Aspects and “Grundvorstellungen” of the concepts of derivative and integral – subject matter related didactical perspectives of concept formation

G. Greefrath, R. Oldenburg, H.-S. Siller, V. Ulm & H.-G. Weigand

Abstract

This paper discusses *aspects* and *Grundvorstellungen* in the development of concepts of derivative and integral, which are considered central to the teaching of calculus in senior high school. We will focus on perspectives that are relevant when these concepts are first introduced.

In the context of a subject matter didactical debate, the ideas are separated into two classes: firstly, more mathematically motivated aspects such as the limit of difference quotients or local linearization within the concept of derivative, as well as the product sum, antiderivative, and measure aspects of integration; secondly, the Grundvorstellungen associated with the concepts of derivative and integral.

We consider finding a comprehensive description of aspects and Grundvorstellungen to be an important objective of subject matter didactics. This description should clarify both the differences and the relationships between these perspectives, including both mathematically motivated aspects and Grundvorstellungen which are central to the students’ perspective. The primary objectives of this paper include a specification of the concepts of aspects and Grundvorstellungen in the context of differentiation and integration and a discussion of the relationships between the aspects and Grundvorstellungen associated with the concepts of derivative and integral.

We begin by presenting the characteristic properties of aspects and Grundvorstellungen, including an account of related concepts and the current state of research. These two concepts are then analyzed, based on a subject matter didactical analysis of the concepts of derivative and integral. We conclude with an account of how these insights can be beneficially exploited for introducing differentiation and integration in real-life environments, within the framework of a theory of concept understanding and subject matter didactics.

Aspekte und Grundvorstellungen zum Ableitungs- und Integralbegriff – stoffdidaktische Perspektiven zur Begriffsbildung

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Abstract

Der Beitrag befasst sich mit *Aspekten* und *Grundvorstellungen* bei der Entwicklung des Ableitungs- und Integralbegriffs, die als zentrale Begriffe des Analysis Unterrichts der Sekundarstufe II angesehen werden. Wir konzentrieren uns dabei auf Sichtweisen, die bei der Einführung dieser Begriffe von Bedeutung sind.

In der stoffdidaktischen Diskussion werden einerseits verschiedene stärker fachwissenschaftlich fundierte Aspekte wie etwa der Grenzwert des Differenzenquotienten oder die lokale Linearisierung beim Ableitungsbegriff bzw. Produktsummen-, Stammfunktions- oder Maßaspekt beim Integralbegriff unterschieden. Andererseits werden mit dem Ableitungs- und dem Integralbegriff Grundvorstellungen verbunden.

Eine wichtige Aufgabe der Stoffdidaktik sehen wir in der umfassenden Beschreibung von Aspekten und Grundvorstellungen sowie in der Klärung der Unterschiede und Zusammenhänge dieser Sichtweisen, also zwischen den eher fachmathematisch zu begründenden Aspekten sowie den aus der Perspektive der Lernenden zentralen Grundvorstellungen. Die Präzisierung der Begriffe *Aspekt* und *Grundvorstellung* im Kontext der betrachteten Begriffe Ableitung und Integral sowie die Diskussion der Beziehungen zwischen Aspekten und Grundvorstellungen am Beispiel des Ableitungs- und Integralbegriffs sind zentrale Ziele des Beitrags.

Im Beitrag werden zunächst unter Berücksichtigung verwandter Konzepte und des aktuellen Forschungsstandes die charakteristischen Eigenschaften von Aspekten und Grundvorstellungen vorgestellt. Dann werden diese auf der Basis einer stoffdidaktischen Analyse für den Ableitungs- und Integralbegriff analysiert. Abschließend geben wir einen Ausblick, wie diese Erkenntnisse unter den aktuellen Rahmenbedingungen im Rahmen von Begriffsverständnis und Stoffdidaktik zur Einführung des Ableitungs- und Integralbegriffs gewinnbringend genutzt werden können.

1 Aspects and Grundvorstellungen of mathematical concepts

According to Vollrath (1984) and Weigand (2014), in order to understand a mathematical concept, students need to acquire knowledge of some features or properties of the concept, together with the relations between them – the *concept content*. They need to acquire an overview of all objects that are subsumed under that concept – the *concept scope.* They should be aware of relations between this concept and others – the *concept network*. Students should also acquire some knowledge of concept *applications* and the *ability to manipulate* the concept. The aim of mathematics lessons is to provide students with a comprehensive understanding of central mathematical concepts in the sense outlined above. The teacher's role is to plan, initiate, support, guide and monitor the process of concept formation.
With respect to the concepts of calculus, there are numerous didactic ideas, suggested teaching methods, empirical studies, and practical investigations within the framework of curricula and mathematical textbooks (see e.g. Rasmussen & Borba 2014). Nevertheless, Rasmussen et al. (2014) come to the following conclusion:

> "While the past several decades of research in calculus has contributed to better understanding of mathematical thinking, learning, and teaching in areas such as limit, derivative, and integral, too much research remains isolated and uncoordinated." (ibid., p. 508)

The present article provides a foundation for overcoming this problem. To this effect, the mathematical aspects of both differentiation and integration are identified, together with their associated Grundvorstellungen (Hofe, v. 1995, Blum et al. 2004, Blum et al. 2005), and the relationships between them. This structures the complex subject matter of derivatives and integrals, allowing these two central concepts in calculus to be considered in their full scope, and subsequently studied from a didactic perspective.

1.1 The concepts of "aspects" and "Grundvorstellungen"

Students' *Grundvorstellungen* of mathematical concepts have been discussed in German-language pedagogy and mathematical didactics for more than 200 years, for example, by Pestalozzi, Herbart, Kühnel, Breidenbach, Oehl and Griesel (see Hofe, v. 1995, p. 22). Grundvorstellungen give meaning to content-based *aspects* of a mathematical concept, providing relations to meaningful contexts. This is a crucial prerequisite for being able to work meaningfully with a concept. We define the two expressions that are basic for the present article:

An *aspect* of a mathematical concept is a subdomain of the concept that can be used to characterize it on a basis of contents.

A *Grundvorstellung* of a mathematical concept is a conceptual interpretation that gives it meaning.

Kilpatrick et al. (2005) establish the need to study the concept of "meaning", arguing that it influences the process of learning mathematics. For them, the meaning of a concept is constituted firstly by its mathematical significance, which should not be separated from its genesis: "To understand the meaning of a concept, theorem or mathematical idea, it is important to appreciate the process through which this entity has evolved. … Thus, the constructive processes, rather than the referential elements, provide meaning." (ibid., p. 2). Secondly, a subjective or individual viewpoint underlies the process of constructing meaning. This is the interpretation of the concept for the individual, the integration of the concept into the individual's personal worldview (see e.g. Koller 2008).

1.2 Distinction: universal and individual Grundvorstellungen

The concept of Grundvorstellungen can be used in both a prescriptive and a descriptive sense (see e.g. Hofe, v. et al. 2005):

Universal Grundvorstellungen are the answer to the subject-didactic question: how should students generally and ideally think of a given mathematical concept? These Grundvorstellungen result from a subject-didactical analysis of the mathematical concept in question. Supporting students in developing these Grundvorstellungen is one of the objectives of mathematical teaching. Thus, they provide teachers with guidance for organizing lessons.

Individual Grundvorstellungen answer the subject-didactic question of how a given student thinks about a given mathematical concept. Individual Grundvorstellungen are the result of personal learning processes. By observing students as they work, and analyzing the oral and written output, teachers can attempt to gain insight into students' individual Grundvorstellungen. These can then serve as a starting point for teaching and support activities within a given learning group, so that individual Grundvorstellungen can be developed towards universal Grundvorstellungen, if necessary.

Whenever this article refers to "Grundvorstellungen", it is referring to universal Grundvorstellungen. Through a subject-didactic analysis of the concepts of derivative and integral, central aspects and associated universal Grundvorstellungen will be established, and the relations between these two viewpoints will be highlighted.

