GOAL-ORIENTED ADAPTIVITY IN CONTROL CONSTRAINED OPTIMAL CONTROL OF PARTIAL DIFFERENTIAL EQUATIONS*

M. HINTERMÜLLER[†] AND R. H. W. HOPPE[‡]

Abstract. Dual-weighted goal-oriented error estimates for a class of pointwise control constrained optimal control problems for second order elliptic partial differential equations are derived. It is demonstrated that the constraints give rise to a primal-dual weighted error term representing the mismatch in the complementarity system due to discretization. The paper also contains a posteriori error estimators for the L^2-norm of the error in the state and in the adjoint state.

Key words. adaptive finite element method, a posteriori error estimate, control constraints, goal-oriented adaptivity, PDE-constrained optimization

AMS subject classifications. 49K20, 65K10, 65N30, 65N50

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1. Introduction.

In many computations involving the discretization of (partial) differential equations or variational inequalities, one is interested in the accurate evaluation of some target quantity. This might be the value of the solution of a partial differential equation (PDE) at some reference point in the domain of interest, a physically relevant quantity such as the drag in airfoil design, or, in optimal control, the value of the objective function at the solution of the underlying minimization problem. Highly accurate numerical evaluations of these targets can be guaranteed by using uniform meshes with a small mesh size h. This, however, usually represents a significant computational challenge due to the resulting large scale of the discrete problems. Therefore, one seeks to adaptively refine the meshes with the goal of achieving a desired accuracy in the evaluation of the output quantity of interest while keeping the computational cost as small as possible.

For this purpose, recently for (systems of) PDEs an approach based on dual-weighted residual-based error estimates was proposed. Here we point to the pioneering work summarized in [1, 3] and the references therein; see also [7] for related literature. It essentially relies on employing the dual problem of the underlying system with the target on the right-hand side. In fact, let A denote some possibly nonlinear partial differential operator and let f be some fixed data. Then, in some abstract form, the primal problem (or PDE) is given by

$$A(y) = f. \tag{1.1}$$

Let y_h be the result of a Galerkin finite element discretization of the underlying problem. If $G(\cdot)$ represents some desired target quantity (or goal), then the dual

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[†]Department of Mathematics, University of Sussex, Mantell Building, Falmer, Brighton, BN1 9RE, UK (M.Hintermueller@sussex.ac.uk, michael.hintermueller@uni-graz.at). This author gratefully acknowledges financial support from the Austrian Science Fund FWF under START-program Y305 “Interfaces and Free Boundaries.”

[‡]Institute of Mathematics, University of Augsburg, D-86159 Augsburg, Germany, and Department of Mathematics, University of Houston, Houston, TX 77204-3008 (rohop@math.uh.edu). This author has been partially supported by the NSF under grants DMS-0411403 and DMS-0511611.

approach consists of considering

$$A'(y_h)^\star p_h = G(\cdot) \tag{1.2}$$

from which an a posteriori error estimate of the type

$$|G(y) - G(y_h)| \le \sum_{T\in\mathbb{T}_h} \mathfrak{p}_T(y_h)\mathfrak{d}_T(p_h)$$

is derived. Above, $A'(\cdot)^\star$ is the dual operator of the Frechét-derivative $A'(\cdot)$ of $A(\cdot)$. Further, $\mathbb{T}_h = \{T\}$ denotes a computational mesh consisting of elements T, and $\mathfrak{p}_T$ and $\mathfrak{d}_T$ stand for the primal residual and the dual weight on each cell T, respectively.

In [2] this concept was transferred to PDE-constrained optimal control problems of the type

$$\text{minimize } J(y,u) \quad \text{subject to} \quad A(y) = f + B(u), \tag{P$_0$}$$

where (y,u) denotes the state-control pair and B models the control impact. The first order optimality system of (P_0) can be formally written as

$$A(y) - B(u) = f, \tag{1.3a}$$

$$J_y(y,u) + A'(y)^\star p = 0, \tag{1.3b}$$

$$J_u(y,u) - B'(u)^\star p = 0. \tag{1.3c}$$

Here, J_y and J_u are the partial derivatives of J with respect to y and u. The variable p is called the adjoint state. Often, (1.3c) results in an algebraic equation, while (1.3a)–(1.3b) form a primal-dual pair of PDEs similar to (1.1)–(1.2). Since (1.3a)–(1.3b) represent a system of PDEs, the dual-weighted approach can be readily carried over to this optimal control setting.

The situation, however, changes significantly if, in addition to the PDE constraint in (P_0), one has to account for pointwise almost everywhere (a.e.) constraints on the control variable. In this case, the resulting problem becomes

$$\left\{\begin{array}{ll} \text{minimize} & J(y,u) \\ \text{subject to} & A(y) = f + B(u), \\ & a \le u \le b \quad \text{a.e. on } \Omega_C \subset \Omega, \end{array}\right. \tag{P$_c$}$$

where $\Omega \subset \mathbb{R}^n$ denotes some suitable domain with $\Omega_C \neq \emptyset$ a measurable subset, and where $a < b$ are given bounds. The corresponding first order necessary optimality system now involves a variational inequality as follows:

$$A(y) - B(u) = f, \tag{1.4a}$$

$$J_y(y,u) + A'(y)^\star p = 0, \tag{1.4b}$$

$$\langle J_u(y,u) - B'(u)^\star p, v - u\rangle \ge 0 \quad \forall v \in U^{\mathrm{ad}},\ u \in U^{\mathrm{ad}}, \tag{1.4c}$$

where the set

$$U^{\mathrm{ad}} = \{v : a \le v \le b\}$$

represents the feasible controls, and $\langle\cdot,\cdot\rangle$ denotes a suitable duality pairing. The variational inequality induces some nonsmoothness in the first order optimality system. This can be seen best when defining the Lagrange multiplier λ pertinent to the pointwise constraints via

$$J_u(y,u) - B'(u)^\star p + \lambda = 0 \tag{1.5}$$

and, assuming that λ permits a pointwise interpretation,

$$\lambda \geq 0 \quad \text{a.e. on } \{u = b\}, \quad \lambda \leq 0 \quad \text{a.e. on } \{u = a\}, \quad \lambda = 0 \quad \text{else.} \tag{1.6}$$

The conditions in (1.6) represent the so-called *complementarity system.* It can be written equivalently as

$$\lambda = \min\{0, \lambda + \sigma(u - a)\} + \max\{0, \lambda + \sigma(u - b)\}, \tag{1.7}$$

where $\sigma > 0$ is an arbitrarily fixed real and the max- and min-operations are understood in the pointwise sense. From (1.7) the nonsmoothness involved in the first order necessary optimality conditions becomes apparent. Of course, suitable a posteriori error concepts have to reflect this situation in order to accurately resolve the influence of the constraints on the solution of the optimal control problem.

We note that for pointwisely constrained problems such as variational inequalities of obstacle type, finite element methods based on various concepts in the a posteriori analysis have been considered in the literature. The goal-oriented dual-weighted approach was used in [4], whereas residual-type and hierarchical-type estimators were derived and analyzed in [5, 10, 13, 16]. Although the situation under consideration is different from obstacle-type problems as the pointwise constraints in our case are imposed on the control acting on the right-hand side of the PDE, a common feature in the a posteriori error analysis is the appropriate treatment of the complementarity conditions.

In this paper, our starting point will be a sufficiently general model problem class of the type (P_c). Based on the Lagrange function

$$\mathcal{L}(y, u, p, \lambda) = J(y, u) + \langle A(y) - f - B(u), p\rangle + (u - b, \lambda)$$

of (P_c), for convenience written here for a unilaterally constrained version of the minimization problem, and with the objective function as the goal, we derive an error representation of the type

$$\begin{aligned} J(y, u) - J(y_h, u_h) = -\frac{1}{2}\langle \nabla_{xx}\mathcal{L}(x_h, \lambda_h)(x_h - x), x_h - x\rangle + (u_h - b, \lambda) \\ + \operatorname{osc}_h + r(x_h, x) \end{aligned}$$

with $x = (p, y, u)$ and its discretized version $x_h = (p_h, y_h, u_h)$, respectively, and $(\cdot, \cdot)$ some inner product. Further, osc_h represents data oscillations and r is the remainder term resulting from a Taylor expansion of $\mathcal{L}$. In a second step we then estimate the term due to the inequality constraints and utilize the a posteriori error estimators derived in [8] in order to obtain a computable error representation.

The rest of the paper is organized as follows. In the next section we derive our new dual-weighted residual-based error estimator for a representative control constrained optimal control model problem. Section 3 is devoted to possible extensions. In fact, we study the bilaterally constrained case, a class of nonlinear governing equations, and alternative concepts for obtaining a posteriori estimates pertinent to the complementarity system. In the appendix, for our constrained optimal control problem we derive a new a posteriori error estimate with respect to the L^2-norm. Finally, in section 4 we report on numerical results due to our new error estimator.

Notation. Throughout we use $\|\cdot\|_{0,\Omega}$ and $(\cdot, \cdot)_{0,\Omega}$ for the usual $L^2(\Omega)$-norm and $L^2(\Omega)$-inner product, respectively. For convenience, with respect to the notation we shall not distinguish between the norm (respectively, inner product) for scalar-valued or vector-valued arguments. We also use $(\cdot,\cdot)_{0,\mathcal{S}}$, which is the $L^2(\mathcal{S})$-inner product over a (measurable) subset $\mathcal{S} \subset \Omega$. By $|\cdot|_{1,\Omega}$ we denote the $H^1(\Omega)$-seminorm $|y|_{1,\Omega} = \|\nabla y\|_{0,\Omega}$, which, by the Poincaré–Friedrichs inequality, is a norm on $H_0^1(\Omega)$. The norm in $H^1(\Omega)$ is written as $\|\cdot\|_{1,\Omega}$. By $\mathbb{T}_h = \mathbb{T}_h(\Omega)$ we denote a shape regular finite element triangulation of the domain Ω. The subscript $h = \max\{\text{diam}(T) \mid T \in \mathbb{T}_h\}$ indicates the mesh size of $\mathbb{T}_h$.

