S_n(t) u^0_n = S(t) u^0$ $(n \in \mathbb{N})$ and consequently, taking the above result into account, we arrive at $R^b_t \subseteq$ s-Lim $\inf_E R^b_{t,n}$.

(iv) The assertion is an immediate consequence of (i) and (iii). #

4. CONVERGENCE OF MINIMUM TIMES AND OPTIMAL CONTROLS

Based upon the convergence results for the input maps and reachable sets we are now able to establish convergence of minimum times and optimal controls. The first result concerns the approximability of (C) by (C_n):

Theorem 4.1.

Let $u^1 \in E$ be an approximately controllable state in the sense that (1.3) holds true for some $\varepsilon > 0$, let conditions $(A_1),(A_2)$ [respectively $(A_1),(A_2)$ $(B_2),(B_4)$] be satisfied and assume that (C_n) is asymptotically uniformly exactly controllable. Moreover, let $u_n^o \in E_n$, $u_n^1 \in E_n$, $n \in \mathbb{N}$, such that $\text{s-lim}_E u_n^o = u^o$ $(n \in \mathbb{N})$ and $\text{s-lim}_E u_n^1 = u^1$ $(n \in \mathbb{N})$. Then there holds:

(i) For any $\delta > 0$ there exist $\eta(\delta) > 0$ with $\eta(\delta) \to 0$ as $\delta \to 0$, a sequence $(\varepsilon_n)_{\mathbb{N}}$ of positive real numbers with $\lim_{n\to\infty} \varepsilon_n \le \varepsilon$ and a final piece $\mathbb{N}_1 \subset \mathbb{N}$ such that for each $n \in \mathbb{N}_1$ there is an $u_n \in R_{t_\varepsilon^* + \delta, n}$ satisfying

$$(4.1) \qquad \| u_n^1 - u_n \|_{E_n} \le \eta_n(\delta) = \varepsilon_n + \eta(\delta).$$

(ii) Let additionally condition (A_3) [respectively conditions (A_3), $(B_1),(B_3),(B_5)$] be fulfilled. Then, for any null sequence $(\delta_n)_{\mathbb{N}}$ of positive real numbers, denoting by t_n^* and f_n^*, $n \in \mathbb{N}_1$, the minimum time and an optimal control with respect to $u_n^o, u_n^1, B_{E_n}(u_n^1, \eta_n(\delta_n))$, we have $t_n^* \to t_\varepsilon^*$ $(n \in \mathbb{N}_1)$ and there exist a subsequence $\mathbb{N}' \subset \mathbb{N}_1$ and an optimal control $f^* \in F_{t_\varepsilon^*}$ such that $\text{w-lim}_V\, g_n^*(\tau) = f^*(\tau)$ $(n \in \mathbb{N}')$ for almost all $\tau \in [0, t^*]$ where $g_n^*(\tau) = f_n^*(\tau)$, $0 \le \tau \le t_\varepsilon^*$, and $g_n^*(\tau) = 0$ elsewhere.

(iii) In the case of boundary control, if in addition to the assumptions in (i),(ii) conditions $(B_7),(B_8)$ are met, then the same statements as in (ii) hold true with discrete weak replaced by discrete strong convergence.

Proof. Let $u^*(t)$, $0 \le t \le t_\varepsilon^*$, be the optimal trajectory with respect to u^0, u^1 and $B_E(u^1, \varepsilon)$. Since $S(t)u \to u$ as $t \to 0^+$, for given $\delta > 0$ there exists $\eta(\delta) > 0$ such that

$$(4.2) \qquad \| S(\delta)u^*(t_\varepsilon^*) - u^*(t_\varepsilon^*) \|_E < \eta(\delta).$$

