Growth of Jacobi Fields and Divergence of Geodesics

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Introduction

A point p of a connected, complete riemannian manifold M is called a pole if the conjugate locus of p is empty. When the manifold is simply connected, this is equivalent to saying that no two distinct geodesics beginning at p meet again. In general the distance function between two such geodesics c_1, c_2 can be quite arbitrary. It can grow to infinity on, for example, a paraboloid of revolution or become arbitrarily small on a surface of revolution which looks like a sphere with a contracting tube attached. However, because of its importance in determining stability and instability properties of the geodesic flow, much research has been devoted to investigating the behavior of $d(c_1(t), c_2(t))$ as $t \to \infty$, usually under the assumption that the manifold have no conjugate points, i.e., that every point be a pole. See, for example [14] and [10]. The most general result for surfaces was proved by Green in [5]. He showed that on any simply connected surface without conjugate points and with Gaussian curvature bounded from below, the distance between any two geodesics rays going out from the same point diverges to infinity. In [6], he tried to extend this to higher dimensions but his proof was incomplete and the problem has not yet been completely solved.

In this paper we are mainly interested in the behavior of the distance function between geodesic rays which go out from a pole of a simply connected manifold. Following Eberlein, we first study the infinitesimal problem of finding estimates for the growth of all Jacobi fields which vanish at the initial points of the rays. Such Jacobi fields can be considered simultaneously by working with solutions of a $\binom{1}{1}$ tensor differential equation along the geodesics. We call such solutions Jacobi tensors. Jacobi tensors from which all solutions of the Jacobi equation can be derived are of particular interest. Such tensors exist globally on a geodesic ray c if and only if c is a proper subset of a ray without conjugate points, and one of the most important examples is the so called stable Jacobi tensor (Proposition 3). The behavior of this tensor as one varies c is closely related to the growth of the Jacobi tensors which vanish at the initial point of c (Theorem 1, Proposition 4).

Eberlein was the first to notice this close relationship. In Remark 2.10 of [2] he sketched a proof of Part (i) of our Theorem 1. Our proof differs somewhat from his.

The hypothesis that the sectional curvatures of a manifold be bounded from below leads only to information about the growth of Jacobi fields along individual geodesic rays but without uniform information about the Jacobi fields which vanish at a pole one cannot integrate to investigate the distance function between two rays. However, in a large number of cases one can find such uniform information and Green's assertion about divergence of rays then holds (Proposition 6, Corollary 1, ff). On the other hand, without uniform information, one can show that the area of certain hypersurfaces of a simply connected manifold which are far away from a pole must be very large (Theorem 2) and as a corollary of this one gets the theorem of Green for surfaces.

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1. The Jacobi Equation in Tensor Form

1.1. Endomorphisms of Vector Spaces

We begin by stating some general facts about endomorphisms of a finite dimensional vector space E endowed with an inner product $\langle\ ,\ \rangle$. Let T be such an endomorphism. Then $\|T\|$, the *norm* of T is defined by

$$\begin{aligned}\|T\| &= \operatorname{Max}\{\langle Tx, Tx\rangle^{\frac{1}{2}} : \langle x, x\rangle = 1\}\\ &= \operatorname{Max}\{\|Tx\| : \|x\| = 1\}.\end{aligned}$$

The *lower norm* of T, $((T))$, is defined by

$$((T)) = \operatorname{Min}\{\|Tx\| : \|x\| = 1\}.$$

If T is invertible, it is easy to show that $((T)) = \|T^{-1}\|^{-1}$ and if T is not invertible, $((T)) = 0$. For a product of two endomorphisms T_1, T_2, the lower norm satisfies the opposite inequality to the usual norm, i.e.

$$((T_1 T_2)) \geqq ((T_1))\,((T_2)).$$

When T is symmetric,

$$\begin{aligned}\|T\| &= \operatorname{Max}\{|\langle Tx, x\rangle| : \|x\| = 1\}\\ &= \operatorname{Max}\{|\lambda| : \lambda \text{ is an eigenvalue of } T\}\end{aligned}$$

and a similar statement, with Max replaced by Min, holds for $((T))$.

One can define an order relation $\geqq$ on the set of symmetric endomorphisms of E by writing $T_1 \geqq T_2$ whenever $T_1 - T_2$ is a positive semi-definite endomorphism, i.e., whenever $\langle T_1 x, x\rangle \geqq \langle T_2 x, x\rangle$ for all unit vectors $x \in E$. A sequence $\{T_n\}$ of symmetric endomorphisms is said to be increasing iff $T_{n+1} \geqq T_n$ for all n, and bounded above iff there exists a symmetric endomorphism T such that $T \geqq T_n$ for all n.

Since any symmetric endomorphism T is determined by $\{\langle Tx, x\rangle: \|x\|=1\}$, it follows easily that an increasing sequence of symmetric endomorphisms of E which is bounded above converges to a symmetric endomorphism.

1.2. $\binom{1}{1}$ *Tensor Fields along Geodesic*

Now let M be a riemannian manifold of dimension $m+1$, let $c\colon [a, b]\to M$ be a geodesic and let Nc be the normal bundle of c, i.e.

$$Nc=\{x\in T_{c(t)}M\colon x\perp c'(t),\ a\leqq t\leqq b\}.$$

A *Fermi frame field* along c consists of m parallel vector fields $E_1, E_2, \ldots, E_m$ along c such that, for each t, $\{E_1(t), \ldots, E_m(t), c'(t)\}$ is an orthonormal basis for $T_{c(t)}M$. By a $\binom{1}{1}$ *tensor field* Z along c we will mean a continuous mapping $Z\colon t\to Z(t)$ for $a\leqq t\leqq b$, where for each t, $Z(t)$ is an endomorphism of N_tc, the fiber of Nc over $c(t)$. The tensor field Z can be described in terms of its coefficients $(Z^i{}_j)$ with respect to some Fermi frame field $\mathscr{E}$ along c and, if Z is C^1, the covariant derivative of Z is the $\binom{1}{1}$ tensor field Z' along c whose coefficients with respect to $\mathscr{E}$ are the ordinary time derivatives of the coefficients of Z. All the rules for ordinary differentiation of time dependent matrices hold, therefore, for the covariant differentiation of $\binom{1}{1}$ tensor fields along c.