2. Residual-based error estimate. For deriving the structure of the new error estimate due to the inequality constraints, we consider the model problem

$$\text{(P)} \qquad \begin{cases} \text{minimize} & J(y,u) := \frac{1}{2}\|y-z\|_{0,\Omega}^2 + \frac{\alpha}{2}\|u\|_{0,\Omega}^2 \\ \text{over} & (y,u) \in H_0^1(\Omega) \times L^2(\Omega) \\ \text{subject to} & -\Delta y = u + f, \\ & u \le b \quad \text{a.e. in } \Omega, \end{cases}$$

which is a particular instance of (P_c). The domain $\Omega \in \mathbb{R}^2$ is assumed to be bounded and polygonal with boundary $\Gamma := \partial\Omega$. For the data we assume $z, b, f \in L^2(\Omega)$ and $\alpha > 0$. It is well known that (P) admits a unique solution $(y^*, u^*) \in H_0^1(\Omega) \times L^2(\Omega)$ (cf., e.g., [12]). Moreover, the optimal solution is characterized by the existence of an adjoint state $p^* \in H_0^1(\Omega)$ and a Lagrange multiplier $\lambda^* \in L^2(\Omega)$ which satisfy the first order necessary (and in this case, also sufficient) conditions

$$-\Delta y^* = u^* + f, \tag{2.1a}$$
$$-\Delta p^* + y^* = z, \tag{2.1b}$$
$$\alpha u^* + \lambda^* - p^* = 0, \tag{2.1c}$$
$$u^* \le b, \quad \lambda^* \ge 0, \quad (u^* - b, \lambda^*)_{0,\Omega} = 0. \tag{2.1d}$$

We define the Lagrange functional $\mathcal{L} : H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \to \mathbb{R}$ pertinent to (P) as

$$\mathcal{L}(y,u,p,\lambda) = J(y,u) + (\nabla y, \nabla p)_{0,\Omega} - (u+f, p)_{0,\Omega} + (u-b, \lambda)_{0,\Omega}. \tag{2.2}$$

For convenience we use $x := (p,y,u)$, $x^* = (p^*, y^*, u^*)$, and $X = P \times Y \times L = H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$. Obviously, the weak form of (2.1a)–(2.1b) and (2.1c) of the optimality system (2.1) is equivalent to

$$\nabla_x \mathcal{L}(x^*, \lambda^*)(\varphi) = 0 \quad \forall \varphi \in X. \tag{2.3}$$

Let $X_h \subset X$, with $X_h = P_h \times Y_h \times L_h$, denote a finite dimensional subspace with the subscript h indicating the mesh size of the discretization obtained by a standard Galerkin method, let $\lambda_h \in L_h \subset L^2(\Omega)$ denote the discrete (finite dimensional) counterpart of λ (analogously for λ^*), and let $f_h, b_h, z_h \in L_h$ be the L^2-projections of f, b, z onto L_h. The finite dimensional version of (2.1) reads

$$\nabla_x \mathcal{L}_h(x_h^*, \lambda_h^*)(\varphi_h) = 0 \quad \forall \varphi_h \in X_h, \tag{2.4a}$$
$$u_h^* \le b_h, \quad \lambda_h^* \ge 0, \quad (u_h^* - b_h, \lambda_h^*)_{0,\Omega} = 0, \tag{2.4b}$$

where the discrete Lagrange function is given by

$$\begin{aligned} \mathcal{L}_h(x_h, \lambda_h) &= J_h(y_h, u_h) + (\nabla y_h, \nabla p_h)_{0,\Omega} - (u_h + f_h, p_h)_{0,\Omega} \\ &\quad + (u_h - b_h, \lambda_h)_{0,\Omega} \end{aligned} \tag{2.5}$$

with $J_h(y_h, u_h) = \frac{1}{2}\|y_h - z_h\|^2_{0,\Omega} + \frac{\alpha}{2}\|u_h\|^2_{0,\Omega}$. Observe that the pointwise representation (2.1c) in the discrete setting reads

$$\alpha u_h^* + \lambda_h^* - M_h p_h^* = 0, \tag{2.6}$$

where M_h represents a projection operator from P_h onto L_h.

Further note that for $x \in X$, $\lambda \in L^2(\Omega)$ and $x_h \in X_h$, $\lambda_h \in L_h$,

$$\mathcal{L}(x, \lambda_h) = \mathcal{L}(x, \lambda) + (u - b, \lambda_h - \lambda)_{0,\Omega}, \tag{2.7}$$

$$\nabla_x \mathcal{L}(x_h, \lambda_h)(\varphi_h) = \nabla_x \mathcal{L}(x_h, \lambda)(\varphi_h) + (\delta u_h, \lambda_h - \lambda)_{0,\Omega} \tag{2.8}$$

for all $(\delta p_h, \delta y_h, \delta u_h) = \varphi_h \in X_h$. Moreover, for our model problem (P) the second derivative of $\mathcal{L}$ with respect to x does not depend on x and λ. Thus, we can write $\nabla_{xx}\mathcal{L}(\varphi, \hat{\varphi})$ instead of $\nabla_{xx}\mathcal{L}(x, \lambda)(\varphi, \hat{\varphi})$. Similar observations hold true for $\mathcal{L}_h$. Due to $X_h \subset X$, we have for $\varphi_h = (\delta p_h, \delta y_h, \delta u_h) \in X_h$,

$$\begin{aligned} 0 &= \nabla_x \mathcal{L}(x^*, \lambda^*)(\varphi_h) \\ &= \nabla_x \mathcal{L}(x_h^*, \lambda^*)(\varphi_h) + \nabla_{xx}\mathcal{L}(x^* - x_h^*, \varphi_h) \\ &= \nabla_x \mathcal{L}(x_h^*, \lambda_h^*)(\varphi_h) + (\delta u_h, \lambda^* - \lambda_h^*)_{0,\Omega} + \nabla_{xx}\mathcal{L}(x^* - x_h^*, \varphi_h) \\ &= \nabla_x \mathcal{L}_h(x_h^*, \lambda_h^*)(\varphi_h) - (f - f_h, \delta p_h)_{0,\Omega} - (z - z_h, \delta y_h)_{0,\Omega} \\ &\quad + (\delta u_h, \lambda^* - \lambda_h^*)_{0,\Omega} + \nabla_{xx}\mathcal{L}(x^* - x_h^*, \varphi_h) \\ &= (\delta u_h, \lambda^* - \lambda_h^*)_{0,\Omega} + \nabla_{xx}\mathcal{L}(x^* - x_h^*, \varphi_h) - (f - f_h, \delta p_h)_{0,\Omega} \\ &\quad - (z - z_h, \delta y_h)_{0,\Omega}. \end{aligned} \tag{2.9}$$

From this we further derive the relations

$$\begin{aligned} &\nabla_{xx}\mathcal{L}(x_h^* - x^*, x_h^* - x^*) \\ &\qquad = \nabla_{xx}\mathcal{L}(x_h^* - x^*, x_h^* - x^* + \varphi_h) - (\delta u_h, \lambda^* - \lambda_h^*)_{0,\Omega} \\ &\qquad\quad + (f - f_h, \delta p_h)_{0,\Omega} + (z - z_h, \delta y_h)_{0,\Omega}, \end{aligned} \tag{2.10}$$

$$\nabla_x \mathcal{L}(x_h^*, \lambda^*)(x^* - x_h^* - \varphi_h) = \nabla_{xx}\mathcal{L}(x_h^* - x^*, x^* - x_h^* - \varphi_h) \tag{2.11}$$

and also

$$\begin{aligned} &\nabla_x \mathcal{L}(x_h^*, \lambda_h^*)(x^* - x_h^* - \varphi_h) \\ &\quad = \nabla_x \mathcal{L}(x^*, \lambda_h^*)(x^* - x_h^* - \varphi_h) + \nabla_{xx}\mathcal{L}(x_h^* - x^*, x^* - x_h^* - \varphi_h) \\ &\quad = (\lambda_h^* - \lambda^*, u^* - u_h^* - \delta u_h)_{0,\Omega} + \nabla_{xx}\mathcal{L}(x_h^* - x^*, x^* - x_h^* - \varphi_h). \end{aligned} \tag{2.12}$$

These preliminary results are now used to prove the following theorem.

THEOREM 2.1. *Let $(x^*, \lambda^*) \in X \times L^2(\Omega)$ and $(x_h^*, \lambda_h^*) \in X_h \times L_h$ denote the solution of* (2.1) *and its finite dimensional counterpart* (2.4). *Then*

$$\begin{aligned} J(y^*, u^*) - J_h(y_h^*, u_h^*) \quad &= -\frac{1}{2}\nabla_{xx}\mathcal{L}(x_h^* - x^*, x_h^* - x^*) \\ &\qquad + (u_h^* - b, \lambda^*)_{0,\Omega} + \mathrm{osc}_h(x_h^*), \end{aligned} \tag{2.13}$$

where the oscillations $\mathrm{osc}_h(x_h^*)$ *are given by*

$$\mathrm{osc}_h(x_h^*) = (y_h^* - z_h, z_h - z)_{0,\Omega} + \frac{1}{2}\|z - z_h\|^2_{0,\Omega} + (f_h - f, p_h^*)_{0,\Omega}.$$

Proof. Observe that $J(y^*, u^*) = \mathcal{L}(x^*, \lambda^*)$ and $J_h(y_h^*, u_h^*) = \mathcal{L}_h(x_h^*, \lambda_h^*)$. Using Taylor expansions and (2.7)–(2.8), we obtain

$$\begin{aligned}
J(y^*, u^*) - J_h(y_h^*, u_h^*) &= \mathcal{L}(x^*, \lambda^*) - \mathcal{L}_h(x_h^*, \lambda_h^*) \\
&= \mathcal{L}(x^*, \lambda^*) - \mathcal{L}_h(x^*, \lambda_h^*) - \nabla_x \mathcal{L}_h(x^*, \lambda_h^*)(x_h^* - x^*) \\
&\quad - \frac{1}{2}\nabla_{xx}\mathcal{L}_h(x_h^* - x^*, x_h^* - x^*) \\
&= J(y^*, u^*) - J_h(y^*, u^*) + (f_h - f, p^*)_{0,\Omega} - (u^* - b_h, \lambda_h^*)_{0,\Omega} \\
&\quad - \nabla_x \mathcal{L}_h(x^*, \lambda_h^*)(x_h^* - x^*) - \frac{1}{2}\nabla_{xx}\mathcal{L}_h(x_h^* - x^*, x_h^* - x^*) \\
&= \operatorname{osc}_h(x_h^*) - (u^* - b_h, \lambda_h^*)_{0,\Omega} - \nabla_x \mathcal{L}(x^*, \lambda_h^*)(x_h^* - x^*) \\
&\quad - \frac{1}{2}\nabla_{xx}\mathcal{L}_h(x_h^* - x^*, x_h^* - x^*) \\
&= \operatorname{osc}_h(x_h^*) - (u^* - u_h^*, \lambda_h^*)_{0,\Omega} + (\lambda^* - \lambda_h^*, u_h^* - u^*)_{0,\Omega} \\
&\quad - \frac{1}{2}\nabla_{xx}\mathcal{L}_h(x_h^* - x^*, x_h^* - x^*) \\
&= \operatorname{osc}_h(x_h^*) + (\lambda^*, u_h^* - b)_{0,\Omega} - \frac{1}{2}\nabla_{xx}\mathcal{L}_h(x_h^* - x^*, x_h^* - x^*),
\end{aligned}$$

where we also used the complementarity relations (2.1d) and (2.4b) as well as (2.3) and (2.4a). ∎

Assume, for the moment, that $\lambda^* = 0$ and $\lambda_h^* = 0$; i.e., the continuous and the discrete control constraints are inactive. Then we infer from (2.10) that for all $\varphi_h \in X_h$ there holds

$$\begin{aligned}
\nabla_{xx}\mathcal{L}(x_h^* - x^*, x_h^* - x^*) &= \nabla_{xx}\mathcal{L}(x_h^* - x^*, x_h^* - x^* + \varphi_h) \\
&\quad + (f - f_h, \delta p_h)_{0,\Omega} + (z - z_h, \delta y_h)_{0,\Omega}
\end{aligned}$$

as well as

$$\begin{aligned}
J(y^*, u^*) - J_h(y_h^*, u_h^*) &= \frac{1}{2}\nabla_x \mathcal{L}_h(x_h, \lambda_h)(x^* - x_h^* - \varphi_h) \\
&\quad + \frac{1}{2}(f_h - f, p^* - p_h^*)_{0,\Omega} + \frac{1}{2}(z_h - z, y^* - y_h^*)_{0,\Omega} \qquad (2.14) \\
&\quad + \operatorname{osc}_h(x_h^*)
\end{aligned}$$

due to (2.12). This corresponds to the result in [2, Proposition 4.1] for the unconstrained version of (P).

