# Element-oriented and edge-oriented local error estimators for nonconforming finite element methods

**Ronald H. W. Hoppe, Barbara Wohlmuth**

Ronald H. W. Hoppe

Barbara Wohlmuth

## Element-oriented and edge-oriented local error estimators for nonconforming finite element methods

# ELEMENT-ORIENTED AND EDGE-ORIENTED LOCAL ERROR ESTIMATORS FOR NONCONFORMING FINITE ELEMENT METHODS (*)

by Ronald H. W. HOPPE ([1]) and Barbara WOHLMUTH ([1])

Résumé. — Nous considérons des estimateurs fiables et facilement calculables pour le traitement de l'approximation des problèmes aux limites elliptiques linéaires au second ordre par la méthode des éléments finis non conformes. En particulier, nous construisons des estimateurs a posteriori sur les éléments du maillage ainsi que sur les arêtes, et nous obtenons par là des bornes inférieures et supérieures pour l'erreur globale de discrétisation. Les contributions locales de ces estimateurs sont utilisées dans un processus adaptatif, comme critères de raffinement local des triangulations.
AMS (MOS) : 65 N15, 65 N30, 65 N50.

Abstract. — We consider easily computable and reliable error estimators for the approximation of linear elliptic boundary value problems by nonconforming finite element methods. In particular, we develop both element-oriented and edge-oriented estimators providing lower and upper bounds for the global discretization error. The local contributions of these estimators may serve as indicators for local refinement within an adaptive framework.

Key words : elliptic boundary value problems, nonconforming finite elements, local error estimators.

## 1. INTRODUCTION

Local error estimators play a decisive role in the development of adaptive finite element methods for the numerical solution of elliptic boundary value problems. In particular, an appropriate error estimator should be efficiently computable and provide reliable information on the global discretization error which is used for local refinements of the triangulations. There is a wide variety of specific approaches differing mainly in the refinement techniques and the choice of the error indicators which are based either on the local residual or on the solution of suitable local subproblems. Pioneering work has been done by Babuška, Rheinboldt and others (for an exhaustive bibliography see e.g. the recent monograph by Szabó and Babuška [11]).

For highly nonuniform triangular meshes, generated by the meanwhile standard refinement process of Bank and others [2], [3], appropriate error

estimators have been developed by Bank and Weiser [4]. These error estimators which provide sharp lower and upper bounds for the global discretization error are element-oriented in the sense that they are based on the elementwise solution of suitable low-dimensional subproblems. An alternative approach developed by Deuflhard, Leinen and Yserentant [9] is based on the same refinement process as in [3] but uses a hierarchical splitting of the finite element space of continuous, piecewise quadratics for the error equation. The resulting error estimator is edge-oriented, since it can be computed by the solution of only scalar equations associated with the midpoints of the edges of the triangulation.

The above error estimators have been established in the framework of conforming finite element techniques. In this paper we will focus on related error indicators in the case of nonconforming methods for second order elliptic boundary value problems based on the use of the lowest order Crouzeix-Raviart nonconforming finite elements [8]. This is also of interest with regard to the application of mixed finite element techniques, since it is well-known [1], [7] that by an appropriate post-processing such methods are closely related to nonconforming discretizations. In particular, we will develop both element-oriented and edge-oriented error estimators similar to the approaches used in [4] and [9]. In both cases the error equation will be approximated in the conforming finite element space of continuous, piecewise quadratics, since there is no canonical choice for a nonconforming piecewise quadratic ansatz.

The paper is organized as follows : in Section 2 we introduce the nonconforming finite element approximation of linear second order elliptic boundary value problems as well as some preliminary results including equivalent discrete expressions for some norms and seminorms of piecewise linear and piecewise quadratic functions. Section 3 is devoted to the construction of element-oriented error estimators much along the lines of [4]. But in contrast to the conforming case the nonconformity must be taken into account by introducing an appropriate projection mapping the Crouzeix-Raviart nonconforming finite element space onto the conforming space of continuous, piecewise linear functions. Finally, in Section 4 we shall deal with an edge-oriented error estimator which can be derived by combining the techniques used in the conforming case in [9] with a suitable tool measuring the discontinuity of nonconforming finite element functions across the interior edges of the triangulation.

## 2. THE NONCONFORMING SETTING AND PRELIMINARY RESULTS

We consider the linear elliptic boundary value problem

$$Lu(x) := - \nabla \cdot (a(x) \nabla u(x)) + b(x) u(x) = f(x), \quad x \in \Omega, \qquad (2.1a)$$
$$u(x) = 0, \qquad x \in \Gamma = \partial\Omega \qquad (2.1b)$$

where $\Omega$ is a bounded polygonal domain in $\mathbf{R}^2$, $f \in L^2(\Omega)$ and $a(\ .\ )$ and $b(\ .\ )$ are piecewise continuous functions on $\overline{\Omega}$ satisfying

$$0 < \underline{a} \leq a(x) \leq \overline{a}, \quad 0 \leq b(x) \leq \overline{b}, \quad x \in \Omega. \tag{2.2}$$

Note that only for simplicity we have chosen homogeneous Dirichlet boundary conditions and a scalar coefficient function $a(\ .\ )$. All the results of this paper can be extended to cover the case of Neumann or mixed boundary conditions as well as such problems where $a(\ .\ )$ is supposed to be a symmetric uniformly positive definite $2 \times 2$ matrix.

For the numerical solution of (2.1a-b) we will use a nonconforming finite element method based on the lowest order triangular Crouzeix-Raviart elements with respect to a (possibly) highly nonuniform triangulation $\mathscr{T}_j$ of $\overline{\Omega}$. In particular, we may think of $\mathscr{T}_j$ as the final triangulation of a sequence $(\mathscr{T}_k)_{k=0}^j$ of triangulations generated from an initial coarse triangulation $\mathscr{T}_0$ by the refinement process of Bank *et al.* (*cf.* e.g. [2], [3]). Then each triangle of any triangulation $\mathscr{T}_k$ is geometrically similar either to a triangle in $\mathscr{T}_0$ or at least to a subtriangle of a triangle in $\mathscr{T}_0$ obtained by bisection, and the triangulations $\mathscr{T}_k$, $0 \leq k \leq j$, share the property of local quasi-uniformity.

Given $\mathscr{T} = \mathscr{T}_j$, in some estimates it will be more convenient to replace the global bounds $\underline{a}$, $\overline{a}$ in (2.2) by their local counterparts $\underline{a}_\tau$, $\overline{a}_\tau$ when considering $a(\ .\ )$ on $\tau \in \mathscr{T}$. We suppose that $\kappa > 0$ is a constant such that $\overline{a}_\tau / \underline{a}_\tau \leq \kappa$ for all $\tau \in \mathscr{T}$. We further denote by $(\ .,.\ )_{0,\tau}$, $\|\ .\ \|_{0,\tau}$ the standard $L^2$-inner product and $L^2$-norm on $L^2(\tau)$ and by $|\ .\ |_{m,\tau}$, $\|\ .\ \|_{m,\tau}$ the standard $H^m$-seminorm and $H^m$-norm on $H^m(\tau)$, $m \in \mathbf{N}$, respectively. Moreover, on $L^2(\Omega)$ we define the $L^2$-inner product $(v, w)_{0,\mathscr{T}} := \sum_{\tau \in \mathscr{T}} (v, w)_{0,\tau}$ with associated norm $\|\ .\ \|_{0,\mathscr{T}}$, and we refer to

$$H_{\mathscr{T}}^m(\Omega) := \left\{ v \in L^2(\Omega) \,|\, v|_\tau \in H^m(\tau), \tau \in \mathscr{T} \right\}, \quad m \in \mathbf{N}$$

as the space of piecewise $H^m$-functions equipped with the broken seminorm $|v|_{m,\mathscr{T}} := \left( \sum_{\tau \in \mathscr{T}} |v|_{m,\tau}^2 \right)^{1/2}$, and the broken norm $\|v\|_{m,\mathscr{T}} := \left( \sum_{\tau \in \mathscr{T}} \|v\|_{m,\tau}^2 \right)$ respectively.

For the finite element solution of (2.1a-b) we define a bilinear form $a(\ .,.\ ) : H_{\mathscr{T}}^1(\Omega) \times H_{\mathscr{T}}^1(\Omega) \to \mathbf{R}$ by

$$a(v, w) := \sum_{\tau \in \mathscr{T}} a_\tau(v, w), \quad a_\tau(v, w) := \int_\tau a \, \nabla v \cdot \nabla w \, dx + \int_\tau b v w \, dx.$$

Then, given an appropriate subspace $V \subset H^1_{\mathcal{G}}(\Omega)$ we aim to compute $u \in V$ satisfying

$$a(u, v) = (f, v)_{0, \mathcal{G}}, \quad v \in V. \tag{2.3}$$

If the bilinear form $a(\;.\;,\;.\;)$ is $V$-elliptic, the existence and the uniqueness of a solution is guaranteed by the Lax-Milgram Lemma. In this case $\|\|v\|\| := a(v, v)^{1/2}$ defines a norm on $V$ which is said to be the (broken) energy norm. Note that in case $V = H^1_0(\Omega)$ the variational equation (2.3) represents the weak formulation of (2.1a-b) and its unique solution $u \in H^1_0(\Omega)$ is called the weak solution.