We can also obtain other $\binom{1}{1}$ tensor fields along c from Z by means of integration. For example, if $t_0\in[a, b]$, there is a $\binom{1}{1}$ tensor field X along c whose coefficients with respect to $\mathscr{E}$ are given by

$$X^i{}_j(t)=\int_{t_0}^{t} Z^i{}_j(t)\,dt.$$

For this tensor field X we write

$$X(t)=\int_{t_0}^{t} Z(t)\,dt.$$

If Z is a symmetric tensor field, then $\int Z(t)\,dt$ is also symmetric and

$$\begin{aligned}\left\|\int Z(t)\,dt\right\| &= \operatorname{Max}\{|\langle\textstyle\int Z(t)\,x\,dt, x\rangle|\colon \|x\|=1\}\\ &= \operatorname{Max}\{|\textstyle\int\langle Z(t)\,x, x\rangle\,dt|\colon \|x\|=1\}\\ &\leqq \textstyle\int \operatorname{Max}\{|\langle Z(t)\,x, x\rangle|\colon \|x\|=1\}\,dt\\ &= \textstyle\int \|Z(t)\|\,dt.\end{aligned}$$

If Z is positive definite and symmetric, the lower norm satisfies

$$((\textstyle\int Z(t)\,dt))\geqq \textstyle\int((Z(t)))\,dt.$$

1.3. Tangent Bundle and Jacobi Fields

Let M be a connected complete riemannian manifold, let TM be its tangent bundle and let $\pi\colon TM \to M$ be the projection map. TTM, the tangent bundle of TM, decomposes naturally under the Levi-Civita connection into the direct sum $\mathscr{H} \oplus \mathscr{V}$ of a horizontal subbundle $\mathscr{H}$ and a vertical subbundle $\mathscr{V}$. At each point $x \in TM$, the vertical part $\mathscr{V}_x$ consists of all vectors which are tangent to the fiber $T_{\pi x}M$. $\mathscr{V}$ is therefore the kernel of the mapping $\pi_*\colon TTM \to TM$. The horizontal part at x, $\mathscr{H}_x$, consists of all vectors which are tangent at x to horizontal curves in TM that begin at x. Such curves are obtained by parallel translation of x along smooth curves in M. The horizontal subspace $\mathscr{H}$ is the kernel of the so called *connection map* $K\colon TTM \to TM$. If a vector $w \in TTM$ is the initial tangent vector to a curve $X(t) \in TM$ and $\pi_* w \neq 0$, then

$$Kw = K\left(\frac{dX}{dt}(0)\right) = \frac{D}{dt} X(0).$$

At each x, π_* is a linear isomorphism of the horizontal subspace $\mathscr{H}_x$ onto $T_{\pi x}M$ and K is a linear isomorphism of the vertical subspace $\mathscr{V}_x$ onto $T_{\pi x}M$. In view of this we will sometimes identify, via these mappings, the horizontal and vertical subspaces at x with $T_{\pi x}M$, and then we will write $w_H = \pi_* w$, $w_V = Kw$ for $w \in T_x TM$. The mapping $\pi_* \oplus K\colon TTM \to \pi^*(TM \oplus TM)$ where $\pi_* \oplus K(w) = (\pi_* w, Kw)$ maps TTM diffeomorphically onto $\pi^*(TM \oplus TM)$, the pullback via π of $TM \oplus TM$ to a bundle over TM. For a more complete description of the connection map, see Section 2.4 of [8]. The riemannian metric on M gives rise, via the mappings π_* and K, to a riemannian metric on TM known as the *Sasaki metric*. In this metric the inner product of two vectors $v, w \in T_x TM$ is given by

$$\langle v, w\rangle = \langle \pi_* v, \pi_* w\rangle + \langle Kv, Kw\rangle.$$

We will also use the term Sasaki metric to refer to the restriction of the above metric to SM, the unit tangent bundle of M. All horizontal lifts to TM of geodesics of M are geodesics of TM relative to the Sasaki metric. In particular, if c is a constant speed geodesic of M, then c' is a geodesic of TM. See [16] and [17].

The *geodesic flow* of M is a one parameter family $\{\phi_t\}_{t \in \mathbb{R}}$ of diffeomorphisms of TM where, for $x \in TM$, $\phi_t x = c'(t)$, c being the constant speed geodesic whose initial velocity vector is x. A *Jacobi field* Y along a geodesic c of M is a vector field that solves the differential equation

$$\nabla_{c'} \nabla_{c'} Y + R(Y, c')\, c' = 0$$

where R is the curvature tensor of M. It is well known that Jacobi fields along c are the variation vector fields of variations of c through geodesics. There is therefore a close relationship between Jacobi fields and the differential of the geodesic flow. Indeed, if $x \in TM$ and $w \in T_x TM$, then

$$Y_w(t) = \pi_* \phi_{t*} w$$

is a Jacobi field along the constant speed geodesic $c = c_x$ determined by x and the covariant derivative of Y_w along c is given by

$$\nabla_{c'} Y_w(t) = K\phi_{t*} w.$$

The mapping $w \mapsto Y_w$ is a linear isomorphism between $T_x TM$ and the space of Jacobi fields along c_x. In what follows we will be interested only in Jacobi fields which are perpendicular to unit speed geodesics c. These Jacobi fields correspond under the isomorphism described above to vectors w which are tangent at $c'(0)$ to the unit tangent bundle of M and which are perpendicular (in the Sasaki metric) to the vector v whose horizontal and vertical components are given by $v_H = c'(0)$, $v_V = 0$. If Nc' is the normal bundle of the geodesic c' of SM, then the set of all such w is the fiber, $N_0 c'$, of Nc' over $c'(0)$. See Proposition 1.7 of [2].

The riemannian metric on M gives rise to an isomorphism between TM and T^*M, the cotangent bundle of M. Under this isomorphism an element $x \in T_p M$ is mapped to the unique element $\theta_x \in T_p^* M$ which satisfies $\theta_x(y) = \langle x, y \rangle$ for all $y \in T_p M$. The canonical symplectic structure on T^*M pulls back, via this isomorphism, to a symplectic form ω on TM. For $v, w \in T_x TM$, we have

$$\omega(v, w) = \langle \pi_* v, Kw \rangle - \langle Kv, \pi_* w \rangle.$$

The symplectic form ω is invariant under the geodesic flow. To see this let Y_v and Y_w be the Jacobi fields along c_x associated to v and w. Then

$$\omega(\phi_{t*} v, \phi_{t*} w) = \langle Y_v(t), Y_w'(t) \rangle - \langle Y_v'(t), Y_w(t) \rangle$$

and it follows easily that the derivative of the right hand side with respect to t vanishes.