If $b_h \leq b$ a.e. in Ω, then (2.13) implies

$$J(y^*, u^*) \leq J_h(y_h^*, u_h^*) + \operatorname{osc}_h(x_h^*).$$

Next we interpret the new, second term in the right-hand side of (2.13). For this purpose we define the active set $\mathcal{A}^*$ and the inactive set $\mathcal{I}^*$ at the optimal solution (x^*, λ^*) of (P) by

$$\mathcal{A}^* := \{x \in \Omega : u^*(x) = b(x)\}, \quad \mathcal{I}^* := \Omega \setminus \mathcal{A}^*. \qquad (2.15)$$

Analogously we define the discrete counterparts $\mathcal{A}_h^*$ and $\mathcal{I}_h^*$. Obviously, $u^* < b$ a.e. in $\mathcal{I}^*$. By (2.1d), this implies $\lambda^* = 0$ a.e. in $\mathcal{I}^*$. Therefore, the term $(u_h^* - b, \lambda^*)_{0,\Omega}$ satisfies

$$(u_h^* - b, \lambda^*)_{0,\Omega} = (u_h^* - b_h, \lambda^*)_{0,\mathcal{A}^* \cap \mathcal{I}_h^*} + (b_h - b, \lambda^*)_{0,\mathcal{A}^*}.$$

The right-hand side above reflects the *error in complementarity.* In fact, the second term represents the data oscillation in the bound in the active set weighted by the continuous Lagrange multiplier. For this term we introduce the notation

$$\mathrm{osc}_h^{\mathcal{A}^*}(b;\lambda^*) := (b_h - b, \lambda^*)_{0,\mathcal{A}^*}.$$

The first term captures a *primal-dual weighted mismatch in complementarity in* $\mathcal{A}^* \cap \mathcal{I}_h^*$.

Let $i_h := (i_h^p, i_h^y, i_h^u)$ be an interpolation operator such that $i_h x \in X_h$ for $x \in X$. Moreover, for $y, p \in H_0^1(\Omega)$ there exist i_h^p and i_h^y such that $\max\{\|i_h^p p - p\|_{H^1}, \|i_h^y y - y\|_{H^1}\} \to 0$ for $h \to 0$. In connection with Theorem 2.1 we have the following result.

THEOREM 2.2. *Let the assumptions of Theorem* 2.1 *be satisfied. Then*

$$\begin{aligned}
&J(y^*, u^*) - J_h(y_h^*, u_h^*) \\
&\qquad = -\frac{1}{2}\Big((y_h^* - z_h, i_h^y y^* - y^*)_{0,\Omega} + (\nabla(i_h^y y^* - y^*), \nabla p_h^*)_{0,\Omega} \\
&\qquad\qquad + (\nabla(i_h^p p^* - p^*), \nabla y_h^*)_{0,\Omega} - (u_h^* + f_h, i_h^p p^* - p^*)_{0,\Omega} \\
&\qquad\qquad + (M_h p_h^* - p_h^*, i_h^u u^* - u^*)_{0,\Omega}\Big) \qquad (2.16)\\
&\qquad + \frac{1}{2}\big[(u_h^* - b, \lambda^*)_{0,\Omega} + (b_h - u^*, \lambda_h^*)_{0,\Omega}\big] + \frac{1}{2}(f - f_h, p_h^* - p^*)_{0,\Omega} \\
&\qquad + \frac{1}{2}(z - z_h, y_h^* - y^*)_{0,\Omega} + \mathrm{osc}_h(x_h^*).
\end{aligned}$$

Proof. Utilizing (2.10)–(2.11) and considering $\varphi_h = (\delta p_h, \delta y_h, \delta u_h) \in X_h$, we obtain

$$\begin{aligned}
J(y^*, u^*) - J_h(y_h^*, u_h^*) &= \frac{1}{2}\nabla_{xx}\mathcal{L}(x, \lambda_h^*)(x^* - x_h^*, x_h^* - x^* + \varphi_h) \\
&\quad + \frac{1}{2}(\delta u_h, \lambda^* - \lambda_h^*)_{0,\Omega} + \frac{1}{2}(f_h - f, \delta p_h)_{0,\Omega} + \frac{1}{2}(z_h - z, \delta y_h)_{0,\Omega} \\
&\quad + (u_h^* - b, \lambda^*)_{0,\Omega} + \mathrm{osc}_h(x_h^*) \\
&= -\frac{1}{2}\nabla_x\mathcal{L}(x_h^*, \lambda_h^*)(x_h^* - x^* + \varphi_h) + \frac{1}{2}(\lambda_h^* + \lambda^*, u_h^* - u^*)_{0,\Omega} \\
&\quad + \frac{1}{2}(f_h - f, \delta p_h)_{0,\Omega} + \frac{1}{2}(z_h - z, \delta y_h)_{0,\Omega} + \mathrm{osc}_h(x_h^*) \\
&= -\frac{1}{2}\nabla_x\mathcal{L}_h(x_h^*, \lambda_h^*)(x_h^* - x^* + \varphi_h) + \frac{1}{2}(\lambda_h^* + \lambda^*, u_h^* - u^*)_{0,\Omega} \\
&\quad + \frac{1}{2}(f - f_h, p_h^* - p^*)_{0,\Omega} + \frac{1}{2}(z - z_h, y_h^* - y^*)_{0,\Omega} + \mathrm{osc}_h(x_h^*).
\end{aligned}$$

Choosing $\varphi_h = (i_h^p p^* - p_h^*,\ i_h^y y^* - y_h^*,\ i_h^u u^* - u_h^*) \in X_h$ and using complementary slackness, we continue with

$$\begin{aligned}
J(y^*, u^*) - J_h(y_h^*, u_h^*) &= -\frac{1}{2}\nabla_x\mathcal{L}_h(x_h^*, \lambda_h^*)(i_h x^* - x^*) \\
&\quad + \frac{1}{2}\big[(\lambda_h^*, b_h - u^*)_{0,\Omega} + (\lambda^*, u_h^* - b)_{0,\Omega}\big] \\
&\quad + \frac{1}{2}(f - f_h, p_h^* - p^*)_{0,\Omega} + \frac{1}{2}(z - z_h, y_h^* - y^*)_{0,\Omega} \\
&\quad + \mathrm{osc}_h(x_h^*).
\end{aligned}$$

The assertion now follows from (2.2) and $\alpha u_h^* - M_h p_h^* + \lambda_h^* = 0$ a.e. in Ω. ◻

This result is interesting in several ways as follows:

(i) For $\|M_h p_h - p_h\|_{0,\Omega} \to 0$ as $h \to 0$ sufficiently fast, only the convergence properties implied by i_h^p and i_h^y are required for obtaining an a posteriori error estimate based on (2.16). Since y^* and p^* solve elliptic PDEs, they usually enjoy more regularity than u^* and λ^*.

(ii) The term in brackets on the right-hand side in (2.16) is again related to errors coming from complementary slackness. The first term of the sum can be interpreted as before, while the second term of the sum reflects the symmetric case, i.e.,

$$(b_h - u^*, \lambda_h^*)_{0,\Omega} = (b - u^*, \lambda_h^*)_{0,\mathcal{A}_h^* \cap \mathcal{I}^*} + (b_h - b, \lambda_h^*)_{0,\mathcal{A}_h^*}.$$

Hence, the first term of the right-hand side above represents the *primal-dual weighted mismatch in complementarity in* $\mathcal{I}^* \cap \mathcal{A}_h^*$, while the second term denotes the data oscillation on $\mathcal{A}_h^*$ weighted by the discrete multiplier, i.e.,

$$\operatorname{osc}_h^{\mathcal{A}_h^*}(b; \lambda_h^*) := (b_h - b, \lambda_h^*)_{0,\mathcal{A}_h^*}.$$

Of course, (2.16) is not immediately amenable to numerical realization since u^* and λ^* are involved. Before we tackle this point, let us first state a posteriori error bounds for the control and the adjoint state which were derived in [8]. A coarser estimate was established in [14]. Recall that U^{ad} denotes the set of admissible controls, and let U_h^{ad} be its discretization. Then the following a posteriori error estimates hold true:

$$\max(\|\lambda^* - \lambda_h^*\|_{0,\Omega}^2, \|u^* - u_h^*\|_{0,\Omega}^2) \le C_1^2\eta_1^2 + C_2^2\eta_2^2 + C_b^2\mu_h^2(b), \tag{2.17a}$$

$$|p^* - p_h^*|_{1,\Omega}^2 \le C_2^2\eta_2^2 + C_z^2 \operatorname{osc}_h^2(z). \tag{2.17b}$$

In what follows we also use

$$C_3^2\eta_3^2 := C_1^2\eta_1^2 + C_2^2\eta_2^2 + C_b^2\mu_h^2(b) \quad \text{and} \quad C_4^2\eta_4^2 := C_2^2\eta_2^2 + C_z^2 \operatorname{osc}_h^2(z).$$

Here and below, $C_i > 0$, $i = 1, 2, 3, 4$, denote constants which depend on α, Ω, and the shape regularity of $\mathbb{T}_h$. The error bounds η_1 and η_2 are defined as

$$\eta_1^2 = \sum_T \int_T h_T^2 \left(p_h^* - M_h p_h^*\right)^2, \tag{2.18}$$

$$\eta_2^2 = \sum_T \int_T h_T^2 \left(f + u_h^* + \Delta y_h^*\right)^2 + \sum_F \int_F h_F [\nabla y_h^* \cdot n]^2 + \sum_T \int_T h_T^2 \left(z - y_h^* + \Delta p_h^*\right)^2 + \sum_F \int_F h_F [\nabla p_h^* \cdot n]^2. \tag{2.19}$$

Further, the data oscillations

$$\mu_h^2(b) = \sum_{T \in \mathbb{T}_h} \|b - b_h\|_{0,T}^2, \tag{2.20}$$

$$\operatorname{osc}_h^2(z) = \sum_{T \in \mathbb{T}_h} h_T^2 \|z - z_h\|_{0,T}^2 \tag{2.21}$$

are involved.

Above, T denotes an element of the triangulation $\mathbb{T}_h$ of Ω. Further, F denotes a face of T, and h_F is the maximal diameter of the face F. Moreover, $[\nabla y_h^* \cdot n]$ is the normal derivative jump over an interior face F. As noted before, the operator M_h represents the projection of a mesh function in P_h ($= Y_h$, typically in our context) onto L_h. If L_h is given by

$$L_h = \{u_h \in L^2(\Omega) : u_h|_T \in P_0(T),\, T \in \mathbb{T}_h\},$$

i.e., the function u_h is piecewise constant on $\mathbb{T}_h$, then the action of M_h in T is given by

$$(M_h p_h)|_T = \frac{1}{|T|}\int_T p_h(x)\,dx, \quad T \in \mathbb{T}_h.$$

A final observation concerns the unconstrained case, which is $U^{ad} = L^2(\Omega)$. In this situation we have $\lambda^* = 0$ a.e. in Ω. From (2.18)–(2.19) we see that the error estimator remains unaffected.