Setting $v_n = R_n u^*(t_\varepsilon^*)$, $n \in \mathbb{N}$, we obviously have $\text{s-lim}_E v_n = u^*(t_\varepsilon^*)$ $(n \in \mathbb{N})$. But $S_n(\delta) \to S(\delta)$ $(n \in \mathbb{N})$ and hence, there is an $n_1(\delta) \in \mathbb{N}$ such that for all $n \ge n_1(\delta)$

$$(4.3) \qquad \| S_n(\delta)v_n - v_n \|_{E_n} \le \eta(\delta).$$

On the other hand, by Theorem 3.6(iii) there exists $(f_n)_{\mathbb{N}}$, $f_n \in F_{t_\varepsilon^*, n}$, $n \in \mathbb{N}$, such that $\text{s-lim}_E w_n = u^*(t_\varepsilon^*)$ $(n \in \mathbb{N})$ where $w_n = S_n(t_\varepsilon^*)u_n^0 + L_{t_\varepsilon^*, n} f_n$. Consequently, there exists $n_2(\delta) \in \mathbb{N}$ such that for all $n \ge n_2(\delta)$

$$(4.4) \qquad \| S_n(\delta)(w_n - v_n) \|_{E_n} \le \gamma_0(\delta)$$

and thus, for $n \ge n_2(\delta)$ we find $g_n \in F_{\delta, n}$ satisfying $S_n(\delta)(w_n - v_n) = L_{\delta, n} g_n$. If we take $u_n = S_n(\delta)v_n$ and $h_n(\tau) = f_n(\tau)$, $0 \le \tau \le t_\varepsilon^*$, $h_n(\tau) = g_n(\tau - t_\varepsilon^*)$, $t_\varepsilon^* < \tau \le t_\varepsilon^* + \delta$, then $h_n \in F_{t_\varepsilon^* + \delta, n}$ and $u_n = S_n(t_\varepsilon^* + \delta)u_n^0 + L_{t_\varepsilon^* + \delta, n} h_n$, i.e. $u_n \in R_{t_\varepsilon^* + \delta, n}$. Moreover,

$$\| u_n^1 - u_n \|_{E_n} \le \| u_n^1 - v_n \|_{E_n} + \| v_n - u_n \|_{E_n}.$$

But $\| u_n^1 - v_n \|_{E_n} \to \| u^1 - u^*(t^*) \|_E \le \varepsilon$ as $n \to \infty$ and thus, taking (4.3) into account, (4.1) holds true for $n \ge \max(n_1(\delta), n_2(\delta))$ with

$\varepsilon_n = \|u_n^1 - v_n\|_{E_n}$.

(ii) It follows directly from part (i) of the proof that $t_n^* \le t_\varepsilon^* + \delta_n$ for sufficiently large $n \in \mathbb{N}$, let's say $n \in \mathbb{N}_1 \subset \mathbb{N}$, and thus $\limsup t_n^* \le t_\varepsilon^*$. On the other hand, if $u_n^*(t)$, $0 \le t \le t_n^*$, is the trajectory corresponding to $f_n^* \in F_{t_n^*,n}$ and $\hat{t}_\varepsilon = \liminf t_n^*$, then $g_n^* |_{[0,\hat{t}_\varepsilon]} \in F_{\hat{t}_\varepsilon,n}$ and $u_n^*(\hat{t}_\varepsilon) \in R_{\hat{t}_\varepsilon,n}$. By Theorem 3.6(i) there exist a subsequence $\mathbb{N}' \subset \mathbb{N}_1$ and an $\hat{f} \in F_{\hat{t}_\varepsilon}$ such that $\text{w-lim}_V\, g_n^*(\tau) = \hat{f}(\tau)$ $(n \in \mathbb{N}')$ a.e. and $\text{w-lim}_E\, u_n^*(\hat{t}_\varepsilon) = u(\hat{t}_\varepsilon)$ $(n \in \mathbb{N}')$ where $u(t)$, $0 \le t \le \hat{t}_\varepsilon$, is the trajectory corresponding to $\hat{f}$. Since discrete strong implies discrete weak convergence, we also have $\text{w-lim}_E\, u_n^1 = u^1$ $(n \in \mathbb{N})$ and hence, $\text{w-lim}_E\, (u_n^1 - u_n^*(\hat{t}_\varepsilon)) = u^1 - u(\hat{t}_\varepsilon)$ $(n \in \mathbb{N}')$ whence

$$\|u^1 - u(\hat{t}_\varepsilon)\|_E \le \liminf \|u_n^1 - u_n^*(\hat{t}_\varepsilon)\|_{E_n} \le \varepsilon .$$

It follows that $t_\varepsilon^* \le \hat{t}_\varepsilon = \liminf t_n^*$ and consequently, combined with that what we have shown before, we have $t_n^* \to t_\varepsilon^*$ $(n \in \mathbb{N}_1)$.