1.4. Jacobi Tensors

When $\dim M = m+1 > 2$, it is often more advantageous to work with m-dimensional spaces of perpendicular Jacobi fields along a unit speed geodesic c, rather than to work with individual ones. This leads naturally to the concept of a Jacobi tensor. Again let Nc be the normal bundle of c, let Nc' be the normal bundle of c' regarded as a geodesic of SM, let $f\colon N_0 c \to N_0 c'$ be any injective linear mapping and let $P_t\colon N_0 c \to N_t c$ denote parallel translation from $c(0)$ to $c(t)$. Then we define Z_f, *the Jacobi tensor associated to f*, to be the $\binom{1}{1}$ tensor field along c which makes the following diagram commutative:

$$\begin{array}{ccccc}
N_0 c' & & \xrightarrow{\phi_{t*}} & & N_t c' \\
\Big\uparrow{\scriptstyle f} & & & & \Big\downarrow{\scriptstyle \pi_*} \\
N_0 c & \xleftarrow[P_t^{-1}]{} & N_t c & \xrightarrow[Z_f]{} & N_t c.
\end{array}$$

When there is no danger of confusion we will identify $N_0 c$ and $N_t c$ by means of P_t. With this identification in mind, we can say that, for each $x \in N_0 c$, $Y_x(t) = Z_f(t)\, x$ is the Jacobi field along c which satisfies $Y_x(0) = \pi_* f(x)$, $Y_x'(0) = Kf(x)$. The riemannian curvature tensor gives rise to a $\binom{1}{1}$ tensor field $R = R(t)$ along c where $R(t)\, y = R(y, c'(t))\, c'(t)$ for all $y \in N_t c$. Since $Y_x = Z_f x$ is a Jacobi field for each

$x \in N_0 c$, we can now conclude easily that Z_f is a solution of the following covariant differential equation of tensor fields along c:

$$Z'' + RZ = 0. \tag{1}$$

Conversely, given a non-degenerate solution Z of (1), i.e. a Jacobi tensor Z along c which satisfies rank $(Z(0), Z'(0)) = m$, then we can find an injective homomorphism $f\colon N_0 c \to N_0 c'$ such that $Z = Z_f$. Indeed if $x \in N_0 c$, $f(x)$ is the unique element of $N_0 c'$ for which $\pi_* \phi_{t*} f(x) = Z(t)\, x$ holds. We sum up with the following

Proposition 1. *Let c be a unit speed geodesic segment of M. Then the non-degenerate solutions of the covariant tensor differential Equation* (1) *and the Jacobi tensors along c are identical and they are in* 1:1 *correspondence with the injective homomorphisms $f\colon N_0 c \to N_0 c'$.*

Suppose now that Z_1 and Z_2 are two Jacobi tensors along c. The tensor

$$W(Z_1, Z_2) = Z_1'^* Z_2 - Z_1^* Z_2'$$

is called the Wronskian of Z_1 and Z_2. Since the symplectic form ω is invariant on TM under the geodesic flow, it follows that the covariant derivative of $W(Z_1, Z_2)$ along c vanishes and therefore $W(Z_1, Z_2)$ is a constant tensor. A Jacobi tensor $Z = Z_f$ along c is called a *Lagrange tensor* iff $W(Z, Z) = 0$. It is easy to see that Z_f is a Lagrange tensor if and only if $f(N_0 c)$ is a Lagrange subspace of $T_{c'(0)} TM$, i.e. a subspace on which the symplectic form ω vanishes.

Suppose Z is a Jacobi tensor along c which is non singular for t in some interval $[a, b]$. Then if Z_1 is any other Jacobi tensor along c, there exists a tensor field X so that

$$Z_1(t) = Z(t)\, X(t) \qquad \text{for } a \leqq t \leqq b.$$

Further, X satisfies the differential equation

$$X'' + 2 Z^{-1} Z' X' = 0.$$

The tensor field X' satisfies, therefore, a first order differential equation and when Z is a Lagrange tensor one can check by differentiation that the solutions of this equation take the form

$$X'(t) = C (Z^* Z)^{-1}(t)$$

where C is some constant tensor. Consequently, for $a \leqq t \leqq b$

$$X(t) = C_1 \int_{t_0}^{t} (Z^* Z)^{-1}(u)\, du + C_2$$

where $t_0 \in [a, b]$ can be chosen arbitrarily and C_1 and C_2 are constant $\binom{1}{1}$ tensors determined by the initial conditions on Z_1. We have now established the following

Proposition 2. *Let Z be a Lagrange tensor along a geodesic c and suppose that Z is everywhere non-singular on some interval J. Then if Z_1 is any other Jacobi tensor* *along c, it follows that,*

$$Z_1(t)=Z(t)\left[C_1\int_{t_0}^{t}(Z^*Z)^{-1}(u)\,du+C_2\right]$$

for all $t\in J$, where $t_0\in J$ is arbitrary and C_1 and C_2 are constant $\binom{1}{1}$ tensors along c.

The initial covariant derivative $Z'(0)$ of any Lagrange tensor Z along c which satisfies $Z(0)=I$ is symmetric since $W(Z,Z)=0$. Let Z_1, Z_2 be two such tensors with $Z_1'(0)\leqq Z_2'(0)$ and suppose that Z_1 is everywhere non-singular on an interval $[0,b]$. It follows from the previous proposition that for $t\in[0,b]$

$$Z_2(t)=Z_1(t)\left[C\int_0^t(Z_1^*Z_1)^{-1}(u)\,du+I\right]$$

where $C=Z_2'(0)-Z_1'(0)$ is positive definite and symmetric. Consequently Z_2 is also non-singular at every point of $[0,b]$. Equivalently, the first singular point of Z_1 must occur before the first singular point of Z_2.