   We denote by $\mathcal{P}_I, \mathcal{E}_I$ the sets of interior vertices and interior edges of $\mathcal{T}$ and by $\mathcal{P}_\Gamma, \mathcal{E}_\Gamma$ the sets of vertices and edges on the boundary $\Gamma$, and we set $\mathcal{P} := \mathcal{P}_I \cup \mathcal{P}_\Gamma, \mathcal{E} := \mathcal{E}_I \cup \mathcal{E}_\Gamma$. If $e \in \mathcal{E}$, then $m_e$ stands for the midpoint of $e$. Further, we refer to $\mathcal{M}_I, \mathcal{M}_\Gamma$ as the sets of the midpoints of the interior edges and the edges on $\Gamma$, and we set $\mathcal{M} := \mathcal{M}_I \cup \mathcal{M}_\Gamma$. Finally, we define $P^k(\tau)$ as the set of polynomials of degree at most $k \in \mathbb{N}$ on $\tau \in \mathcal{T}$. We are interested in the computation of an approximation to the weak solution $u \in H^1_0(\Omega)$ using the Crouzeix-Raviart nonconforming finite elements of lowest order, i.e., in (2.3) we choose $V = N_{\mathcal{G}}(\Omega)$ where

$$N_{\mathcal{G}}(\Omega) := \big\{ v \in L^2(\Omega) | v|_\tau \in P^1(\tau), \tau \in \mathcal{T},$$

$$v|_{\tau_1}(m_e) = v|_{\tau_2}(m_e), \tau_1 \cap \tau_2 = e \in \mathcal{E}_I,$$

$$v|_\tau(m_e) = 0, \tau \cap \Gamma = e \in \mathcal{E}_\Gamma \big\}.$$

For further reference we also define

$$L_{\mathcal{G}}(\Omega) := \big\{ v \in L^2(\Omega) | v|_\tau \in P^1(\tau), \tau \in \mathcal{T},$$

$$v|_\tau(m_e) = 0, \tau \cap \Gamma = e \in \mathcal{E}_\Gamma \big\},$$

$$Q_{\mathcal{G}}(\Omega) := \big\{ v \in L^2(\Omega) | v|_\tau \in P^2(\tau), \tau \in \mathcal{T},$$

$$v|_\tau(m_e) = 0, \tau \cap \Gamma = e \in \mathcal{E}_\Gamma \text{ and}$$

$$\int_e v|_\tau \, d\sigma = 0, e \in \mathcal{E}_\Gamma \big\},$$

and we refer to $L^0_{\mathcal{G}}(\Omega) := \{ v \in C_0(\Omega) | v|_\tau \in P^1(\tau), \tau \in \mathcal{T} \}$ and $Q^0_{\mathcal{G}}(\Omega) := \{ v \in C_0(\Omega) | v|_\tau \in P^2(\tau), \tau \in \mathcal{T} \}$ as the conforming finite element spaces of continuous piecewise linear and continuous piecewise qua-

dratic finite element functions with respect to $\mathcal{T}$, respectively. The nonconforming piecewise linear basis functions spanning $N_{\mathcal{T}}(\Omega)$ will be denoted by $\psi_m^N$ with supporting points $m \in \mathcal{M}_I$, the conforming piecewise linear and piecewise quadratic basis functions spanning $L_{\mathcal{T}}^0(\Omega)$ and $Q_{\mathcal{T}}^0(\Omega)$ by $\psi_p^L$ and $\psi_p^Q$, $\psi_m^Q$ with supporting points $p \in \mathcal{P}_I$ and $p \in \mathcal{P}_I$, $m \in \mathcal{M}_I$. If we choose $V = N_{\mathcal{T}}(\Omega)$, $V = L_{\mathcal{T}}^0(\Omega)$ or $V = Q_{\mathcal{T}}^0(\Omega)$, then (2.3) is uniquely solvable and we denote the corresponding solutions by $u_N$, $u_L$ and $u_Q$, respectively. We further assume that $U \in N_{\mathcal{T}}(\Omega)$ is some approximation of $u_N$ computed for example by preconditionned $cg$-iterations using the conforming BPX-type preconditioner for nonconforming $P_1$ elements as proposed by Oswald in [10].

The local error estimators for the global discretization error $u - U$ to be constructed in the subsequent sections will rely on the assumption that the piecewise quadratic approximation $u_Q$ approximates $u$ of higher accuracy than the nonconforming approximation $u_N$. To state that assumption in a more precise form we denote by $|\,.\,|_{\mathscr{E}}$ the $L^2$-norm on $L^2(\mathscr{E})$ given by $|v|_{\mathscr{E}} := \left( \sum_{e \in \mathscr{E}} \int_e v^2 \, d\sigma \right)^{1/2}$. In Section 3 we will frequently consider the traces $v|_e$ and $\partial v/\partial n|_e$, $e \in \mathscr{E}$, of piecewise $H^m$-functions. In particular, we will be interested in the $|\,.\,|_{\mathscr{E}}$-norms of the functions $[v]_I$, $[\partial v/\partial n]_I$, $I \in \{A, J\}$, where $[v]_A$ and $[v]_J$ stand for the average and the jump on the edges defined as follows: if $e \in \mathscr{E}_I$ and $e = \tau_i \cap \tau_0$, then $[v]_A|_e := 1/2(v|_{\tau_0} + v|_{\tau_i})|_e$ and $[v]_J|_e := (v|_{\tau_0} - v|_{\tau_i})|_e$ while $[v]_A|_e := 1/2(v|_\tau)|_e$ and $[v]_J|_e := (-v|_\tau)|_e$ if $e \in \mathscr{E}_\Gamma$ and $e = \tau \cap \Gamma$. Note that the sign of $[v]_J|_e$ depends on the specification of $\tau_i$ and $\tau_0$. Further, the functions $[\partial v/\partial n]_A$ and $[\partial v/\partial n]_J$ can be defined analogously if $n$ is chosen as the outer unit normal of $\tau_i$ in case $e = \tau_i \cap \tau_0 \in \mathscr{E}_I$ and of $\tau$ if $e = \tau \cap \Gamma \in \mathscr{E}_\Gamma$. (Note that $[\partial v/\partial n]_J|_e = 0$ in the latter case.)

Now, setting $e_N := u - u_N$ and $e_Q := u - u_Q$ in the subsequent sections we suppose the existence of a function $\beta_N = \beta_N(h_{\mathcal{T}})$, $h_{\mathcal{T}} := \max_{\tau \in \mathcal{T}} h_\tau$, $h_\tau := \operatorname{diam} \tau$, with $\beta_N(h_{\mathcal{T}}) \to 0$ as $h_{\mathcal{T}} \to 0$ such that

$$\||e_Q\|| + \left| h_e^{1/2} \left[ \frac{\partial(e_Q)}{\partial n} \right]_A \right|_{\mathscr{E}} \leq \beta_N \||e_N\|| \tag{2.4}$$

where $h_e$ stands for the length of $e \in \mathscr{E}$. This assumption is supported by the fact that under appropriate regularity conditions on the data of the problem the left-hand side is of order $O(h_{\mathcal{T}}^2)$ whereas $\||e_N\||$ is only of order $O(h_{\mathcal{T}})$. As a consequence of (2.4) we obtain the following result.

LEMMA 2.1 : *Under the assumption (2.4) there exists a constant $c_e > 0$ independent of $\mathcal{T}$ such that :*

$$\left| h_e^{1/2} \left[ a \frac{\partial(u - U)}{\partial n} \right]_A \right|_{\mathcal{E}} \leqslant C_1 \|u_N - U\| + C_2 \|u - U\| \tag{2.5}$$

*where*  $C_1 := \max(1, c_e) \beta_N$  *and*  $C_2 := c_e + C_1.$

*Proof :* The triangle inequality gives

$$\left| h_e^{1/2} \left[ a \frac{\partial(u - U)}{\partial n} \right]_A \right|_{\mathcal{E}} \leqslant \left| h_e^{1/2} \left[ a \frac{\partial(u - u_Q)}{\partial n} \right]_A \right|_{\mathcal{E}} +$$

$$+ \left| h_e^{1/2} \left[ a \frac{\partial(u_Q - U)}{\partial n} \right]_A \right|_{\mathcal{E}} . \tag{2.6}$$

Since the first term on the right-hand side of (2.6) already appears on the left-hand side in (2.4), we only have to estimate the second term. We find

$$\sum_{e \in \mathcal{E}} \left| h_e^{1/2} \left[ a \frac{\partial(u_Q - U)}{\partial n} \right]_A \right|_e^2 \leqslant \sum_{e \in \mathcal{E}} \frac{h_e}{2} \bar{a}_e^2 \left( \left| \left( \frac{\partial(u_Q - U)}{\partial n} \right) \right|_{\tau_0} \right|_e^2 \right.$$

$$\left. + \left| \left( \frac{\partial(u_Q - U)}{\partial n} \right) \right|_{\tau_i} \right|_e^2 \right) \tag{2.7}$$

Using the elementary inequality

$$|\partial v/\partial n|_{\mathcal{E}_\tau}^2 \leqslant \alpha(h_\tau^{-1} \|\nabla v\|_{0,\tau}^2 + h_\tau \|\nabla^2 v\|_{0,\tau}^2) \quad \forall v \in H^2(\tau)$$

where $\mathcal{E}_\tau$ is the set of all edges of $\tau \in \mathcal{T}$ and $\alpha > 0$ is independent of $\tau$, as well as the inverse inequality

$$\|\nabla^2 v\|_{0,\tau} \leqslant \beta h_\tau^{-1} \|\nabla v\|_{0,\tau}, \quad v \in P^2(\tau)$$

with $\beta > 0$ independent of $\tau$, both terms on the right-hand side in (2.7) can be bounded by $\alpha(1 + \beta^2) \|\nabla(u_Q - U)\|_{0,\tau}^2$, $\tau \in \{\tau_i, \tau_0\}$. We thus obtain

$$\left| h_e^{1/2} \left[ a \frac{\partial(u_Q - U)}{\partial n} \right]_A \right|_{\mathcal{E}} \leqslant c_e(\|u - U\| + \|u_Q - u\|) \tag{2.8}$$

where $c_e := (1/2 \, \kappa \bar{a} \alpha(1 + \beta^2))^{1/2}$. Using (2.4) and (2.8) in (2.6) gives the assertion. ∎

For the computation of norms or seminorms of piecewise linear or quadratic functions it is useful to have equivalent discrete expressions at hand. The following two results which will be needed in the subsequent sections are easy to prove and therefore, the proof is omitted.