Our investigations concentrate now on the term

$$\frac{1}{2}\big[(u_h^* - b, \lambda^*)_{0,\Omega} + (b_h - u^*, \lambda_h^*)_{0,\Omega}\big] =: \Psi^*(\Omega), \tag{2.22}$$

which contains u^* and λ^*. A simple manipulation yields

$$\Psi^*(\Omega) = \frac{1}{2}\big[(\lambda_h^* - \lambda^*, b_h - u^*)_{0,\Omega} + (\lambda^* - \lambda_h^*, u_h^* - b)_{0,\Omega} + (\lambda^* + \lambda_h^*, b_h - b)_{0,\Omega}\big].$$

From first order optimality we recall

$$u^* \le b,\ \lambda^* \ge 0, \quad (u^* - b, \lambda^*)_{0,\Omega} = 0, \quad \alpha u^* - p^* + \lambda^* = 0, \tag{2.23}$$

$$u_h^* \le b_h,\ \lambda_h^* \ge 0, \quad (u_h^* - b_h, \lambda_h^*)_{0,\Omega} = 0, \quad \alpha u_h^* - M_h p_h^* + \lambda_h^* = 0. \tag{2.24}$$

Obviously, we have

$$\Psi^*(\mathcal{I}^* \cap \mathcal{I}_h^*) = 0, \tag{2.25a}$$

$$\Psi^*(\mathcal{A}^* \cap \mathcal{A}_h^*) = \frac{1}{2}(\lambda^* - \lambda_h^*, b_h - b)_{0,\mathcal{A}^*\cap\mathcal{A}_h^*} + (\lambda_h^*, b_h - b)_{0,\mathcal{A}^*\cap\mathcal{A}_h^*}, \tag{2.25b}$$

where $\Psi^*(\mathcal{S}) = \frac{1}{2}\left[(u_h^* - b, \lambda^*)_{0,\mathcal{S}} + (b_h - u^*, \lambda_h^*)_{0,\mathcal{S}}\right]$. In the right-hand side of (2.25b), typically the latter term dominates. It is nonpositive if $b_h \le b$ a.e. in $\mathcal{A}_h^*$. Note that if $b_h = b$ a.e. in Ω, then $\Psi^*(\mathcal{A}^* \cap \mathcal{A}_h^*) = 0$. From the structure of $\Psi^*(\mathcal{A}^* \cap \mathcal{A}_h^*)$ we can see that it represents a dual-weighted data oscillation on $\mathcal{A}^* \cap \mathcal{A}_h^*$. Subsequently we use

$$\operatorname{osc}_h^{\mathcal{S}}(b; \lambda^* + \lambda_h^*) := (b_h - b, \lambda_h^* + \lambda^*)_{0,\mathcal{S}}. \tag{2.26}$$

Note that $\operatorname{osc}_h^{\mathcal{I}^*\cap\mathcal{I}_h^*}(b; \lambda^* + \lambda_h^*) = 0$.

Utilizing (2.23)–(2.26), for $\mathcal{C}_1^* = \mathcal{A}^* \cap \mathcal{I}_h^*$ and $\mathcal{C}_2^* = \mathcal{I}^* \cap \mathcal{A}_h^*$ we obtain

$$\Psi^*(\mathcal{C}_1^*) = \frac{\alpha}{2}\|u_h^* - u^*\|_{0,\mathcal{C}_1^*}^2 + \frac{1}{2}(p^* - M_h p_h^*, u_h^* - u^*)_{0,\mathcal{C}_1^*}, \tag{2.27a}$$

$$\Psi^*(\mathcal{C}_2^*) = \frac{1}{2}(b_h - \alpha^{-1}p^*, \lambda_h^*)_{0,\mathcal{C}_2^*}. \tag{2.27b}$$

On the respective sets we get the following estimates:

(i) In $\mathcal{C}_1^*$ we have $u^*_{|\mathcal{C}_1^*} = b_{|\mathcal{C}_1^*}$. Thus,

$$|\Psi^*(\mathcal{C}_1^*)| \leq \frac{1}{2}\left(\|M_h p_h^* - p_h^*\|_{0,\mathcal{C}_1^*} + \|p_h^* - p^*\|_{0,\mathcal{C}_1^*} + \alpha\|u_h^* - b\|_{0,\mathcal{C}_1^*}\right)\|u_h^* - b\|_{0,\mathcal{C}_1^*}.$$

Given $\mathcal{C}_1^*$ and the discrete control u_h^* and adjoint state p_h^*, the first and third terms in parentheses above are computable a posteriori. We therefore study $\|p_h^* - p^*\|_{0,\mathcal{C}_1^*}$ next. Since $p_h^*, p^* \in H_0^1(\Omega)$ and, for $n \geq 2$, $H_0^1(\Omega) \subset L^s(\Omega)$ for some $s \in (2, +\infty)$, from Hölder's inequality we obtain

$$\|p_h^* - p^*\|_{0,\mathcal{C}_1^*} \leq \text{meas}(\mathcal{C}_1^*)^{r(s)}|p^* - p_h^*|_{1,\mathcal{C}_1^*} \leq C_4\,\text{meas}(\mathcal{C}_1^*)^{r(s)}\eta_4 \tag{2.28}$$

with $r(s) := \frac{1}{2} - \frac{1}{s} > 0$. Hence, we get

$$\|p_h^* - p^*\|_{0,\mathcal{C}_1^*} \leq \min\left(C_0^p\eta_{0,p}, C_4\,\text{meas}(\mathcal{C}_1^*)^{r(s)}\eta_4\right) =: C^p(\mathcal{C}_1^*), \tag{2.29}$$

where $\eta_{0,p}$ denotes the a posteriori estimator for $\|p^* - p_h^*\|_{0,\Omega}$ (see Appendix A for its derivation) and $C_0^p > 0$ is a constant. This yields

$$\begin{aligned}|\Psi^*(\mathcal{C}_1^*)| \leq \frac{1}{2}&\left(\|M_h p_h^* - p_h^*\|_{0,\mathcal{C}_1^*} + C^p(\mathcal{C}_1^*) + \alpha\|u_h^* - b\|_{0,\mathcal{C}_1^*}\right)\\ &\cdot\|u_h^* - b\|_{0,\mathcal{C}_1^*} =: \mu_1(\mathcal{C}_1^*).\end{aligned} \tag{2.30}$$

(ii) In $\mathcal{C}_2^*$ we use the identities $\lambda_h^* = M_h p_h^* - \alpha u_h^*$ and $p^* = \alpha u^*$. From this, and assuming $b_h \in L^t(\Omega)$, $2 \leq t \leq s$, we infer

$$\begin{aligned}2|\Psi^*(\mathcal{C}_2^*)| &= |(u_h^* - u^*, \lambda_h^*)_{\mathcal{C}_2^*}|\\ &\leq \text{meas}(\mathcal{C}_2^*)^{r(t)}\|b_h - \alpha^{-1}p^*\|_{t,\mathcal{C}_2^*}\|\lambda_h^*\|_{0,\mathcal{C}_2^*}\\ &\leq \text{meas}(\mathcal{C}_2^*)^{r(t)}\left(\|b_h - \alpha^{-1}p_h^*\|_{t,\mathcal{C}_2^*} + \alpha^{-1}|p_h^* - p^*|_{1,\Omega}\right)\|\lambda_h^*\|_{0,\mathcal{C}_2^*}\\ &\leq \text{meas}(\mathcal{C}_2^*)^{r(t)}\left(\|b_h - \alpha^{-1}p_h^*\|_{t,\mathcal{C}_2^*} + \alpha^{-1}C_4\eta_4\right)\|\lambda_h^*\|_{0,\mathcal{C}_2^*}\end{aligned}$$

with $r(t) \geq 0$. Alternatively, we may use (2.17a) for estimating $\|u_h^* - u^*\|_{0,\mathcal{M}_2^*}$. Hence, setting

$$C^u(\mathcal{C}_2^*) := \min\left(\text{meas}(\mathcal{C}_2^*)^{r(t)}\left(\|b_h - \alpha^{-1}p_h^*\|_{t,\mathcal{C}_2^*} + \alpha^{-1}C_4\eta_4\right), C_3\eta_3\right),$$

we obtain

$$|\Psi^*(\mathcal{C}_2^*)| \leq \frac{1}{2}C^u(\mathcal{C}_2^*)\|\lambda_h^*\|_{0,\mathcal{C}_2^*} := \mu_2(\mathcal{C}_2^*). \tag{2.31}$$

Since $\lambda_h^* = 0$ in $\mathcal{I}_h^*$, we obviously have $\mu_2(\mathcal{I}_h^*) = 0$.

In both cases above we assume $\mu_1(\emptyset) = 0$ and $\mu_2(\emptyset) = 0$. Summarizing, we obtain

$$\begin{aligned}|\Psi^*(\Omega)| &= |\Psi^*(\mathcal{A}^* \cap \mathcal{A}_h^*) + \Psi^*(\mathcal{C}_1^*) + \Psi^*(\mathcal{C}_2^*)|\\ &\leq \frac{1}{2}\left|\text{osc}_h^{\mathcal{A}^*\cap\mathcal{A}_h^*}(b; \lambda^* + \lambda_h^*)\right| + \mu_1(\mathcal{C}_1^*) + \mu_2(\mathcal{C}_2^*).\end{aligned}$$

An alternative (and possibly coarse) estimate of $\Psi^*(\Omega)$ uses only the error estimate η_3 and $\|\lambda_h^*\|_{0,\mathcal{A}_h^*}$ as follows:

$$|\Psi^*(\Omega)| = \frac{1}{2}|(\lambda_h^* + \lambda^*, u^* - u_h^*)| \leq \frac{1}{2}C_3\eta_3(C_3\eta_3 + 2\|\lambda_h^*\|_{0,\mathcal{A}_h^*}) =: \mu_3(\Omega).$$

If the original problem is unconstrained with respect to u, then $\lambda^* = 0$. As a consequence, the first order conditions yield $\alpha u^* = p^*$, i.e., u^* inherits the regularity of $p^* \in H_0^1(\Omega)$. Then we may choose the same ansatz when discretizing p *and* u. Thus, we obtain $\eta_1 = 0$, since M_h becomes the identity operator, and—up to data oscillations—$\eta_2 = \eta_3$, and further, $\|M_h p_h^* - p_h^*\|_{0,\mathcal{C}_1^*} = 0$ in μ_1.