(iii) The assertion follows along the same lines by applying Theorem 3.6(ii). #

The next result provides a partial characterization of the sets of approximately controllable states in E:

Theorem 4.2.

Under conditions $(A_1),(A_3)$ [respectively $(A_1),(A_3),(B_1),(B_3),(B_5)$] let $(u_n^o)_{\mathbb{N}}$ be a sequence of initial states $u_n^o \in E_n$, $n \in \mathbb{N}$, such that $\text{w-lim}_E\, u_n^o = u^o$ $(n \in \mathbb{N})$, and let $(u_n^1)_{\mathbb{N}}$ be a sequence of approximately controllable states $u_n^1 \in E_n$, $n \in \mathbb{N}$, such that (1.12) holds true for any sequence $(\varepsilon_n)_{\mathbb{N}}$ of positive real numbers with $\lim_{n\to\infty} \varepsilon_n = \varepsilon$ for some $\varepsilon > 0$ and assume that the corresponding sequence $(t_{\varepsilon_n}^*)_{\mathbb{N}}$ of minimum times is bounded.

Then, each $u^1 \in w\text{-}\lim\sup_E \{(u_n^1)_{\mathbb{N}}\}$ is an approximately controllable state in E in the sense that (1.3) holds true. In particular, there exists an $u^1 \in w\text{-}\mathrm{Lim}\sup_E \{(u_n^1)_{\mathbb{N}}\}$ with minimum time $t_\varepsilon^* \le \liminf t_{\varepsilon_n}^*$.

If additionally condition (A_2) [respectively conditions (A_2), (B_2), (B_4)] is satisfied, (C_n) is asymptotically uniformly exactly controllable, $s\text{-}\lim_E u_n^0 = u^0$ $(n \in \mathbb{N})$ and $u^1 \in s\text{-}\mathrm{Lim}\sup_E \{(u_n^1)_{\mathbb{N}}\}$, then for a sequence $\mathbb{N}' \subset \mathbb{N}$ we have $t_{\varepsilon_n}^* \to t_\varepsilon^*$ $(n \in \mathbb{N}')$ and, denoting by $f_n^* \in F_{t_{\varepsilon_n}^*, n}$ corresponding optimal controls, there exist another subsequence $\mathbb{N}'' \subset \mathbb{N}'$ and an optimal control $f^* \in F_{t_\varepsilon^*}$ such that $w\text{-}\lim_V g_n^*(\tau) = f^*(\tau)$ $(n \in \mathbb{N}'')$ a.e. where g_n^* is defined as in Theorem 4.1(ii).
Under assumptions (B_7), (B_8) the same holds true with discrete weak replaced by discrete strong convergence.

Proof. If $u^1 \in w\text{-}\mathrm{Lim}\sup_E\{(u_n^1)_{\mathbb{N}}\}$ then there exists a subsequence $\mathbb{N}' \subset \mathbb{N}$ such that $u^1 = w\text{-}\lim_E u_n^1$ $(n \in \mathbb{N}')$. Since $(t_{\varepsilon_n}^*)_{\mathbb{N}'}$ is bounded, we find $\mathbb{N}' \subset \mathbb{N}'$ with $t_{\varepsilon_n}^* \to \hat{t}_\varepsilon$ $(n \in \mathbb{N}'')$. By Theorem 3.6(iii) there are another subsequence $\mathbb{N}''' \subset \mathbb{N}''$ and an $\hat{f} \in F_{\hat{t}_\varepsilon}$ such that $w\text{-}\lim_E u_n^*(\hat{t}_\varepsilon) = \hat{u}(\hat{t}_\varepsilon)$ $(n \in \mathbb{N}''')$ where $u_n^*(t)$ and $\hat{u}(t)$ are the trajectories corresponding to f_n^* and $\hat{f}$ respectively, and we have

$$\| u^1 - \hat{u}(\hat{t}_\varepsilon) \|_E \le \liminf \| u_n^1 - u_n^*(\hat{t}_\varepsilon) \|_{E_n} \le \varepsilon .$$