LEMMA 2.2 : *There exist constants $0 < \lambda \leq \Lambda$ depending only on the shape regularity of $\mathcal{T}_0$ such that for all $\tau \in \mathcal{T}$ and $v \in P^1(\tau)$*

$$\lambda \sum_{i,j=1}^{3} (v(m_i) - v(m_j))^2 \leq |v|_{1,\tau}^2 \leq \Lambda \sum_{i,j=1}^{3} (v(m_i) - v(m_j))^2$$

*where the $m_i$'s, $1 \leq i \leq 3$, stand for the midpoints of the edges of $\tau$.*

If $\tau \in \mathcal{T}$, we additionally denote by $p_i$, $1 \leq i \leq 3$, the vertices of $\tau$ and define for an arbitrarily, but fixed chosen $i_0 \in \{1, 2, 3\}$ :

$$\delta_\tau^{(1)} := \sum_{i=1}^{3} \left\{ (v(p_i) - v(m_{i_0}))^2 + (v(m_i) - v(m_{i_0}))^2 \right\},$$

$$\delta_\tau^{(2)} := \sum_{i=1}^{3} \left\{ (v(p_i) - v(p_{i_0}))^2 + (v(m_i) - v(p_{i_0}))^2 \right\},$$

$$\Delta_\tau^{(1)} := \delta_\tau^{(1)} + h_\tau^2 v^2(m_{i_0}), \quad \Delta_\tau^{(2)} := \delta_\tau^{(2)} + h_\tau^2 v^2(p_{i_0}).$$

LEMMA 2.3 : *There exist constants $0 < \xi \leq \Xi$ and $0 < \theta \leq \Theta$ depending only on the shape regularity of $\mathcal{T}_0$ such that for all $\tau \in \mathcal{T}$, $v \in P^2(\tau)$ and $1 \leq i \leq 2$*

$$\xi \delta_\tau^{(i)} \leq |v|_{1,\tau}^2 \leq \Xi \delta_\tau^{(i)}, \quad \theta \Delta_\tau^{(i)} \leq \|v\|_{1,\tau}^2 \leq \Theta \Delta_\tau^{(i)}.$$

## 3. ELEMENT-ORIENTED LOCAL ERROR ESTIMATORS

As in the case of conforming finite element methods treated by Bank and Weiser in [4] we will derive element-oriented local error estimators whose computations amount to the elementwise solution of either a linear algebraic system with a $6 \times 6$ coefficient matrix or a linear system with a $3 \times 3$ coefficient matrix plus the computation of a local energy norm which, however, can be easily evaluated in view of Lemma 2.3. Although the basic ideas in deriving these error estimators are the same as in [4], we will need

additional tools which take care of the peculiarities caused by the noncon-
forming setting. In particular, we consider linear operators $I:$
$Q_{\mathcal{T}}(\Omega) \to L_{\mathcal{T}}(\Omega)$ and $Q: N_{\mathcal{T}}(\Omega) \to L_{\mathcal{T}}^0(\Omega)$. The operator $I$ will be used to
specify the distance between a function $v \in Q_{\mathcal{T}}(\Omega)$ and the subspace
$L_{\mathcal{T}}(\Omega)$ while $Q$ will serve as a measure for the discontinuity of the noncon-
forming finite element functions. We require the operator $I$ to satisfy the
following conditions :

• There exists a constant $C_I \geq 1$ independent of $h_{\mathcal{T}}$ such that

$$\sup \left\{ |||Iv|||/|||v||| \mid v \in Q_{\mathcal{T}}(\Omega), |||v||| \neq 0 \right\} \leq C_I ; \qquad (3.1a)$$

• $Iv \in L_{\mathcal{T}}^0(\Omega)$ for all $v \in Q_{\mathcal{T}}^0(\Omega)$ ; $\qquad (3.1b)$
• $Iv = v$ for all $v \in L_{\mathcal{T}}(\Omega)$. $\qquad (3.1c)$

If we define $I^\perp$ as the orthogonal projection of $Q_{\mathcal{T}}(\Omega)$ onto $L_{\mathcal{T}}(\Omega)$ with
respect to the energy inner product $a(\,.\,,\,.\,)$, we may uniquely decompose
each $v \in Q_{\mathcal{T}}(\Omega)$ according to

$$v = v_L + v_{Q^\perp} \qquad (3.2)$$

where $v_L := I^\perp v \in L_{\mathcal{T}}(\Omega)$ and $v_{Q^\perp} := v - I^\perp v$, $v_{Q^\perp} \in Q_{\mathcal{T}}^\perp(\Omega) :=$
$\{ v \in Q_{\mathcal{T}}(\Omega) \mid I^\perp v = 0 \}$. Obviously

$$a(v, Iv) = 0, \quad v \in Q_{\mathcal{T}}^\perp . \qquad (3.3)$$

Note that $I^\perp$ satisfies (3.1a) with $C_{I^\perp} = 1$ but does not satisfy condition
(3.1b). On the other hand we can show :

LEMMA 3.1 : *Assume (3.1a-c). Then there holds*

$$|||Iv||| \leq \sqrt{C_I^2 - 1} \, |||v|||, \quad v \in Q_{\mathcal{T}}^\perp(\Omega) ; \qquad (3.4a)$$

$$|||v - Iv||| \leq C_I |||v - I^\perp v|||, \quad v \in Q_{\mathcal{T}}(\Omega) ; \qquad (3.4b)$$

$$|||v - Iv + w|||^2 \geq 2 \left( 1 - \frac{\sqrt{C_I^2 - 1}}{C_I} \right) |||v - Iv||| \cdot |||w||| \qquad (3.4c)$$

*for all* $v \in Q_{\mathcal{T}}(\Omega)$, $w \in L_{\mathcal{T}}(\Omega)$.

*Proof :* Using (3.1a) and (3.3), for any $\alpha \in \mathbf{R}$ we get

$$(1 + \alpha)^2 |||Iv|||^2 = |||I(v + \alpha Iv)|||^2 \leq C_I^2 |||v + \alpha Iv|||^2$$

$$= C_I^2 ( |||v|||^2 + \alpha^2 |||Iv|||^2 )$$

whence

$$( \alpha^2 ( 1 - C_I^2 ) + 2 \alpha + 1 ) |||Iv|||^2 \leqslant C_I^2 |||v|||^2 .$$

Without loss of generality we may assume $C_I > 1$. The quadratic function $q(\alpha) := ( 1 - C_I^2 ) \alpha^2 + 2 \alpha + 1$ attains its positive maximum $C_I^2$. $( C_I^2 - 1 )^{-1}$ in $\alpha_{max} := ( C_I^2 - 1 )^{-1}$ so that (3.4a) follows with $\alpha = \alpha_{max}$.

Using the decomposition (3.2), the assertion (3.4b) is an immediate consequence of (3.1c), (3.3) and (3.4a) :

$$|||v - Iv|||^2 = |||v_{Q^\perp} - Iv_{Q^\perp}|||^2 = |||v_{Q^\perp}|||^2 + |||Iv_{Q^\perp}|||^2$$

$$\leqslant C_I^2 |||v_{Q^\perp}|||^2 = C_I^2 |||v - I^\perp v|||^2 .$$

Using again the decomposition (3.2) and taking (3.1c) and (3.3) into account, the proof of (3.4c) amounts to the computation of the maximal $\chi_{max}$ among all $\chi \in \mathbf{R}$ satisfying

$$|||v_{Q^\perp}|||^2 + |||w - Iv_{Q^\perp}|||^2 - \chi \sqrt{|||v_{Q^\perp}|||^2 + |||Iv_{Q^\perp}|||^2} . |||w||| \geqslant 0$$

for all $v \in Q_{\mathcal{T}}(\Omega)$ and all $w \in L_{\mathcal{T}}(\Omega)$. By (3.4a) $|||v_{Q^\perp}||| = 0$ implies $|||Iv_{Q^\perp}||| = 0$ and hence, without loss of generality we may assume $|||v_{Q^\perp}||| = 1$. Then, setting $\alpha := |||w|||$ and $\beta := |||Iv_{Q^\perp}|||$ and observing (3.4a) we have that $\chi_{max} \geqslant \tilde{\chi}_{max}$ where $\tilde{\chi}_{max}$ is the maximum of all $\tilde{\chi} \in \mathbf{R}$ for which

$$1 + ( \alpha - \beta )^2 - \tilde{\chi} \alpha \sqrt{1 + \beta^2} \geqslant 0$$

holds true for all $\alpha \geqslant 0$ and all $0 \leqslant \beta \leqslant ( C_I^2 - 1 )^{1/2}$. An easy calculation reveals $\tilde{\chi}_{max} = 2( 1 - ( 1 - C_I^{-2} )^{1/2} )$ which gives the assertion. ∎

By (3.3) the spaces $Q_{\mathcal{T}}^\perp(\Omega)$ and $L_{\mathcal{T}}(\Omega)$ are orthogonal with respect to the inner product $a( . , . )$. If we define $Q_{\mathcal{T}}^I(\Omega) := \{ v \in Q_{\mathcal{T}}(\Omega) | Iv = 0 \}$, then (3.4c) in the preceding Lemma allows us to show that elements in $Q_{\mathcal{T}}^I(\Omega)$ and $L_{\mathcal{T}}(\Omega)$ form an acute angle.

COROLLARY 3.2 : *Under the assumptions (3.1a-c) there holds*

$$|a( v, w )| \leqslant \eta^2 . |||v||| . |||w|||, \quad v \in Q_{\mathcal{T}}^\perp(\Omega), w \in L_{\mathcal{T}}(\Omega) \qquad (3.5)$$

*where* $\quad \eta^2 := C_I^{-1}( C_I^2 - 1 )^{1/2}.$

*Proof :* We may exclude the cases $\|\|v\|\| = 0$ or $\|\|w\|\| = 0$, since then (3.5) becomes trivial. Replacing $v$ in (3.4c) by $\tilde{v} := \pm \|\|v\|\|^{-1}\|\|w\|\| . v$, $v \in Q^I_{\mathcal{T}}(\Omega)$, gives

$$\|\|w + \tilde{v}\ \|\|^2 = 2( \|\|w\|\|^2 \pm \|\|v\|\|^{-1}\|\|w\|\| . a(v, w)) \geqslant 2(1 - \eta^2)\|\|w\|\|^2$$

whence

$$- \eta^2\|\|v\|\| . \|\|w\|\| \leqslant a(v, w) \leqslant \eta^2\|\|v\|\| . \|\|w\|\| \quad \blacksquare$$

The following results can be easily deduced from (3.4a).