1.3 Relation to the idea "Concept Image – Concept Definition"

Grundvorstellungen (see 1.1) of mathematical concepts can also be considered within the theoretical framework of "Concept Image – Concept Definition". These terms have been used in mathematical didactics since the early 1980s to distinguish between technical issues of a concept and the associated mental images (e.g. Vinner & Hershkowitz 1980, Tall & Vinner 1981, p. 151, Bingolbali & Monaghan 2008, p. 31). "Concept Definition" refers to the formal or explicit definition, which can itself be categorized as an aspect of a particular concept, and "Concept Image" refers to all mental images identified with the concept that have developed over the years, parallel to the concept itself. Thus, Grundvorstellungen are a central component of the "Concept Image" (for a detailed analysis, see Rembowski 2013).

The relation "Concept Image – Concept Definition" has been studied in many different contexts, such as "irrational numbers" (Sirotic & Zazkis 2007, p. 49) or "limits of sequences" (Roh 2008, p. 218). A recurring problem addressed in these publications is that the Concept Image associated with a given Concept Definition is very narrow. In addition, students are in danger of drawing conclusions about the Concept Definition by generalizing a Concept Image that focuses exclusively on certain special cases only (Vinner 2011, p. 248). This danger is particularly present in the basic concepts of calculus, given that in specific classroom environments and especially in exam assignments, there is a bias towards calculation-oriented exercises, which are easily practiced beforehand on a formal, symbolic level. Törner et al. (2014, p. 547) remark that: "However in some nations, teaching of calculus in the classroom is rather traditional, focusing on procedural aspects of knowledge."

The relations "Concept Image – Grundvorstellung" and *"Concept Definition – Aspect"* can be described as follows: A Concept Image may contain several Grundvorstellungen that conceptualize different perspectives of that concept. These Grundvorstellungen give meaning to mathematical concepts that may be studied under various aspects. Each of these aspects may be realized in the various Concept Definitions that one reads in textbooks. This network is shown in the figure below, which explains that the relation between Concept Definition and Concept Image is highly non-trivial.

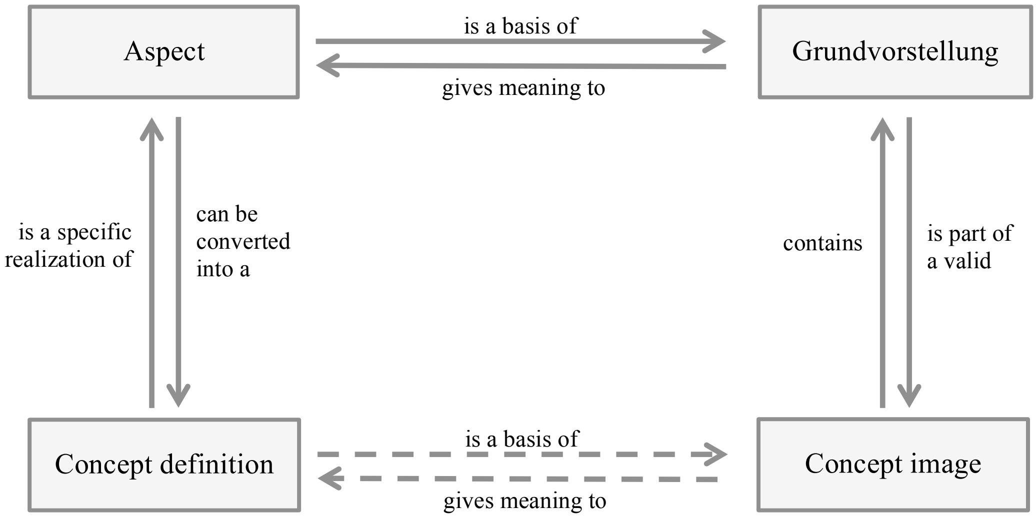

Fig. 1: Relations between Aspect, Grundvorstellung, Concept Definition and Concept Image

2. The idea of infinity and concept of limit – a basic element of teaching calculus

The concept of limit is a basic one in calculus and may even be the most important basis of the concepts of derivative and integral. It has to be seen – especially in terms of historical development – in close interrelationship with the concept of infinity. Aristoteles (384-322 BC) already distinguished between two kinds of infinity: *Potential infinity* exists only in somebody's mind or perception and can be perceived as an infinitely often on-going process, e. g. over time, by continuously counting or continuously dividing a spatial object. This infinitely dynamic process cannot be completed in a finite timeframe and in this sense, infinity is not real. *Actual infinity* can be seen as the result of a continuous process, such as the length of a line segment, as the result of summing an infinite number of line segments that are becoming shorter or as a number, the limit, as a sum of an infinite count of numbers.

The formalization of the limit concept took a long time in the history of mathematics and reached a mature status especially under the influence of Karl Weierstraß (1815-1897), when infinitely small or large numbers were excluded from mathematics and substituted by processes which can be presented by finite operations. This leads to the now common definition of the limit of a sequence:

Given is a real sequence $(a_n)_{n\in\mathbb{N}}$. $A \in \mathbb{R}$ is called limit of the sequence, if:

$$\forall \epsilon > 0 \ \exists n_0 \in \mathbb{N} \ \forall n \in \mathbb{N} \text{ with } n > n_0 : \ |A - a_n| < \epsilon$$

This formal definition is nowadays of minor importance in the German high school curricula. The German learning standards (KMK 2012) emphasize the conception of the intuitive limit concept. It is stated that students should use „ limit concepts on the basis of a preliminary limit concept especially for the determination of differentiation and integration“ (p. 22). The notion of a sequence no longer appears in these standards. In many curricula of the German states, its significance had been considerably reduced or it has even been removed entirely from mathematics classrooms. Moreover, the recent tendency towards a greater consideration of realistic mathematics lessons and more modeling, as well as an emphasis on "preformal argumentations" have reinforced the abandonment of a formal and thus inner mathematically oriented approach to the sequence concept in mathematics classrooms.
Especially the transition to a more formal approach to the limit concept has to be a goal in an education that is based on understanding the central mathematical basics of calculus. Törner et al. (2014) also emphasize this in their metastudy on calculus lessons in European countries: „From the literature review, the progression from informal to formal knowledge seems to be a goal for calculus teaching.“ (p. 558) Furthermore, they argue: „However, the informal introduction of the concepts, especially of the concept of the limit, does not seem to encourage the progression to the formal definitions later at

the university" (ibid.) Therefore it is the goal of mathematics lessons to develop aspects and basic ideas of calculus in a way that it is possible to proceed to strongly formal notions.

3 Aspects and Grundvorstellungen of the concept of derivative

Following the method of subject matter didactics ("Stoffdidaktik"), we start by analyzing the mathematical definition of a derivative. Differentiation can be defined in different ways and we elaborate on this by defining two aspects of this concept in the first sub-section. The second sub-section makes the passage from these aspects to the notions that students should develop.

3.1 The aspect "limit of difference quotient"

When considering the aspect of differentiation as the limit of a difference quotient, the first step is to analyze the difference quotient $\frac{f(x)-f(x_0)}{x-x_0}$. Starting from the absolute change $f(x)-f(x_0)$, by forming a ratio, one obtains the average rate of change over the interval $[x_0, x]$. In certain situations, it is not the rate of change within an interval that is of interest, but rather the local rate of variation at a point x_0. This yields the limit value that is used in the definition of the differential quotient.

> ***Differentiation as the limit of a quotient of differences***
> *Let f be a real-valued function defined in some neighborhood of the point $x_0 \in \mathbb{R}$. We say that f is differentiable at the point x_0 if the limit $f'(x_0) \coloneqq lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ (or equivalently $f'(x_0) \coloneqq lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$) exists. We refer to this limit the derivative of f at x_0.*

3.2 The aspect "local linearization"

Another aspect stems from the fundamental idea of approximation; we attempt to approximate the values of the function by a linear function in an (arbitrarily) small neighborhood of x_0. If the function is indeed linear, then any function value $f(x)$ can be computed from the $f(x_0)$ value at some fixed x_0 by adding a multiple of the distance $x-x_0$. The multiplication factor is the slope $m \in \mathbb{R}$, thus we have $f(x) = f(x_0) + m \cdot (x-x_0)$. If the function is not really linear, one could try to approximate it linearly. To do this, we modify the slope by an additive correction function: $f(x) = f(x_0) + (m + \delta(x-x_0)) \cdot (x-x_0)$. The linear part is a good approximation locally if the correction vanishes near x_0.