Now, let $\mathbb{N}' \subset \mathbb{N}$ be such that $t_{\varepsilon_n}^* \to \hat{t}_\varepsilon = \liminf t_{\varepsilon_n}^*$ $(n \in \mathbb{N}')$. But $(u_n^1)_{\mathbb{N}'}$ is bounded and hence, due to the discrete weak compactness of bounded sequences of elements in E_n, there exist $\mathbb{N}'' \subset \mathbb{N}'$ and $u^1 \in E$ with $w\text{-}\lim_E u_n^1 = u^1$ $(n \in \mathbb{N}'')$. By the same reasoning as above we conclude that u^1 is an approximately controllable state with minimum time $t_\varepsilon^* \le \hat{t}_\varepsilon$.

Under the additional assumptions there exist $\mathbb{N}' \subset \mathbb{N}$ and a sequence $(\tilde{\varepsilon}_n)_{\mathbb{N}'}$ of positive real numbers with $\lim \tilde{\varepsilon}_n = \varepsilon$ $(n \in \mathbb{N}')$ such that

$$\| u_n^1 - u_n^*(t_\varepsilon^*) \|_{E_n} \leq \tilde{\varepsilon}_n.$$

Let us assume $t_\varepsilon^* < \liminf t_{\varepsilon_n}^*$ $(n \in \mathbb{N}')$. If $\tilde{\varepsilon}_n \leq \varepsilon_n$ this contradicts the time minimality of $t_{\varepsilon_n}^*$. On the other hand, if $\tilde{\varepsilon}_n > \varepsilon_n$, we find $z_n \in \partial B_{E_n}(u_n^1, \varepsilon_n)$ such that $\| z_n - u_n^*(t_\varepsilon^*) \|_{E_n} \to 0$ $(n \in \mathbb{N}')$. Then, arguing as in the proof of Theorem 4.1(i), for any $\delta > 0$ and sufficiently large $n \in \mathbb{N}'$ there exist controls $g_n \in F_{t_\varepsilon^* + \delta, n}$ and corresponding trajectories $u_n(t)$, $0 \leq t \leq t_\varepsilon^* + \delta$, such that

$$\| u_n^1 - u_n(t_\varepsilon^* + \delta) \|_{E_n} \leq \| u_n^1 - z_n \|_{E_n} + \eta(\delta)$$

which, letting $\delta \to 0$, also contradicts the time minimality of $t_{\varepsilon_n}^*$. #

5. APPLICATIONS

As an example let us consider the parabolic equation

$$\text{(5.1a)} \qquad \frac{\partial}{\partial t} u(x,t) + Au(x,t) = 0 \ , \quad x \in \Omega \ , \quad t > 0$$

$$\text{(5.1b)} \qquad u(x,0) = u^0(x) \ , \quad x \in \Omega$$

where Ω is a smooth bounded domain in Euclidean space R^d, $d \in \mathbb{N}$, and A is a second order uniformly elliptic operator given by

$$A = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} \right) + \sum_{i=1}^{d} b_i(x) \frac{\partial}{\partial x_i} + c(x)$$

with sufficiently smooth real-valued coefficients $a_{ij} = a_{ji}$, b_i and c. We consider boundary control either in the Dirichlet data

$$(5.1c)_D \qquad u(x,t) = g(x,t) \quad , x \in \Gamma = \partial\Omega \quad , \quad t > 0$$

or in the Neumann data

$$(5.1c)_N \qquad \frac{\partial}{\partial \nu} u(x,t) = g(x,t) \quad , x \in \Gamma = \partial\Omega \ , \quad t > 0$$

where $\frac{\partial}{\partial \nu} = \sum_{i,j=1}^{d} n_i a_{ij} \frac{\partial}{\partial x_j}$.