COROLLARY 3.3 : *Assume (3.1a-c). Then we have*

$$\|\|u_Q - u_L\|\| \leqslant C_I\|\|u_Q - u_N\|\| . \tag{3.6}$$

*If we additionally suppose (2.4), then there holds*

$$\|\|u - u_L\|\| \leqslant \tilde{C}_I\|\|u - u_N\|\| \tag{3.7}$$

*where* $\quad \tilde{C}_I := C_I + \beta_N( 1 + C_I).$

*Proof :* For all $v_L \in L^0_{\mathcal{T}}(\Omega)$ we have

$$a(u_Q - u_L, u_Q - u_L) = a(u_Q - u_L, u_Q - v_L) .$$

In view of (3.1b) we may choose $v_L = Iu_Q$ and hence, using (3.4a) we get

$$\|\|u_Q - u_L\|\| \leqslant \|\|u_Q - Iu_Q\|\| \leqslant C_I\|\|u_Q - I^\perp u_Q\|\|$$

$$= C_I \inf_{v \in L_{\mathcal{T}}(\Omega)} \|\|u_Q - v\|\| \leqslant C_I\|\|u_Q - u_N\|\| .$$

Setting $u - u_L = (u - u_Q) + (u_Q - u_L)$, (3.7) is an immediate consequence of the triangle inequality and (2.4), (3.4a) and (3.5). $\quad\blacksquare$

A natural choice for an operator $I$ satisfying (3.1a-c) is to define $I$ locally as the Lagrangian interpolation operator

$$Iv|_\tau := \sum_{i=1}^{3} v|_\tau(p_{\tau, i}) . \psi^L_{p_{\tau, i}}$$

where $p_{\tau, i}$, $1 \leqslant i \leqslant 3$, are the vertices of the triangle $\tau \in \mathcal{T}$.

In the sequel another main tool will be a projection $Q :$ $N_{\mathcal{T}}(\Omega) \mapsto L^0_{\mathcal{T}}(\Omega)$ which is subjected to the following conditions :

● There exists a constant $C_Q \geq 1$ independent of $h_{\mathcal{T}}$ such that

$$\sup \left\{ |||Qv|||/|||v||| \mid v \in N_{\mathcal{T}}(\Omega), |||v||| \neq 0 \right\} \leq C_Q \qquad (3.8a)$$

● $Qv = v$ for all $v \in L_{\mathcal{T}}^0(\Omega)$. $\qquad (3.8b)$

LEMMA 3.4 : *Assume (2.4), (3.1a-c) and (3.8a-b). Then there holds*

$$|||U - QU||| \leq C_3 |||u - U||| + C_4 |||U - u_N||| \qquad (3.9)$$

*where* $C_3 := (1 + C_Q)(1 + \tilde{C}_I)$, *and* $C_4 := \tilde{C}_I(1 + C_Q)$ *with* $\tilde{C}_I$ *from (3.7).*

*Proof:* Using (3.7) and (3.8a-b) we have

$$|||U - QU||| = |||U - u_L - Q(U - u_L)||| \leq (1 + C_Q) |||U - u_L|||$$

$$\leq (1 + C_Q)(|||U - u||| + \tilde{C}_I |||u - u_N|||)$$

$$\leq (1 + C_Q)((1 + \tilde{C}_I)|||U - u||| + \tilde{C}_I|||U - u_N|||) . \qquad \blacksquare$$

If we define $Q$ as the pseudo-interpolant proposed in the well known thesis of Xu [13 ; section 4.5], condition (3.8a) is guaranteed [13 ; Lemma 4.9]. But it is easy to see that (3.8b) is not satisfied.

A natural choice for an operator $Q$ satisfying (3.8a-b) is the quasi-interpolation operator as proposed by Oswald in [10] :

$$Qv := \sum_{p \in \mathcal{P}_I} \lambda_p(v) \, \psi_p^L \qquad (3.10)$$

$$\lambda_p(v) := v_p^{-1} \sum_{l=1}^{v_p} v|_{\tau_{l,p}}(p) $$

where $v_p$ is the number of triangles containing $p$ as a vertex and $\tau_{l,p}$, $1 \leq l \leq v_p$, are the triangles having $p$ as a vertex. Since the nonconforming finite element functions are defined by their values at the midpoints of the edges, it is more convenient to express $\lambda_p(v)$ by these values

$$\lambda_p(v) = v_p^{-1} \sum_{l=1}^{v_p} (2 \, v(m_{l,p}^a) - v(m_{l,p}^b)) \qquad (3.11)$$

where $m_{l,p}^a$ are the midpoints of the edges emanating from $p$ while $m_{l,p}^b$ are the midpoints of the edges opposite to $p$ (*cf. fig.* 3.1 below).
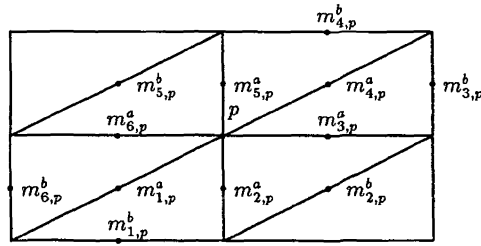
**Figure 3.1. — Position of midpoints** $m_{i,p}^a, m_{i,p}^b$ .

It is immediately clear by (3.10) that $Q$ satisfies (3.8$b$). The fact that $Q$ also satisfies (3.8$a$) has been established by Oswald in [10].

After these prerequisites we are now able to derive two element-oriented local error estimators which are based on the variational equation satisfied by the global discretization error $u - U$. As in [4] throughout the rest of this section we will assume $a \in C(\overline{\Omega})$.

LEMMA 3.5 : *Let* $l^{(1)} : Q_{\mathcal{J}}(\Omega) \mapsto \mathbf{R}$ *be the functional given by*

$$l^{(1)}(v) := \sum_{\tau \in \mathcal{J}} \int_\tau (f - L(U)) \cdot v \, dx$$

$$+ \sum_{e \in \mathcal{E}} \int_K \left[ a \frac{\partial U}{\partial n} \right]_J \cdot [v]_A \, d\sigma, \quad v \in Q_{\mathcal{J}}(\Omega) . \tag{3.12}$$

*Then there holds*

$$a(u_Q - U, v_Q + v_N) = l^{(1)}(v_Q + v_N) + a(u_Q - u_N, v_N)$$

$$+ \sum_{e \in \mathcal{E}} \int_e \left[ a \frac{\partial U}{\partial n} \right]_A [v_N]_J \, d\sigma,$$

$$v_Q \in Q_{\mathcal{J}}^0(\Omega), v_N \in N_{\mathcal{J}}(\Omega) , \tag{3.13a}$$

$$a(u - U, v) = l^{(1)}(v) + \sum_{e \in \mathcal{E}} \int_e \left[ a \frac{\partial (U - u)}{\partial n} \right]_A \cdot [v]_J \, d\sigma \quad v \in Q_{\mathcal{J}}(\Omega) . \tag{3.13b}$$

*Proof :* In view of (2.3) it follows that

$$a(u_Q, v_Q + v_N) = (f, v_Q + v_N)_{0, \mathcal{T}} + a(u_Q - u_N, v_N),$$

$$a(u, v) = (f, v)_{0, \mathcal{T}} - \sum_{e \in \mathcal{E}} \int_e \left[ a \frac{\partial u}{\partial n} \right]_A \cdot [v]_J \, d\sigma.$$

Then, both equations (3.12a) and (3.12b) can be easily established by an application of Green's formula for each $\tau \in \mathcal{T}$. ∎

As in [4] we now define $e_1 \in Q_{\mathcal{T}}(\Omega)$ in such a way that $e_1|_\tau$ satisfies

$$a_\tau(e_1|_\tau, v) = l_\tau^{(1)}(v - Iv), \quad v \in Q_{\mathcal{T}}(\Omega), \tag{3.14a}$$

$$\int_\tau e_1 \, dx = 0, \text{ if } \tau \in \Omega \text{ such that } b|_\tau \equiv 0, \tag{3.14b}$$

where $l_\tau^{(1)}(v) := l^{(1)}(v|_\tau)$, $\tau \in \mathcal{T}$. The computation of $e_1|_\tau$ amounts to the solution of a linear algebraic system $A_\tau e_1|_\tau = b_\tau^{(1)}$ where $A_\tau$ and $b_\tau^{(1)}$ are the local stiffness matrix and local load vector, respectively. In particular, $A_\tau$ is a $6 \times 6$ matrix, if $\tau \in \Omega$, and of lower dimension, if $\tau \cap \Gamma \neq \emptyset$. Note that $A_\tau$ has full rank unless $\tau \in \Omega$ and $b|_\tau \equiv 0$ in which case rank $(A_\tau, b_\tau^{(1)}) = 5$ ensuring the existence of a solution while the uniqueness is then implied by the additional condition (3.14b).

As we shall see later on, the energy norm of $e_1$ does only provide a lower bound for the global discretization error $u - U$, since it does not account for the discontinuity of $U$ across the edges. Therefore, we have to look for an appropriate modification. For that purpose we introduce $l^{(2)} : Q_{\mathcal{T}}(\Omega) \mapsto \mathbf{R}$ as the functional given by

$$l^{(2)}(v) := l^{(1)}(v - Iv) + a(QU - U, v), \quad v \in Q_{\mathcal{T}}(\Omega) \tag{3.15}$$

and define $e_2 \in Q_{\mathcal{T}}(\Omega)$ locally as the solution of

$$a_\tau(e_2|_\tau, v) = l_\tau^{(2)}(v), \quad v \in Q_{\mathcal{T}}(\Omega), \tag{3.16a}$$

$$\int_\tau e_2 \, dx = \int_\tau (QU - U) \, dx, \text{ if } \tau \in \Omega \text{ and } b|_\tau \equiv 0, \tag{3.16b}$$

where again $l_\tau^{(2)}(v) := l^{(2)}(v|_\tau)$, $\tau \in \mathcal{T}$. By the same arguments as before, $e_2|_\tau$ is uniquely determined and can be computed as the solution of a linear algebraic system of dimension equal to or lower than 6 with the same local stiffness matrix as in (3.14).