Finally, we express μ_1 and μ_2 such that we obtain cell-oriented error estimates. Let us first consider $\mu_1(\mathcal{C}_1^*)$. We have

$$\begin{aligned} \mu_1(\mathcal{C}_1^*) &= \frac{1}{2}\big(C^p(\mathcal{C}_1^*) + \|M_h p_h^* - p_h^*\|_{0,\mathcal{C}_1^*} + \alpha\|u_h^* - b\|_{0,\mathcal{C}_1^*}\big)\|u_h^* - b\|_{0,\mathcal{C}_1^*} \\ &= \frac{1}{2}\left(\hat{C}^p(\mathcal{C}_1^*) + \hat{C}_5(\mathcal{C}_1^*)\|M_h p_h^* - p_h^*\|_{0,\mathcal{C}_1^*}^2 + \alpha\|u_h^* - b\|_{0,\mathcal{C}_1^*}^2\right). \end{aligned}$$

Above, we use

$$\hat{C}_0^p := \begin{cases} C_0^p \frac{\|u_h^* - b\|_{0,\mathcal{C}_1^*}}{\eta_{0,p}} & \text{if } \operatorname{meas}(\mathcal{C}_1^*) \neq 0 \text{ and } \eta_{0,p} > 0, \\ 0 & \text{if } \operatorname{meas}(\mathcal{C}_1^*) = 0, \end{cases}$$

as well as

$$\hat{C}_4(\mathcal{C}_1^*) := \begin{cases} C_4 \frac{\|u_h^* - b\|_{0,\mathcal{C}_1^*}}{\eta_4} & \text{if } \operatorname{meas}(\mathcal{C}_1^*) \neq 0 \text{ and } \eta_4 > 0, \\ 0 & \text{if } \operatorname{meas}(\mathcal{C}_1^*) = 0, \end{cases}$$

and further,

$$\hat{C}_5(\mathcal{C}_1^*) := \begin{cases} \frac{\|u_h^* - b\|_{0,\mathcal{C}_1^*}}{\|M_h p_h^* - p_h^*\|_{0,\mathcal{C}_1^*}} & \text{if } \operatorname{meas}(\mathcal{C}_1^*) \neq 0 \text{ and } \|M_h p_h^* - p_h^*\|_{0,\mathcal{C}_1^*} > 0, \\ 0 & \text{if } \operatorname{meas}(\mathcal{C}_1^*) = 0. \end{cases}$$

We therefore have

$$\hat{C}^p(\mathcal{C}_1^*) = \min\left(\hat{C}_0^p \eta_{0,p}^2, \hat{C}_4(\mathcal{C}_1^*) \operatorname{meas}(\mathcal{C}_1^*)^{r(s)} \eta_4^2\right).$$

Finally, we turn to $\mu_2(\mathcal{C}_2^*)$. We obtain

$$\mu_2(\mathcal{C}_2^*) = \frac{1}{2}\hat{C}^u(\mathcal{C}_2^*),$$

with

$$\hat{C}_i(\mathcal{C}_2^*) := \begin{cases} C_i \frac{\|\lambda_h^*\|_{0,\mathcal{C}_2^*}}{\eta_i} & \text{if } \operatorname{meas}(\mathcal{C}_2^*) \neq 0 \text{ and } \eta_i > 0, \\ 0 & \text{if } \operatorname{meas}(\mathcal{C}_2^*) = 0, \end{cases}$$

for $i = 3, 4$, and

$$\begin{aligned} \hat{C}^u(\mathcal{C}_2^*) := \min\Big(&\operatorname{meas}(\mathcal{C}_2^*)^{r(t)}\big(\|b_h - \alpha^{-1} p_h^*\|_{t,\mathcal{C}_2^*}\|\lambda_h^*\|_{0,\mathcal{C}_2^*} \\ &+ \alpha^{-1}\hat{C}_4(\mathcal{C}_2^*)\eta_4^2\big), \hat{C}_3(\mathcal{C}_2^*)\eta_3^2\Big). \end{aligned}$$

We summarize our above findings in the following proposition.

PROPOSITION 2.1. *Let the assumptions of Theorem* 2.1 *be satisfied. Then*

$$|\Psi^*(\Omega)| \leq \min\left(\frac{1}{2}\left|\operatorname{osc}_h^{\mathcal{A}^* \cap \mathcal{A}_h^*}(b; \lambda^* + \lambda_h^*)\right| + \mu_1(\mathcal{C}_1^*) + \mu_2(\mathcal{C}_2^*), \mu_3(\Omega)\right). \tag{2.32}$$

We denote the right-hand side in (2.32) by $\hat{\nu}$. In the case where the solution of (P) satisfies $u^* < b$ a.e. on Ω, we expect that $\hat{\nu} \approx 0$. Indeed, for sufficiently small h we have $\lambda_h^* \approx 0$ (or even $\lambda_h^* = 0$). Thus, $\mu_2(\mathcal{C}_2^*) \approx 0$ (or $\mu_2(\mathcal{C}_2^*) = 0$) holds true. Further, $\mu_1(\mathcal{C}_1^*) = 0$ since $\mathcal{A}^* = \emptyset$. Then (2.32) yields $\hat{\nu} \approx 0$ (or $\hat{\nu} = 0$). If (P) involves no inequality constraints on u, which means that we can set $b \equiv +\infty$ on Ω, then we naturally obtain $\hat{\nu} = 0$. Hence, we recover the error estimator for unconstrained optimal control problems; compare [2, 14].

For deriving the full error estimate, it remains to consider the first term in parentheses on the right-hand side of (2.16) in Theorem 2.2. This term is independent of the control constraints and corresponds to the usual expression obtained for (unconstrained) optimal control problems; see [2, 14]. A standard argument yields

$$
\begin{aligned}
&|(\nabla y_h^*, \nabla(i_h^p p^* - p^*))_{0,\Omega} - (u_h^* + f_h, i_h^p p^* - p^*)_{0,\Omega}| \\
&\quad \le \sum_T \|-\Delta y_h^* - u_h^* - f_h\|_{0,T} \|p^* - i_h^p p^*\|_{0,T} \\
&\qquad + \sum_F \left\| \left[\frac{\partial y_h^*}{\partial n} \right] \right\|_{0,F} \|p^* - i_h^p p^*\|_{0,F} =: \eta_2^p
\end{aligned} \tag{2.33}
$$

for the primal equation,

$$
\begin{aligned}
&|(y_h^* - z_h, i_h^y y^* - y^*)_{0,\Omega} + (\nabla(i_h^y y^* - y^*), \nabla p_h^*)_{0,\Omega}| \\
&\quad \le \sum_T \|-\Delta p_h^* + y_h^* - z_h\|_{0,T} \|y^* - i_h^y y^*\|_{0,T} \\
&\qquad + \sum_F \left\| \left[\frac{\partial p_h^*}{\partial n} \right] \right\|_{0,F} \|y^* - i_h^y y^*\|_{0,F} =: \eta_2^d
\end{aligned} \tag{2.34}
$$

for the dual equation, and

$$
|(M_h p_h^* - p_h^*, i_h^u u^* - u^*)_{0,\Omega}| =: \eta_2^u. \tag{2.35}
$$

The overall residual- and complementarity-based error estimate is given in the following theorem.

THEOREM 2.3. *Let the assumptions of Theorem* 2.1 *be satisfied. Then we have the following error estimate:*

$$
\begin{aligned}
|J(y^*, u^*) - J(y_h^*, u_h^*)| &\le \frac{1}{2}(\eta_2^p + \eta_2^d + \eta_2^u) + \hat{\nu} \\
&\quad + \frac{1}{2}\left[C_0^p \eta_{0,p} \|f - f_h\|_{0,\Omega} + C_0^y \eta_{0,y} \|z - z_h\|_{0,\Omega}\right] \\
&\quad + |\mathrm{osc}_h(x_h^*)|
\end{aligned} \tag{2.36}
$$

with η_2^p, η_2^d, η_2^u, *and* $\hat{\nu}$ *defined by* (2.33), (2.34), (2.35), *and* (2.32), *respectively. Further,* $C_0^y > 0$ *is a constant and* $\eta_{0,y}$ *denotes an error estimate for* $\|y_h^* - y^*\|_{0,\Omega}$. *For the definition of* $\eta_{0,p}$ *and* $\eta_{0,y}$ *see* (A.10) *and* (A.11) *in Appendix A.*

The numerical evaluation of (2.36) depends on estimates of $\|i_h^y y^* - y^*\|_{0,T}$, $\|i_h^y y^* - y^*\|_{0,F}$, and analogously, for $i_h^p p^* - p^*$. When discretizing the state and the adjoint state in two dimensions by continuous piecewise linear finite elements, the following averaging technique, replacing η_2^p and η_2^d in (2.33) and (2.34), respectively, is appropriate:

$$\eta_{2,h}^p := \frac{1}{3}\sum_T \left(h_T \|-\Delta y_h^* - u_h^* - f_h\|_{0,T} \sum_{F(T)} h_F^{1/2} \left\| \left[\frac{\partial p_h^*}{\partial n} \right] \right\|_{0,F} \right) + \sum_F h_F \left\| \left[\frac{\partial y_h^*}{\partial n} \right] \right\|_{0,F} \left\| \left[\frac{\partial p_h^*}{\partial n} \right] \right\|_{0,F} \tag{2.37}$$

for the primal equation, and

$$\eta_{2,h}^d := \frac{1}{3}\sum_T \left(h_T \|-\Delta p_h^* + y_h^* - z_h\|_{0,T} \sum_{F(T)} h_F^{1/2} \left\| \left[\frac{\partial y_h^*}{\partial n} \right] \right\|_{0,F} \right) + \sum_F h_F \left\| \left[\frac{\partial p_h^*}{\partial n} \right] \right\|_{0,F} \left\| \left[\frac{\partial y_h^*}{\partial n} \right] \right\|_{0,F} \tag{2.38}$$

for the dual equation, where $F(T)$ denotes the edges pertinent to triangle T. Notice that (2.37) and (2.38) yield typically sharper estimates than residual-based estimators for our model problem; compare (2.17) and [8]. Further observe that we can only expect boundedness of $\|i_h^u u^* - u^*\|_{0,\Omega}$ in general. However, typically $\|M_h p_h^* - p_h^*\|_{0,\Omega}$ is small, or, when using the same ansatz for discretizing u^* as well as p^*, it is even zero.

For the numerical evaluation of $\hat{\nu}$ observe that $\mathcal{I}_h^* \setminus \mathcal{A}^* \subset \mathcal{I}^*$, and hence $\lambda_h^* = 0$ and $\lambda^* = 0$ on this set. Consequently, we obtain

$$\Psi^*(\mathcal{I}_h^* \setminus \mathcal{A}^*) = 0.$$

Next observe that $\mathcal{I}_h^* = \mathcal{C}_1^* \,\dot\cup\, (\mathcal{I}_h^* \setminus \mathcal{A}^*)$. Therefore, we have

$$\Psi^*(\mathcal{C}_1^*) = \Psi^*(\mathcal{I}_h^*) - \Psi^*(\mathcal{I}_h^* \setminus \mathcal{A}^*) = \Psi^*(\mathcal{I}_h^*). \tag{2.39}$$

If $b_h = b$, then we obtain $\Psi^*(\mathcal{A}_h^* \setminus \mathcal{I}^*) = 0$, and further,

$$\Psi^*(\mathcal{C}_2^*) = \Psi^*(\mathcal{A}_h^*) - \Psi^*(\mathcal{A}_h^* \setminus \mathcal{I}^*) = \Psi^*(\mathcal{A}_h^*). \tag{2.40}$$

The estimates $\mu_1(\mathcal{C}_1^*)$ and $\mu_1(\mathcal{C}_2^*)$, however, do not satisfy relations analogous to (2.39)–(2.40) even when $b_h = b$. Hence, $\hat{\nu}$ is not a posteriori. In order to have a fully a posteriori estimate, we replace $\hat{\nu}$ in (2.36) by

$$\hat{\nu}^a = \min\left(\mu_1(\mathcal{I}_h^*), \mu_3(\Omega)\right) + \min\left(\mu_2(\mathcal{A}_h^*), \mu_3(\Omega)\right). \tag{2.41}$$

An alternative technique based on set estimation is the subject of section 3.2.

3. Extensions. Now we consider possible extensions of the concept derived in the previous section. We focus on two aspects as follows: (i) effects due to nonlinear PDEs and/or bilateral constraints; and (ii) alternative ways of making $\hat{\nu}$ fully a posteriori.