2. Stable Jacobi Fields

Again let M be a connected, complete riemannian manifold of dimension $m+1$ and let $c\colon [0,\alpha)\to M$ be a unit speed geodesic ray without conjugate points, i.e. no non-trivial Jacobi field along c vanishes twice. Then the Jacobi tensor A along c which satisfies $A(0)=0$, $A'(0)=I$, where I is the identity endomorphism of N_0c, is a Lagrange tensor and $A(t)$ is non-singular for all $t>0$. Conversely, the existence of a Jacobi tensor along a geodesic ray which vanishes initially and is non-singular for $t>0$ implies, via the Sturm separation theorem, that no two points of that ray are conjugate. This is also clear by Proposition 2. Since there are no conjugate points on c, there exists, for each $s>0$, a unique Jacobi tensor D_s along c satisfying $D_s(0)=I$, $D_s(s)=0$. For $t>0$, it follows from Proposition 2 that

$$D_s(t)=A(t)\left[C_1\int_s^t(A^*A)^{-1}(u)\,du+C_2\right].$$

From $D_s(s)=0$ we obtain $C_2=0$ and a calculation of the Wronskian $W(A,D_s)$ at $t=0$ and $t=s$ yields $C_1=-I$. Therefore

$$D_s(t)=A(t)\int_t^s(A^*A)^{-1}(u)\,du. \tag{2}$$

From the theory of differential equations it follows that as $s\to\infty$, D_s converges to a Jacobi tensor D called the *stable Jacobi tensor* along c, if and only if $\lim D_s'(0)$ exists. The following proposition gives a purely geometric answer to the question of when this limit exists. Compare Green [7] and Hopf [11].

Proposition 3. *If c is a geodesic ray, the following are equivalent.*

(i) *There exists an everywhere non-singular Lagrange tensor along c.*

(ii) *c is a proper subset of a geodesic ray k without conjugate points.*

(iii) *The stable Jacobi tensor D exists along c.*

Proof. (i)$\Rightarrow$(ii). Let Z be such a Lagrange tensor with $Z(0)=I$. There exists $\alpha>0$ such that we can extend c to a unit speed geodesic $k\colon [-\alpha, \infty]\to M$ and Z to an everywhere non-singular Lagrange tensor along k. From Proposition 2

$$A_{-\alpha}(t)=Z(t)\int_{-\alpha}^{t}(Z^*Z)^{-1}(u)\,du$$

defines a Jacobi tensor along k which vanishes at $t=-\alpha$ and is non-singular for $t>-\alpha$. Therefore, k has no conjugate points.

(ii)$\Rightarrow$(iii). Suppose c can be extended to a unit speed geodesic ray $k\colon[-\alpha,\infty)\to M$ such that no two points of k are conjugate. Then if $s\geqq-\alpha$ and $s\neq 0$, there exists a unique Jacobi tensor D_s along k satisfying $D_s(0)=I$, $D_s(s)=0$. Since $W(D_s, D_s)=0$, it follows that $D_s'(0)$ is a symmetric endomorphism of $N_0k=N_0c$. For $0<r<s$, it follows from (2) that

$$D_s'(0)-D_r'(0)=\int_r^s(A^*A)^{-1}(u)\,du$$

and therefore, since A^*A is everywhere positive definite, $D_s'(0)\geqq D_r'(0)$. Now let $x\in N_0c$ and consider the broken Jacobi field Y where

$$Y(t)=\begin{cases}D_{-\alpha}(t)\,x & \text{for } -\alpha\leqq t\leqq 0\\ D_s(t)\,x & \text{for } 0\leqq t\leqq s.\end{cases}$$

Let $I_{-\alpha}^s$ be the index form on $k|_{[-\alpha,s]}$. Since there are no conjugate points on k, the index form is positive definite on piecewise smooth vector fields along $k|_{[-\alpha,s]}$ which vanish at $k(-\alpha)$ and $k(s)$. See [8], pp. 142–145. Therefore

$$0<I_{-\alpha}^s(Y,Y)=\langle D_{-\alpha}'(0)\,x,x\rangle-\langle D_s'(0)\,x,x\rangle$$

and so $D_{-\alpha}'(0)\geqq D_s'(0)$. We have now shown that the sequence $\{D_s'(0)\colon s>0\}$ of symmetric endomorphisms of N_0c increases with s and is bounded above by $D_{-\alpha}'(0)$. Therefore by Section 1.1, $\lim_{s\to\infty} D_s'(0)$ exists and hence, as $s\to\infty$, D_s converges to the Jacobi tensor D along c which satisfies $D(0)=I$, $D'(0)=\lim_{s\to\infty} D_s'(0)$.

(iii)$\Rightarrow$(i). If D exists, it follows from (2) that

$$D(t)=A(t)\int_t^{\infty}(A^*A)^{-1}(u)\,du \tag{3}$$

and therefore $D(t)$ is non-singular for all $t\geqq 0$. Since $W(D_s, D_s)=0$ for all $s>0$, D must be a Lagrange tensor.

Since the initial covariant derivative $Z'(0)$ of a Lagrange tensor Z which is initially equal to the identity is symmetric, it follows that we can define an order relation on the set of all such Lagrange tensors by writing $Z_1\leqq Z_2$ whenever

$Z_1'(0) \leqq Z_2'(0)$. When the stable Jacobi tensor D along c exists, it is characterized by the following fact.

Proposition 3′. *D is the smallest everywhere non-singular Lagrange tensor which is initially equal to the identity.*

Proof. We must show that if Z is any everywhere non-singular Lagrange tensor along c with $Z(0)=I$, then the symmetric endomorphism $C=Z'(0)-D'(0)$ is positive semi-definite. For $t \geqq 0$ we have

$$Z(t)=D(t)+A(t)\,C=A(t)\left[\int_t^{\infty}(A^*A)^{-1}(u)\,du+C\right]$$

$$=A(t)\,[X(t)+C], \text{ say.}$$

By Section 1.1 it follows (modulo parallel translation) that, for each unit vector $y \in N_0 c$,

$$\langle X(t)\,y, y\rangle \geqq ((X(t))) \geqq \int_t^{\infty} \|A(u)\|^{-2}\,du.$$

Since $A(0)=0$, $A'(0)=I$, it follows that for u in some neighborhood of 0, $\|A(u)\| \leqq 2u$ and, consequently, that $\|A(u)\|^{-2} \geqq 1/4u^2$. Therefore $\langle X(t)\,y, y\rangle \to \infty$ as $t \to 0$ and so the symmetric endomorphism $X(t)+C$ is positive definite for all sufficiently small t. Now, if C had a negative eigenvalue, there would exist a unit vector $x \in N_0 c$ such that $\langle (X(t)+C)\,x, x\rangle < 0$ for all sufficiently large t, because $\lim_{t\to\infty} X(t)=0$. But then, for some $t_0>0$, $X(t_0)+C$ would have a zero eigenvalue and therefore zero determinant. This would contradict the fact that $Z(t)$ is non-singular for all $t \geqq 0$. Consequently $C=Z'(0)-D'(0)$ is positive semi-definite and the proposition is established.

