Amorphous surface growth via a level set approach

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1. Introduction

Amorphous surface growth problems have been considered by Barabasi and Stanley [4] from a physical point of view. Particles leave from a source and impinge on a substrate perpendicularly, which causes a buildup in the profile. Consider an interface characterized by its height $h(x, t)$, and assume that $h(x, t)$ is a single-valued function, i.e., there are no "overhangs". The equation has the form (see also [12]).

$$\frac{\partial h(x,t)}{\partial t} = a_1 \nabla^2 h + a_2 \nabla^4 h + a_3 \nabla^2 (\nabla h)^2 + a_4 (\nabla h)^2 + a_5 (\nabla^2 h)^2 + F + \eta,$$

where F is the constant mean deposition flux, a_i, $1 \leq i \leq 5$, are experimentally determined coefficient functions, and $\eta(x, t)$ represents spatio-temporal Gaussian white noise as given by

$$\langle \eta(x,t) \rangle_\eta = 0, \quad \left\langle \eta(x,t)\eta(x',t') \right\rangle_\eta = 2D\delta(x - x')\delta(t - t').$$

Here $\langle . \rangle_\eta$ denotes the ensemble average. Note that the equation of the surface growth was obtained by standard symmetry principles (cf. [4]).

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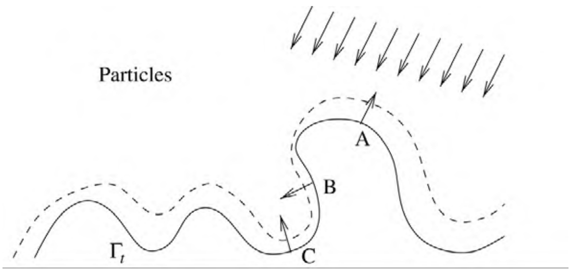

Fig. 1. Deposition process.

In the present paper, we consider the amorphous surface growth by the level set approach in case the deposition direction is not perpendicular to the substrate. Also, the substrate will be considered as an initial hypersurface and therefore, the height function $h(x,t)$ is no longer a single-valued function. The surface Γ_t at time t is considered as the zero-level set of some function u (cf. [3,8,9]), i.e.,

$$\Gamma_t := \{x \in \mathbb{R}^d \mid u(x,t) = 0\}.$$

The problem is as follows:

Let Γ_0 be a smooth hypersurface in $\mathbb{R}^d$, $d \geq 2$. Let U_0 be a bounded open set in $\mathbb{R}^d$ whose boundary is given by Γ_0. As time progresses we allow the surface to evolve by moving each point at a velocity depending on some physical effects which will be called “deposition”. Assuming this evolution is smooth, for each $\{\Gamma_t\}_{t>0}$ we thereby define a new hypersurface Γ_t. The primary problem is then to study geometric properties of $\{\Gamma_t\}_{t>0}$.

The uniformly parallel stream of particles impinges on the hypersurface $\{\Gamma_t\}_{t\geq 0}$ from a source with the unit direction e (cf. Fig. 1). We denote by $\beta > 0$ the strength of this source and define the vector $b := \beta e$.

Let $\nu = \nu(x,t)$ denote the unit outward normal vector field to Γ_t, $t \geq 0$. Consider the point $A \in \Gamma_t$ in Fig. 1. The rate of milling of Γ_t at A is $-b.\nu > 0$, the direction of ν is the direction of $-b$. Therefore, the deposition velocity of Γ_t at A is $-b.\nu$. Next, we observe that at the point $B \in \Gamma_t$, we have $-b.\nu < 0$ but the deposition velocity of the surface there is zero, as the point B is screened from the source. We can combine these cases to note that the deposition velocity of the surface at both points A, B is given by $(-b.\nu)_+$, the subscript $+$ denoting the positive part. However, at the point C we have $-b.\nu > 0$. Since there are no particles depositing on this position, the deposition velocity at this point is zero. Consequently, let us introduce a parameter χ which is zero, if a given point is not visible from the source with direction e, and 1 otherwise. Then, at each of the points A, B, C, and therefore everywhere on Γ_t, we obtain the deposition velocity of the motion by

$$\chi(-b.\nu)_+. \tag{1}$$

On the other hand, the velocity of the motion of a hypersurface depends on the surface diffusion (see, e.g., [4]). For instance, the Edwards–Wilkinson equation describes deposition processes with surface relaxation. The arriving particles from the source do not deposit perpendicularly to the substrate orientation. This implies that there are more particles arriving at positions, where we have negative mean curvature, than at positions with positive mean curvature. This can be understood in the sense that the motion of a particle does not depend on the local position of the surface, but only on the number of bonds that must be broken for diffusion to take place. If the mean curvature is negative, the particle has a large number of neighbors, and moving away from the site will be difficult. In contrast, if the mean curvature is positive, the particle has a few neighbors, and is able to diffuse easily. Therefore, the velocity of the surface diffusion is proportional to the negative mean curvature

$$-\lambda H, \tag{2}$$

where H denotes the mean curvature of the surface and $\lambda > 0$. We combine (1) and (2) to find that the velocity V of the interface is given by

$$V = \chi(-b.\nu)_+ - \lambda H. \tag{3}$$

Our paper is organized as follows:
In Section 2, we formalize a level set method [1–3,8,9] to the surface evolution equations. In Section 3, we define the notion of a viscosity solution, state some properties of viscosity solutions, and prove the existence and uniqueness of a viscosity solution to the surface deposition equation.

Hereafter, $\mathbb{R}^d$ denotes the d-dimensional Euclidean space, for $x \in \mathbb{R}^d$, $x = (x_1, \ldots, x_d)$, $x_i \in \mathbb{R}$, $i = 1, \ldots, d$. We denote $|x| = (x_1^2 + \cdots + x_d^2)^{\frac{1}{2}}$, $x \in \mathbb{R}^d$ the norm of x in $\mathbb{R}^d$, and $x.y$ the inner scalar product of x and y in $\mathbb{R}^d$. $B(x, r)$ and $B[x, r]$ denote the open and closed ball of radius r centered at x in $\mathbb{R}^d$, $B_r := B(0, r)$. $S^{d\times d}$ denotes the set of all symmetric $(d \times d)$-matrices. ∇u and $\nabla^2 u$ stand for the gradient and the Hessian of u, while $\mathrm{D}u$ and $\mathrm{D}^2 u$ stand for the first order and the second order distributional derivatives.

2. The level set method

From an initial hypersurface $\Gamma_0 := \Gamma\ (t = 0)$, the hypersurface Γ_t $(t \geq 0)$ evolves according to its normal vector field with velocity given by (3). The main idea is to express this propagating interface as the zero-level set of a higher dimensional function u. Namely, we determine an equation for the evolving function $u(x, t)$ which contains the embedded motion of Γ_t as the zero-level set $\{u = 0\}$. Let $(x(t), t)$ be the path of a point on the propagating front, i.e., $x\ (t = 0)$ is a point on the initial front Γ_0. Since the evolving function u is always zero on the propagating hypersurface, we must have

$$u(x(t), t) = 0, \quad t \geq 0. \tag{4}$$

By the chain rule,

$$u_t + \nabla u(x(t), t).x'(t) = 0. \tag{5}$$

Since $x'(t).\nu = V$, where $\nu := \frac{\nabla u}{|\nabla u|}$, we then have

$$u_t + V|\nabla u| = 0, \tag{6}$$

with the initial condition

$$u(x, 0) = u_0. \tag{7}$$

As we have mentioned before, the interface Γ_t $(t \geq 0)$ is considered as the zero-level set of u, i.e.,