> ***Differentiation as local linearization***
> *Let f be a real-valued function defined in some neighborhood of the point $x_0 \in \mathbb{R}$. We say that f is differentiable at the point x_0 if there exists some number m such that $f(x) = f(x_0) + (m + \delta(x-x_0)) \cdot (x-x_0)$, where the slope correction $\delta(x-x_0)$ is such that: $lim_{x \to x_0} \delta\,(x-x_0) = 0$. We call this number m the derivative $f'(x_0)$.*

Equivalently, one may define $r(x-x_0) \coloneqq (x-x_0) \cdot \delta(x-x_0)$ that satisfies $\lim_{x \to x_0} \frac{r(x-x_0)}{x-x_0} = 0$ and yields the linear approximation in the form $f(x) = f(x_0) + m \cdot (x-x_0) + r(x-x_0)$.
The approaches in these two definitions differ, and different mathematical operations are performed in order to arrive at the concept of a derivative. For example, local linearization allows for a generalization to higher dimensions, and to potentially simpler proofs of the rules for calculating derivatives. On the other hand, approaching the concept via the limit value of difference quotients has practical advantages, in particular for calculating the derivative of simple power functions.

These two definitions define the same mathematical concept, but they reveal different fundamental properties. They shed light on two aspects of the concept of derivative. From a didactic perspective, the fact that both aspects of differentiation correspond to different concept images for students, is important. The aspect of differentiation as the limit value of a difference quotient supports concepts of speed and rates of change.

The interpretation of differentiation as a local linear approximation supports understanding the error between the optimally approximating linear function and the original function, and about the possibility of describing the function as linear in a small neighborhood—its graph appears as a straight line when zoomed in at a particular point (function microscope, see Kirsch (1979)).

3.3 The Grundvorstellung "local rate of change"

In the lower grades of secondary education, various concepts describing processes of change are introduced, including absolute and relative variation, percentage change and monotonicity, etc. This prompts the question of how fast a variation can occur as a function of an argument, consequently leading to the idea of considering the rate over an interval (see Hahn & Prediger, 2008). For a dependent entity *f*, the change in value over the interval $[x, x_0]$ as a function of the argument value is described by the expression $\frac{f(x)-f(x_0)}{x-x_0}$ —the difference quotient.

Studies (e.g. Herbert & Pierce 2012) have shown that the concepts of rate or rate of change are complex, yielding multiple perspectives (e.g. Thompson 1994, Ubuz 2007), and "[...] students at various levels have difficulty conceptualizing the idea rate of change" (see Teuscher and Reys 2010, p. 519). Herbert and Pierce (2012) demonstrate that for some students, the rate of change is a singular object, whereas for others, it is composed of two distinct changes. For others still, their concept image is primarily associated with the expression used in calculations. Yet other students assume that rates of change are always constant—possibly due to generalizing from linear graphs (see Herbert & Pierce 2012, p. 94). In (Thompson and Thompson, 1994 and 1996) it is shown that teaching this Grundvorstellung puts non-trivial demands on teachers.

The instantaneous rate of change can be obtained as the limit of the difference quotient (first aspect). The interpretation of the content of this aspect (particularly by means of dependent entities) when the differences are regarded as changes, is the Grundvorstellung "local rate of change".
A comprehensive, explicit Grundvorstellung of the local rate of change should include:

- the conception of the instantaneous rate of change of a process (the conception of instantaneous velocity can be used as a prototype)
- the conception that the change in a dependent entity is given by $\Delta y \approx f'(x) \cdot \Delta x$

Using technology, it may be useful to give students the opportunity to experience rates of change in virtual words and this may have positive effects (Thompson, Byerly & Hatfield, 2013).
If the equation $\Delta y \approx f'(x) \cdot \Delta x$ is considered algebraically, and reformulated, one immediately obtains an approximation of the derivative $\frac{\Delta y}{\Delta x} \approx f'(x)$. In particular, when reading off a slope value (from slope triangles), the representation $\frac{\Delta y}{\Delta x}$ is used. This shows that local rates of change are not the only Grundvorstellung that need to be discussed; there are multiple other possibilities.

3.4 The Grundvorstellung "tangent slope"

Students become acquainted with the concept of slope during secondary education (Crawford & Scott, 2000) that is associated with an extension of the concept of tangent (see Friedrich, 2001). Using what is known as the "touch point property" ("Berührpunkteigenschaft", see Möller, 2013, p. 20; Tall 2012, p. 307), the concept image that the tangent of a graph does not re-intersect with the graph is established (see Büchter 2014, p. 45)—as shown in Figure 2. Büchter (2012, p. 171) explains this approach: "In [...] school textbooks, concept development for circle tangents is essentially reduced to the property of having 'exactly one common point'". Conceptual change (see Tirosh & Tsamir 2004, Vosniadou & Verschaffel 2004) is necessary to convert this into a conception that is viable in calculus.

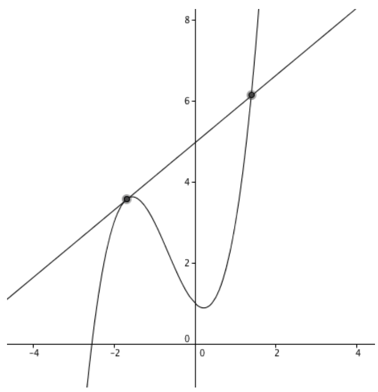

Fig. 2: Tangent to a curve

This interpretation has historical roots. Even in ancient Greek mathematics, in Book III of Euclid's Elements (see Deak, 2010), an analogous interpretation of the concept of tangent can be found: "We say that a straight line touches a circle (is tangent to that circle) if it encounters the circle, but its extension does not." If this concept image is extended to arbitrary curves without critical thinking, concept development becomes incompatible with the concepts of calculus. This concept image of tangents must be modified (Stump, 1999), in particular so that it does not contradict the first definition of the differential expression.

On the other hand, the interpretation of tangents as locally tangential lines (see Blum & Törner 1983 p. 93, Danckwerts & Vogel 2006) is appropriate and consistent. In this case, tangents are understood as lines that are *locally* tangential to the graph (Blum & Törner 1983, Biza 2011, Tall 1987). In order to judge whether a line is or is not a tangent, it is sufficient to look at an arbitrarily small interval around the point in question. Whatever may happen outside this interval is not relevant to answering this question. This local viewpoint is a key idea of calculus that is new to students.
Here, an important difference between the concept images of tangents, as discussed by Vinner (1991), and the Grundvorstellung of a tangent occurs. We discuss the Grundvorstellung "derivative as slope of the tangent". So the tangent is subordinate to the derivative. Thus, Vinner's concept image allows the line with equation $x = 0$ to be seen as tangent to the graph of $\sqrt{|x|}$, which is not part of our Grundvorstellung.
Understanding the derivative as the slope of the tangent has the positive effect that one can easily understand why the slope of $x \mapsto k \cdot f(x)$ and $x \mapsto f(k \cdot x)$ is k times higher than that of f. That is, a slope triangle attached to the tangent gets "stretched" in the y-direction or "compressed" in the y-direction, resulting in an amplification of the slope by the factor. Similar geometric arguments clarify the situation for $x \mapsto a + f(x + b)$. As the graph of the inverse function is the mirror image with respect to the line $y = x$, one may moreover imagine the tangent and a slope triangle mirrored as well, in order to understand the rule for the derivative of the inverse function.
This Grundvorstellung should include the conception

- of tangents as locally tangential lines.
- of the slope of a line at a point.
- that a tangent gives the local direction of a curve.