For each Jacobi tensor Z along c which is non-singular on some interval J, the tensor $U=Z'Z^{-1}$ along c is a solution of the Riccati equation

$$U'+U^2+R=0. \tag{4}$$

If the sectional curvatures of all two planes which contain vectors tangent to c are bounded below by $-r^2$ for some $r>0$, i.e., if the symmetric tensor field R along c satisfies $R(t) \geqq -r^2 I$ for all $t \geqq 0$, then Sturm comparison techniques can be used to find upper and lower bounds for symmetric solutions of (4). In fact, a symmetric solution U which is defined for $t>0$ satisfies

$$-rI \leqq U(t) \leqq r \coth rt.\,I$$

and therefore

$$\begin{aligned} \|U(t)\| &\leqq r \coth rt \quad \text{for all } t>0 \\ &\leqq 2r \quad \text{for } t \geqq T=(1/r)\operatorname{arcoth} 2. \end{aligned} \tag{5}$$

In particular, subject to the curvature condition, the tensors $A'A^{-1}$ and $D'D^{-1}$ satisfy (5). If we set

$$X(t)=\int_t^{\infty}(A^*A)^{-1}(u)\,du,$$

then $D(t)=A(t)\,X(t)$ and, using the fact that $W(A,D)=I$, we obtain easily that

$$A'A^{-1}-D'D^{-1}=(A^{-1})^*\,X^{-1}A^{-1}.$$

Therefore, for $t\geqq T$ and $x\in N_0c$ (recall that N_0c and N_tc are identified via parallel translation), we have

$$|\langle (A^{-1})^*\,X^{-1}A^{-1}x,x\rangle|\leqq 4r.$$

On the other hand

$$\begin{aligned}|\langle (A^{-1})^*\,X^{-1}A^{-1}x,x\rangle| &= |\langle X^{-1}A^{-1}x, A^{-1}x\rangle|\\ &\geqq ((X^{-1}))\,\|A^{-1}x\|^2=\|X\|^{-1}\cdot\|A^{-1}x\|^2\end{aligned}$$

by Section 1.1, since X is symmetric. Consequently,

$$\|A^{-1}(t)\|\leqq (4r\,\|X(t)\|)^{1/2}\qquad \text{for } t\geqq T. \tag{6}$$

For a detailed proof, see Proposition 2.9 of [2].

Now let K be any compact subset of SM, the unit tangent bundle of M, and suppose that for each $z\in K$ the stable Jacobi tensor D_z exists along the geodesic ray c_z whose initial tangent vector is z. Then c_z certainly has no conjugate points and therefore the Jacobi tensor A_z along c_z satisfying $A_z(0)=0$, $A_z'(0)=I$, *is* non-singular if $t>0$. For $t>0$, set

$$\phi_z(t)=((A_z(t)))=\|A_z^{-1}(t)\|^{-1}. \tag{7}$$

Any Jacobi field Y_z along c_z satisfying $Y_z(0)=0$, $\|\nabla Y_z(0)\|=1$ is of the form $Y_z(t)=A_z(t)\,x$ for some $x\in N_0c$, $\|x\|=1$. Therefore, it follows from Section 1.1 that $\phi_z(t)=\inf\|Y_z(t)\|$ over all such Jacobi fields Y_z. Let

$$\phi_K(t)=\inf\{\phi_z(t)\colon z\in K\}.$$

For each $z\in K$ the stable Jacobi tensor D_z determines an m-dimensional space of Jacobi fields along c_z which are also called stable. The limit of a convergent sequence of such stable Jacobi fields is again a stable Jacobi field if and only if the mapping $z\to D_z'(0)$, $t\in K$, is continuous. Eberlein was the first to notice that there is a close relationship between the continuity of this mapping and the growth of the function ϕ_K when the sectional curvatures of M are bounded from below. In Remark 2.10 of [2] he states that $\lim\limits_{t\to\infty}\phi_K(t)=\infty$ for a compact subset K of the unit tangent bundle of a manifold without conjugate points. This fact also follows immediately from Part (i) of the following theorem.

Theorem 1. *Let M be a complete riemannian manifold whose sectional curvatures are bounded from below by $-r^2$ for some $r>0$. Let K be a compact subset of SM and suppose that for each vector $z\in K$ the stable Jacobi tensor D_z exists along the geodesic ray c_z whose initial tangent vector is z. Then the following is true.*

(i) *If the mapping $z\to D_z'(0)$, $z\in K$, is continuous, there exists a continuous function f which increases monotonically to infinity such that $\phi_K(t)\geqq f(t)$ for all $t\geqq 0$.*

(ii) *If $1/\phi_K\in L^2\,[1,\infty)$, then the mapping $z\to D_z'(0)$ is continuous.*

Proof of (i). Clearly the mapping $(t, z) \to A_z^{-1}(t)$ is continuous on $(0, \infty) \times K$ and so is the mapping $(t, z) \to D_z(t)$ since we have assumed that $z \to D_z'(0)$ is continuous on K. Therefore, since

$$X_z(t) = \int_t^\infty (A_z^* A_z)^{-1}(u)\, du = A_z^{-1}(t)\, D_z(t),$$

it follows that the mapping σ: $(t, z) \to (4r \| X_z(t) \|)^{-1/2}$ is continuous on $(0, \infty) \times K$. For each fixed $z \in K$, the function σ_z defined by $\sigma_z(t) = \sigma(t, z)$ is continuous and strictly monotonically increasing in t. Since $\lim_{t \to 0^+} \| X_z(t) \|^{-1} = 0$ and $\lim_{t \to \infty} \| X_z(t) \|^{-1} = \infty$, it follows that σ_z has a continuous inverse which is strictly monotonically increasing and which maps $(0, \infty)$ onto $(0, \infty)$. Thus we obtain a continuous function τ: $(0, \infty) \times K \to (0, \infty)$ defined by $\tau(t, z) = \sigma_z^{-1}(t)$. Set $\alpha(t) = \max_{z \in K} \tau(t, z)$ and $\beta(t) = \min_{z \in K} \sigma(t, z)$. Then β increases monotonically, is continuous, and takes on arbitrarily large values because $\sigma_z(\alpha(t)) \geqq t$ for all $z \in K$. Set $f(t) = \beta(t)$ for $t \geqq T = (1/r)\operatorname{arcoth} 2$ and extend f to a continuous monotone increasing function on $[0, \infty)$ such that $f(t) \leqq \| A_z^{-1}(t) \|^{-1}$ for $0 \leqq t \leqq T$ and for all $z \in K$. Part (i) of the theorem now follows from (6) and (7).