3.1. Semilinear PDEs and bilateral constraints. Next we assume that the underlying PDE is semilinear as follows:

$$A(y) = Bu + f, \tag{3.1}$$

where the operators A and B induce a semilinear form $\mathfrak{a}(\cdot)(\cdot)$ and a bilinear form $\mathfrak{b}(\cdot,\cdot)$, respectively. Hence, the weak form of (3.1) becomes

$$\mathfrak{a}(y)(v) = (f,v)_{0,\Omega} + \mathfrak{b}(u,v) \quad \forall v \in Y.$$

For our arguments to follow, we assume that A (respectively, $\mathfrak{a}$) is sufficiently often differentiable. Further, we suppose that the control is subject to bilateral constraints, i.e.,

$$a \le u \le b \text{ a.e. in } \Omega.$$

The Lagrange function corresponding to the associated minimization problem has the structure

$$\mathcal{L}(x,\lambda_a,\lambda_b) = J(y,u) + \mathfrak{a}(y)(p) - (f,p)_{0,\Omega} - \mathfrak{b}(u,p) + (a-u,\lambda_a)_{0,\Omega} + (u-b,\lambda_b)_{0,\Omega},$$

where $\lambda_a, \lambda_b \in L^2(\Omega)$ represent the Lagrange multipliers pertinent to the bilateral pointwise constraints. The first order necessary optimality conditions are given by

$$A(y^*) - Bu^* = f, \tag{3.2a}$$

$$A'(y^*)^\star p^* + J_y(y^*,u^*) = 0, \tag{3.2b}$$

$$J_u(y^*,u^*) + \lambda_b^* - \lambda_a^* - B^\star p^* = 0, \tag{3.2c}$$

$$u^* \ge a,\ \lambda_a^* \ge 0,\ (u^*-a,\lambda_a^*)_{0,\Omega} = 0, \tag{3.2d}$$

$$u^* \le b,\ \lambda_b^* \ge 0,\ (u^*-b,\lambda_b^*)_{0,\Omega} = 0. \tag{3.2e}$$

As the pointwise control constraints are affine, the error estimator for the nonlinear case is similar to the linear case. This parallels the situation in [2], where the unconstrained case was considered. Due to essentially the same proof arguments as in [2, Proposition 6.1], the following result holds true. In what follows, we use

$$\mathcal{L}_0(x) = J(y,u) + \mathfrak{a}(y)(p) - (f,p)_{0,\Omega} - \mathfrak{b}(u,p),$$

and use $\mathcal{L}_{0,h}(x)$ for its discrete counterpart.

THEOREM 3.1. *For a Galerkin finite element discretization of the first order necessary optimality conditions* (3.2)*, the following relation holds true:*

$$\begin{aligned} J(y^*,u^*) - J_h(y_h^*,u_h^*) &= \frac{1}{2}\nabla_x \mathcal{L}_{0,h}(x_h^*)(x^* - i_h x^*) \\ &+ \frac{1}{2}\big[(u_h^* - b,\lambda_b^*)_{0,\Omega} + (b_h - u^*,\lambda_{b,h}^*)_{0,\Omega}\big] \\ &+ \frac{1}{2}\big[(a - u_h^*,\lambda_a^*)_{0,\Omega} + (u^* - a_h,\lambda_{a,h}^*)_{0,\Omega}\big] \\ &+ \frac{1}{2}\left((f - f_h, p_h^* - p^*)_{0,\Omega} + (z - z_h, y_h^* - y^*)_{0,\Omega}\right) + \mathrm{osc}_h(x_h^*) \\ &+ r(x^*,x_h^*), \end{aligned}$$

where $r(x^*,x_h^*)$ denotes the remainder term of a Taylor expansion of $\mathcal{L}_0$ about x_h^*. It is bounded by

$$|r(x^*,x_h^*)| \le \sup_{\bar{x}\in[x_h^*,x^*]} |\nabla_x^3 \mathcal{L}_0(\bar{x})[x^* - x_h^*]^3|.$$

3.2. Alternative a posteriori estimate for $\hat{\nu}$. At the end of section 2 we derived an a posteriori estimate for $\hat{\nu}$; recall $\hat{\nu}^a$ in (2.41), where we replaced $\mathcal{C}_1^*$ by $\mathcal{I}_h^*$ and $\mathcal{C}_2^*$ by $\mathcal{A}_h^*$, respectively. This may give rise to an overestimation of the error term pertinent to the complementarity system. In the following we provide an alternative approach based on set estimation.

Assuming, without loss of generality, $b_h = b$, we focus on the unilaterally constrained case and start by considering $\hat{\mu}_1(\mathcal{C}_1^*)$. For this purpose recall that $\mathcal{C}_1^* = \mathcal{I}_h^* \cap \mathcal{A}^*$. Similarly to [11, section 3.3] we estimate the continuous active set $\mathcal{A}^*$ by

$$\chi_h^{\mathcal{A}^*} = 1 - \frac{b - u_h^*}{\gamma\, h^r + b - u_h^*},$$

where γ denotes some (possibly small) positive constant, and $r > 0$ is fixed. Note that $\chi_h^{\mathcal{A}^*} = 1$ in $\mathcal{A}_h^*$. Further, let $\chi(\mathcal{S})$ denote the characteristic function of a set $\mathcal{S} \subset \Omega$. We briefly argue that our approximation is useful. In fact, assume that $T \subset \mathcal{A}^*$. Then

$$\|\chi(\mathcal{A}^*) - \chi_h^{\mathcal{A}^*}\|_{0,T} = \left\| \frac{b - u_h^*}{\gamma h^r + b - u_h^*} \right\|_{0,T} \leq \min\{1, \gamma^{-1} h^{-r} \|u^* - u_h^*\|_{0,T}\},$$

which tends to zero whenever $\|u^* - u_h^*\|_{0,T} = \mathcal{O}(h^q)$ with $q > r$. If $T \in \mathcal{I}^*$, then we distinguish two cases as follows:

(i) $T \subset \{b - u_h^* > \gamma h^{\epsilon r}\}$ for some $0 \leq \epsilon < 1$. Then

$$\|\chi(\mathcal{A}^*) - \chi_h^{\mathcal{A}^*}\|_{0,T} = \left\| \frac{\gamma h^r}{\gamma h^r + b - u_h^*} \right\|_{0,T} \leq h^{(1-\epsilon)r} \to 0 \text{ as } h \to 0.$$

(ii) Finally, in the case where $T \in \{b - u_h^* \leq \gamma h^{\epsilon r}\}$, we use $T \subset \mathcal{I}^*$ and $\|u^* - u_h^*\|_{0,\Omega} \to 0$ to conclude that the measure of this set tends to zero as $h \to 0$.

We therefore use the following approximation of $\chi(\mathcal{C}_1^*)$:

$$\chi(\mathcal{C}_1^*) \approx \chi(\mathcal{I}_h^*)\chi_h^{\mathcal{A}^*} =: \chi_h^{\mathcal{C}_1^*}.$$

In the definition of $\mu_1(\mathcal{C}_1^*)$, we then use

$$\|\chi_h^{\mathcal{C}_1^*}(u_h^* - b)\|_{0,\Omega} \quad \text{instead of} \quad \|u_h^* - b\|_{0,\mathcal{C}_1^*}$$

and analogously for $\|M_h p_h^* - p_h^*\|_{0,\mathcal{C}_1^*}$. Further, the measure of $\mathcal{C}_1^*$ is approximated by

$$\text{meas}(\mathcal{C}_1^*) \approx \int_\Omega \chi_h^{\mathcal{C}_1^*}\, dx.$$

The definition of μ_2 involves the set $\mathcal{C}_2^* = \mathcal{A}_h^* \cap \mathcal{I}^*$. Here we employ the approximation

$$\chi_h^{\mathcal{C}_2^*} := \chi(\mathcal{A}_h^*)\chi_h^{\mathcal{I}^*}$$

with $\chi_h^{\mathcal{I}^*} = 1 - \chi_h^{\mathcal{A}^*}$. Then we replace $\|\lambda_h^*\|_{0,\mathcal{C}_2^*}$ by $\|\chi_h^{\mathcal{C}_2^*}\lambda_h^*\|_{0,\Omega}$, $\|b - \alpha p_h^*\|_{t,\mathcal{C}_2^*}$ by $\|\chi_h^{\mathcal{C}_2^*}(b - \alpha^{-1}p_h^*)\|_{t,\Omega}$, and obtain

$$\text{meas}(\mathcal{C}_2^*) \approx \int_\Omega \chi_h^{\mathcal{C}_2^*}\, dx.$$

The extension of this concept to the bilaterally constrained case is straightforward.

4. Numerics. For the practical realization of the goal-oriented dual-weighted approach, we follow the cycle SOLVE, ESTIMATE, MARK, and REFINE known from adaptive finite element methods. Here, SOLVE stands for the numerical solution of the discrete optimal control problem which is taken care of by a primal-dual active set strategy [9]. The following step, ESTIMATE, is devoted to the computation of the edge and element residuals of the error estimator η_h, the local components of the consistency error $\hat{\nu}_h$, and the data oscillations. We note that

$$\eta_h := \eta_{2,h}^p + \eta_{2,h}^d + \eta_{2,h}^u. \tag{4.1}$$

Here, $\eta_{2,h}^p$ and $\eta_{2,h}^d$ are given by (2.38) and (2.37). Moreover, $\eta_{2,h}^u$ is given by (2.35) with $i_h^u u^* - u^*$ replaced by $u_h^* - \bar{u}_h^*$, where $\bar{u}_h^*|_T := |T|^{-1} \int_T u_h^* \, dx$, $T \in \mathbb{T}_h$. We refer to $\eta_{2,T}^p$, $\eta_{2,T}^d$, $\eta_{2,T}^u$, $T \in \mathbb{T}_h$, as the elementwise contributions to $\eta_{2,h}^p$, $\eta_{2,h}^d$, and $\eta_{2,h}^u$, respectively, so that

$$\eta_h = \sum_{T \in \mathbb{T}_h} \eta_T, \qquad \eta_T := \eta_{2,T}^p + \eta^d 2, T + \eta_{2,T}^u.$$

Likewise, we have

$$\hat{\nu}_h = \sum_{T \in \mathbb{T}_h} \hat{\nu}_T^a, \tag{4.2}$$

where $\hat{\nu}_T^a$, $T \in \mathbb{T}_h$, are the elementwise contributions to $\hat{\nu}^a$ that can be easily deduced from (2.32). Finally, we summarize the remaining terms of (2.36) in Theorem 2.3 according to

$$\mathrm{osc}_h := \frac{1}{2}[C_0^p \eta_{0,p} \|f - f_h\|_{0,\Omega} + C_0^y \eta_{0,y} \|z - z_h\|_{0,\Omega}] + |\mathrm{osc}_h(x_h^*)| \tag{4.3}$$

and observe

$$\mathrm{osc}_h = \sum_{T \in \mathbb{T}_h} \mathrm{osc}_T,$$

where again osc_T, $T \in \mathbb{T}_h$, refers to the elementwise contribution to osc_h. In the step MARK of the adaptive cycle, we specify constants $\Theta_i \in (0,1)$ and select subsets $\mathbb{M}_i \subset \mathbb{T}_h$, $1 \le i \le 3$, by means of the bulk criteria

$$\Theta_1 \sum_{T \in \mathbb{T}_h} \eta_T \le \sum_{T \in \mathbb{M}_1} \eta_T, \tag{4.4a}$$

$$\Theta_2 \sum_{T \in \mathbb{T}_h} \hat{\nu}_T^a \le \sum_{T \in \mathbb{M}_2} \hat{\nu}_T^a, \tag{4.4b}$$

$$\Theta_3 \sum_{T \in \mathbb{T}_h} \mathrm{osc}_T \le \sum_{T \in \mathbb{M}_3} \mathrm{osc}_T \tag{4.4c}$$

known from the convergence analysis of adaptive finite element methods (cf., e.g., [6, 15]). The bulk criteria can be realized by a greedy algorithm (cf., e.g., [8]). The final step REFINE of the adaptive loop is devoted to the creation of a new refined mesh based on longest edge bisection of any element $T \in \mathbb{T}_h$ that has been marked, i.e., $T \in \bigcup_{i=1}^3 \mathbb{M}_i$.