$$\Gamma_t = \{x \in \mathbb{R}^d \mid u(x, t) = 0\}. \tag{8}$$

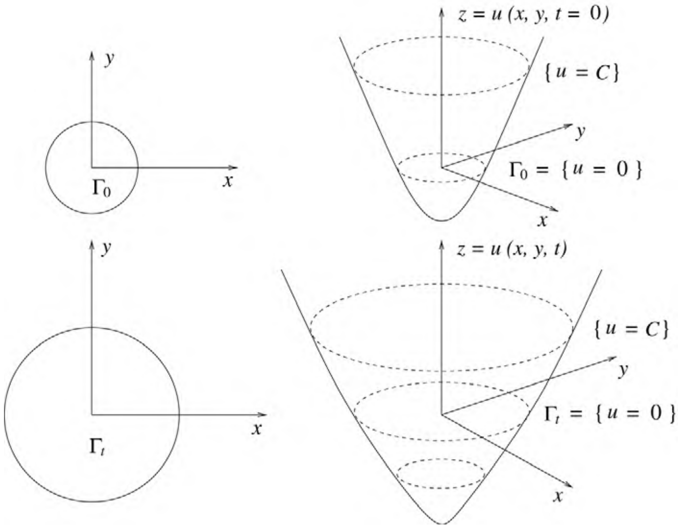

Fig. 2. The level set approach.

Let us initialize u by selecting any smooth function $u_0 : \mathbb{R}^d \longrightarrow \mathbb{R}$ so that

$$\Gamma_0 := \{x \in \mathbb{R}^d \mid u_0(x) = 0\}, \quad U_0 := \{x \in \mathbb{R}^d \mid u_0(x) > 0\}. \tag{9}$$

This level set approach is illustrated by Fig. 2.
Since Γ_t is a zero-level set of u for $t \geq 0$,

$$\nu = \frac{\nabla u}{|\nabla u|},$$

and the mean curvature of Γ_t is

$$H = -\mathrm{div}(\nu) = -\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right).$$

On the other hand, the normal velocity of the level set Γ_t is

$$-\frac{u_t}{|\nabla u|}.$$

Thus, we have

$$-\frac{u_t}{|\nabla u|} = -\lambda\,\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) + \chi\left(\frac{b.\nabla u}{|\nabla u|}\right)_+,$$

and so

$$u_t = \lambda\left(\delta_{ij} - \frac{u_{x_i}u_{x_j}}{|\nabla u|^2}\right)u_{x_ix_j} - \chi(b.\nabla u)_+.$$

We wish to write χ in terms of u by setting

$$\chi = \chi(x,t,u) = \begin{cases} 1 & \text{if } u(x-sb,t) < u(x,t) \quad \text{for all } s > 0 \\ 0 & \text{if } u(x-sb,t) > u(x,t) \quad \text{for some } s > 0. \end{cases} \tag{10}$$

Without loss of generality, we may assume that $\lambda = 1$ and finally have the equation of motion

$$u_t = \left(\delta_{ij} - \frac{u_{x_i} u_{x_j}}{|\nabla u|^2}\right) u_{x_i x_j} - \chi(x, t, u)(b.\nabla u)_+ \quad \text{in } \mathbb{R}^d \times [0, T], \tag{11}$$

where T denotes some fixed positive time and the initial condition is

$$u = u_0 \quad \text{on } \mathbb{R}^d \times \{t = 0\}. \tag{12}$$

3. Viscosity solutions

We are going to define the notion of a viscosity solution (see [5–10]). In particular, we will follow Evans and Spruck [8] as well as Jensen [10] and define our viscosity solution in terms of its pointwise behavior with respect to a smooth test function. We further follow Adalsteinsson et al. [3] to introduce the following functions which we need to define our viscosity solutions

$$\chi^+(x, t, u) = \begin{cases} 1 & \text{if } u(x - sb, t) \le u(x, t) \quad \text{for all } s > 0 \\ 0 & \text{if } u(x - sb, t) > u(x, t) \quad \text{for some } s > 0, \end{cases} \tag{13}$$

$$\chi^-(x, t, u) = \begin{cases} 1 & \text{if } u(x - sb, t) < u(x, t) \quad \text{for all } s > 0 \\ 0 & \text{if } u(x - sb, t) \ge u(x, t) \quad \text{for some } s > 0. \end{cases} \tag{14}$$

We note that $\chi^- \le \chi^+$.

Definition 3.1. A function $u \in C(\mathbb{R}^d \times [0, T])$ is a viscosity subsolution of (11) provided there holds:

If for each $\phi \in \mathbb{R}^{d+1}$ the function $u - \phi$ has a local maximum at a point $(x_0, t_0) \in \mathbb{R}^d \times (0, T]$, then

$$\begin{cases} \phi_t(x_0, t_0) \le \left(\delta_{ij} - \dfrac{\phi_{x_i}\phi_{x_j}}{|\nabla \phi|^2}\right) \phi_{x_i x_j}(x_0, t_0) - \chi^-(x_0, t_0, u)(b.\nabla\phi(x_0, t_0))_+ \\ \text{if } \nabla\phi(x_0, t_0) \ne 0, \end{cases} \tag{15}$$

and

$$\begin{cases} \phi_t(x_0, t_0) \le (\delta_{ij} - \eta_i \eta_j)\phi_{x_i x_j}(x_0, t_0) \\ \text{for some } \eta \in \mathbb{R}^d \text{ with } |\eta| \le 1, \text{ if } \nabla\phi(x_0, t_0) = 0. \end{cases} \tag{16}$$

Definition 3.2. A function $u \in C(\mathbb{R}^d \times [0, T])$ is a viscosity supersolution of (11) provided there holds:

If for each $\phi \in \mathbb{R}^{d+1}$ the function $u - \phi$ has a local minimum at a point $(x_0, t_0) \in \mathbb{R}^d \times (0, T]$, then

$$\begin{cases} \phi_t(x_0, t_0) \ge \left(\delta_{ij} - \dfrac{\phi_{x_i}\phi_{x_j}}{|\nabla \phi|^2}\right) \phi_{x_i x_j}(x_0, t_0) - \chi^+(x_0, t_0, u)(b.\nabla\phi(x_0, t_0))_+ \\ \text{if } \nabla\phi(x_0, t_0) \ne 0, \end{cases} \tag{17}$$

and

$$\begin{cases} \phi_t(x_0, t_0) \ge (\delta_{ij} - \eta_i \eta_j)\phi_{x_i x_j}(x_0, t_0) \\ \text{for some } \eta \in \mathbb{R}^d \text{ with } |\eta| \le 1, \text{ if } \nabla\phi(x_0, t_0) = 0. \end{cases} \tag{18}$$

Definition 3.3. A function $u \in C(\mathbb{R}^d \times [0, T])$ is a viscosity solution of (11) provided u is both a viscosity subsolution and a viscosity supersolution.

Remark. (i) It is easy to see that a classical solution of (11) is a viscosity solution of (11) (cf. [8]), and:

(ii) If u is a viscosity solution of (11) and u is twice differentiable at a point (x_0, t_0), then u satisfies Eq. (11) at (x_0, t_0).

We derive an equivalent definition of viscosity solutions by the notions of sub-differential and super-differential. We use the following notations: $z := (x, t)$, $z_0 := (x_0, t_0)$, and $S^{(d+1)\times(d+1)}$ is the set of all symmetric $(d + 1) \times (d + 1)$-matrices.