Proof of (ii). For each $z \in K$ let D_{z1} be the Jacobi tensor along c_z which satisfies $D_{z1}(0) = I$, $D_{z1}(1) = 0$. Since the mapping $z \to D_{z1}'(0)$ is continuous, we are done if we can show that the mapping $z \to D_z'(0) - D_{z1}'(0)$ is continuous. From (2) and (3) we have

$$D_z'(0) - D_{z1}'(0) = \int_1^\infty (A_z^* A_z)^{-1}(u)\, du$$

Let $\{z_n\}$ be a sequence in K which converges to z, let $\{x_n\}$ be a sequence of unit vectors, $x_n \in N_0(c_{z_n})$, which converges to a unit vector $x \in N_0(c_z)$, and set $A_n = A_{z_n}$. Then $\langle (A_n^* A_n)^{-1}(u)\, x_n, x_n \rangle$ converges to $\langle (A_z^* A_z)^{-1}(u)\, x, x \rangle$ for each $u \geqq 1$. By assumption $\| A_z^{-1}(u) \| \leqq 1/\phi_K(u)$ for all $z \in K$, and so it follows that

$$\int_1^\infty \langle (A_n^* A_n)^{-1}(u)\, x_n, x_n \rangle\, du \leqq \int_1^\infty (1/\phi_K^2(u))\, du.$$

Therefore by the Lebesgue convergence theorem for integrals,

$$\lim_{n \to \infty} \int_1^\infty \langle (A_n^* A_n)^{-1}(u)\, x_n, x_n \rangle\, du = \int_1^\infty \langle (A_z^* A_z)^{-1}(u)\, x, x \rangle\, du$$

and, since $(A_z^* A_z)^{-1}$ is symmetric for all $z \in K$, it follows that the mapping $z \to D_z'(0) - D_{z1}'(0)$ is continuous. This completes the proof.

In the next proposition we obtain a more explicit lower bound for ϕ_K in the case where the norms of the stable Jacobi tensors along geodesic rays going out from K are uniformly bounded.

Proposition 4. *Let M and K be as in the previous theorem. Suppose there exists a constant $\rho > 0$ such that $\| D_z(t) \| \leqq \rho$ for all $z \in K$ and for all $t \geqq 0$. Then the mapping $z \to D_z'(0)$ is continuous and, further, there exists a number $T > 0$ such that $\phi_K(t) \geqq$ (const.) $t^{\frac{1}{2}}$ for $t \geqq T$.*

Proof. As before we have, for each $z \in K$,

$$X_z(t) = \int_t^\infty (A_z^* A_z)^{-1}(u)\,du = A_z^{-1}(t)\,D_z(t).$$

Since the stable Jacobi tensor D_z is a Lagrange tensor, we can use Proposition 2 and the initial conditions on A_z and D_z to get

$$A_z(t) = D_z(t) \int_0^t (D_z^* D_z)^{-1}(u)\,du$$

Using the methods of Sections 1.1 and 1.2, we obtain from the latter equation and the fact that $\|D_z(t)\| \leqq \rho$

$$\|X_z(t)\|^{-1} = ((X_z^{-1}(t))) = ((D_z^{-1}(t)\,A_z(t))) \geqq \rho^{-2}\,t.$$

Since the above inequality does not depend on z, it now follows from (6) and (7) that

$$\phi_K(t) \geqq (4r\rho^2)^{-\frac{1}{2}}\,t^{\frac{1}{2}} \qquad \text{for } t \geqq (1/r)\operatorname{arcoth} 2.$$

For $z \in K$ and $s > 0$, let D_{zs} be the Jacobi tensor along the geodesic ray determined by z which satisfies $D_{zs}(0) = I$, $D_{zs}(s) = 0$. From (2) and (3) we have

$$\|D_z'(0) - D_{zs}'(0)\| \leqq \|X_z(s)\| \leqq \rho^2/s.$$

Since the tensors $D_{zs}'(0)$ depend continuously (in fact, smoothly) on z, it now follows easily that the mapping $z \to D_z'(0)$ is continuous. This completes the proof.

An immediate consequence of this proposition is that if the norms of the stable Jacobi tensors are uniformly bounded on a complete manifold without conjugate points whose sectional curvatures are bounded from below, then there exist positive numbers α and T such that, for $t \geqq T$, the inequality

$$\|Y(t)\| \geqq \alpha\, t^{\frac{1}{2}}$$

is satisfied uniformly by all Jacobi fields which vanish initially and have initial covariant derivative of length 1. An interesting class of riemannian manifolds for which $\|D(t)\| \leqq 1$ for all stable Jacobi tensors D is the class of manifolds without focal points. These manifolds are characterized by the fact that the length of any Jacobi field which vanishes initially is an increasing function. See Proposition 4 of [15]. Berger recently showed, without any curvature assumptions, that, on surfaces without focal points $Y(t) \geqq \frac{1}{2} t$ for all Jacobi fields Y which satisfy $Y(0) = 0$, $\|\nabla Y(0)\| = 1$. See Section 3 of [1].

3. Divergence of Geodesics

Throughout this section M will be a connected, complete, simply connected riemannian manifold and we will be mainly interested in the behavior of the geodesic rays going out from a pole p of M. We recall that a pole of a riemannian manifold is a point whose conjugate locus is empty. On M this is equivalent to saying that no two distinct geodesics rays going out from p meet again. In [6] Green introduced the following concept of *uniform divergence of* geodesic rays. A sequence $\{p_n\}$ of points of M is said to be divergent it it has no limit point and a geodesic ray going out from p is said to be a *limit ray* of such a sequence if the geodesic segments c_n joining p to p_n converges to c, i.e., if $c'(0)=\lim_{n\to\infty} c_n'(0)$. The geodesic rays going out from a pole p of M are said to diverge uniformly if each geodesic ray c going out from p having bounded distance from a divergent sequence $\{p_n\}$ (i.e., $d(p_n, c)$ is bounded) is a limit ray of that sequence. Uniform divergence at p clearly implies that any two geodesic rays c_1 and c_2 going out from p diverge, i.e., that $\lim_{t\to\infty} d(c_1(t), c_2(t))=\infty$. For surfaces the converse is an easy consequence of the Jordan curve theorem. In general, for manifolds of higher dimension there is no reason to believe that divergence of the geodesic rays going out from p implies their uniform divergence, although in special cases this is so, e.g., on the universal riemannian covering of a compact manifold which admits a metric of strictly negative curvature. This latter fact follows from the lemma on page 71 of [12]. The concept of uniform divergence of geodesic rays seems, when compared with divergence, to be rather complicated. As an example of it's usefulness we prove the following proposition which can not be deduced as a consequence of divergence alone.