Denoting by $a(.,.): H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ the bilinear form

$$a(u,v) = \sum_{i,j=1}^{d} \int_\Omega a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx + \sum_{i=1}^{d} \int_\Omega b_i \frac{\partial u}{\partial x_i} v \, dx + \int_\Omega cuv \, dx,$$

clearly, $a(.,.)$ is bounded, i.e.

$$(5.2) \qquad |a(u,v)| \le C \ |u|_{1,\Omega} \ |v|_{1,\Omega} \ , \qquad u,v \in H^1(\Omega) \ .$$

Moreover, let us assume for simplicity that $a(.,.)$ is strictly $H^1(\Omega)$ - coercive, i.e.

$$(5.3) \qquad a(u,u) \ge \gamma \ |u|^2_{1,\Omega} \ , \quad u \in H^1(\Omega) \quad , \quad \gamma > 0 \ .$$

It follows that the spectrum of the operators $A_D : D(A_D) \subset L^2(\Omega) \to L^2(\Omega)$, $D(A_D) = \{u \in H^2(\Omega) \mid u \ |_\Gamma = 0\}$ respectively $A_N : D(A_N) \subset L^2(\Omega) \to L^2(\Omega)$, $D(A_N) = \{u \in H^2(\Omega) \mid \frac{\partial}{\partial \nu} u \ |_\Gamma = 0\}$ is confined to the positive half-axis and $-A_D$ respectively $-A_N$ generate strongly continuous semigroups $S_D(t): L^2(\Omega) \to L^2(\Omega)$, $t \ge 0$, respectively $S_N(t): L^2(\Omega) \to L^2(\Omega)$, $t \ge 0$, of type (M,ω)

for some $\omega < 0$.
Finally, let us denote by D the Dirichlet map $u = Dg$ where

$$(5.4)_D \qquad Au(x) = 0 \ , \quad x \in \Omega \ , \quad u(x) = g(x) \ , \quad x \in \Gamma$$

and by N the Neumann map $u = Ng$ where

$$(5.4)_N \qquad Au(x) = 0 \ , \quad x \in \Omega \ , \quad \frac{\partial}{\partial \nu} u(x) = g(x) \ , \quad x \in \Gamma \ .$$

It is well known that under the foregoing assumptions $D : L^2(\Gamma) \to L^2(\Omega)$ and $N : L^2(\Gamma) \to L^2(\Omega)$ are compact linear operators satisfying (1.7) for some $0 < \theta < 1$. Consequently, the parabolic boundary control problems (5.1a), (5.1b), $(5.1c)_D$ respectively $(5.1c)_N$ can be cast in the abstract setting of Section 1 with $E = (L^2(\Omega), |.|_{0,\Omega})$, $V = (L^2(\Gamma), |.|_{0,\Gamma})$, $A = -A_D$ respectively $A = -A_N$ and D denoting the Dirichlet and Neumann map respectively.
We remark that the adjoint operators $-A_D^*$ and $-A_N^*$ with $D(A_D^*) = \{u \in H^2(\Omega) \mid u|_\Gamma = 0\}$ respectively $D(A_N^*) = \{u \in H^2(\Omega) \mid (\frac{\partial}{\partial \nu} + n.b)u|_\Gamma = 0\}$ likewise generate strongly continuous semigroups $S_D^*(t)$ and $S_N^*(t)$, $t \geq 0$.
Furthermore, it folloxs easily from Green's formula

$$(5.5) \qquad (Au,v)_{0,\Omega} - (u,A^*v)_{0,\Omega} = \int_\Gamma u(\frac{\partial}{\partial \nu} v + n.bv)\, d\sigma - \int_\Gamma \frac{\partial}{\partial \nu} u \, v \, d\sigma$$