Definition 3.4. A function $u \in C(\mathbb{R}^d \times [0, T])$ is a viscosity subsolution of (11) provided that if whenever $(x_0, t_0) \in \mathbb{R}^d \times (0, T]$ and

$$\begin{aligned} u(x,t) \le\ & u(x_0, t_0) + p.(x - x_0) + q(t - t_0) \\ & + \frac{1}{2}(z - z_0)^T M(z - z_0) + o(|z - z_0|^2) \quad \text{as } z \to z_0, \end{aligned} \tag{19}$$

for some $p \in \mathbb{R}^d$, $q \in \mathbb{R}$, $M = (m_{ij}) \in S^{(d+1)\times(d+1)}$, then

$$q \le \left(\delta_{ij} - \frac{p_i p_j}{|p|^2}\right) m_{ij} - \chi^-(x_0, t_0, u)(b.p)_+ \quad \text{if } p \ne 0, \tag{20}$$

and

$$q \le (\delta_{ij} - \eta_i \eta_j) m_{ij}, \tag{21}$$

for some $\eta \in \mathbb{R}^d$, $|\eta| \le 1$, if $p = 0$.

Definition 3.5. A function $u \in C(\mathbb{R}^d \times [0, T])$ is a viscosity supersolution of (11) provided that if whenever $(x_0, t_0) \in \mathbb{R}^d \times (0, T]$ and

$$\begin{aligned} u(x,t) \ge\ & u(x_0, t_0) + p.(x - x_0) + q(t - t_0) \\ & + \frac{1}{2}(z - z_0)^T M(z - z_0) + o(|z - z_0|^2) \quad \text{as } z \to z_0, \end{aligned} \tag{22}$$

for some $p \in \mathbb{R}^d$, $q \in \mathbb{R}$, $M = (m_{ij}) \in S^{(d+1)\times(d+1)}$, then

$$q \ge \left(\delta_{ij} - \frac{p_i p_j}{|p|^2}\right) m_{ij} - \chi^+(x_0, t_0, u)(b.p)_+, \quad \text{if } p \ne 0, \tag{23}$$

and

$$q \ge (\delta_{ij} - \eta_i \eta_j) m_{ij} \tag{24}$$

for some $\eta \in \mathbb{R}^d$, $|\eta| \le 1$, if $p = 0$.

The proof of the equivalences is standard.

Hereafter, our smooth initial function u_0 is assumed to satisfy the following condition:

$$\begin{cases} u_0 \ge -1 & \text{on } \mathbb{R}^d \\ u_0 = -1 & \text{on } \mathbb{R}^d - B(0, R) \quad \text{for some } R > 0. \end{cases} \tag{25}$$

Moreover, we will be mostly interested in tracking the evolution of certain bounded level set of viscosity solutions of (11) and (12). Therefore, we may assume without loss of generality that:

$$\begin{cases} u \geq -1 & \text{on } \mathbb{R}^d \times [0, T] \\ u = -1 & \text{on } (\mathbb{R}^d - B(0, R)) \times [0, T]. \end{cases} \tag{26}$$

3.1. Some properties of viscosity solutions

Theorem 3.1. *Assume u_k is a viscosity subsolution of (11) for $k = 1, 2, \dots$ and $u_k \to u$ bounded and locally uniformly on $\mathbb{R}^d \times [0, T]$. Then u is a viscosity subsolution of (11).*

An analogous assertion holds for viscosity supersolutions and solutions.

Remark. By means of (26) we may always assume that

$$\begin{cases} u, u_k \geq -1 & \text{on } \mathbb{R}^d \times [0, T] \\ u, u_k = -1 & \text{on } (\mathbb{R}^d - B(0, R)) \times [0, T], k = 1, 2, \dots. \end{cases} \tag{27}$$

Proof. Choose $\phi \in C^\infty(\mathbb{R}^{d+1})$ and suppose first that $u - \phi$ has a strict local maximum at some point $(x_0, t_0) \in \mathbb{R}^d \times (0, T]$. Since $u_k \to u$ uniformly near (x_0, t_0), there exists a sequence of points $\{(x_k, t_k)\}_{k=1}^\infty \subset \mathbb{R}^d \times (0, T]$ satisfying

$$\begin{cases} (x_k, t_k) \to (x_0, t_0) \text{ as } k \to \infty \\ u_k - \phi \text{ has a local maximum at } (x_k, t_k) \\ u_k(x_k, t_k) \to u(x_0, t_0) \text{ as } k \to \infty. \end{cases} \tag{28}$$

Since each u_k is a viscosity subsolution of (11), by Definition 3.1 of viscosity subsolutions we either have

$$\begin{cases} \phi_t(x_k, t_k) \leq \left(\delta_{ij} - \dfrac{\phi_{x_i}\phi_{x_j}}{|\nabla\phi|^2}\right)\phi_{x_i x_j}(x_k, t_k) - \chi^-(x_k, t_k, u)(b.\nabla\phi(x_k, t_k))_+ \\ \text{if } \nabla\phi(x_k, t_k) \neq 0, \end{cases} \tag{29}$$

or

$$\begin{cases} \phi_t(x_k, t_k) \leq (\delta_{ij} - \eta_i\eta_j)\phi_{x_i x_j}(x_k, t_k) \\ \text{for some } \eta \in \mathbb{R}^d \text{ with } |\eta| \leq 1, \text{ if } \nabla\phi(x_k, t_k) = 0. \end{cases} \tag{30}$$

Assume next that $\nabla\phi(x_0, t_0) \neq 0$. Then $\nabla\phi(x_k, t_k) \neq 0$ for all sufficiently large k. Hence, the left hand side and the first term on the right hand side of the inequality (29) converge to

$$\phi_t(x_0, t_0) \quad \text{and} \quad \left(\delta_{ij} - \frac{\phi_{x_i}\phi_{x_j}}{|\nabla\phi|^2}\right)\phi_{x_i x_j}(x_0, t_0),$$

respectively. We wish to prove that

$$\chi^-(x_0, t_0, u) \leq \liminf_{k\to\infty} \chi^-(x_k, t_k, u_k). \tag{31}$$

Indeed, if the right-hand side of (31) equals 1, then there is nothing to prove. Otherwise, we may assume that

$$\chi^-(x_k, t_k, u) = 0, \quad k = 1, 2, \dots. \tag{32}$$

According to (14), we have

$$u_k(x_k - s_k b, t_k) \geq u_k(x_k, t_k) \quad \text{for some } s_k > 0, \quad k = 1, 2, \ldots. \tag{33}$$

From (27), we may suppose that $\{s_k\}_{k=1}^{\infty}$ is bounded and

$$s_k \to s \geq 0.$$

In view of (33) and (28), we deduce

$$u(x_0 - sb, t_0) \geq u(x_0, t_0).$$

Now, if $s > 0$, this implies $\chi^-(x_0, t_0, u) = 0$ and assertion (31) follows. Suppose instead $s = 0$. Since $u_k - \phi$ has a maximum at (x_k, t_k) and $s_k \to 0$, we get

$$(u_k - \phi)(x_k, t_k) \geq (u_k - \phi)(x_k - s_k b, t_k).$$

But then

$$\phi(x_k - s_k b, t_k,) - \phi(x_k, t_k) \geq u_k(x_k - s_k b, t_k) - u_k(x_k, t_k) \geq 0.$$

Dividing by s_k and letting $k \to \infty$ (note that $s_k \to 0$), we deduce

$$b.\nabla\phi(x_0, t_0) \leq 0. \tag{34}$$

This inequality yields

$$\chi^-(x_0, t_0, u)(b.\nabla\phi(x_0, t_0))_+ = 0. \tag{35}$$

Letting $k \to \infty$, we obtain

$$\phi_t(x_0, t_0) \leq \left(\delta_{ij} - \frac{\phi_{x_i}\phi_{x_j}}{|\nabla\phi|^2}\right)\phi_{x_i x_j}(x_0, t_0) - \chi^-(x_0, t_0, u)(b.\nabla\phi(x_0, t_0))_+.$$