3.5 The Grundvorstellung "local linearity"

Curves can be approximated by piecewise-linear curves (Blum & Törner, 1983, p. 96). The idea of thinking about smooth curves as locally linear is first introduced in lower secondary education (e.g. using polygons to approximate circles and other curves, see Oldenburg 2012), and can further be developed into a Grundvorstellung of differentiability and of derivatives.
We consider an arbitrary function f at a point x_0 and seek a linear function that locally approximates the behavior of the function as well as possible (see Teague, 1996).
There are infinitely many lines that pass through the point $(x_0 \mid f(x_0))$ on the graph. Each has a characteristic slope m. The equation of each line is given by $g(x) = f(x_0) + (x - x_0) \cdot m$. The difference in function value between f and g near the point x_0 is $r(h) = f(x_0 + h) - g(x_0 + h) = f(x_0 + h) - f(x_0) - h \cdot m$ and by continuity, this error approaches zero as $x \to x_0$ for every value of m, but for one specific value, even the relative approximation error satisfies:

$$\lim_{h \to 0} \frac{r(h)}{h} = 0$$

Thus, the line of best approximation has a slope equal to exactly $f'(x_0)$, the derivative of f, that can be understood as the slope of the tangent. This gives an example of a "multiple-linked representation" (see Tall, 1991, p. 33), which "allows a person to use several different representations at the same time, switching from one to another when it is appropriate to do so" (see Mackie, 2002).
The basic idea is to observe the graph with a (metaphorical) microscope, and recognize the straight-line behavior that is present in differentiable functions. To this end, *dx* and *dy* can be defined as the components of a vector in the corresponding direction. Studies performed by Tall (2012, p. 289 ff.) show that students taught from this perspective developed more sound conceptions about tangents. The idea that the graph of a differentiable function is locally linear corresponds with the idea that it can be approximated by a linear spline. The idea of building up curved graphs from simple (linear) but very small components dates back to the inventors of calculus (Jahnke 1999, p. 89 – 129).
A comprehensive, explicit Grundvorstellung of local linearity should include:

- When zooming in very close to a point of the graph of a differentiable function, one sees an almost straight segment.

- For small changes in argument, the function is essentially linear, so that it can be approximated with a linear model.

We now demonstrate how this Grundvorstellung promotes understanding the structure of the chain rule $(u(v(x))' = u'(v(x)) \cdot v'(x)$. Where does the product come from in the expression? The functions are approximated locally by linear functions, the composition of which yields a function whose slope is the product of those of the original two functions.
Moreover, it is clear that at local extrema, the derivative must vanish, as otherwise one would have a direction in which the function values will decrease or increase.

3.6 The Grundvorstellung "amplification factor"

If there is a functional relationship between (two) parameters, changes (or uncertainty, such as errors in measurement) in the independent parameter induce changes in the dependent parameter. The Grundvorstellung of amplification factor for small changes is that they are proportional.
The Grundvorstellung of amplification factors can be used both for the difference quotient and the differential quotient (see also Malle 2003). It can help to bridge the transition and is in a certain sense (as evident in the second definition) closely related to the idea of local linearization.

Most current calculus textbooks do not place any emphasis on this Grundvorstellung although it is very useful. For example, it explains how differentiation can work similarly to a "variation detector". Wherever a function has little variation, the values of the derivative are close to zero, and wherever the changes are significant, the derivative is large. An example of this is given in Figure 3.

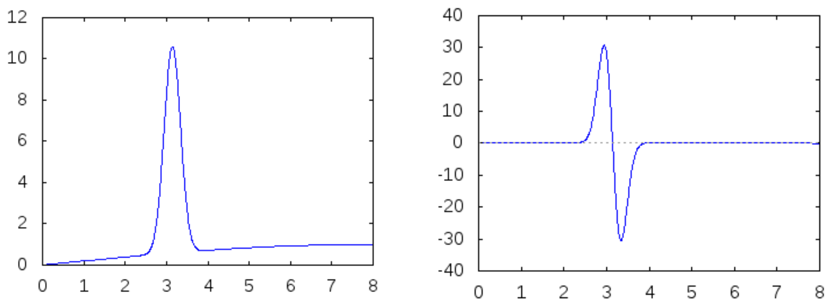

Fig. 3: On the left, a slowly varying function with a strong variation near one point x, on the right its derivative, showing the large effect of small changes near x

Mamolo and Zazkis (2012) record that students experience difficulties with exercises that require this Grundvorstellung. Although they do not explicitly use the notion of amplification factor, their work may indicate that the students they observed had an insufficiently developed Grundvorstellung of differentiation as an amplification factor.

A comprehensive, explicit Grundvorstellung of amplitude factors should include the following components:

- differentiation measures the extent of changes in the dependent parameter induced by small changes in the independent parameter.
- high values of the derivative indicate fast/large variation.
- for small changes, Δx and Δy are approximately directly proportional.

This Grundvorstellung enables an understanding of the structure of the chain rule $(u(v(x)))' = u'(v(x)) \cdot v'(x)$ in the following way: When two variational processes are combined, each multiplies the change by their own amplitude factor, so that overall, factors are multiplied.

3.7 Overview: Aspects and Grundvorstellungen of the concept of derivative

From a subject matter didactic analysis of the concept of derivative, two aspects and four Grundvorstellungen were identified. These are summarized in the diagram below, including the relations between the different aspects and Grundvorstellungen. The connecting lines indicate that the aspect is a basis of the related Grundvorstellung and that the Grundvorstellung gives meaning to the aspect (see section 1.4). For example, the aspect that the derivative is a limit of a difference quotient may be understood either by interpreting it as the limit of average change rates, as the limit of secant slopes or as amplification factor. On the other hand, there is no linking line from this aspect to “local linearity”, as this Grundvorstellung cannot really be built up, based on a definition that emphasizes this aspect.

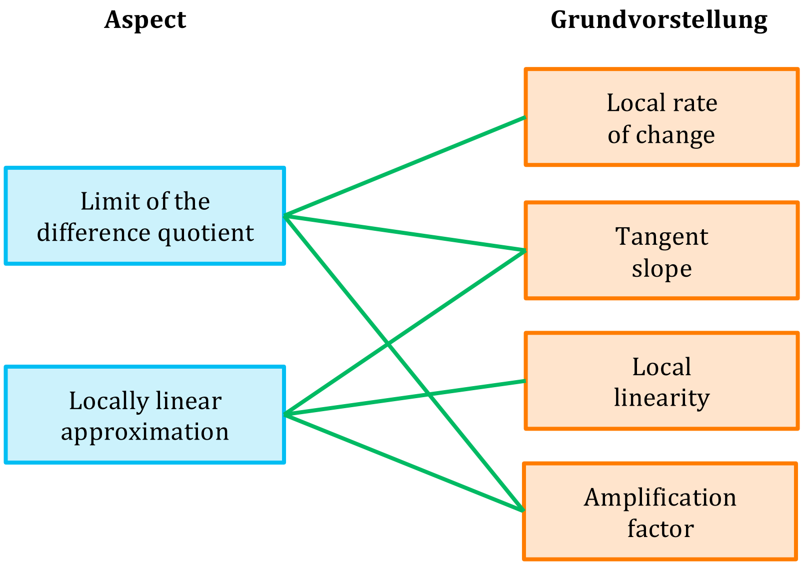

Fig. 4: Aspects and Grundvorstellungen of differentiation

In the above analysis we have given several small examples that show how Grundvorstellungen may help to clarify what is true and why. However, note that each Grundvorstellung has particular limitations and there are situations in which none is sufficient to yield a definitive conclusion. Consider, for example, the function with $f(0) = 0, f(x) = x^2 \sin\left(\frac{1}{x}\right)$ for $x \neq 0$. None of the Grundvorstellungen is precise enough to reveal whether this function is differentiable at the origin (which is the case). However, the Grundvorstellungen may render the property of differentiability plausible; one can zoom into the graph and have the visual sensation of the graph becoming straight, or one may note that the sine function is bounded by 1, so that small changes of x cannot be amplified by more the the amplification of the quadratic function – so one expects the function to be differentiable.

4 Aspects and Grundvorstellungen of the concept of definite integral

In this section, the aspects and Grundvorstellungen associated with the concept of a definite integral are discussed. Firstly, we distinguish between the aspect of a definite integral as a product sum, the aspect of anti-derivatives, and the aspect of measure. The defining property of these aspects is that they can all be used to define or technically characterize the concept of a definite integral. Secondly, we present the Grundvorstellungen of (re)construction, area, average value and of accumulation. Finally, the relations between the different aspects and the different Grundvorstellungen will be discussed, as well as alternative perspectives of the concept of definite integral.