Since

$$a(e_2 - e_1, v) = a(QU - U, v), \quad v \in Q_{\mathcal{J}}(\Omega),   \tag{3.17}$$

it follows that $e_2$ and $e_1$ are related by

$$e_2 = e_1 + QU - U.   \tag{3.18}$$

Further, by (3.14) we have $a(e_1, v) = 0$, $v \in L_{\mathcal{J}}(\Omega)$, and hence, (3.17) implies the orthogonality relation

$$\|\|e_2\|\|^2 = \|\|e_1\|\|^2 + \|\|U - QU\|\|^2.   \tag{3.19}$$

Moreover, in view of (3.18) we get

$$e_2 - Ie_2 = e_1 - Ie_1   \tag{3.20}$$

and thus, taking account of $I^{\perp} e_1 = 0$ and (3.19), Lemma 3.1 gives

$$\|\|e_2 - Ie_2\|\| = \|\|e_1 - Ie_1\|\| \le C_l \|\|e_1\|\| \le C_l \|\|e_2\|\|.   \tag{3.21}$$

With the help of the preceding results we can now show that the energy norm of $e_2$ does provide a lower bound for the global discretization error.

THEOREM 3.6 : *Under the assumptions (2.4), (3.1a-c) and (3.8a-b) there holds*

$$\|\|e_2\|\| \le \gamma_1 \|\|u - U\|\| + \gamma_2 \|\|u_N - U\|\|   \tag{3.22}$$

*where $\gamma_1$ and $\gamma_2$ are positive constants depending only on the shape regularity of $\mathcal{J}_0$ and on the constants $C_i$, $1 \le i \le 4$, $C_l$, $\bar{C}_l$ and $C_Q$.*

*Proof :* Using (3.20) we get

$$\|\|e_1\|\|^2 = l^{(1)}(e_1 - Ie_1) = l^{(1)}(e_2 - Ie_2)$$

$$= a(u - U, e_2 - Ie_2) - \sum_{e \in \mathcal{E}} \int_e \left[ a \frac{\partial(U - u)}{\partial n} \right]_A \cdot [e_2 - Ie_2]_J \, d\sigma.$$

In order to bound the right-hand side we remark that if $v \in H_{\mathcal{J}}(\Omega)$ and

$$\|\|v\|\|_{0, \tau} \le \gamma h_\tau \, a_\tau(v, v)^{1/2}, \quad \tau \in \mathcal{J}$$

with $\gamma > 0$ independent of $\tau$, then it is easy to show that there exists $\Gamma > 0$ depending on $\underline{a}$, $\gamma$ and the shape regularity of $\mathcal{T}_0$ such that

$$|h_e^{-1/2}[v]_J|_{\mathscr{E}} \leq \Gamma \|v\| .$$

Applying the above inequality, Lemma 2.1 and (3.21) it follows that

$$\|e_1\|^2 \leq \|u - U\| \cdot \|e_2 - Ie_2\|$$

$$+ \Gamma \|e_2 - Ie_2\| \cdot ( C_1 \|u_N - U\| + C_2 \|u - U\| )$$

$$\leq C_I ( ( 1 + \Gamma C_2 ) \|u - U\| + C_1 \Gamma \|u_N - U\| ) \cdot \|e_2\| . \tag{3.23}$$

Finally, (3.9) and (3.19) imply

$$\|U - QU\|^2 \leq \|U - QU\| \|e_2\|$$

$$\leq \|e_2\| ( C_3 \|u - U\| + C_4 \|u_N - U\| ) \tag{3.24}$$

and hence, using (3.23) and (3.24) in (3.19) gives the assertion. $\blacksquare$

On the other hand, the energy norm of $e_2$ also provides an upper bound for the error.

THEOREM 3.7 : *Under the same assumptions as in Theorem 3.6 we have*

$$( 1 - \beta_N ) \|u - U\| \leq \|e_2\| + ( C_I C_Q + \beta_N ) \|u_N - U\| . \tag{3.25}$$

*Proof :* Assumption (2.4) implies

$$\|u - U\| \leq \|u_Q - u\| \|u_Q - U\|$$

$$\leq \beta_N \|u - u_N\| + \|u_Q - U\|$$

whence

$$( 1 - \beta_N ) \|u - U\| \leq \beta_N \|u_N - U\| + \|u_Q - U\| . \tag{3.26}$$

Moreover, by (3.1c), (3.8b) and (3.13a) we have

$$\|\|u_Q - U\|\|^2 = a(u_Q - U, u_Q - Iu_Q) + a(u_Q - U, Iu_Q - U)$$

$$= l^{(1)}(u_Q - U - I(u_Q - U)) + a(u_Q - U, Iu_Q - QU)$$

$$+ a(u_Q - U, QU - U)$$

$$= a(e_1 + QU - U, u_Q - U) + a(u_Q - U, QI(u_Q - U))$$

$$= a(e_2, u_Q - U) + a(u_N - U, QI(u_Q - U))$$

$$\leq (\|\|e_2\|\| + C_I C_Q \cdot \|\|u_N - U\|\|) \|\|u_Q - U\|\| . \tag{3.27}$$

Using (3.27) in (3.26) gives the assertion. ∎

We will now concentrate on the construction of an error estimator requiring less computational work. For that purpose we first define $e_3|_\tau$, $\tau \in \mathcal{T}$, as the unique solution of

$$a_\tau(e_3|_\tau, v) = l_\tau^{(1)}(v), \quad v \in Q_{\mathcal{T}}^I(\Omega) . \tag{3.28}$$

Indeed, (3.28) represents a linear algebraic system with a symmetric positive definite coefficient matrix which is $3 \times 3$, if $\tau \in \Omega$, and of lower dimension otherwise.

In view of

$$a(e_3, v) = l^{(1)}(v) = l^{(1)}(v - Iv) = a(e_1, v), \quad v \in Q_{\mathcal{T}}^I(\Omega) , \tag{3.29}$$

it follows that $e_3$ is the elliptic projection of $e_1$ onto $Q_{\mathcal{T}}^I(\Omega)$ whence $\|\|e_3\|\| \leq \|\|e_1\|\|$. More than that we have the equivalence of $e_1$ and $e_3$.

LEMMA 3.8 : *Assume (3.1a-c). Then there holds*

$$(1 - \eta^2)^{1/2}\|\|e_1\|\| \leq \|\|e_3\|\| \leq \|\|e_1\|\| . \tag{3.30}$$

*Proof :* Only the first inequality remains to be shown. For that purpose we decompose $e_1$ according to $e_1 = e_L + e_I$ where $e_L \in L_{\mathcal{T}}(\Omega)$ and $e_I \in Q_{\mathcal{T}}^I(\Omega)$. By (3.1c) we have $a(e_1, e_L) = 0$ and hence, $a(e_L, e_L) = - a(e_L, e_I)$ giving $\|\|e_L\|\| \leq \|\|e_I\|\|$. Using Corollary 3.2 it follows that

$$\|\|e_1\|\|^2 = a(e_L, e_I) + a(e_I, e_I)$$

$$\geq \|\|e_I\|\| \cdot (\|\|e_I\|\| - \eta^2 \|\|e_L\|\|) \geq (1 - \eta^2) \|\|e_I\|\|^2$$

which yields

$$\|\!|e_1|\!\|^2 = a(e_1, e_I) + a(e_3, e_I)$$

$$\leq \|\!|e_3|\!\| \cdot \|\!|e_I|\!\| \leq (1 - \eta^2)^{-1/2} \|\!|e_3|\!\| \cdot \|\!|e_1|\!\| . \quad \blacksquare$$

With regard to this equivalence we cannot expect $e_3$ to provide a two-sided estimate of the global discretization error. However, if we define

$$e_4 := e_3 + QU - U , \qquad (3.31)$$

then we can prove equivalence of $e_2$ and $e_4$.

LEMMA 3.9 : *Under the assumptions (3.1a-c) we have*

$$(1 - \eta^2) \|\!|e_2|\!\| \leq \|\!|e_4|\!\| \leq \sqrt{(1 + \eta^2)} \|\!|e_2|\!\| . \qquad (3.32)$$

*Proof :* Corollary 3.2, (3.19) and Lemma 3.8 give

$$\|\!|e_4|\!\|^2 = \|\!|e_3|\!\|^2 + 2\, a(e_3, QU - U) + \|\!|QU - U|\!\|^2$$

$$\leq (1 + \eta^2)(\|\!|e_3|\!\|^2 + \|\!|QU - U|\!\|^2)$$

$$\leq (1 + \eta^2)(\|\!|e_1|\!\|^2 + \|\!|QU - U|\!\|^2) = (1 + \eta^2) \|\!|e_2|\!\|^2 .$$

Likewise

$$\|\!|e_4|\!\|^2 \geq (1 - \eta^2)(\|\!|e_3|\!\|^2 + \|\!|QU - U|\!\|^2)$$

$$\geq (1 - \eta^2)^2 (\|\!|e_1|\!\|^2 + \|\!|QU - U|\!\|^2) = (1 - \eta^2)^2 \|\!|e_2|\!\|^2 . \quad \blacksquare$$

Summarizing the results of Theorem 3.6, Theorem 3.7 and Lemma 3.9 we obtain :

THEOREM 3.10 : *Assume (2.4), (3.1a-c) and (3.8a-b). Then there holds*

$$(1 - \beta_N) \|\|u - U\|\| \leq (1 - \eta^2)^{-1} \|\|e_4\|\| + (C_I C_Q + \beta_N) \|\|u_N - U\|\| \qquad (3.33a)$$

$$(1 + \eta^2)^{-1/2} \|\|e_4\|\| \leq \gamma_1 \|\|u - U\|\| + \gamma_2 \|\|u_N - U\|\| . \qquad (3.33b)$$