Next, suppose that $\nabla\phi(x_0, t_0) = 0$. We set

$$\xi^k := \begin{cases} \dfrac{\nabla\phi}{|\nabla\phi|}(x_k, t_k) & \text{if } \nabla\phi(x_k, t_k) \neq 0 \\ \eta^k & \text{if } \nabla\phi(x_k, t_k) = 0. \end{cases}$$

Passing, if necessary, to a subsequence we may assume $\xi^k \to \eta$. Then $|\eta| \leq 1$. Utilizing now (30), we deduce as well

$$\phi_t(x_0, t_0) \leq (\delta_{ij} - \eta_i\eta_j)\phi_{x_i x_j}(x_0, t_0).$$

The requirement that $u - \phi$ has a strict local maximum at (x_0, t_0) can be removed by an approximation. Hence, u is a viscosity subsolution of (11). □

Theorem 3.2. *Assume $\theta : \mathbb{R} \to \mathbb{R}$ is C^1 with $\theta' > 0$. If u is a viscosity subsolution (resp. viscosity supersolution) of (11), then $\hat{u} = \theta(u)$ is a viscosity subsolution (resp. viscosity supersolution) of (11).*

The proof of this theorem is standard (cf., e.g., [3,8]).

3.2. Uniqueness of viscosity solutions

Definition 3.6. Let $w : \Omega_T := \mathbb{R}^d \times [0, T] \longrightarrow \mathbb{R}$ be a continuous function. For each $\epsilon > 0$, we write

$$w^\epsilon(x, t) := \sup_{(y,s)\in\Omega_T} \left\{ w(y, s) - \frac{1}{\epsilon}(|x - y|^2 + (t - s)^2) \right\}, \tag{36}$$

and

$$w_\epsilon(x, t) := \inf_{(y,s)\in\Omega_T} \left\{ w(y, s) + \frac{1}{\epsilon}(|x - y|^2 + (t - s)^2) \right\} \tag{37}$$

for $(x, t) \in \Omega_T$.

w^ϵ and w_ϵ are called **Sup** and **Inf convolution** of w, respectively. Note that if w is continuous and bounded, the "sup" and "inf" above can be replaced by "max" and "min". We follow Evans and Spruck (see [8]) to introduce some properties of the inf and sup convolution:

Lemma 3.3. *Let $w : \Omega_T \longrightarrow \mathbb{R}$ be a bounded continuous function. Then we have*

(i) $w_\epsilon \le w \le w^\epsilon$ *on* Ω_T.
(ii) *If* $(y, s) \in \Omega_T$, *and* $w^\epsilon(x, t) = w(y, s) - \frac{1}{\epsilon}(|x - y|^2 + (t - s)^2)$, *then*

$$|x - y|, |t - s| \le C\epsilon^{\frac{1}{2}} =: \sigma(\epsilon). \tag{38}$$

A similar assertion holds for w_ϵ.
(iii) $w^\epsilon, w_\epsilon \to w$ *as* $\epsilon \to 0^+$, *locally uniformly on* Ω_T.
(iv) *The mapping*

$$(x, t) \mapsto w^\epsilon(x, t) + \frac{1}{\epsilon}(|x|^2 + t^2) \tag{39}$$

is convex, and the mapping

$$(x, t) \mapsto w_\epsilon(x, t) - \frac{1}{\epsilon}(|x|^2 + t^2) \tag{40}$$

is concave.
(v) *Assume u is a viscosity subsolution of* (11) *in* Ω_T. *Then* u^ϵ *is a viscosity subsolution of* (11) *on* $\mathbb{R}^d \times (\sigma(\epsilon), T]$. *Similarly, if u is a viscosity supersolution of* (11) *in* Ω_T *then* u_ϵ *is a viscosity supersolution of* (11) *on* $\mathbb{R}^d \times (\sigma(\epsilon), T]$.

Proof. Assertions (i)–(iv) are standard. We only prove (v):

For $\phi \in C^\infty(\mathbb{R}^{d+1})$, assume that $u^\epsilon - \phi$ has a local maximum at point (x_0, t_0) with $t_0 > \sigma(\epsilon)$. We then employ (26) and (36) to choose $(y_0, s_0) \in \Omega_T$ so that

$$u^\epsilon(x_0, t_0) = w(y_0, s_0) - \frac{1}{\epsilon}(|x_0 - y_0|^2 + (t_0 - s_0)^2).$$

Set

$$\psi(x, t) := \phi(x + x_0 - y_0, t + t_0 - s_0). \tag{41}$$

Since $u^\epsilon - \phi$ has a local maximum at (x_0, t_0), we compute

$$w(y_0, s_0) - \frac{1}{\epsilon}(|x_0 - y_0|^2 + (t_0 - s_0)^2) - \phi(x_0, t_0)$$

$$= u^\epsilon(x_0, t_0) - \phi(x_0, t_0) \geq u^\epsilon(x, t) - \phi(x, t)$$
$$\geq u(y, s) - \frac{1}{\epsilon}(|x - y|^2 + (t - s)^2) - \phi(x, t)$$

for all (x, t) near (x_0, t_0) and all $(y, s) \in \Omega_T$. Fix (y, s) close to (y_0, s_0) and set $x := y + x_0 - y_0, t := s + t_0 - s_0$ as above, we have

$$u(y_0, s_0) - \phi(x_0, t_0) \geq u(y, s) - \phi(y + x_0 - y_0, s + t_0 - s_0).$$

Using (41), we get

$$u(y_0, s_0) - \psi(y_0, s_0) \geq u(y, s) - \psi(y, s) \tag{42}$$

for all (y, s) near (y_0, s_0). Hence, $u - \psi$ has a local maximum at (y_0, s_0).

Since u is a viscosity subsolution of (11),

$$\begin{cases} \psi_t(y_0, s_0) \leq \left(\delta_{ij} - \dfrac{\psi_{x_i}\psi_{x_j}}{|\nabla\psi|^2}\right)\psi_{x_ix_j}(y_0, s_0) - \chi^-(y_0, s_0, u)(b.\nabla\psi(y_0, s_0))_+ \\ \text{if} \quad \nabla\psi(y_0, s_0) \neq 0, \end{cases}$$

and

$$\begin{cases} \psi_t(y_0, s_0) \leq (\delta_{ij} - \eta_i\eta_j)\psi_{x_ix_j}(y_0, s_0) \\ \text{for some } \eta \in \mathbb{R}^d \text{ with } |\eta| \leq 1, \text{ if } \nabla\psi(y_0, s_0) = 0. \end{cases}$$

Moreover,

$$\nabla\psi(y_0, s_0) = \nabla\phi(x_0, t_0), \quad \psi_t(y_0, s_0) = \phi_t(x_0, t_0),$$
$$D^2\psi(y_0, s_0) = D^2\phi(x_0, t_0).$$

It remains to prove that

$$\chi^-(y_0, s_0, u) \geq \chi^-(x_0, t_0, u^\epsilon). \tag{43}$$

If $\chi^-(y_0, s_0, u) = 1$, then there is nothing to prove. Otherwise, $\chi^-(y_0, s_0, u) = 0$. By definition (14), there exists $\alpha > 0$ such that

$$u(y_0 - \alpha b, s_0) \geq u(y_0, s_0).$$

From (36), we see that

$$\begin{aligned} u^\epsilon(x_0 - \alpha b, t_0) &\geq u(y_0 - \alpha b, s_0) - \frac{1}{\epsilon}(|x_0 - y_0|^2 + (t_0 - s_0)^2) \\ &\geq u(y_0, s_0) - \frac{1}{\epsilon}(|x_0 - y_0|^2 + (t_0 - s_0)^2) \\ &= u^\epsilon(x_0, t_0). \end{aligned}$$