4.1 The “product sum” aspect

By a product sum, we mean an expression of the type: $a_1 \cdot b_1 + a_2 \cdot b_2 + \cdots + a_n \cdot b_n$. The Riemann integral can be defined and calculated using product sums. To define the definite integral, we consider a function f defined and bounded on a closed real interval $[a, b]$. The interval is split into n subintervals $[t_{i-1}, t_i]$, where $t_0, t_1, \ldots, t_n$ are finitely many points such that $a = t_0 < t_1 < t_2 < \ldots < t_n = b$. Within each subinterval, the supremum and infimum of f are both finite, as f is bounded. Let the supremum of f on the subinterval $[t_{i-1}, t_i]$ be denoted M_i, and the infimum be denoted m_i. $Z = [t_0 = a, t_1, t_2, \ldots, t_n = b]$ $(t_i \in \mathbb{R}, 1 \leq i \leq n, n \in \mathbb{N},)$ is then a partition of the interval $[a, b]$. The product sum $\sum_{i=1}^{n} M_i \cdot (t_i - t_{i-1})$ is the *upper sum*, and the product sum $\sum_{i=1}^{n} m_i \cdot (t_i - t_{i-1})$ is the *lower sum* of f for the partition Z. If, for a given function f, bounded on the closed interval *[a; b]*, the supremum S of all lower sums is equal to the infimum I of all upper sums, then this number is defined to be the *Riemann integral of f on the interval [a; b]* (see Walter 2004, p. 197 ff.).

In addition to upper and lower sums, it is also possible to consider intermediate sums, which are also product sums. In many applications in which definite integrals are calculated, the product sums of two distinct entities are considered, yielding a third entity. A possible example is the product sum comprising time and velocity, which gives the physical entity of distance. Whenever the definite integral is interpreted as the limit of upper, lower or intermediate sums, or if specific product sums are calculated from given numbers, the emphasis is on the product sum aspect of integration.
There are two possible interpretations of definite integrals when considering product sums. As product sums involve both addition and multiplication, one can choose which of the two operations to emphasize. The first interpretation is the idea of a generalized sum, which belongs to the Grundvorstellung of accumulation. But the definite integral can also be thought of as a generalized product. This focuses more on the result of the calculation, as in the case of physical work, the definite integral has units of the corresponding product of force and distance. Blum and Törner (1983, p. 158 ff.) also describe these two viewpoints of product sums, i.e. *integration as generalized summation*, and *integration as generalized multiplication (of physical quantities)*, as basic ways of understanding the definite integral concept.

4.2 The "anti-derivative" aspect

Definite integrals can also be defined using the concept of anti-derivatives. Given a function $f: I \longrightarrow \mathbb{R}$ defined on an interval I, we say that $F: I \longrightarrow \mathbb{R}$ is an *anti-derivative* of f if F is differentiable and $F' = f$. This form of inverse differentiation makes it possible to find anti-derivatives of elementary functions, e.g. power functions, polynomials, trigonometric functions and exponential functions. It is then possible to *define* a certain class of definite integrals in the following way:

> *Let $f: I \longrightarrow \mathbb{R}$ be a function defined on an interval I with anti-derivative F. For $a, b \in I$, we define:*
> $\int_a^b f(x)dx \;:=\; F(b) - F(a)$

It remains to be determined whether that this definition is well-defined, i.e. that the value does not depend on the choice of anti-derivative. However, if G is another anti-derivative of f, then F and G can only differ by an additive constant, as $(F - G)' = F' - G' = 0$.
This approach leads readily to a method for calculating certain types of definite integral. By defining definite integrals as anti-derivatives, it becomes clear that the operations of differentiation and integration are mutually opposing. Anti-derivatives can then be interpreted meaningfully as a "revision of differentiation" (Bender 1990a, p. 75) and used „as a base for constructing the *function matching* symbolic form" (Jones 2013, S. 138). This aspect becomes even clearer when treating the integral with a variable upper limit as a function. However, in this context we only consider the definite integral.
With the above definition of the definite integral, the Fundamental Theorem of Calculus is used as a definition to link together the concepts of derivative and definite integral. The assumption of function continuity that is a necessary hypothesis in the theorem is somewhat obscured. In additions, questions of existence of definite integrals cannot be adequately discussed within the framework given by a definition of this type, as "Riemann integrability" and "having an anti-derivative" are non-equivalent properties of functions. It is possible to give examples of functions that possess anti-derivatives and yet are not integrable, and conversely, there are functions that are integrable but do not have an anti-derivative.
When using the „concept of anti-derivative and its use for calculating definite integrals" (Kouropatov & Dreyfus, 2013, p. 646), which is based on the Fundamental Theorem of Calculus, instead of the definition above, the issues described previously do not occur. As a result, the aspect could be introduces later on in the course and, moreover, analytically connect some unrelated topics like the gradient of tangents and the area below curves.

4.3 The "measure" aspect

Definite integrals are used for measuring length, area and volume in the context of their measure aspect, and are interpreted as a measure for certain representatives of these physical quantities. The mathematical meaning of measure is an ordering of suitable sets of real numbers, which – depending

on the nature of the set – can be interpreted as lengths, areas or volumes. In particular, measure is non-negative, monotonous and additive. The definite integral also satisfies these fundamental properties of measure whenever it is applied for measuring areas, lengths (e.g. curve lengths) and volumes (e.g. bodies of rotation). It should be noted that the value of the definite integral is not inherently non-negative, but only acquires this property when used in a certain way, such as measuring the area above the x-axis. Interpreted in this way, the definite integral is a measure of areas, lengths, and volumes. This perspective is a generalization of the area perception of integration, which includes lengths, volumes, and potentially higher-dimension quantities.
Measure theory and the Lebesgue integral can be considered as the mathematical foundation of the measure aspect of integration. This concept of definite integral allows functions to be integrated over arbitrary measure spaces. It is a generalization of the Riemann integral, i.e. there are Lebesgue integrable functions which are not Riemann integrable. However, the "measure" aspect should not imply a focus on the Lebesgue integral, as it is also possible to calculate length, areas and volumes with the help of the Riemann integral.

The aspects described above provide interpretations of three separate subthemes of the concept of definite integral. Whereas the product sum aspect primarily emphasizes developing the concept of definite integral from the Riemannian approach, the anti-derivative aspect highlights the Fundamental Theorem of Calculus and thus underlines the link between integration and differentiation. The measure aspect, on the other hand, illustrates the application of integration in making measurements, and the link to the Lebesgue integral. Narrowing down viewpoints to one or more specific aspects of integration is generally seen as questionable (Huang 2012, p. 167).
In the next section, we will describe the Grundvorstellungen associated with the above aspects, which can be understood as meaning-constructing interpretations of the concept of definite integral.

4.4 The Grundvorstellung "area"

The Grundvorstellung of definite integrals as area emphasizes one of the applications of definite integrals within the framework of its measure aspect. Whereas area is always non-negative, measuring areas with integration naively yields the *oriented area*, i.e. areas under the x-axis contribute negatively. Thus, it is the net area that is measured. The actual area can be measured by considering multiple subregions between points of intersection with the x-axis or with the graph of another function. The interpretation of definite integrals as an area is a Grundvorstellung and not an aspect, as measuring area links the concept of definite integral with physical experiences from everyday life ("the action of measuring area has and always will have explicit real-life elements to it…" Bender 1991, p. 51).

Example: approximating an area
Approximate the area between the function of f with $f(x) = \frac{10}{2-x}$ and the x-axis over the interval [-1; 1]. In order to do so, calculate the area of rectangles that approximate the graph from below. How could this method be optimized to calculate the area more accurately?