## 4. EDGE-ORIENTED ERROR ESTIMATOR

In this section we will follow the approach in [9] to derive an error estimator which can be computed locally by the solution of scalar equations associated with the midpoints of the interior edges. The first step is that we replace the exact solution $u$ in the global discretization error $u - U$ by its piecewise quadratic approximation $u_Q$. It is immediately clear that if $u_Q$ approximates $u$ of higher accuracy than $U$ and $\tilde{\varepsilon}_Q$ is some approximation of $\varepsilon_Q := u_Q - U$ providing a two-sided estimate of $\|\|\varepsilon_Q\|\|$, then this also results in a two-sided estimate of the global discretization error $\|\|u - U\|\|$. Using the orthogonal projection $Q^\perp : N_{\mathcal{T}}(\Omega) + Q^0_{\mathcal{T}}(\Omega) \mapsto Q^0_{\mathcal{T}}(\Omega)$ given by $a(Q^\perp v, v_Q) = a(v, v_Q)$, $v_Q \in Q^0_{\mathcal{T}}(\Omega)$, we split $\varepsilon_Q$ into a « continuous » part $u_Q - Q^\perp U$ and a « discontinuous » part $Q^\perp U - U$ and obtain

$$\|\|U - u_Q\|\|^2 = \|\|Q^\perp U - u_Q\|\|^2 + \|\|Q^\perp U - U\|\|^2 . \qquad (4.1)$$

We will estimate the two terms on the right-hand side in (4.1) separately. It turns out that $\|\|Q^\perp U - u_Q\|\|$ can be estimated in much the same way as in the conforming case (*cf.* [9]) while the estimation of $\|\|Q^\perp U - U\|\|$ requires some extra tools.

We begin with the two-level splitting

$$Q^0_{\mathcal{T}}(\Omega) = L^0_{\mathcal{T}}(\Omega) \oplus V^Q_{\mathcal{T}}(\Omega) \qquad (4.2)$$

of $Q^0_{\mathcal{T}}(\Omega)$ into its linear part $L^0_{\mathcal{T}}(\Omega)$ spanned by the nodal basis functions $\psi^L_p$, $p \in \mathcal{P}_I$, and the quadratic part $V^Q_{\mathcal{T}}(\Omega)$ spanned by the quadratic nodal basis functions $\psi^Q_m$ associated with the midpoints $m \in \mathcal{M}_I$ of the interior edges. With respect to (4.2) the associated stiffness matrix $A_Q$ can be represented as a block $2 \times 2$ matrix

$$A_Q = \begin{pmatrix} A_{LL} & A_{LQ} \\ A_{QL} & A_{QQ} \end{pmatrix} .$$

Splitting $v \in Q_{\mathcal{J}}^0(\Omega)$ according to $v = v_L + v_Q$, $v_L \in L_{\mathcal{J}}^0(\Omega)$, $v_Q \in V_{\mathcal{J}}^Q(\Omega)$, and identifying finite element functions and vectors it is not hard to see that there exists a positive constant $\gamma_0$ depending only on the shape regularity of $\mathcal{J}_0$ and on the ellipticity of $a(\,.\,,\,.\,)$ such that

$$\gamma_0\, v^T \begin{pmatrix} A_{LL} & 0 \\ 0 & D_{QQ} \end{pmatrix} v \leqslant v^T \begin{pmatrix} A_{LL} & A_{LQ} \\ A_{QL} & D_{QQ} \end{pmatrix} v \leqslant 4\, v^T \begin{pmatrix} A_{LL} & 0 \\ 0 & D_{QQ} \end{pmatrix} v \quad (4.3)$$

where $D_{QQ} := \operatorname{diag}(A_{QQ})$.

Indeed, the upper estimate is easily established while the lower estimate can be deduced from Lemma 2.3 and Corollary 3.2 as follows :

$$v^T A_Q\, v = \| \sum_{p \in \mathscr{P}_l} v(p)\, \psi_p^L + \sum_{m \in \mathscr{M}_l} v(m)\, \psi_m^Q \|^2 \geqslant$$

$$\geqslant (1 - \eta^2) \left( \| \sum_{p \in \mathscr{P}_l} v(p)\, \psi_p^L \|^2 + \| \sum_{m \in \mathscr{M}_l} v(m)\, \psi_m^Q \|^2 \right)$$

$$\geqslant (1 - \eta^2) \left( \| \sum_{p \in \mathscr{P}_l} v(p)\, \psi_p^L \|^2 + \sum_{\tau \in \mathcal{J}} \underline{a}_\tau | \sum_{m \in \mathscr{M}_l \cap \tau} v(m)\, \psi_m^Q |_{1, \tau}^2 \right)$$

$$\geqslant (1 - \eta^2) \left( \| \sum_{p \in \mathscr{P}_l} v(p)\, \psi_p^L \|^2 + \right.$$

$$\left. + \Theta^{-1}\, \xi \min(\kappa^{-1}, \underline{a/b}) \sum_{m \in \mathscr{M}_l} \| v(m)\, \psi_m^Q \|^2 \right)$$

$$\geqslant \gamma_0\, v^T \begin{pmatrix} A_{LL} & 0 \\ 0 & D_{QQ} \end{pmatrix} v,$$

where $\gamma_0 := (1 - \eta^2) \min(1, \Theta^{-1}\, \xi \min(\kappa^{-1}, \underline{a/b}))$.

If $\mathcal{J} = \mathcal{J}_j$ is the final triangulation obtained from an initial coarse triangulation $\mathcal{J}_0$ by the refinement process of Bank *et al.* we may further

replace $A_{LL}$ by its *BPX*-preconditioner $H_{BPX}$. As has been shown in [5], [6], [14] there exist positive constants $c_0$ and $c_1$ depending only on the shape regularity of $\mathscr{T}_0$ and the ellipticity of $a( \, . \, , . \, )$ such that

$$c_0^{-1}(j+1)^{-1} v_L^T H_{BPX} v_L \leqslant v_L^T A_{LL} v_L \leqslant c_1 v_L^T H_{BPX} v_L \, . \qquad (4.4)$$

Denoting the residual by

$$r( v ) := (f, v )_{0, \mathscr{T}_j} - a( U, v ), \quad v \in Q_{\mathscr{T}}^0 ( \Omega ) \qquad (4.5)$$

and representing the associated vector as a block vector according to $r = ( r_L, r_Q )^T$, we obtain :

THEOREM 4.1 : *With the constants* $\gamma_0$, $c_0$ *and* $c_1$ *from (4.3), (4.4) and the residual vector r from (4.5) there holds*

$$\|\|u_Q - Q^\perp U\|\|^2 \leqslant \gamma_0^{-1} \max ( 1, c_0(j+1) ) \, r^T \begin{pmatrix} H_{BPX}^{-1} & 0 \\ 0 & D_{QQ}^{-1} \end{pmatrix} r \, , \quad (4.6a)$$

$$\|\|u_Q - Q^\perp U\|\|^2 \geqslant \min ( 1/4, 1/( 2 \, c_1 ) ) \, r^T \begin{pmatrix} H_{BPX}^{-1} & 0 \\ 0 & D_{QQ}^{-1} \end{pmatrix} r \, .$$
$$\qquad (4.6b)$$

*Remark :* If $a( U, v ) = a( u_N, v )$, $v \in L_{\mathscr{T}}^0( \Omega )$, we have $r_L \equiv 0$ and then the computation of the lower and upper bounds reduces to the solution of scalar equations associated with the midpoints of the interior edges.

As far as the estimation of $\|\|Q^\perp U - U\|\|$ is concerned it is quite clear that the computation of $Q^\perp U$ is as costly as the computation of the piecewise quadratic approximation $u_Q$ itself and therefore, we have to look for an appropriate simplification. For that purpose we introduce a further projection $Q_P : N_{\mathscr{T}}( \Omega ) + Q_{\mathscr{T}}^0( \Omega ) \mapsto Q_{\mathscr{T}}^0( \Omega )$ which we require to satisfy the following assumptions :

● There exists a constant $C_P \geqslant 1$ independent of $h_{\mathscr{T}}$ such that

$$\|\|Q_P v\|\| \leqslant C_P \|\|v\|\|, \quad v \in N_{\mathscr{T}}( \Omega ) \, ; \qquad (4.7a)$$

● $Q_P v = v$ for all $v \in Q_{\mathscr{T}}^0( \Omega )$. $\qquad\qquad\qquad\qquad\qquad (4.7b)$

As a consequence of Corollary 3.2 and (4.7a-b) we note that

$$\|\|Q_P v\|\| \leqslant \overline{C}_P \|\|v\|\|, \quad v \in N_{\mathscr{T}}( \Omega ) + Q_{\mathscr{T}}^0( \Omega ) \qquad (4.8)$$

where $\quad \overline{C}_P := ( 2( 1 - \eta^2 )^{-1} C_P^2 )^{1/2}.$

Indeed, we may uniquely split $v$ according to $v = v_N + v_I$ where $v_N \in N_{\mathcal{T}}(\Omega)$, $v_I \in Q'_{\mathcal{T}}(\Omega) \cap Q^0_{\mathcal{T}}(\Omega)$, and then get

$$\|Q_P v\|^2 = \|Q_P v_N + v_I\|^2 \leq 2 \, C^2_P(\|v_N\|^2 + \|v_I\|^2) \, ,$$

$$\|v\|^2 = \|v_N\|^2 + 2 \, a(v_N, v_I) + \|v_I\|^2$$

$$\geq (1 - \eta^2)(\|v_I\|^2 + \|v_N\|^2) \, .$$

Using the previous results it is easy to see that in $\|Q^{\perp} U - U\|$ we may replace $Q^{\perp}$ by $Q_P$.