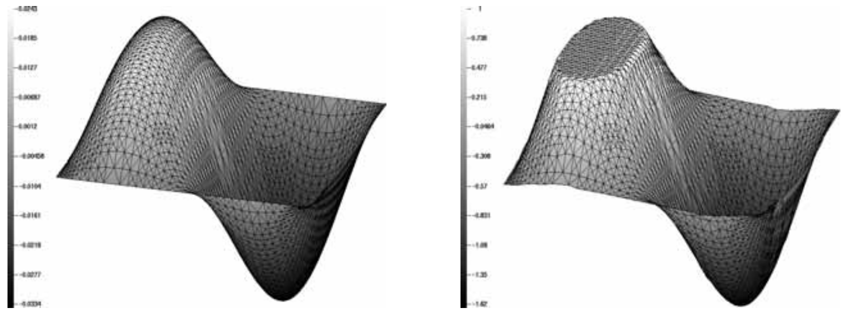

FIG. 4.1. *Example* 1: *Optimal state (left) and optimal control (right).*

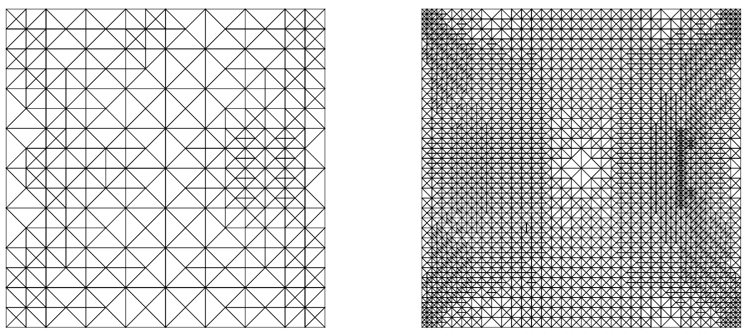

FIG. 4.2. *Example* 1: *Adaptively refined grid after* 6 *(left) and* 10 *(right) refinement steps* ($\Theta_i = 0.6$, $1 \le i \le 3$).

Finally, we provide documentation of numerical results illustrating the performance of the goal-oriented dual-weighted approach for two representative distributed optimal control problems that have been considered in [8] in the framework of an error analysis of residual-type a posteriori error estimators for control constrained optimal control problems.

Example 1 (constant obstacle). This first example features a constant obstacle. The data are as follows:

$$\Omega := (0,1)^2, \quad z := \begin{cases} 200x_1x_2(x_1-0.5)^2(1-x_2) & \text{if } 0 \le x_1 \le 0.5, \\ 200(x_1-1)x_2(x_1-0.5)^2(1-x_2) & \text{if } 0.5 < x_1 \le 1, \end{cases}$$
$$\alpha := 0.01, \quad b := 1, \quad f := 0.$$

Figures 4.1 and 4.2 show a visualization of the optimal state and the optimal control as well as the adaptively refined grids after 6 and 10 refinement steps in the case when $\Theta_i = 0.6$, $1 \le i \le 3$ in the bulk criteria (4.4). The active region is an ellipse (cf. the plateau in Figure 4.1 (right)). The convergence history of the adaptive loop is displayed in Table 4.1, containing the total number of degrees of freedom N_{DOF}, the error $\delta_h := |J(y^*, u^*) - J_h(y_h^*, u_h^*)|$ in the objective functional, the error estimator η_h, the consistency error $\hat{\nu}_h^a$, and the data oscillations osc_h. Finally, Figure 4.3 shows the error δ_h as a function of the total number of degrees of freedom in the case of adaptive refinement (solid line) and uniform refinement (dotted line). Since in this example the optimal state and adjoint state are smooth, there is only a slight benefit gained when using the adaptive process.

TABLE 4.1
Example 1: Convergence history of the goal-oriented dual-weighted approach.

ℓ	N_{dof}	δ_h	η_h	$\hat{\nu}_h^a$	osc_h
0	12	2.73e-03	1.47e-02	0.00e+00	1.17e-01
1	25	8.57e-04	2.03e-02	2.04e-03	6.23e-02
2	42	5.09e-04	1.42e-02	4.86e-03	3.44e-02
3	80	2.54e-04	7.63e-03	3.13e-03	2.17e-02
4	138	1.52e-04	4.61e-03	1.66e-04	1.27e-02
5	282	7.32e-05	2.30e-03	1.62e-05	7.26e-03
6	478	4.24e-05	1.35e-03	3.67e-05	4.20e-03
7	928	1.77e-05	6.45e-04	1.43e-05	5.24e-03
8	1706	9.91e-06	3.67e-04	4.27e-06	2.08e-03
9	3236	5.13e-06	1.85e-04	1.54e-06	1.20e-03
10	6237	2.52e-06	9.95e-05	3.82e-07	6.60e-04
11	11292	1.42e-06	5.25e-05	1.56e-07	3.73e-04
12	22639	5.92e-07	2.74e-05	1.63e-07	1.63e-04
13	38549	4.20e-07	1.53e-05	4.41e-08	1.12e-04
14	81325	1.57e-07	7.57e-06	7.60e-09	5.05e-05
15	136571	1.17e-07	4.38e-06	6.78e-09	3.24e-05
16	299028	4.65e-08	2.05e-06	1.32e-09	1.58e-05

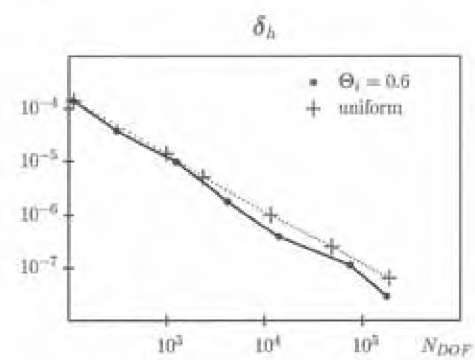

FIG. 4.3. *Example 1: Adaptive refinement (solid line) versus uniform refinement (dotted line).*

Example 2 (variable obstacle). This example is constructed in such a way that there is a lack of strict complementarity. It differs from the general setting insofar as the term containing the control in the objective functional additionally includes a fixed shift control $w \in L^2(\Omega)$ as follows:

$$J(y,u) := \frac{1}{2}\|y-z\|_{0,\Omega}^2 + \frac{\alpha}{2}\|u-w\|_{0,\Omega}^2.$$

The data are as follows:

$$\Omega := (0,1)^2, \quad z := 0, \quad w := \hat{u} + \alpha^{-1}\left(\hat{\sigma} + \Delta^{-2}\hat{u}\right),$$
$$b := \begin{cases} (x_1-0.5)^8 & \text{if } (x_1,x_2) \in \Omega_1, \\ (x_1-0.5)^2 & \text{otherwise,} \end{cases} \qquad \alpha := 0.1, \quad f := 0.$$

Here, $\hat{u}$ and $\hat{\sigma}$ are given by

$$\hat{u} := \begin{cases} b(x_1,x_2) & \text{if } (x_1,x_2) \in \Omega_1 \cup \Omega_2, \\ -1.01\, b(x_1,x_2) & \text{otherwise} \end{cases}$$

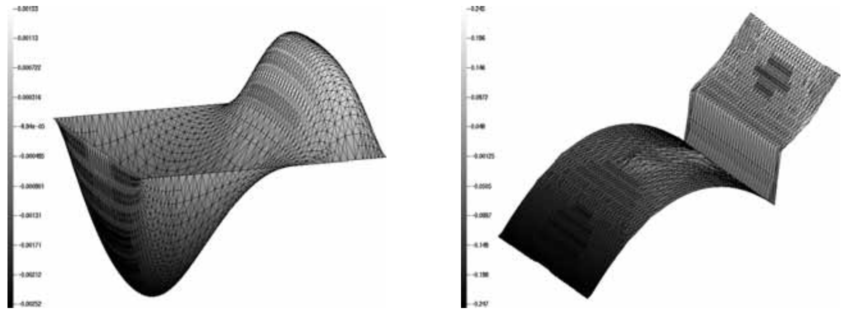

FIG. 4.4. *Example* 2: *Optimal state (left) and optimal control (right).*

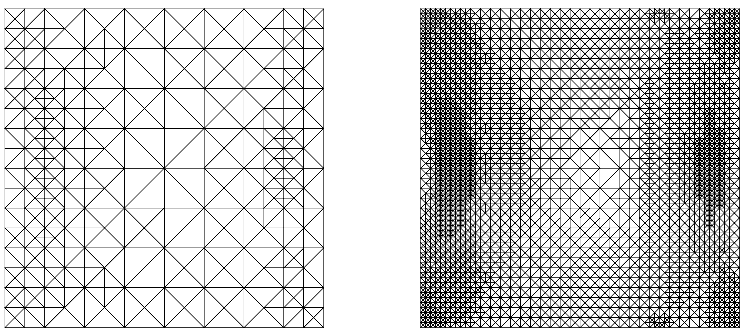

FIG. 4.5. *Example* 2: *Adaptively refined grid after* 6 *(left) and* 10 *(right) refinement steps* ($\Theta_i = 0.6$, $1 \le i \le 3$).

and

$$\hat{\sigma} := \begin{cases} 2.25\,(x_1 - 0.75) \cdot 10^{-4} & \text{if } (x_1, x_2) \in \Omega_2, \\ 0 & \text{otherwise} \end{cases}$$

with Ω_1 and Ω_2 specified as follows:

$$\Omega_1 := \{(x_1, x_2) \in \Omega \mid ((x_1 - 0.5)^2 + (x_2 - 0.5)^2)^{1/2} \le 0.15\},$$
$$\Omega_2 := \{(x_1, x_2) \in \Omega \mid x_1 \ge 0.75\}.$$

We note that Ω_2 corresponds to the strongly active set (where strict complementarity holds true, i.e., $\lambda^* > 0$ a.e. in Ω_2), whereas the set $\Omega_1 \cup \{(x_1, x_2) \in \Omega \mid x_1 = 0.5\}$ represents the weakly active set, where strict complementarity does not hold true, i.e., $\lambda^* = 0$ a.e. in this set.

The shift control $w \in L^2(\Omega)$ is approximated by $w_h \in L_h$, giving rise to an additional term in the data oscillations $\text{osc}_h(x_h^*)$.

Figure 4.4 displays the computed optimal state and optimal control. Figure 4.5 shows the adaptively refined grids after 6 and 10 refinements steps, where we have chosen $\Theta_i = 0.6$, $1 \le i \le 3$ in the bulk criteria (4.4). Table 4.2 reflects the convergence history of the refinement process in terms of the same data as in the first example, and Figure 4.6 shows the comparison between adaptive and uniform refinement. In this example, the benefits of adaptive refinement are more pronounced than in the previous one.