by choosing $u = Dg$, $v = (A_D^*)^{-1} w$ respectively $u = Ng$, $v = (A_N^*)^{-1} w$ that the adjoints D^* and N^* are given by

$$(5.6)_D \qquad D^* w = -(\frac{\partial}{\partial \nu} + n.b)(A_D^*)^{-1} w|_\Gamma \ ,$$

$$(5.6)_N \qquad N^* w = (A_N^*)^{-1} w|_\Gamma \ .$$

We first consider the approximate solution of Neumann boundary control. Given a null sequence $(h_n)_{\mathbb{N}}$ of positive real numbers $h_n < 1$, $n \in N$, we choose $E_n = S_{h_n}^{2,1}(\Omega)$, where $S_h^{r,k}(\Omega)$, $r>k\geq 0$, denotes a regular (r,k)-system, i.e. for every $u \in H^l(\Omega)$, $l\geq 0$, and every s with $0\leq s \leq \min(l,k)$ there exists $v \in S_h^{r,k}(\Omega)$ such that

$$(5.7) \qquad |v - u|_{s,\Omega} \leq C\, h^{\mu}\, |u|_{l,\Omega} \ , \quad \mu = \min(r-s, l-s) \ .$$

Moreover, we assume E_n to satisfy the inverse assumption

$$(5.8) \qquad |u_n|_{1,\Omega} \leq C\, h_n^{-1}\, |u_n|_{0,\Omega} \ , \quad u_n \in E_n \ .$$

Then, denoting by R_n^E the orthogonal projection of $L^2(\Omega)$ onto E_n with respect to the inner product in $L^2(\Omega)$, we have

$$(5.9) \qquad |R_n^E u - u|_{0,\Omega} \leq C h^r\, |u|_{r,\Omega} \ , \quad 0\leq r\leq 2 \ , \quad u \in H^r(\Omega)$$

and consequently, $(H^r(\Omega), \Pi E_n, R^E)$, $0<r\leq 2$, define discrete cn- approximations with discrete convergence being equivalent to norm convergence in $L^2(\Omega)$. But $H^r(\Omega)$ is dense in $L^2(\Omega)$ and thus,these cn-approximations uniquely induce a cn-approximation $(L^2(\Omega), \Pi E_n, R^E)$ (cf. [6]).
We choose V_n as the space generated by the traces of functions in $S_{h_n}^{2,1}(\Omega)$. V_n is known to define an $S_{h_n}^{3/2,1/2}(\Gamma)$ system and hence, denoting by R_n^V the orthogonal projection of $L^2(\Gamma)$ onto V_n with respect to $(.,.)_{0,\Gamma}$ we have

$$(5.10) \qquad |R_n^V u - u|_{0,\Gamma} \leq C\, h^{\mu}\, |u|_{r,\Gamma} \ , \quad \mu = \min(3/2-r, r) \ .$$

By the same argument as above, we thus obtain a cn-approximation $(L^2(\Gamma), \Pi V_n, R^V)$.

We define $A_{N,n}: E_n \to E_n$ by

(5.11) $$(A_{N,n}u_n, v_n)_{0,\Omega} = a(u_n, v_n) \quad , \quad u_n, v_n \in E_n$$

and discrete Neumann maps $N_n : V_n \to E_n$ by

(5.12) $$a(N_n g_n, v_n) = (g_n, v_n)_{0,\Gamma} \quad , \quad v_n \in E_n \quad .$$

By (5.2) and (5.3) it follows that for every $\lambda \geq 0$ the sequence $(\lambda I_n + A_{N,n})_{\mathbb{N}}$ is both stable and inversely stable. In particular, $-A_{N,n}$, $n \in N$, generates a semigroup $S_{N,n}(t): E_n \to E_n$, $t \geq 0$, of the same type as $S_N(t)$. Moreover, denoting by R_n^H the elliptic projection of $H = H^1(\Omega)$ onto E_n, i.e.