LEMMA 4.2 : *Assume (3.1a-c) and (4.7a-b). Then there holds*

$$\tilde{C}_P\|U - Q_P U\| \leq \|U - Q^{\perp} U\| \leq \|U - Q_P U\| \tag{4.9}$$

*where*      $\tilde{C}_P := (1 + \overline{C}^2_P)^{-1/2}$.

*Proof :* Since $Q_P U \in Q^0_{\mathcal{T}}(\Omega)$, the second inequality trivially holds true. On the other hand, using (4.8) we have

$$\|U - Q_P U\|^2 = \|U - Q^{\perp} U\|^2 + \|Q^{\perp} U - Q_P U\|^2$$

$$= \|U - Q^{\perp} U\|^2 + \|Q_P(U - Q_{\perp} U)\|^2$$

$$\geq (1 + \overline{C}^2_P) \|U - Q^{\perp} U\|^2 \, . \quad \blacksquare$$

Observing assumption (4.7*b*) and the continuity of functions $v \in N_{\mathcal{T}}(\Omega)$ in $m \in \mathcal{M}_I$, a possible choice of the projection $Q_P$ is as follows

$$Q_P v := \sum_{m \in \mathcal{M}_I} v(m) \cdot \psi^Q_m + \sum_{p \in \mathcal{P}} \lambda_p(v) \, \psi^Q_p \tag{4.10}$$

where $\lambda_p(v)$ is given as in (3.10). Therefore, we may view $(Q_P v - v)(p)$, $p \in \mathcal{P}$, as a measure for the deviation from the mean value. In order to prove the continuity of $Q_P$ on $N_{\mathcal{T}}(\Omega)$ we denote by $p^i_\tau$, $1 \leq i \leq 3$, the vertices of $\tau \in \mathcal{T}$ and by $v_{\tau, i}$ the number of triangles containing $p^i_\tau$ as a vertex. Then, fixing such a triangle $\tau$, we set $\tau_{i, 1} := \tau$ and

number the remaining triangles in the mathematical positive sense with obvious modifications in case $p_\tau^i$ is a vertex on the boundary $\Gamma$. Applying Lemma 2.2 and Lemma 2.3 it follows that

$$\|v - Q_P v\|^2 \le \max(\bar{a}, \bar{b}) \, \|v - Q_P v\|_{1,\mathcal{T}}^2 \le$$

$$\le \Theta \max(\bar{a}, \bar{b}) \sum_{\tau \in \mathcal{T}} \sum_{i=1}^{3} \left( v|_\tau(p_\tau^i) - \frac{1}{v_{\tau,i}} \sum_{l=1}^{v_{\tau,i}} v|_{\tau_{i,l}}(p_\tau^i) \right)^2$$

$$\le \Theta \max(\bar{a}, \bar{b}) \sum_{\tau \in \mathcal{T}} \sum_{i=1}^{3} \frac{1}{v_{\tau,i}^2} \left( \sum_{l=1}^{v_{\tau,i}-1} (v_{\tau,i} - l)(v|_{\tau_{i,l}}(p_\tau^i) - v|_{\tau_{i,l+1}}(p_\tau^i)) \right)^2$$

$$\le \Theta \max(\bar{a}, \bar{b}) \sum_{\tau \in \mathcal{T}} \sum_{i=1}^{3} \frac{v_{\tau,i}-1}{v_{\tau,i}^2} \left( \sum_{l=1}^{v_{\tau,i}-1} (v_{\tau,i} - l)^2 (v|_{\tau_{i,l}}(p_\tau^i) - v|_{\tau_{i,l+1}}(p_\tau^i))^2 \right)$$

$$\le \frac{\Theta}{\lambda} \max(\bar{a}, \bar{b}) \sum_{\tau \in \mathcal{T}_n} \sum_{i=1}^{3} (v_{\tau,i}) \left( \sum_{l=1}^{v_{\tau,i}} |v|_{1,\tau_{i,l}}^2 \right)$$

$$\le 3 \, v_{max}^2 \frac{\Theta}{\lambda} \frac{1}{\underline{a}} \max(\bar{a}, \bar{b}) \, \|v\|^2$$

where $v_{max}$ depends on the minimum interior angle of $\mathcal{T} = \mathcal{T}_j$ and thus only on the shape regularity of $\mathcal{T}_0$.

Although $Q_P U$ can be cheaply computed, the projection $Q_P$ is not a useful tool to obtain an edge-oriented error estimator. The reason is that $Q_P U - U$ represents a discontinuous piecewise quadratic function which, however, is continuous at the midpoints of the interior edges where it attains the value zero. Therefore, we have to compute $Q_P U - U$ elementwise by evaluating it at the vertices. Anyway, no matter how the operator $Q_P$ is chosen we do not get a purely edge-oriented error estimator, since $Q_P v - v$ vanishes at all vertices if and only if $v$ is continuous. Instead of representing the deviation from the mean value at the vertices we are better off by considering the discontinuity across the interior edges. For that purpose we introduce another operator $Q_E : N_\mathcal{T}(\Omega) \mapsto Q_\mathcal{T}(\Omega)$ which is locally given by

$$Q_E v|_\tau := v|_\tau + \sum_{i=1}^{3} (v|_\tau(p_\tau^{(i+1)\bmod 3}) - v|_{\tau_i}(p_\tau^{(i+1)\bmod 3})) \cdot \psi_{m_{\tau,i}}^Q \qquad (4.11)$$

where $m_{\tau,i}$ is the midpoint of the edge $e_{\tau,i}$ of $\tau \in \mathcal{T}$ opposite to the vertex $p_\tau^i$ and $\tau_i$ is the adjoint triangle with $\tau_i \cap \tau = e_{\tau,i}$, $1 \le i \le 3$ (*cf. fig. 4.1* below).
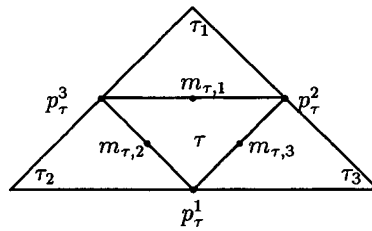
Figure 4.1. — Location of $\tau$, $\tau_i$, $p_\tau^i$ and $m_{\tau,i}$, $1 \leq i \leq 3$.

Note that in case $e_{\tau,i} \subset \Gamma$ we formally define $\tau_i$ by reflection of $\tau$ at $\Gamma$ and set $v|_{\tau_i}(p_\tau^{(i+1)\bmod 3}) := 0$.

In view of $p_\tau^{(i+1)\bmod 3} = p_{\tau_i}^{(i-1)\bmod 3}$ it is easy to see that $Q_E v$ is continuous in $m \in \mathcal{M}_I$. Further, $Q_E v - v$ represents a continuous piecewise quadratic function with zero values at the vertices of the triangles, and we can prove that the energy norms of $Q_E U - U$ and $Q_P U - U$ are equivalent.

LEMMA 4.3 : Assume that $Q_P$ and $Q_E$ are given by (4.10), (4.11), respectively. Then there exist constants $0 < c_E \leq C_E$ depending only on the shape regularity of $\mathcal{T}_0$ and the ellipticity of $a(\,.\,,.\,)$ such that

$$c_E \|\!| U - Q_P U \|\!| \leq \|\!| U - Q_E U \|\!| \leq C_E \|\!| U - Q_P U \|\!| . \qquad (4.12)$$

Proof : With regard to Lemma 2.3 we obtain

$$\|\!| U - Q_E U \|\!|^2 \leq 2 \max (\bar{a}, \bar{b}) \, \Theta \sum_{m \in \mathcal{M}_I} ( U|_{\tau_{m,1}}(p_m) - U|_{\tau_{m,2}}(p_m) )^2 \qquad (4.13a)$$

$$\|\!| U - Q_E U \|\!|^2 \geq 2 \, \underline{a} \xi \sum_{m \in \mathcal{M}_I} ( U|_{\tau_{m,1}}(p_m) - U|_{\tau_{m,2}}(p_m) )^2 \qquad (4.13b)$$

where $\tau_{m,1}$ and $\tau_{m,2}$ are those triangles having $m \in \mathcal{M}_I$ as the midpoint of their common edge and $p_m$ is a vertex situated on the same edge as $m$. Note that $p_m$ is not uniquely determined but its choice does not change the right-hand sides in (4.13a) and (4.13b). Likewise, denoting by $\overline{M}_p$ the mean value

$$\overline{M}_p := v_p^{-1} \sum_{l=1}^{v_p} U|_{\tau_l,p}(p), \quad p \in \mathcal{P} ,$$

we get

$$\|\|U - Q_P \, U\|\|^2 \leqslant \max{(\overline{a}, \overline{b})} \, \Theta \sum_{p \in \mathscr{P}} \sum_{l=1}^{v_p} (\, U|_{\tau_{l,p}}(p) - \overline{M}_p)^2 \,, \tag{4.14a}$$

$$\|\|U - Q_P \, U\|\|^2 \geqslant \underline{a} \xi \sum_{p \in \mathscr{P}} \sum_{l=1}^{v_p} (\, U|_{\tau_{l,p}}(p) - \overline{M}_p)^2 \,. \tag{4.14b}$$

Since $v_p \leqslant v_{\max}$, $p \in \mathscr{P}$, with $v_{\max}$ depending only on the shape regularity of $\mathcal{T}_0$, there exist constants $0 < c(v_{\max}) \leqslant C(v_{\max})$ such that for all $p \in \mathscr{P}$

$$c(v_{\max}) \sum_{l=1}^{v_p} (\, U|_{\tau_{l,p}}(p) - \overline{M}_p)^2 \leqslant \sum_{l=1}^{v_p} (\, U|_{\tau_{l,p}}(p) - U|_{\tau_{(l+1)\,\mathrm{mod}\,v_p,p}}(p))^2$$

$$\leqslant C(v_{\max}) \sum_{l=1}^{v_p} (\, U|_{\tau_{l,p}}(p) - \overline{M}_p)^2 \,. \tag{4.15}$$

Combining (4.13)-(4.15) gives the assertion.   ∎

## 5. NUMERICAL RESULTS

In this section, we present some numerical results illustrating the refinement process as well as the performance of both the element-oriented and the edge-oriented *a posteriori* error estimator. The following second order elliptic boundary value problems have been chosen as test examples :