By definition (14), we have

$$\chi^-(x_0, t_0, u^\epsilon) = 0.$$

Therefore, we finally obtain

$$\begin{cases} \phi_t(x_0, t_0) \leq \left(\delta_{ij} - \dfrac{\phi_{x_i}\phi_{x_j}}{|\nabla\phi|^2}\right)\phi_{x_ix_j}(x_0, t_0) - \chi^-(x_0, t_0, u^\epsilon)(b.\nabla\phi(x_0, t_0))_+ \\ \text{if} \quad \nabla\phi(x_0, t_0) \neq 0, \end{cases}$$

and

$$\begin{cases}\phi_t(x_0,t_0) \le (\delta_{ij} - \eta_i\eta_j)\phi_{x_ix_j}(x_0,t_0)\\ \text{for some } \eta \in \mathbb{R}^d \text{ with } |\eta| \le 1, \text{ if } \nabla\phi(x_0,t_0) = 0.\end{cases}$$

Thus, u^ϵ is a viscosity subsolution of (11). □

Theorem 3.4. *Assume that u is a viscosity subsolution and v is a viscosity supersolution of* (11) *in* $\Omega_T = \mathbb{R}^d \times [0, T]$. *Suppose further that*

$$u \le v \quad \text{on } \mathbb{R}^d \times \{t = 0\}. \tag{44}$$

Then

$$u \le v \quad \text{in } \Omega_T. \tag{45}$$

Remark. Recall that we always assume

$$\begin{cases} u, v \ge -1 & \text{on } \mathbb{R}^d \times [0, T],\\ u, v = -1 & \text{on } (\mathbb{R}^d - B(0, R)) \times [0, T].\end{cases} \tag{46}$$

Proof. As a contradiction to the assertion, we assume that

$$\max_{\Omega_T}(u - v) =: a > 0.$$

Consequently, for α small enough,

$$\max_{\Omega_T}(u - v - \alpha t) \ge \frac{a}{2} > 0.$$

We further note that $u^\epsilon = u, v_\epsilon = v$ on $(\mathbb{R}^d - B(0, R)) \times [0, T]$, and $u^\epsilon \to u, v_\epsilon \to v$ as $\epsilon \to 0^+$ uniformly on Ω_T. Hence, for sufficiently small $\epsilon > 0$

$$\max_{\Omega_T}(u^\epsilon - v_\epsilon - \alpha t) \ge \frac{a}{4} > 0. \tag{47}$$

Given $\delta > 0$, for $x, y \in \mathbb{R}^d$ and $t, t + s \in [0, T]$ we define

$$\Phi(x, y, t, s) := u^\epsilon(x + y, t + s) - v_\epsilon(x, t) - \alpha t - \frac{1}{\delta}(|y|^4 + s^4). \tag{48}$$

In view of (47), we see that

$$\max_{(x,t),(x+y,t+s)\in\Omega_T} \Phi(x, y, t, s) \ge \frac{a}{4}. \tag{49}$$

We now choose $(x_1, t_1), (x_1 + y_1, t_1 + s_1) \in \Omega_T$ so that

$$\Phi(x_1, y_1, t_1, s_1) = \max_{(x,t),(x+y,t+s)\in\Omega_T} \Phi(x, y, t, s) > 0. \tag{50}$$

Moreover, (x_1, y_1, t_1, s_1) can be chosen from the set of maxima $M(\epsilon, \delta)$ of Φ in $\mathbb{R}^d \times [0, T]$ in such a way that

$$b.x_1 = \min\{b.x \mid (x, y, t, s) \in M(\epsilon, \delta)\}.$$

It follows

$$\begin{cases}(x_1 - \tau b, y, t, s) \notin M(\epsilon, \delta),\\ \text{for all } \tau > 0,\, y \in \mathbb{R}^d,\, 0 \le s, t \le T.\end{cases} \tag{51}$$

Thus,

$$\Phi(x_1 - \tau b, y_1, t_1, s_1) < \Phi(x_1, y_1, t_1, s_1), \quad \tau > 0. \tag{52}$$

From (48) and (50) we find

$$|y_1|, |s_1| \le C\delta^{\frac{1}{4}}, \tag{53}$$

where C is a constant independent of δ.

We claim next that if $\epsilon, \delta > 0$ are chosen small enough, we have

$$t_1 > \sigma(\epsilon) \tag{54}$$

with $\sigma(\epsilon)$ defined by (38). Indeed, if $t_1 \le \sigma(\epsilon)$, then

$$\begin{aligned}\frac{a}{4} &\le \Phi(x_1, y_1, t_1, s_1)\\ &\le u^\epsilon(x_1 + y_1, t_1 + s_1) - v_\epsilon(x_1, t_1)\\ &= u(x_1 + y_1, t_1 + s_1) - v(x_1, t_1) + o(1) \quad \text{as } \epsilon \to 0\\ &= u(x_1 + y_1, s_1) - v(x_1, 0) + o(1) \quad \text{as } \epsilon \to 0\\ &= u(x_1, 0) - v(x_1, 0) + o(1) \quad \text{as } \epsilon, \delta \to 0\\ &\le o(1) \quad \text{as } \epsilon, \delta \to 0.\end{aligned}$$

For sufficiently small $\epsilon, \delta > 0$ this is a contradiction, whence $t_1 > \sigma(\epsilon)$.

We will prove that

$$y_1 \ne 0. \tag{55}$$

Assume for contradiction that in fact $y_1 = 0$. Then (48) and (50) imply

$$\begin{aligned}&u^\epsilon(x_1, t_1 + s_1) - v_\epsilon(x_1, t_1) - \alpha t_1 - \frac{1}{\delta}s_1^4\\ &\qquad \ge u^\epsilon(x + y, t + s) - v_\epsilon(x, t) - \alpha t - \frac{1}{\delta}(|y|^4 + s^4)\end{aligned} \tag{56}$$

for all $(x, t), (x + y, t + s) \in \Omega_T$. We choose $x = x_1$ and $t = t_1$ as above, and simplify to obtain the inequality

$$u^\epsilon(x_1 + y, t_1 + s) \le u^\epsilon(x_1, t_1 + s_1) + \frac{1}{\delta}|y|^4 + \frac{1}{\delta}(s^4 - s_1^4)$$

for $(x_1 + y, t_1 + s) \in \Omega_T$. We see that

$$\begin{aligned}u^\epsilon(x_1 + y, t_1 + s) \le\ & u^\epsilon(x_1, t_1 + s_1) + \frac{4}{\delta}s_1^3(s - s_1) + \frac{6}{\delta}s_1^2(s - s_1)^2\\ &+ o(|s - s_1|^3 + |y|^4) \quad \text{as } (y, s) \to (0, s_1).\end{aligned}$$

Since u^ϵ is a viscosity subsolution of (11) near $(x_1, t_1 + s_1)$, we may invoke (19) and (21) with $x_0 = x_1, t_0 = t_1 + s_1, p = 0, q = \frac{4}{\delta}s_1^3, m_{ij} = 0$ $(i, j = 1, \ldots, d)$. This gives

$$\frac{4}{\delta}s_1^3 \le 0. \tag{57}$$

Going back and inserting $y = x_1 - x$ and $s = t_1 + s_1 - t$ into (56), we have

$$v_\epsilon(x,t) \geq v_\epsilon(x_1,t_1) + \left(\frac{4}{\delta}s_1^3 - \alpha\right)(t - t_1) + \frac{6}{\delta}s_1^2(t-t_1)^2 + o(|x-x_1|^4 + |t-t_1|^3) \quad \text{as } (x,t) \to (x_1,t_1).$$

Now, since v_ϵ is a viscosity supersolution of (11) near (x_1, t_1), we may invoke (22) and (24) with $x_0 = x_1, t_0 = t_1, p = 0, q = \frac{4}{\delta}s_1^3 - \alpha, m_{ij} = 0$ $(i, j = 1, \dots, d)$. This gives

$$\frac{4}{\delta}s_1^3 - \alpha \geq 0, \tag{58}$$

representing a contradiction to (57), since $\alpha > 0$. This establishes (55).