$$a(R_n^H u, v_n) = a(u, v_n) \quad , \quad v_n \in E_n \quad ,$$

it turns out immediately that $A_{N,n} R_n^H = R_n^E A$. Since

(5.13) $$|R_n^H u - u|_{0,\Omega} \leq Ch^r \quad |u|_{r,\Omega} \quad , \quad 1 \leq r \leq 2 \quad , \quad u \in H^r(\Omega) \quad ,$$

the pair $(\lambda I + A_N), (\lambda I_n + A_{N,n})_{\mathbb{N}}$ is consistent for every $\lambda \geq 0$, and we thus obtain the biconvergence $(\lambda I_n + A_{N,n}) \to (\lambda I + A_N)$ $(n \in \mathbb{N})$ and $(\lambda I_n + A_{N,n})^{-1} \to (\lambda I + A_N)^{-1}$ $(n \in \mathbb{N})$. It is easy to check that the same statements hold true for the adjoint operators A_N^* , $A_{N,n}^*$, $n \in \mathbb{N}$. Consequently, assumptions $(A_1), (A_2), (A_3)$ and $(B_4), (B_5)$ of Section 3 are satisfied in this case.

As far as the Neumann maps are concerned, it follows from (5.12) that

$N_n = R_n^H N|_{V_n}$ and $N_n^* = R_n^V N^*|_{E_n}$ implying that $(N_n)_{\mathbb{N}}$ is stable and that the pairs $N,(N_n)_{\mathbb{N}}$ and $N^*,(N_n^*)_{\mathbb{N}}$ are consistent thus establishing conditions (B_1) (with $\theta_n=\theta$), (B_2) and (B_3).

The discrete compactness of $(N_n)_{\mathbb{N}}$ (condition (B_7)) follows readily from the compactness of N. To establish a-regularity of $N,(N_n)_{\mathbb{N}}$ let $(g_n)_{\mathbb{N}}, g_n \in V_n$, $n \in \mathbb{N}$, be bounded and assume $N_n g_n \to u$ $(n \in \mathbb{N})$ for some $u \in L^2(\Omega)$. Then there exist a subsequence $\mathbb{N}' \subset \mathbb{N}$ and a $g \in L^2(\Gamma)$ such that $g_n \rightharpoonup g$ $(n \in \mathbb{N}')$ and $N_n g_n \to Ng$ $(n \in \mathbb{N}')$ whence $u=Ng$. Since $(N_n)_{\mathbb{N}}$ is inversely stable, we further have $g_n \to g$ $(n \in \mathbb{N}')$.

For the approximate solution of Dirichlet boundary control we choose $E_n \subset H_0^1(\Omega)$ as a (2,1)-regular system $S^{2,1}_{h_n,0}(\Omega)$ and $V_n \subset H^{1/2}(\Gamma)$ as a (3/2, 1/2)-regular system $S^{3/2,1/2}_{h_n}(\Gamma)$ generated by the traces of a (2,1)-regular system $F_n = S^{2,1}_{h_n}(\Omega) \subset H^1(\Omega)$. Again, we denote by R_n^E respectively R_n^V the orthogonal projections of $L^2(\Omega)$ respectively $L^2(\Gamma)$ onto E_n respectively V_n and by R_n^H respectively $R_n^{H}{}_O$ the elliptic projections of $H=H^1(\Omega)$ respectively $H_O=H_0^1(\Omega)$ onto F_n respectively E_n. We define $A_{D,n}: E_n \to E_n$ by

$$(5.14) \qquad (A_{D,n}u_n, v_n)_{0,\Omega} = a(u_n, v_n) \quad , \quad u_n, v_n \in E_n$$

and discrete Dirichlet maps $D_n: V_n \to E_n$ by $D_n = R_n^E D_n^1$ where $D_n^1: V_n \to F_n$ is given by

$$(5.15) \qquad a(D_n^1 g_n, v_n) = 0 \ , \ v_n \in E_n \ , \quad D_n^1 g_n|_\Gamma = g_n \ .$$

In view of $A_{D,n}R_n^E = R_n^{H}{}_O A_D$, $D_n^1 = R_n^H D|_{V_n}$ and $D_n^* = R_n^V D^*|_{E_n}$, conditions

(A_1)-(A_3),(B_1)-(B_5) ean be established in a similar way as in the Neumann case. The discrete compactness of $(D_n)_{\mathbb{N}}$ (condition (B_7)) follows from the compactness of D which can also be used to deduce the a-regularity of the pair $D,(D_n)_{\mathbb{N}}$ (condition (B_8)).