Proposition 5. *Let M be the universal riemannian covering of a compact manifold N and let $\pi: M \to N$ be the projection map. Then if the geodesic rays going out from a point p of M are uniformly divergent, the initial tangent vectors to the (not necessarily smoothly) closed unit speed geodesic loops at p form a dense subset of $S_{\pi p}N$.*

Proof. Let F be a compact fundamental domain in M of diameter d containing the point p and let $z\in S_pM$. To prove the proposition it clearly suffices to show that we can approximate z arbitrarily closely by the initial tangent vectors of unit speed geodesics joining p to points congruent to p by deck transformations. Let $\{q_n\}$ be a divergent sequence of points lying on the geodesic ray c_z whose initial tangent vector is z. For each n there exists a deck transformation g_n such that $q_n\in g_nF$. So

$$d(g_n p, c_z) \leqq d(g_n p, q_n) \leqq d.$$

and therefore, since the geodesic rays going out from p diverge uniformly, it follows that the geodesic segments joining p to $g_n p$ converge to c_z. This completes the proof.

We now proceed to use the infinitesimal tools developed in the previous section to derive some facts about uniform divergence of geodesic rays. Our notation is the same as in Section 2. In particular, if p is a pole of M, then for $t>0$,

$$\phi_p(t)=\inf\{((A_z(t))): z\in S_pM\}=\inf \|Y(t)\|$$

where the latter infimum is taken over all Jacobi fields which vanish at p and have initial covariant derivative of length 1.

Proposition 6. *Let p be a pole of a complete, connected and simply connected riemannian manifold M. If $\lim_{t\to\infty} \phi_p(t)=\infty$, then the geodesic rays going out from p diverge uniformly.*

Proof. Let $\{p_n\}$, $\{q_n\}$ be two divergent sequences of points of M with $d(p_n, q_n)$ bounded above by some constant α. Let c_n, k_n be the unit speed geodesic segments which join p to p_n and p to q_n, respectively. We will prove the slightly stronger result that the angle $\theta_n = \langle c_n'(0), k_n'(0)\rangle$ between c_n and k_n tends to zero as $n \to \infty$. Since p is a pole, the mapping $\psi\colon S_pM \times \mathbb{R}^+ \to M - \{p\}$ where $\psi(z,t) = \exp tz$ is a diffeomorphism. If $w = (w_1, w_2) \in T_{(z,t)}(S_pM \times \mathbb{R}^+)$, it follows from the Leibniz formula that

$$\psi_* w = (\exp_*)_{tz}\, t w_1 + dt(w_2)\, c_z'(t)$$

where c_z is the geodesic ray going out from p with initial velocity vector z. For the first term on the right hand side above we get, using the canonical identification between $T_v(T_pM)$ and $T_p(M)$,

$$(\exp_*)_{tz}\, t w_1 = A_z(t) \cdot w_1 = Y(t)$$

where Y is the Jacobi field along c_z satisfying $Y(0)=0$, $\nabla Y(0) = w_1$. See [8], page 132. Moreover, by the Gauss lemma, the two terms are mutually orthogonal and so we obtain

$$\|\psi_* w\| \geqq \|Y(t)\| \geqq \phi_p(t)\, \|w_1\|. \tag{8}$$

Now let $a_n\colon [0,1] \to M$ be a minimizing geodesic segment joining p_n to q_n. If n is sufficiently large, p cannot lie on a_n and so for all such n there exist curves z_n: $[0,1] \to S_pM$ and $t_n\colon [0,1] \to \mathbb{R}^+$ such that $a_n = \psi(z_n, t_n)$. If u denotes arc length on the curve z_n, we can parametrize a_n by writing

$$a_n(u) = \psi(z_n(u), t_n(u)).$$

The parameter u will lie in some interval $[0, b_n]$ and b_n must be greater than or equal to θ_n because the minimizing geodesic on the unit sphere S_pM joining $c_n'(0)$ to $k_n'(0)$ has length θ_n. Since $z_n'(u)$ has length 1, it follows from (8) that

$$\|a_n'(u)\| \geqq \phi_p(t_n(u)).$$

Set

$$\bar{\phi}_n = \inf\{\phi_p(t_n(u))\colon u \in [0, b_n]\}.$$

From our assumptions on $\{p_n\}$ and $\{q_n\}$, it follows that $d(p, a_n) \to \infty$ as $n \to \infty$. Therefore, since $\lim\limits_{t\to\infty} \phi_p(t) = \infty$, we have $\lim\limits_{n\to\infty} \bar{\phi}_n = \infty$. From

$$\alpha \geqq L(a_n) \geqq \int_0^{b_n} \|a_n'(u)\|\, du \geqq \bar{\phi}_n \theta_n,$$

we obtain $\theta_n \leqq \alpha/\bar{\phi}_n$ and therefore $\theta_n \to 0$ as $n \to \infty$. This completes the proof.

If we combine Theorem 1 with the above proposition, we obtain immediately the following corollary:

Corollary 1. *Let p be a pole of a complete, connected, simply connected manifold M whose sectional curvatures are bounded from below. If the stable Jacobi tensors D_z exist for each $z \in S_pM$ and depend continuously on z then the geodesic rays going out from p diverge uniformly.*

From the above corollary it follows that the assertion of Green (Theorem 1 of [6]), that the geodesic rays going out from each point of a simply connected, complete riemannian manifold without conjugate points whose sectional curvatures are bounded from below diverge uniformly, is true for a large class of such manifolds; namely, those for which the stable Jacobi tensors vary continuously with the geodesic rays. In particular, uniform divergence holds, subject to the curvature condition, on simply connected manifolds without conjugate points for which $D(t) \leqq \rho$, ρ constant, for all stable Jacobi tensors D and for all $t \geqq 0$ (Proposition 4). For more details on these manifolds, see [3] and [4]. One interesting class of such manifolds, where $\|D(t)\|$ is actually $\leqq 1$, consists of the universal coverings of manifolds without focal points. By Proposition 4 and the result of Berger (3.5 of [1]), uniform divergence holds without curvature assumptions on surfaces without focal points. Another interesting class consists of universal coverings of compact manifolds with geodesic flow of Anosov type. Klingenberg [13] showed that these manifolds cannot have conjugate points. They may, however, have focal points (Gulliver [9]).