This example for approximating an area can be solved with the help of the Grundvorstellung "area", as it is the aim of this task to calculate the area more accurately, which is possible by showing that the number of rectangles can be increased. Furthermore, it is possible to determine the area more precisely by additionally approximating the graph from above. The issue of approximating the area of a function below the x-axis is not covered by the task.
Potential applications of the definite integral, such as in physics and economics, can be rendered more difficult if the emphasis is on the Grundvorstellung of area, as these applications are considerably more diverse and involve much more than merely calculating areas (Hall 2010, p. 6). "Such an inappropriate 'area-conception' of definite integral stems from the special case, $f(x) \geq 0$, as its construction is based on a generalisation of the special case" (Bezuidenhout & Olievier 2000, p. 78). The area perception is not incorrect, but is in some cases less target-oriented, or insufficient for understanding the concept of definite integral (Jones 2013, p. 138; Sealey 2006, p. 52) or the concept of integral in multivariable calculus (Dray & Manogue 2006, p. 5). There is a widespread misconception that "integral is area and area is always positive" (Kouropatov & Dreyfus 2013, p. 643). Nevertheless, the Grundvorstellung "area" is often used in textbooks, due to the advantage of connecting to secondary school students' prior knowledge. However, students may need more time and a deeper understand-

ing in order to apply the Grundvorstellung appropriately, and to be able to link the aspects "area below a curve" and Riemann sum (Engelke & Sealey 2009, p. 4). Accordingly "area" should not be used as the first Grundvorstellung for introducing integrals in school lessons.

4.5 The Grundvorstellung "(re)construction"

By construction or reconstruction in the context of integration, we mean both the (re)construction of a quantity from the given information about rates or speed, and the (re)construction of one of the anti-derivatives of a given function. Existence and variation are important categories for both of these contexts (see Hahn & Prediger 2008, p. 178).
The distinction between construction and reconstruction resides in the interpretation of the values of the original function. If one considers them to be the values of a known quantity, then integration amounts to constructing a new functional relationship. If one interprets them as rates of change of an existing functional relationship, then integration amounts to reconstructing this relationship (Bender 1990b).

Example: Spirometer
A so-called spirometer is used by doctors to obtain a curve of the rate of air flow in and out of the lung. Thus, a mouthpiece is used for breathing in and out of the spirometer. This procedure creates a pressure difference in the machine, so that the flow rate of the respired air (in liters per second) can be determined.
The following table shows data for a spirometry measure taken from a resting person in the phase of breathing in (Schmidt 2007).

Time in seconds	*0*	*0.3*	*0.6*	*1.0*	*1.5*	*1.8*	*2.1*	*2.5*	*2.75*	*3*
Flow rate in liter / second	*0*	*0.18*	*0.32*	*0.45*	*0.50*	*0.48*	*0.42*	*0.28*	*0.15*	*0*

How much air has been breathed in during the three-second interval illustrated in the table?
What is the average flow rate (in liters per second) during the recorded time of three seconds?

When working on the "Spirometer" example, it is first of all necessary to create a mathematical model for the flow rate in order to expand the given data to the interval appropriately from 0 to 3 seconds. We choose a piecemeal linear model as an easy one. For this purpose, adjacent data points are connected by lines. As a result, it is possible to approximately calculate the lung volume that has been inhaled. By taking the maximum flow rate for a short interval and further multiplying it by the length of the interval, we obtain an upper approximation for the inhaled air during that time. Summing the result to the total interval from 0 to 3 seconds, we then obtain an upper bound for the inhaled air. In this context, the upper sum and the lower sum serve as an illustration, because they are each identified as an upper and lower approximation of the inhaled air volume. The integral is equal to the limit of those upper and lower sums and, hence, represents the air volume in question.
Well-known examples of contexts involving this Grundvorstellung in the sense of quantity reconstruction include the reconstruction of distance travelled from velocity data (e.g. see Hußmann 2001, p. 60), or the reconstruction of the net amount of water left over using data on inflow and outflow for a given container. In reality, in many cases it is difficult to differentiate between construction and reconstruction. The following claim unites both perspectives: "The function F obtained by accumulation is the same regardless of whether the accumulation is thought of as a new construction from another given function f ... or as the reconstruction of an anti-derivative..." (Tietze et al. 2000, p. 287).
The Grundvorstellung of an definite integral, as the reconstruction of a quantity, is often accompanied by the summation of partial products, and so is linked to the product sum aspect and the accumulation perception of the definite integral. In the case of area construction or reconstruction, there is a link to both the measure aspect and the area perception of the definite integral concept.
The second perspective of the reconstruction concept contains the idea of (re)construction of an anti-derivative from the data of a given function. The idea of reconstructing anti-derivatives for specific functions can be illustrated geometrically (Herford & Reinhard 1980, p. 98).
The interpretation of the substance of definite integrals from the viewpoint of construction or reconstruction of quantities or anti-derivatives is particularly diverse, and has links to all other aspects of

integration. It is therefore regarded as particularly important for understanding the subject matter of integration (Büchter & Henn 2010, p.92; Danckwerts & Vogel 2006, p. 98 ff.). It is also seen as containing links to interpretations of differentiation: “A fundamental grasp of differentiation as a local rate of change is crucial for the aspect of integration as reconstruction” (Danckwerts & Vogel 2006, p. 125, see also section 2).

4.6 The Grundvorstellung “average value”

The definite integral can also be used to determine average values. The technical basis of the Grundvorstellung of average values is the mean value theorem (see Walter 2004, p. 208): if f is continuous over the interval $[a, b]$ then, $\xi \in [a, b]$ such that $f(\xi) = \frac{1}{b-a}\int_a^b f(x)\,dx$. Due to a conceptual change, this context can be seen as a generalization of the arithmetic mean to continuous functions.
Considered geometrically, the oriented region under the graph of f over the interval $[a, b]$ and the rectangle of width $b - a$ and height $f(\xi)$ have equal area. The value of the definite integral of a given function over a given interval divided by the length of the interval thus gives the “average value” of the function over that interval. This Grundvorstellung is therefore associated with the idea of forming a rectangle with the same area as a given region delimited by a curve.
The mean value theorem cannot, however, be used to define definite integrals. It is therefore a topic of integration that is not an aspect, but rather a substance-based interpretation that gives meaning to the concept of definite integrals. The average value perception of integration can be thought of as the next step in generalizing the arithmetic mean. For example, suppose that we wish to determine the daily average temperature from the data given by an automatic thermometer. The mean of the temperature values $T_i, i = 1,2,3, \ldots$ at times $s_i, i = 1,2,3, \ldots$ can be calculated discretely as follows:

$$T_M = \frac{T_1 \cdot s_1 + T_2 \cdot s_2 + \cdots}{s_1 + s_2 + \cdots}.$$

If we now assume that more data is available, and reduce the time lapse between successive readings, we obtain an expression as a sum for calculating the mean value more precisely. However, there are still fixed amounts of elapsed time between each individual measurement:

$$T_M = \frac{1}{\sum_{i=1,2,3,\ldots} \Delta s_i} \sum_{i=1,2,3,\ldots} T_i \Delta s_i.$$

If every point in time s is assigned a temperature T(s), then we finally obtain the generalized average using integrals:

$$\bar{T} = \frac{1}{s_b - s_a} \int_{s_a}^{s_b} T(s)\,\mathrm{d}s.$$

Emphasizing the average value perception forges a stronger relationship between calculating integrals and stochastic systems, as the expected value of an integrable random variable X, which is defined as $E(X) = \int_{\Omega} X\,dP$, is the generalization of this type of average (Danckwerts & Vogel 1986, p. 69). Emphasizing this Grundvorstellung too strongly however, may render other possible applications more difficult to understand. The average value perception has been judged as of secondary importance (Tietze et al. 2000, Danckwerts & Vogel 2006). It might therefore be necessary to view the Grundvorstellung in association with the “average rate”, as it thus becomes more important. Bezuidenhout, Human & Olivier (1998, p. 101) found in a study that students have inadequate intuition about the “average rate” and “average value” concepts. Therefore, we consider this conception to be very helpful in several cases. The second question of the spirometer task serves as an example.

4.7 The Grundvorstellung “accumulation”

The Grundvorstellung of accumulation builds on a suitable interpretation of product sum that tend towards the definite integral as their limit. In general, accumulation is understood as referring to a process of aggregation, or of collection combined with storage. In the context of integration, the intended meaning is the aggregation or cumulative summation of partial products to form a product sum. An example shows the task “area”, in which it is necessary to sum the areas of many rectangles in order to determine the value of the integral.
One example to support the accumulation perception is the generalization of the definition of physical work given below, which can, for the time being, be thought of as the scalar product of force and dis-

tance vectors: $W = \vec{F} \cdot \vec{s}$. If the force is now made a function of the distance, in order to calculate the approximate value of this scalar product, the force is assumed to be constant along small segments of the path, and these contributions are summed together. This gives the product sum: $= \overrightarrow{F_1} \cdot \overrightarrow{s_1} + \overrightarrow{F_2} \cdot \overrightarrow{s_2} + \overrightarrow{F_3} \cdot \overrightarrow{s_3} + \cdots + \overrightarrow{F_n} \cdot \overrightarrow{s_n}$. Taking this idea one step further, one can start with a path-dependent force function, and consider the integral as the generalized product sum: $W = \int_{s_1}^{s_2} \vec{F}(\vec{s})\, \mathrm{d}\vec{s}$.