Note next that in general if $f : \mathbb{R}^m \longrightarrow \mathbb{R}$ is convex, then so is the mapping $(w,z) \mapsto f(w+z)$ on $\mathbb{R}^{2m}$. Consequently, Lemma 3.3(iv) asserts that the mapping

$$(x,y,t,s) \mapsto u^\epsilon(x+y,t+s) + \frac{1}{\epsilon}(|x+y|^2 + (t+s)^2)$$

is convex. As

$$(x,t) \mapsto -v_\epsilon(x,t) + \frac{1}{\epsilon}(|x|^2 + t^2)$$

is convex as well, we see that for some sufficiently large constant $C = C(\epsilon, \delta)$

$$(x,y,t,s) \mapsto \Phi(x,y,t,s) + C(|x|^2 + |y|^2 + t^2 + s^2)$$

is convex near (x_1, y_1, t_1, s_1). Since Φ additionally attains its maximum at (x_1, y_1, t_1, s_1), we may invoke Jensen (cf. [10]): There exist points $\{(x_1^k, y_1^k, t_1^k, s_1^k)\}_{k=1}^\infty$ such that

$$(x_1^k, y_1^k, t_1^k, s_1^k) \to (x_1, y_1, t_1, s_1), \tag{59}$$

$$\Phi,\ u^\epsilon \text{ and } v_\epsilon \text{ are each twice differentiable at } (x_1^k, y_1^k, t_1^k, s_1^k),\ k = 1, 2, \dots, \tag{60}$$

$$\nabla_{x,y,t,s}\Phi(x_1^k, y_1^k, t_1^k, s_1^k) \to 0, \tag{61}$$

$$D^2_{x,y,t,s}\Phi(x_1^k, y_1^k, t_1^k, s_1^k) \leq o(1)I_{2d+2} \quad \text{as } k \to \infty. \tag{62}$$

Using (48) and (60), we see

$$\begin{aligned}
\nabla_x\Phi(x_1^k, y_1^k, t_1^k, s_1^k) &= \nabla u^\epsilon(x_1^k + y_1^k, t_1^k + s_1^k) - \nabla v_\epsilon(x_1^k, t_1^k) \\
&=: p^k - \overline{p}^k, \\
\nabla_y\Phi(x_1^k, y_1^k, t_1^k, s_1^k) &= \nabla u^\epsilon(x_1^k + y_1^k, t_1^k + s_1^k) - \frac{4}{\delta}|y_1^k|^2 y_1^k \\
&= p^k - \frac{4}{\delta}|y_1^k|^2 y_1^k.
\end{aligned}$$

Since $y_1^k \to y_1$, we deduce from (61) that

$$p^k, \overline{p}^k \to \frac{4}{\delta}|y_1|^2 y_1 =: p \quad \text{in } \mathbb{R}^d. \tag{63}$$

Assertion (55) tells us $p \neq 0$ and so $p^k, \overline{p}^k \neq 0$ for k large enough.

Again, employing (48) and (60) we note

$$\begin{aligned}
\Phi_t(x_1^k, y_1^k, t_1^k, s_1^k) &= u_t^\epsilon(x_1^k + y_1^k, t_1^k + s_1^k) - v_{\epsilon t}(x_1^k, t_1^k) - \alpha \\
&=: q^k - \overline{q}^k - \alpha.
\end{aligned}$$

As u^ϵ and v_ϵ are Lipschitz continuous, upon passing to a subsequence and reindexing if necessary, we may assume that

$$q^k \to q, \quad \overline{q}^k \to \overline{q} \quad \text{in } [0, T].$$

Then, (61) ensures

$$q - \overline{q} = \alpha > 0. \tag{64}$$

Moreover, (48) and (60) give

$$\begin{aligned} D_x^2 \Phi(x_1^k, y_1^k, t_1^k, s_1^k) = {} & D^2 u^\epsilon(x_1^k + y_1^k, t_1^k + s_1^k) - D^2 v_\epsilon(x_1^k, t_1^k) \\ =: {} & M^k - \overline{M}^k. \end{aligned}$$

Now, (62) implies

$$M^k - \overline{M}^k \leq \epsilon_k I_d, \quad (\epsilon_k \to 0).$$

Furthermore, Lemma 3.3(iv) shows that $M^k \geq -C I_d$ and $\overline{M}^k \leq C I_d$, for $C = C(\epsilon)$. Hence,

$$-C I_d \leq M^k \leq \overline{M}^k + \epsilon_k I_d \leq C I_d.$$

Consequently, passing if necessary to a subsequence, we may suppose that

$$M^k \to M, \quad \overline{M}^k \to \overline{M} \quad \text{in } S^{d \times d}$$

with

$$M \leq \overline{M}. \tag{65}$$

We recall that (60) holds true and $p^k := \nabla u^\epsilon(x^k + y^k, t^k + s^k)$ and $\overline{p}^k := \nabla v_\epsilon(x^k, t^k)$ are nonzero for large k. Since u^ϵ is a viscosity subsolution near $(x_1 + y_1, t_1 + s_1)$ and v_ϵ is a viscosity supersolution near (x_1, t_1) of (11), for all large k we thus have

$$q^k \leq \left(\delta_{ij} - \frac{p_i^k p_j^k}{|p^k|^2} \right) m_{ij}^k - \chi^-(x_k + y_k, t_k + s_k, u^\epsilon)(b.p^k)_+$$

and

$$\overline{q}^k \geq \left(\delta_{ij} - \frac{\overline{p}_i^k \overline{p}_j^k}{|\overline{p}^k|^2} \right) \overline{m}_{ij}^k - \chi^+(x_k, t_k, v_\epsilon)(b.\overline{p}^k)_+.$$

We will see that

$$\liminf_{k \to \infty} \chi^-(x_k + y_k, t_k + s_k, u^\epsilon) \geq \chi^-(x_1 + y_1, t_1 + s_1, u^\epsilon) \tag{66}$$

and

$$\limsup_{k \to \infty} \chi^+(x_k, t_k, v_\epsilon) \leq \chi^+(x_1, t_1, v_\epsilon). \tag{67}$$

Indeed, if the left-hand side of (66) equals 1, then there is nothing to prove. Otherwise, we may assume that

$$\chi^-(x_k + y_k, t_k + s_k, u^\epsilon) = 0, \quad k = 1, 2, \ldots.$$

By definition (14), there exists $s_0 > 0$ such that

$$u^\epsilon(x_k + y_k - s_0 b, t_k + s_k) \geq u^\epsilon(x_k + y_k, t_k + s_k), \quad k = 1, 2, \ldots.$$