**Problem 1.** Equation (2.1a) is considered on the octagon $\Omega$ with corners $\left(\cos\frac{\pi}{8}, \sin\frac{\pi}{8}\right)$, $\left(\cos\frac{3\,\pi}{8}, \sin\frac{3\,\pi}{8}\right)$, $\left(\cos\frac{5\,\pi}{8}, \sin\frac{5\,\pi}{8}\right)$, $\left(\cos\frac{7\,\pi}{8}, \sin\frac{7\,\pi}{8}\right)$, $\left(\cos\frac{9\,\pi}{8}, \sin\frac{9\,\pi}{8}\right)$, $\left(\cos\frac{11\,\pi}{8}, \sin\frac{11\,\pi}{8}\right)$, $\left(\cos\frac{13\,\pi}{8}, \sin\frac{13\,\pi}{8}\right)$, $\left(\cos\frac{15\,\pi}{8}, \sin\frac{15\,\pi}{8}\right)$. The coefficient $a(x, y)$ is piecewise constant with the values 1 and 100 on alternate triangles of the initial triangulation (*cf. fig.* 5.1) and $b \equiv 0$. The right-hand side $f$ and the Dirichlet boundary conditions are chosen according to the solution $u(x, y) = (\alpha x^2 - y^2)(\alpha y^2 - x^2)/a$, $\alpha = \left(\tan\frac{3\,\pi}{8}\right)^2$. Note that the solution is continuous and has a jump discontinuity of the first derivatives at the interfaces.

**Problem 2.** Equation (2.1a) with $a = 1$ and $b = 100$ on the unit square $\Omega = (-0.5, 0.5)^2$. Again the right-hand side $f$ and the Dirichlet data are chosen according to the solution $u(x, y) = \exp(-100(x^2 - y^2)^2)$ which has an interior layer along the lines $x = y$ and $x = -y$.
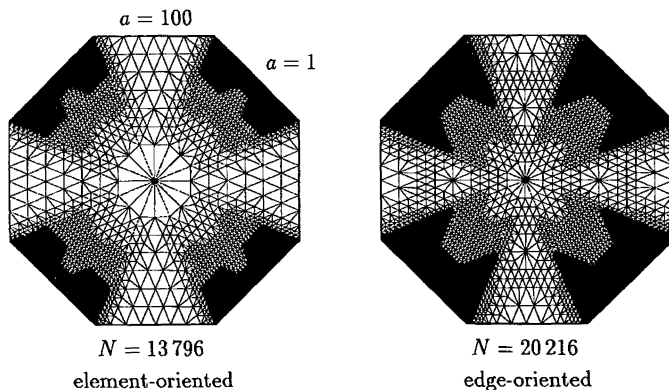
$a = 100$

$a = 1$



$N = 13\,796$          $N = 20\,216$

element-oriented          edge-oriented

**Figure 5.1. — Adaptively generated triangulation on level 6 (Problem 1).**

The problems have been discretized by means of the standard nonconforming $P1$ approximation with respect to a hierarchy of simplicial triangulations which has been adaptively generated using the error estimators described in the preceding section. The discretized problems have been solved by multilevel preconditioned $cg$-iterations with a multilevel preconditioner of *BPX*-type designed for nonconforming approximations (*cf.* [12]).

Starting from an initial coarse triangulation, on each refinement level $k$ the local error contributions $\varepsilon_\tau$, $\tau \in \mathcal{T}_k$ and $\varepsilon_e$, $e \in \mathcal{E}_k$ have to be calculated, respectively. We use the standard refinement process of Bank and others [3]. In addition to the simple mean value strategy we take the area of the triangles into account. In case of the element-oriented error estimator we obtain $\varepsilon_\tau^2|\Omega| \geq \sigma|\tau| \sum_{\tau \in \mathcal{T}_k} \varepsilon_\tau^2$ as refinement criteria, where $|\Omega|$ and $|\tau|$ stand for the area of $\Omega$ and $\tau$, respectively. $\sigma$ denotes a safety factor which is chosen as $\sigma = 0.95$. The choice $\sigma \leq 1$ guarantees uniform refinement if the weighted local contributions are all the same.

Figures 5.1 and 5.2 represent the triangulations on level 6 generated by the described adaptive refinement process. In Problem 1, the resolution of the interface between the areas of large and small diffusion coefficient is only sharp if we use the element-oriented error estimator. In Problem 2, we observe the expected local refinement in the regions of the layer (*cf. fig.* 5.2).

We observe that for almost all test examples the edge-oriented error estimator generates more nodes $N$ per refinement step.

In figure 5.3 and 5.4 the efficiency index $\eta := \dfrac{\varepsilon_{\text{est}}}{\varepsilon_{\text{true}}} - 1$ is shown as a function of the total number of nodes. The estimated error is denoted by $\varepsilon_{\text{est}}$ and $\varepsilon_{\text{true}}$ stands for the true error in the energy norm. In case $\eta > 0$ the true error is overestimated, in case $\eta < 0$ the error estimation is too optimistic.
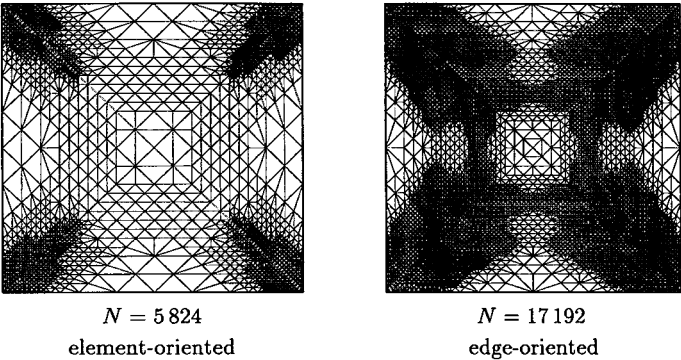
$N = 5\,824$                                  $N = 17\,192$
element-oriented                            edge-oriented

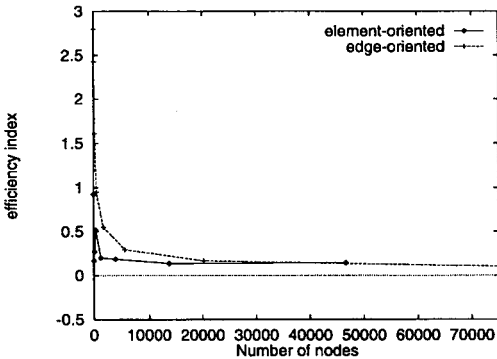**Figure 5.2. — Adaptively generated triangulation on level 6 (Problem 2).**



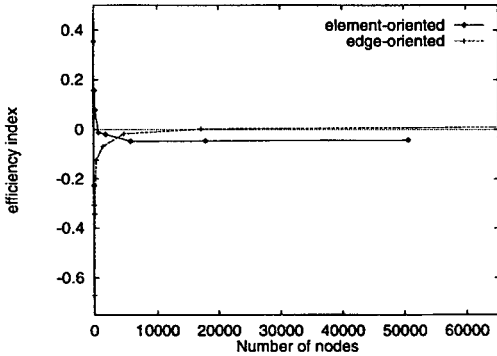**Figure 5.3. — Error Estimation for Problem 1.**



**Figure 5.4. — Error Estimation for Problem 2.**

Both presented error estimators give a good asymptotic approximation of the true error in the energy norm. At the beginning of the refinement process we observe over- or underestimation, depending on the chosen error estimator and the test problem.

## REFERENCES

[1] D. N. ARNOLD and F. BREZZI, 1985, Mixed and nonconforming finite element methods : implementation, post-processing and error estimates, *Math. Modelling Numer. Anal.*, **19**, pp. 7-35.

[2] R. E. BANK, 1990, *PLTMG - A software package for solving elliptic partial differential equations, User's Guide 6.0.*, SIAM, Philadelphia.

[3] R. E. BANK, A. H. SHERMAN and A. WEISER, 1983, *Refinement algorithm and data structures for regular local mesh refinement*, Scientific Computing, R. Stepleman *et al.* (eds.), Amsterdam, IMACS North-Holland, pp. 3-17.

[4] R. E. BANK and A. WEISER, 1985, Some *a posteriori* error estimators for elliptic partial differential equations. *Math. Comp.*, **44**, pp. 283-301.

[5] F. BORNEMANN, 1991, *A sharpened condition number estimate for the BPX preconditioner of elliptic finite element problems on highly nonuniform triangulations*, Konrad-Zuse-Zentrum Berlin, Preprint SC 91-9.

[6] J. H. BRAMBLE, J. E. PASCIAK, J. XU, 1990, Parallel multilevel preconditioners. *Math. Comp.*, **55**, pp. 1-22.

[7] F. BREZZI and M. FORTIN, 1991, *Mixed and hybrid finite element methods*, Springer, Berlin-Heidelberg-New York.

[8] P. G. CIARLET, 1978, *The finite element method for elliptic problems*, North-Holland, Amsterdam.

[9] P. DEUFLHARD, P. LEINEN and H. YSERENTANT, 1989, Concepts of an adaptive hierarchical finite element code, *IMPACT Comut. Sci. Engrg.*, **1**, pp. 3-35.

[10] P. OSWALD, 1991, *On a BPX-preconditioner for P1 elements*, Preprint, FSU Jena.

[11] B. SZABÓ and I. BABUŠKA, 1991, *Finite element analysis*, John Wiley & Sons, New York.

[12] B. WOHLMUTH and R. H. W. HOPPE, 1994, Multilevel approaches to nonconforming finite element discretizations of linear second order elliptic boundary value problems, to appear in Journal of Computation and Information, 4, pp. 73-86.

[13] J. XU, 1989, *Theory of multilevel methods*, Department of Mathematics Pennstate, Report No. AM 48.

[14] H. YSERENTANT, 1990, Two preconditioners based on the multilevel splitting of finite element spaces, *Numer. Math*, **58**, pp. 163-184.