In the context of the accumulation Grundvorstellung, the definite integral can thus be thought of as a product sum, obtained by accumulating or aggregating multiple partial products. This view of summation emphasizes the process of integration more than the result of the calculation. The geometric perception of the accumulation Grundvorstellung corresponds to the "integral as the limit of (the area of) a series of rectangular regions, as the size of the steps becomes arbitrarily small in the limit" (Blum & Törner 1983). This Grundvorstellung is similar to the area perception, but is more general, as it also evokes other ideas (Blum & Kirsch 1996). From a general perspective, every integral can be considered as an accumulation (Sealey & Oehrtmann 2005, p. 1) and the vector example demonstrates the generality of this Grundvorstellung. Bender (1990a) sees the accumulation Grundvorstellung as providing a relation between the anti-derivative aspect and the Grundvorstellung of area. Firstly, forming sums of products and taking their limit is the converse operation to forming rates of change, and secondly, finding areas can also be achieved by cumulating sub-regions. Therefore, the Grundvorstellung "accumulation" occupies an outstanding position (Thompson & Silverman 2007, p. 117). This importance is thus particularly relevant for the "concepts of the definite and indefinite integrals and its connection with the concept of the derivate" (Kouropatov & Dreyfus 2013, p. 644). The technical essence of the interpretation of integration as accumulation lies in the product sum aspect. It also allows for a link between the process of accumulation and the product to be established.

4.8 Overview: aspects and Grundvorstellungen of the concept of definite integral

The figure shown below gives the links between aspects and Grundvorstellungen established from the discussion in the previous sections, which together describe the concept of definite integral from both technical mathematical and subject-matter didactical perspectives. The left column lists the aspects, and the right column lists the Grundvorstellungen. The links shown correspond subject matter relations between aspects and Grundvorstellungen. For example, connecting of the concepts of product sum and area means that areas can be determined with the help of product sums according to the Grundvorstellung "area". The missing connection between "measure" and "average value" should state that from a subject-specific point of view, the measure aspect has no meaning for the Grundvorstellung "average value".

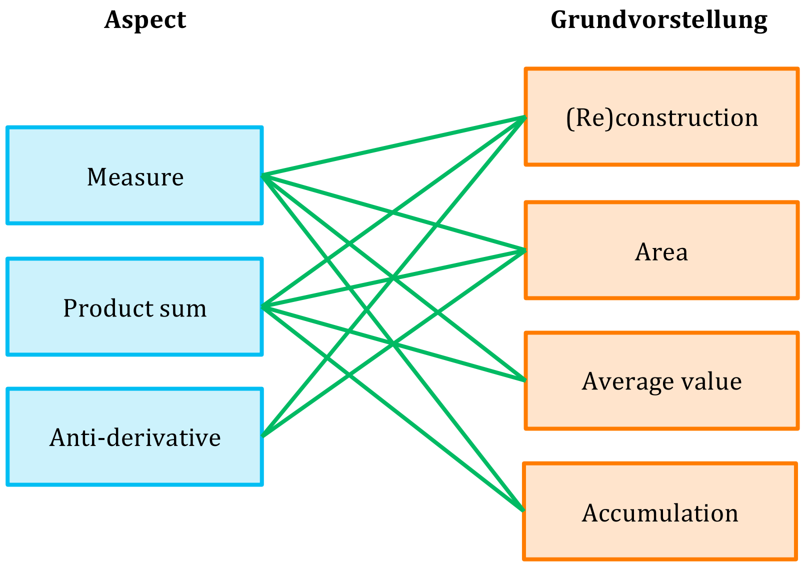

Fig. 5: Aspects and Grundvorstellungen of integration

The characterization of aspects and Grundvorstellungen for integration are not used consistently across the literature. In particular, there is often no differentiation between aspects and Grundvorstellungen. Hußmann (2001) distinguishes for example between *aspects* that permit “different angles for viewing the concept of integral” and Grundvorstellungen, among which he includes both *accumulation* and the *total effect* as do Danckwerts & Vogel (2006, p. 96 ff.), calling them “underlying conceptions”. Tietze et al. (2000) uses the term *Grundverständnis* (basic understanding), and states, regarding the calculation of definite integrals: “The integral $\int_a^b f$ of a function f over an interval [a; b] is a number resulting from a limiting process, giving the result of an accumulation.” (Tietze et al. 2000, p. 281). The Grundvorstellung of accumulation is thus accorded special importance for definite integral calculations. Accumulation is also considered to have many links to the various different aspects: “Antiderivatives arise from accumulation”, and “Finding the area under the graph is accumulation.” (Bender 1990b, p. 114 ff.). It is seen as a concept or an idea (Thompson & Silverman 2007, p. 117, Kouropatov & Dreyfus 2013, p. 643). Büchter and Henn (2010, p. 92) on the other hand, focus on a *fundamental idea*, and regard *reconstruction* as the fundamental idea behind integration.

In addition to the aspects and conceptions mentioned above Hußmann (2001) also described the approximation aspect, both “with regard to approximating areas and with regard to approximating the function” (Hußmann 2001, p. 56). Approximation is also seen in an exemplary curriculum as the “initial activity for introducing the concept of the definite integral” (Kouropatov & Dreyfus 2013, p. 646).
In the theory described here, approximation is neither an aspect nor a Grundvorstellung of integration. Approximation on its own cannot be used for defining definite integrals. The definition occurs instead within the framework of the product-sum aspect described above, as the limit of upper and lower sums. Thus, product sums have been described as the relevant aspect, not approximation. The idea of approximation can occur in all of the previously mentioned Grundvorstellungen. Therefore, it is not useful to specify “approximation” as a Grundvorstellung on its own.
The idea of approximation is important for developing a conceptual understanding of the definite integral (Sealey & Oehrtman 2005, p. 83). Integrating, as a means of determining areas under curvilinear boundaries, also cannot be considered without the idea of approximation (Dankwerts & Vogel 2006, p. 100 ff.).
In any case, it is important in the mathematics classroom that there be varied views on the definite integral, so as “to build a more insightful concept” (Rasslan & Tall 2002, p. 8). Huang (2012, p. 167) regards narrowing down viewpoints to one or more specific aspects or Grundvorstellungen of integration as questionable.

5 Aspects and Grundvorstellungen in the context of concept understanding and subject matter didactics

In this section, the *aspects* and *Grundvorstellungen* of differentiation and integration that were established in the previous sections are classified according to the process of concept development. We propose two competence models of the concepts of derivative and integral based on our subject matter related analysis concerning aspects and Grundvorstellungen.

5.1 The relation between aspects, Grundvorstellungen and concept understanding for the concept of derivative

Fig. 6 shows the relation between aspects and Grundvorstellungen in differentiation. Overall, this can been represented by a 2 x 4 matrix (with 6 non-empty cells). If the “dimension” of concept understanding is added for each of the 4 categories, this can be represented by a 3d model.