The main obstacle to proving Green's assertion in dimension higher than 2 is the following: The assumption that the sectional curvatures of a manifold without conjugate points are bounded from below gives, a priori, no uniform information about the growth of the lower norms of Jacobi tensors which vanish initially. Since the stable Jacobi tensor exists along each geodesic ray of a manifold without conjugate points, $\lim_{t \to \infty} X_z(t) = 0$, and therefore, from (6) and (7), one has $\lim_{t \to \infty} ((A_z(t))) = \infty$ for each fixed $z \in SM$. However, in order to establish uniform divergence of the geodesic rays going out from p, one would need such information uniformly for all unit tangent vectors at p. Nevertheless, with only non-uniform information at hand, one can say something about the area of hypersurfaces of a simply connected manifold M which are far away from a pole p.

Suppose M has dimension $m+1$. Since p is a pole, the exponential mapping $\exp: S_pM \times \mathbb{R}^+ \to M - \{p\}$ where $(z, t) \to \exp tz$ is a diffeomorphism. Contraction of the riemannian volume dV of M relative to the radial vector field $\partial/\partial t$ gives rise to an m-form η on $M - \{p\}$ where, if $x_1, x_2, \ldots, x_m$ are elements of TM,

$$\eta(x_1, x_2, \ldots, x_m) = dV(\partial/\partial t, x_1, x_2, \ldots, x_m).$$

Let U be an open subset of S_pM. An imbedding $\sigma: U \to M - \{p\}$ of the form $\sigma(z) = \exp t(z)\, z$ gives rise to a hypersurface Σ on M which we will call a *cap shaped hypersurface over U*. Let dV_σ be the riemannian volume element induced on Σ by the metric on M. If we restrict the form η to Σ and pull back to U via the imbedding map σ, we have the following inequality:

$$\sigma^* dV_\sigma \geqq |\sigma^* \eta|.$$

For $z \in U$, $w \in T_zU$, the Leibniz formula yields

$$(\sigma_*)_z\, w = (\exp_*)_{t(z)\, z}\, t(z)\, w + dt(w)\, c_z'(t(z))$$

where c_z is the unit speed geodesic ray in M determined by z. As in Proposition 6, we identify the first term on the right hand side above as

$$(\exp_*)_{t(z)\, z}\, t(z)\, w = A_z(t(z)) \cdot w = Y(t)$$

where Y is the Jacobi field along c_z satisfying $Y(0)=0$, $Y(0)=w$. Therefore, if $\{w_1, w_2, \ldots, w_m\}$ is an orthonormal basis for $T_z U$, it follows that

$$|\sigma^* \eta(w_1, w_2, \ldots, w_m)| = \det A_z(t(z)) \geqq ((A_z(t(z))))^m.$$

In other words, if dz is the Euclidean volume element on $S_p M$ and $a(z)=((A_z(t(z))))^m$, we have

$$|\sigma^* \eta| \geqq a(z)\, dz.$$

By combining inequalities we get therefore,

$$\text{Area } \Sigma = \int_\Sigma dV_\sigma = \int_U \sigma^* dV_\sigma \geqq \int_U a(z)\, dz.$$

We are now in a position to prove the following:

Theorem 2. *Let M^{m+1} be a complete, connected, simply connected riemannian manifold. Let p be a pole of M and suppose that for each $z \in S_p M$, $\lim_{t\to\infty} ((A_z(t))) = \infty$. Let Σ_n be a sequence of cap shaped hypersurfaces over open subsets U_n of $S_p M$ with $\lim_{n\to\infty} d(p, \Sigma_n) = \infty$. Then if $\lim_{n\to\infty} \text{Area } \Sigma_n \neq \infty$, $\bigcap_{n=1}^{\infty} U_n$ must have measure zero.*

Proof. Let $O = \bigcap_{n=1}^{\infty} U_n$. Let a_n be the function a of the previous paragraph defined for the hypersurface Σ_n. Then we have

$$\text{Area } \Sigma_n \geqq \int_{U_n} a_n(z)\, dz \geqq \int_O a_n(z)\, dz.$$

If $\liminf_{n\to\infty} \text{Area } \Sigma_n$ is finite, then there exists a real number $\alpha > 0$ such that

$$\alpha \geqq \liminf_{n\to\infty} \int_O a_n(z)\, dz \geqq \int_O \liminf_{n\to\infty} a_n(z)\, dz$$

where the right hand inequality is, of course, Fatou's lemma. Consequently, $\liminf_{n\to\infty} a_n$ is finite on O except, perhaps, on a set of measure zero. So then, if O had positive measure, there would exist at least one $x \in O$ with $\liminf_{n\to\infty} a_n(x)$ finite. Since $\lim_{n\to\infty} d(p, \Sigma_n) = \infty$, this would imply the existence of a positive sequence $s_i \to \infty$ with $\{((A_x(s_i)))\}$ bounded and this would contradict our hypothesis. This completes the proof.

An immediate corollary of the above theorem is the result of Green about uniform divergence of geodesic rays on surfaces ([5], Theorem 3.1).

Corollary 2. *Let M be as in the previous theorem. If M is a surface, then the geodesic rays going out from p diverge uniformly.*

Proof. Let $z \in S_p M$, let c be the geodesic ray determined by z, let $\{p_n\}$ be a divergent sequence of points in M and let k_n be the unit speed geodesic segment joining p to p_n. Suppose that $d(p_n, c)$ remains bounded but that $\lim_{n\to\infty} k_n'(0) = x \neq z$ and let k be the geodesic ray determined by x. Then, for each n, we can find a minimizing geodesic segment c_n connecting p_n to c such that $\{L(c_n)\}$ is bounded. For sufficiently large n, c_n is a cap shaped hypersurface over a segment γ_n of the unit circle S_pM which begins at z. By passing to a subsequence, if necessary, we can assume that all the c_n lie in one of the two wedges into which c divides M. Let $\theta = d(x, z) = \sphericalangle(x, z)$. Since $\lim_{n\to\infty} k_n'(0) = x$, it follows that there exists an integer N such that $\bigcap_{n \geq N} \gamma_n$ is a segment with length greater or equal to $\frac{1}{2}\theta$. By the theorem this is impossible and, therefore, the geodesic rays going out from p deverge uniformly.

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