We finally consider the use of nonconforming methods in Dirichlet boundary control (cf. [14]). We choose E_n as a regular (r,2)-system $S^{r,2}_{h_n}(\Omega)$, $r\geq 4$, satisfying additionally the following inverse assumption

$$(5.16)\qquad |u_n|_{k,\Omega} + |u_n|_{k,\Gamma} \leq C\, h_n^{s-k}\,(\,|u_n|_{s,\Omega} + |u_n|_{s,\Gamma})$$

where $0\leq k\leq 1$ and $0\leq s\leq \min(k,1)$.

We define $\tilde{A}_{D,n}:E_n \to E_n$ by the nonconforming Rayleigh-Ritz Galerkin technique as used by Bramble and Schatz in [2]:

$$(5.17)\qquad (\tilde{A}_{D,n}u_n,v_n)_{0,\Omega} = \tilde{a}(u_n,v_n) + h_n^{-3}(u_n,v_n)_{0,\Gamma}$$

where $\tilde{a}(u_n,v_n) = (Au_n,Av_n)_{0,\Omega}$, $u_n,v_n\in E_n$.

Using the approximation properties of $S^{r,2}_{h_n}(\Omega)$ and the inverse assumptions (5.16),conditions (A_1)-(A_3) and (B_4),(B_5) can be readily verified in view of [2;Thm.4.1].

Instead of choosing $V_n = E_n|_\Gamma$ and $D_n = R^{E}_{n}D^{1}_{n}$ where D^1_n is defined as in the conforming case,we may take a larger class of approximating boundary controls,namely we may choose $V_n\subset L^2(\Gamma)$ as the space of piecewise constant functions on Γ , and we may define discrete Dirichlet maps $\tilde{D}_n: V_n\to E_n$ by using the same nonconforming technique as in the definition of $\tilde{A}_{D,n}$,i.e.

(5.18) $\quad \tilde{a}(\tilde{D}_n g_n, v_n) + h_n^{-3}(g_n - \tilde{D}_n g_n, v_n)_{0,\Gamma} = 0 \ , \ v_n \in E_n$

If follows from [2;Corollary 4.1] that for $4-r \le k \le 1/2$

(5.19) $\quad |\tilde{D}_n g_n - Dg_n|_{k,\Omega} \le C\, h_n^{1/2-k}\, |g_n|_{0,\Gamma}$

which immediately gives stability of $(\tilde{D}_n)$ and consistency of $D,(\tilde{D}_n)$ (conditions $(B_1),(B_2)$). In particular, $\tilde{D}_n \to D(n\in\mathbb{N})$ and discrete compactness of $(\tilde{D}_n)_{\mathbb{N}}$ as well as a-regularity of $D,(\tilde{D}_n)_{\mathbb{N}}$ (conditions (B_7), (B_8)) can be established as in the conforming case using the compactness of D.

Finally, if we choose v_n by $A^*_{D,n} A_{D,n} v_n = w_n$ in (5.18), where $A_{D,n}$ is defined as in the conforming case with respect to an (r,)-regular system $S^{r,2}_{h_n,0}(\Omega)$, we find

$$(\tilde{D}^*_n w_n, g_n)_{0,\Gamma} = -(\tilde{D}_n g_n (\frac{\partial}{\partial\nu} + n.b)\ (A^*_{D,n})^{-1} w_n)_{0,\Gamma}$$

from which we can deduce $\tilde{D}^*_n \to D^*(n\in\mathbb{N})$ and thus condition (B_3).

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