Since u^ϵ is continuous and $(x_k, y_k, t_k, s_k) \to (x_1, y_1, t_1, s_1)$ as $k \to \infty$,

$$u^\epsilon(x_1 + y_1 - s_0 b, t_1 + s_1) \geq u^\epsilon(x_1 + y_1, t_1 + s_1).$$

This implies

$$\chi^-(x_1 + y_1, t_1 + s_1, u^\epsilon) = 0.$$

For the proof of (67), we argue as follows: If the right-hand side of (67) is 1, the assertion is trivial. Otherwise, we assume that

$$\chi^+(x_1, t_1, v_\epsilon) = 0.$$

By definition (13), there exists $\alpha_1 > 0$ such that

$$v_\epsilon(x_1 - \alpha_1 b, t_1) > v_\epsilon(x_1, t_1).$$

Since $(x_k, t_k) \to (x_1, y_1)$ and v_ϵ is continuous, for sufficiently large k

$$v_\epsilon(x_k - \alpha_1 b, t_k) > v_\epsilon(x_k, t_k).$$

Therefore, for all large enough k

$$\chi^+(x_k, t_k, v_\epsilon) = 0.$$

We send k to infinity, recalling (66) and (67) to obtain

$$q \leq \left(\delta_{ij} - \frac{p_i p_j}{|p|^2}\right) m_{ij} - \chi^-(x_1 + y_1, t_1 + s_1, u^\epsilon)(b.p)_+ \tag{68}$$

and

$$\overline{q} \leq \left(\delta_{ij} - \frac{p_i p_j}{|p|^2}\right) \overline{m}_{ij} - \chi^+(x_1, t_1, v_\epsilon)(b.p)_+. \tag{69}$$

Next, we will prove that

$$\chi^-(x_1 + y_1, t_1 + s_1, u^\epsilon) \geq \chi^+(x_1, t_1, v_\epsilon). \tag{70}$$

In fact, if $\chi^-(x_1 + y_1, t_1 + s_1, u^\epsilon) = 1$, then (70) is obviously true. Assume instead that

$$\chi^-(x_1 + y_1, t_1 + s_1, u^\epsilon) = 0.$$

Then, for some $\tau > 0$, we have

$$u^\epsilon(x_1 + y_1 - \tau b, t_1 + s_1) \geq u^\epsilon(x_1 + y_1, t_1 + s_1).$$

Now, (52) implies

$$u^\epsilon(x_1 + y_1 - \tau b, t_1 + s_1) - v_\epsilon(x_1 - \tau b, t_1) < u^\epsilon(x_1 + y_1, t_1 + s_1) - v_\epsilon(x_1, t_1).$$

Thus,

$$\begin{aligned} 0 &\leq u^\epsilon(x_1 + y_1 - \tau b, t_1 + s_1) - u^\epsilon(x_1 + y_1, t_1 + s_1) \\ &< v_\epsilon(x_1 - \tau b, t_1) - v_\epsilon(x_1, t_1), \end{aligned}$$

which implies

$$\chi^+(x_1, t_1, v_\epsilon) = 0.$$

Since the matrix $((\delta_{ij} - \frac{p_i p_j}{|p|^2}))_{i,j=1}^d$ is nonnegative definite, we subtract (69) from (68) and recall (65) and (70) to obtain

$$q - \overline{q} = \alpha \leq 0.$$

This is a contradiction to (64). $\square$

3.3. *Existence of viscosity solutions*

We construct a viscosity solution to problem (11) and (12). We recall that the initial function u_0 always satisfies the condition (25). We will prove that the viscosity solution of (11) with initial condition (12) can be obtained by means of a limit process $\epsilon \to 0^+$ for a family $(u^\epsilon)_{\epsilon>0}$ of approximate solutions. For this purpose, consider

$$u_t^\epsilon = \left((1+\epsilon)\delta_{ij} - \frac{u_{x_i}^\epsilon u_{x_j}^\epsilon}{|\nabla u^\epsilon|^2 + \epsilon^2}\right) u_{x_i x_j}^\epsilon \\ - \beta_\epsilon\left(\max_{\epsilon \leq s \leq S}[u^\epsilon(x - sb, t) - u^\epsilon(x, t)]\right)(b.\nabla u^\epsilon)_+ \text{ in } \mathbb{R}^d \times [0, T], \tag{71}$$

with initial condition

$$u^\epsilon = u_0 \quad \text{on } \mathbb{R}^d \times \{t = 0\}, \tag{72}$$

where,

$$\beta_\epsilon(w) = \begin{cases} 1 & \text{if } w \leq 0, \\ \text{linear} & \text{if } 0 \leq w \leq \epsilon, \\ 0 & \text{if } w \geq \epsilon \end{cases} \tag{73}$$

and

$$S = \frac{2R}{|b|}. \tag{74}$$

Here, R is given by (25) and (26).

We note that the coefficients $\{a_{ij}\}$ with

$$a_{ij}(p) := \left((1+\epsilon)\delta_{ij} - \frac{p_i p_j}{|p|^2 + \epsilon^2}\right)$$

satisfy the uniform ellipticity condition:

$$\epsilon|\xi|^2 \leq a_{ij}(p)\xi_i\xi_j, \quad \xi \in \mathbb{R}^d, \quad p \in \mathbb{R}^d.$$

Therefore, there exists a unique smooth solution u^ϵ of (71) with initial condition (72) in $\mathbb{R}^d \times [0, T]$ (cf., e.g., [11]). Moreover, the solution u^ϵ satisfies the following estimate

$$\|u^\epsilon\|_{L^\infty(\mathbb{R}^d\times[0,T])}, \|\nabla u^\epsilon\|_{L^\infty(\mathbb{R}^d\times[0,T])}, \|u_t^\epsilon\|_{L^\infty(\mathbb{R}^d\times[0,T])} \leq C, \tag{75}$$

where C does not depend on ϵ.

The convergence of u^ϵ to u as $\epsilon \to 0^+$ will be shown in the following theorem.

Theorem 3.5. *There exists a viscosity solution of* (11) *and* (12).

Proof. In view of (75), there exists a subsequence $\{u^{\epsilon_k}\}_{k=1}^{\infty} \subset \{u^{\epsilon}\}_{0<\epsilon<1}$ such that $\epsilon_k \to 0^+$ and $u^{\epsilon_k} \to u$ locally uniformly in $\mathbb{R}^d \times [0, T]$ as $k \to \infty$. The function u is therefore bounded and Lipschitz continuous. We will prove that u is a viscosity solution of (11) and (12).

Assume that $\phi \in C^{\infty}(R^{d+1})$ and $u - \phi$ has a strict local maximum at a point $(x_0, t_0) \in \mathbb{R}^d \times (0, T]$. Since $u^{\epsilon_k} \to u$ uniformly near the point (x_0, t_0), there exist points $(x_k, t_k) \in \mathbb{R}^d \times (0, T]$ such that

$$\begin{cases} (x_k, t_k) \to (x_0, t_0) \quad \text{as } k \to \infty, \quad \text{and} \\ u^{\epsilon_k} - \phi \quad \text{has a local maximum at } (x_k, t_k). \end{cases} \tag{76}$$

Since u^{ϵ_k} and ϕ are smooth,

$$\nabla u^{\epsilon_k} = \nabla \phi, \quad u_t^{\epsilon_k} = \phi_t, \quad \mathrm{D}^2 u^{\epsilon_k} \le \mathrm{D}^2 \phi \quad \text{at } (x_k, t_k).$$