Table 4.2
Example 2: Convergence history of the goal-oriented dual-weighted approach.

ℓ	$N_{\rm dof}$	δ_h	η_h	$\hat{\nu}_h^a$	osc_h
0	5	2.41e-04	2.58e-06	0.00e+00	1.07e-01
1	12	1.61e-04	5.26e-06	2.71e-07	8.11e-02
2	26	7.62e-05	4.78e-06	4.19e-07	5.25e-02
3	43	3.50e-05	3.69e-06	5.82e-07	3.71e-02
4	73	1.54e-05	2.08e-06	0.00e+00	2.89e-02
5	133	8.59e-06	1.29e-06	0.00e+00	2.22e-02
6	253	4.09e-06	6.45e-07	0.00e+00	1.59e-02
7	475	2.38e-06	3.78e-07	8.08e-11	1.17e-02
8	953	1.16e-06	1.79e-07	9.86e-12	8.39e-03
9	1776	6.44e-07	9.86e-08	1.79e-12	6.05e-03
10	3507	3.41e-07	4.87e-08	2.66e-13	4.70e-03
11	6645	1.82e-07	2.64e-08	7.94e-14	3.34e-03
12	12684	1.03e-07	1.33e-08	3.08e-14	2.59e-03
13	24746	5.36e-08	7.06e-09	1.25e-14	1.91e-03
14	45486	2.99e-08	3.71e-09	2.23e-15	1.52e-03
15	90991	1.57e-08	1.91e-09	1.75e-15	1.13e-03
16	165366	8.12e-09	1.05e-09	2.65e-16	9.06e-04

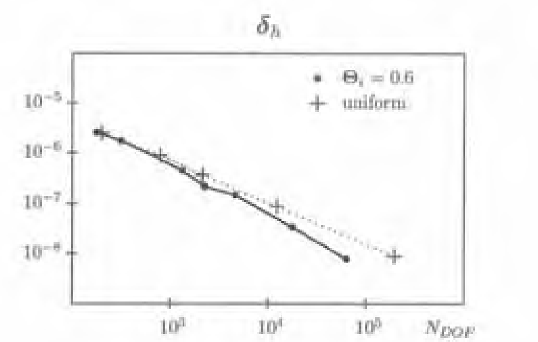

Fig. 4.6. *Example 2: Adaptive refinement (solid line) versus uniform refinement (dotted line).*

It should be noted that in both examples there is comparably less refinement than in the case of the residual-type a posteriori error estimator from [8]. This does not come as a surprise. As we noted before, the error estimation derived from the goal-oriented dual-weighted approach provides a finer estimate, since the residual-type upper bound can be derived from it by further estimation. On the other hand, the residual-type estimator from [8] has been designed for an estimation of the errors in the state, the adjoint state, the control, and the adjoint control. Therefore, a more pronounced refinement has to be expected.

Appendix A. A posteriori estimates in the L^2-norm. In this section we derive a posteriori error estimates for $\|p^* - p_h^*\|_{0,\Omega}$ and $\|y^* - y_h^*\|_{0,\Omega}$. The subsequent proof technique is based on a combination of the approaches in [8] and [17].

In what follows, we assume that Ω is convex and that $b = b_h$ a.e. in Ω, and we use $a(y,w) = (\nabla y, \nabla w)_{0,\Omega}$. Given $u_h^* \in L_h$, by $y(u_h^*), p(u_h^*) \in H_0^1(\Omega)$ we denote the solutions to

$$a(y(u_h^*), v) = (f + u_h^*, v)_{0,\Omega},$$
$$a(p(u_h^*), v) = (z - y(u_h^*), v)_{0,\Omega}$$

for all $v \in H_0^1(\Omega)$. The Poincaré–Friedrichs inequality yields

$$\|p(u_h^*) - p^*\|_{0,\Omega} \le c(\Omega)\|y(u_h^*) - y^*\|_{0,\Omega}, \tag{A.1}$$

$$\|y(u_h^*) - y^*\|_{0,\Omega} \le c(\Omega)\|u_h^* - u^*\|_{0,\Omega}, \tag{A.2}$$

where we assume that $y^* \in H_0^1(\Omega)$ satisfies $a(y^*, v) = (f + u^*, v)_{0,\Omega}$ for all $v \in H_0^1(\Omega)$, and $c(\Omega)$ is a constant depending only on the domain Ω. Hence, for $p^* \in H_0^1(\Omega)$ satisfying $a(p^*, v) = (z - y^*, v)_{0,\Omega}$ for all $v \in H_0^1(\Omega)$, we get

$$\|p^* - p_h^*\|_{0,\Omega} \le \|p(u_h^*) - p_h^*\|_{0,\Omega} + c(\Omega)^2\|u_h^* - u^*\|_{0,\Omega}. \tag{A.3}$$

Next let us assume that u^*, respectively, u_h^*, satisfies the system

$$\alpha u^* - p^* + \lambda^* = 0 \quad \text{and} \quad \alpha u_h^* - M_h p_h^* + \lambda_h^* = 0.$$

Then we obtain

$$\begin{aligned} \alpha\|u^* - u_h^*\|_{0,\Omega}^2 &\le (\lambda_h^* - \lambda^*, u^* - u_h^*)_{0,\Omega} + (p^* - p_h^*, u^* - u_h^*)_{0,\Omega} \\ &\quad + \frac{\alpha}{4}\|u^* - u_h^*\|_{0,\Omega}^2 + \frac{1}{\alpha}\|p_h^* - M_h p_h^*\|_{0,\Omega}^2 \\ &\le (p^* - p_h^*, u^* - u_h^*)_{0,\Omega} + \frac{\alpha}{4}\|u^* - u_h^*\|_{0,\Omega}^2 \\ &\quad + \frac{1}{\alpha}\|p_h^* - M_h p_h^*\|_{0,\Omega}^2 \end{aligned} \tag{A.4}$$

since $(\lambda_h^* - \lambda^*,\, u^* - u_h^*)_{0,\Omega} \le 0$. One also has

$$(p^* - p(u_h^*), u^* - u_h^*)_{0,\Omega} \le 0.$$

Hence, we have

$$\begin{aligned} (p^* - p_h^*, u^* - u_h^*)_{0,\Omega} &\le (p(u_h^*) - p_h^*, u^* - u_h^*)_{0,\Omega} \\ &\le \frac{\alpha}{4}\|u^* - u_h^*\|_{0,\Omega}^2 + \frac{1}{\alpha}\|p_h^* - p(u_h^*)\|_{0,\Omega}^2. \end{aligned}$$

This allows us to continue (A.4) as follows:

$$\|u^* - u_h^*\|_{0,\Omega}^2 \le \frac{2}{\alpha^2}\|p_h^* - p(u_h^*)\|_{0,\Omega}^2 + \frac{2}{\alpha^2}\|p_h^* - M_h p_h^*\|_{0,\Omega}^2. \tag{A.5}$$

Combining the above estimates results in

$$\begin{aligned} \|p^* - p_h^*\|_{0,\Omega} &\le \left(1 + \frac{\sqrt{2}}{\alpha}c(\Omega)^2\right)\|p_h^* - p(u_h^*)\|_{0,\Omega} \\ &\quad + \frac{\sqrt{2}}{\alpha}c(\Omega)^2\|p_h^* - M_h p_h^*\|_{0,\Omega}, \end{aligned} \tag{A.6}$$

$$\begin{aligned} \|y^* - y_h^*\|_{0,\Omega} &\le \|y(u_h^*) - y_h^*\|_{0,\Omega} + \frac{\sqrt{2}}{\alpha}c(\Omega)\big(\|p_h^* - p(u_h^*)\|_{0,\Omega} \\ &\quad + \|p_h^* - M_h p_h^*\|_{0,\Omega}\big). \end{aligned} \tag{A.7}$$

Utilizing standard L^2-estimates (see, e.g., [17, Proposition 3.8]) we infer

$$\|y(u_h^*) - y_h^*\|_{0,\Omega}^2 \le C\left(\sum_T h_T^2\eta_{y,T}^2 + \sum_F h_F^2\eta_{y,F}^2\right) =: C\tilde{\eta}_{0,y}^2, \tag{A.8}$$

$$\|p(u_h^*) - p_h^*\|_{0,\Omega}^2 \le C\left(\sum_T h_T^2\tilde{\eta}_{p,T}^2 + \sum_F h_F^2\eta_{p,F}^2\right) =: C\tilde{\eta}_{0,p}^2, \tag{A.9}$$

where the element and edge residuals are given by

$$\begin{aligned}\eta_{y,T} &:= h_T \|f + u_h^*\|_{0,T},\\ \eta_{y,F} &:= h_F^{1/2} \|n_F \cdot [\nabla y_h^*]\|_{0,F},\\ \tilde{\eta}_{p,T} &:= h_T \|z - y(u_h^*)\|_{0,T},\\ \eta_{p,F} &:= h_F^{1/2} \|n_F \cdot [\nabla p_h^*]\|_{0,F}\end{aligned}$$

with n_F denoting the exterior unit normal of T. The triangle inequality yields

$$\sum_T h_T^4 \|z - y(u_h^*)\|_{0,T}^2 \le C\, h^2 \tilde{\eta}_{0,y}^2 + 2 \sum_T h_T^2 \eta_{p,T}^2$$

with the element residual

$$\eta_{p,T} := h_T^2 \|z - y_h^*\|_{0,T}.$$

Finally, we derive the estimate

$$\begin{aligned}\|p^* - p_h^*\|_{0,\Omega} \le\ & C \left(h^2 \tilde{\eta}_{0,y}^2 + \sum_T h_T^2 \eta_{p,T}^2 + \sum_F h_F^2 \eta_{p,F}^2 \right)^{1/2}\\ & + \frac{\sqrt{2}}{\alpha} c(\Omega)^2 \|p_h^* - M_h p_h^*\|_{0,\Omega} + \mathrm{osc}_{0,h}(z) + \mathrm{osc}_{0,h}(f)\\ =:\ & C_0^p \eta_{0,p} + \mathrm{osc}_{0,h}(z) + \mathrm{osc}_{0,h}(f),\end{aligned} \tag{A.10}$$

where the data oscillations are given by

$$\begin{aligned}\mathrm{osc}_{0,h}(z) &= \left(\sum_T h_T^2\, \mathrm{osc}_T(z)^2 \right)^{1/2},\\ \mathrm{osc}_T(z) &= h_T \|z - z_h\|_{0,T}\end{aligned}$$

and analogously for $\mathrm{osc}_{0,h}(f)$.

The error in the state is estimated a posteriori by

$$\begin{aligned}\|y^* - y_h^*\|_{0,\Omega} \le\ & C \tilde{\eta}_{0,y} + \frac{\sqrt{2}}{\alpha} c(\Omega)\, (\tilde{\eta}_{0,p} + \|p_h^* - M_h p_h^*\|_{0,\Omega})\\ & + \mathrm{osc}_{0,h}(f) + \mathrm{osc}_{0,h}(z)\\ =:\ & C_0^y \eta_{0,y} + \mathrm{osc}_{0,h}(f) + \mathrm{osc}_{0,h}(z).\end{aligned} \tag{A.11}$$

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