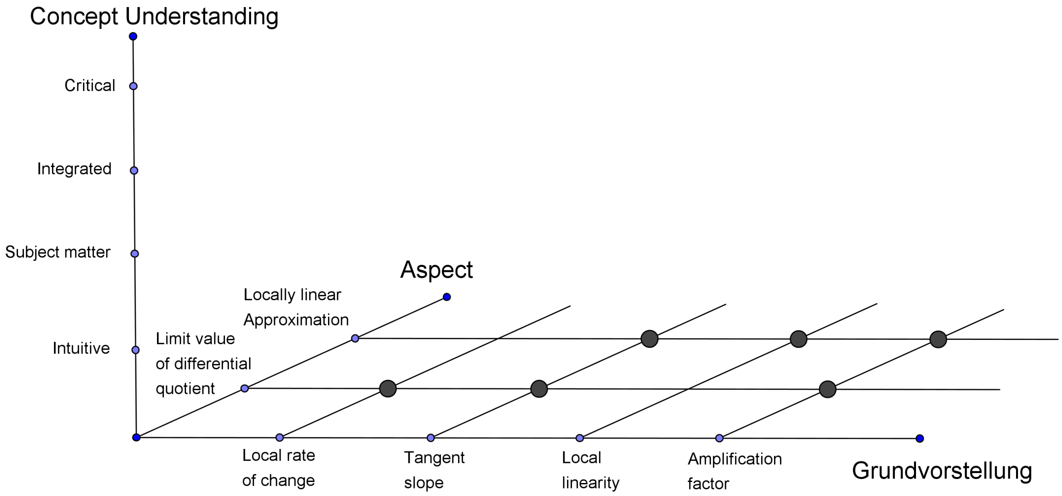

Fig. 6: 3d-matrix representing aspects, Grundvorstellungen and concepts understanding in the concept of derivative

Using this representation, 6 x 4 of the "cells" can be characterized. In the next section, some of these cells will be described as examples.

On the concept of derivative

- At the level of *intuitive concept understanding*, the different pairings of Grundvorstellungen and aspects can be clearly distinguished. The characterization of the cells is given as follows:
    - The Grundvorstellung of local rate of change with the limit aspect encompasses the conception that the average rate of change stabilizes numerically, as the interval declines in size.
    - The same intuitive limit together with the interpretation of differential quotients as the slope of secant lines, yields the Grundvorstellung of tangent slope within the same aspect.
    - Given the aspect of local linearization, the conception of tangent slope is the idea of closely fitting lines to the graph.
    - The mental image of local linearization using a function microscope (see Kirsch, 1979, p. 25) can be viewed as an intuitive understanding of the idea of local linear approximation.
    - The intuitive conception of amplification factors incorporates, among other things, experiences beyond the field of mathematics—small changes of independent parameters are transferred to the dependent parameter. The conception of a generalized factor of proportionality can be identified within the differential quotient aspect. However, within the aspect of linear approximability, the calculation-based perception of a factor as in $\Delta y \approx m \cdot \Delta x$ is dominant.

At the levels of understanding further along the vertical direction, the various different Grundvorstellungen are integrated and combined into one comprehensive overall concept, so that concept understanding is enhanced and extended.

- *Subject-matter concept understanding* involves in particular the application of the concept and its properties to reasoning processes. For example, using the aspect of limits of differential quotients, it is possible to reason (based on a calculation) that tangent slopes combine additively. Using the conception of rate of change within the aspect of local linearization, it is possible to explain the product structure of the chain rule (see Section 2.6).

- Results about monotonicity and local extrema can be interpreted on different levels of representation. However, these different conceptions link together increasingly. Graphical interpretation, for example, links the rates of change and the conception of tangents. Thus, we arrive at an *integrated concept understanding*. If an understanding of this type takes root, it becomes possible to embed the concept of derivative into the wider theory of calculus as a tool for studying functions.

- The formation of a *critical concept understanding* also makes it possible to argue that certain functions are not differentiable, for example because they do not appear to be locally linear, either because there is no unique factor of amplification, or because a limit value does not exist.

This 3d-model can be used in several ways. On the one hand, it could support teachers in introducing this important topic of derivative. Thus, one has an overview of the Grundvorstellungen and aspects at the addressed level of concept understanding. On the other hand this model can also be used for diagnostic intervention. Thus one also has an orientation, while constructing examples for a certain test on – here – derivatives. But this model can guide educators of educators, in order to show pre-service teachers, at a certain point, what needs to be known, so that mathematics education moves forward in its orientation. All the listed facts can also be interpreted for the integral. We are therefore considering the aspects, Grundvorstellungen and concept understanding in analogously.

5.2 The relation between aspects, Grundvorstellungen and concept understanding for the concept of integral

Fig. 7 gives the relation between aspects and Grundvorstellungen for integration. In the plane, this relation is given by a 3 x 4 matrix with 10 non-empty cells. If the "dimension" of concept understanding is added for each of the 4 categories, this can be represented by a 3d model.

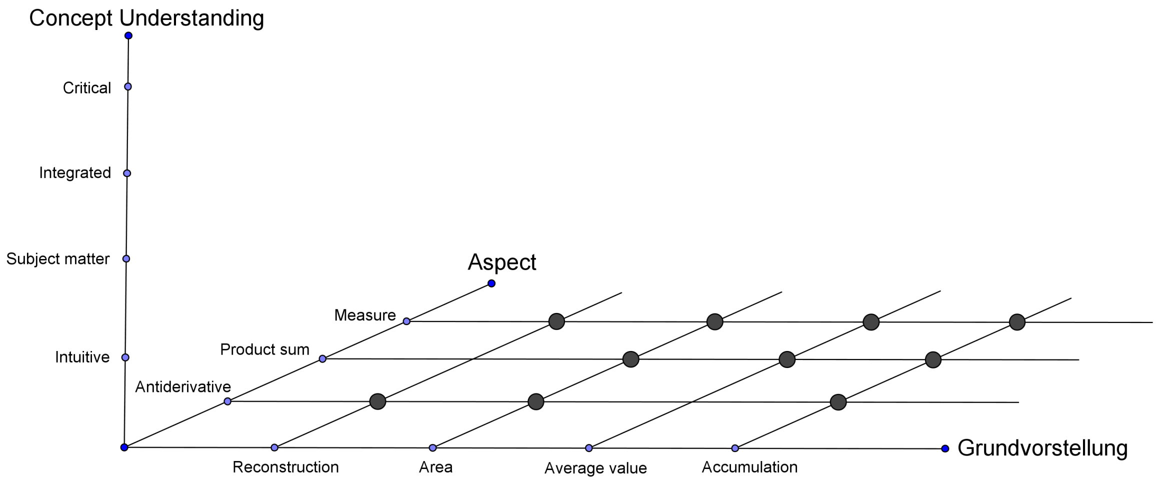

Fig. 7: 3d-matrix of understanding the aspects, Grundvorstellungen and concepts within the concept of integral

- The relations between Grundvorstellungen and aspects in *intuitive concept* understanding can be described with the following example:
    - The Grundvorstellung of area, combined with the measure aspect, includes the conception that integrals can be used to measure the area of regions between the graph of the function and the *x*-axis;
    - The Grundvorstellung of reconstruction, combined with the antiderivative aspect, includes the idea that an antiderivative can be graphically constructed from a given function using piecewise-linear curves.

- At the level of *subject-matter concept understanding*, one can, for example, use the product sum aspect to argue within the context of the average value Grundvorstellung that the integral can be used as the basis for a generalized concept of average value.

- The Fundamental Theorem of Calculus should be seen as part of an *integrated concept understanding*. On the one hand, it can be understood from the angle of the antiderivative aspect, using the accumulation Grundvorstellung. Alternatively, working from the perspective of area together with the measure aspect, also makes sense.

- Forming a *critical concept understanding* makes it possible to reason, using the antiderivative aspect together with the Grundvorstellung of construction, that certain functions do not possess an antiderivative.

6 Final remarks

The aim of this article was to specify the concept of Grundvorstellung for differentiation and integration, and embed it in a model of understanding for these two concepts. This was accomplished by a technical and subject-matter didactic analysis that established the (technical) aspects of these two concepts. The result is a representation in the form of a three-dimensional matrix, whose cells are described by a triple (aspect, Grundvorstellung, level of concept understanding). Each cell can be characterized on the one hand by specific knowledge, abilities and the capacity to manipulate the concepts in question, or alternatively, by the nature of the relation between the technical properties (aspects) and Grundvorstellungen. This leads to a description of the expected competencies for each of the "cells" of the 3d matrix. This 3d matrix can also be interpreted as the basis for a competency model that undertakes to establish a "theoretical delimitation of subject matter and structural description of an area of competency" (Leuders 2014, p. 9), in this case of the concepts of derivative and integral. Furthermore, this can be a first step towards an empirical evaluation of competence models on the basis of aspects and Grundvorstellungen.

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