Thus, we have

$$\begin{aligned} \phi_t \le & \left((1+\epsilon_k)\delta_{ij} - \frac{\phi_{x_i}\phi_{x_j}}{|\nabla\phi|^2 + \epsilon^2} \right) \phi_{x_i x_j}(x_k, t_k) \\ & - \beta_{\epsilon_k} \left(\max_{\epsilon_k \le s \le S} [u^{\epsilon_k}(x_k - sb, t_k) - u^{\epsilon_k}(x_k, t_k)] \right) (b.\nabla\phi(x_k, t_k))_+. \end{aligned} \tag{77}$$

We claim next that

$$\chi^-(x_0, t_0, u) \le \lim_{k\to\infty} \beta_{\epsilon_k} \left(\max_{\epsilon_k \le s \le S} [u^{\epsilon_k}(x_k - sb, t_k) - u_k^{\epsilon}(x_k, t_k)] \right), \tag{78}$$

provided that

$$b.\nabla\phi(x_0, t_0) > 0. \tag{79}$$

Indeed, if $\chi^-(x_0, t_0, u) = 0$, then (78) obviously holds true. Otherwise, $\chi^-(x_0, t_0, u) = 1$, which means that for all $s > 0$, we have

$$u(x_0 - sb, t_0) < u(x_0, t_0). \tag{80}$$

In view of (76), we see that

$$u^{\epsilon_k}(x_k, t_k) - \phi(x_k, t_k) \ge u^{\epsilon_k}(x_k - sb, t_k) - \phi(x_k - sb, t_k). \tag{81}$$

Taking advantage of (79), provided that $0 < s < \delta$, where $\delta > 0$ is chosen small enough and k is sufficiently large, we find

$$\begin{aligned} u^{\epsilon_k}(x_k - sb, t_k) - u^{\epsilon_k}(x_k, t_k) & \le \phi(x_k - sb, t_k) - \phi(x_k, t_k) \\ & = -sb.\nabla\phi(x_k, t_k) + o(s^2) \\ & < 0. \end{aligned}$$

Moreover, (80) implies

$$u^{\epsilon_k}(x_k - sb, t_k) - u^{\epsilon_k}(x_k, t_k) \le 0,$$

if $\delta \le s \le S$ and k is large. Consequently, for sufficiently large k

$$\max_{\epsilon_k \le s \le S} [u^{\epsilon_k}(x_k - sb, t_k) - u_k^{\epsilon}(x_k, t_k)] < 0,$$

whence

$$\beta_{\epsilon_k}\left(\max_{\epsilon_k\le s\le S}[u^{\epsilon_k}(x_k-sb,t_k)-u_k^{\epsilon}(x_k,t_k)]\right)=1.$$

Consequently, (78) follows if $\chi^-(x_0,t_0,u)=1$ and $b.\nabla\phi(x_0,t_0)>0$.

Suppose next that $\nabla\phi(x_0,t_0)\neq 0$. Then $\nabla\phi(x_k,t_k)\neq 0$ for large k. We consequently pass to limits in (77), recalling (78) to deduce

$$\phi_t(x_0,t_0)\le\left(\delta_{ij}-\frac{\phi_{x_i}\phi_{x_j}}{|\nabla\phi|^2}\right)\phi_{x_ix_j}(x_0,t_0)-\chi^-(x_0,t_0,u)(b.\nabla\phi(x_0,t_0))_+.$$

Next, assume instead that $\nabla\phi(x_0,t_0)=0$. We set

$$\eta^k=\frac{\nabla\phi(x_k,t_k)}{(|\nabla\phi(x_k,t_k)|^2+\epsilon_k^2)^{\frac{1}{2}}},$$

so that (77) becomes

$$\phi_t(x_k,t_k)\le((1+\epsilon_k)\delta_{ij}-\eta_i^k\eta_j^k)\phi_{x_ix_j}(x_k,t_k).\tag{82}$$

Since $|\eta^k|\le 1$, upon passing to a subsequence and reindexing if necessary, we may assume that $\eta^k\to\eta$ in $\mathbb{R}^d$ for some $|\eta|\le 1$. Sending k to infinity in (82), we obtain

$$\phi_t(x_0,t_0)\le(\delta_{ij}-\eta_i\eta_j)\phi_{x_ix_j}(x_0,t_0).$$

Consequently, u is a viscosity subsolution of (11).

In order to verify that u is a viscosity supersolution of (11), we again take $\phi\in C^\infty(\mathbb{R}^{d+1})$ and suppose that $u-\phi$ has a strict local minimum at $(x_0,t_0)\in\mathbb{R}^d\times(0,T]$. Since $u^{\epsilon_k}\to u$ uniformly near (x_0,t_0), there exist $(x_k,t_k)\in\mathbb{R}^d\times(0,T]$ such that

$$\begin{cases}(x_k,t_k)\to(x_0,t_0)\quad\text{as } k\to\infty\text{, and}\\ u^{\epsilon_k}-\phi\text{ has a local minimum at }(x_k,t_k).\end{cases}\tag{83}$$

Since u^{ϵ_k} and ϕ are smooth,

$$\nabla u^{\epsilon_k}=\nabla\phi,\quad u_t^{\epsilon_k}=\phi_t,\quad \mathrm{D}^2u^{\epsilon_k}\ge\mathrm{D}^2\phi\quad\text{at }(x_k,t_k).$$

Thus, we deduce

$$\begin{aligned}\phi_t\ge&\left((1+\epsilon_k)\delta_{ij}-\frac{\phi_{x_i}\phi_{x_j}}{|\nabla\phi|^2+\epsilon^2}\right)\phi_{x_ix_j}(x_k,t_k)\\&-\beta_{\epsilon_k}\left(\max_{\epsilon_k\le s\le S}[u^{\epsilon_k}(x_k-sb,t_k)-u_k^{\epsilon}(x_k,t_k)]\right)(b.\nabla\phi(x_k,t_k))_+.\end{aligned}\tag{84}$$

We claim next that

$$\chi^+(x_0,t_0,u)\ge\lim_{k\to\infty}\beta_{\epsilon_k}\left(\max_{\epsilon_k\le s\le S}[u^{\epsilon_k}(x_k-sb,t_k)-u_k^{\epsilon}(x_k,t_k)]\right).\tag{85}$$

Now, if $\chi^+(x_0,t_0,u)=1$, then (85) obviously holds true. Otherwise, $\chi^+(x_0,t_0,u)=0$, which means that for some $s_0>0$ we have

$$u(x_0-s_0b,t_0)>u(x_0,t_0).$$

We further note that $s_0\le S$, otherwise $u(x_0-s_0b,t_0)=-1=\min u$ on $\mathbb{R}^d\times[0,T]$.

Thus, for k sufficiently large, we have $\epsilon_k \le s_0$, and hence,

$$\max_{\epsilon_k \le s \le S}[u^{\epsilon_k}(x_k - sb, t_k) - u_k^{\epsilon}(x_k, t_k)] \ge u^{\epsilon_k}(x_k - s_0 b, t_k) - u_k^{\epsilon}(x_k, t_k) \to u(x_0 - s_0 b, t_0) - u(x_0, t_0) > 0.$$

By definition of β_ϵ, we have

$$\lim_{k\to\infty} \beta_{\epsilon_k}\left(\max_{\epsilon_k \le s \le S}[u^{\epsilon_k}(x_k - sb, t_k) - u_k^{\epsilon}(x_k, t_k)]\right) = 0.$$

The rest of the proof is similar to above, and we conclude that u is a viscosity supersolution of (11). Altogether, we have thus shown that u is a viscosity solution of (11). □

Acknowledgement

This work was supported by the German National Science Foundation (DFG) within the DFG-funded Collaborative Research Field SFB 438 and the Graduate Program Graduiertenkolleg 283.

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