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Horospheres and the Stable Part of the Geodesic Flow

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Introduction

Horospheres have always been a central point of interest in hyperbolic geometry. In modern language, horospheres are defined as enveloping hypersurfaces of all riemannian spheres having a common normal vector in the hyperbolic space. In fact, using this definition, horospheres can be defined for all riemannian manifolds where the cut locus of every point is empty. These are exactly the simply connected manifolds without conjugate points. Those generalized horospheres are hypersurfaces of differentiability class C^1 (see Prop. 1). If the curvature is negative, it is well known that the inner and the outer normal vectors of the horospheres form two foliations of the sphere bundle SM which are transversal to each other and both invariant under the geodesic flow. They are called the stable and the unstable foliation since the geodesic flow contracts the first and expands the second one. (See [12, 1], also Section 7 of this paper.) In the general case it seems that the horospheres do not give rise to foliations of SM. The reason is that the spheres may converge badly to the horospheres. In Section 3 and 4 we look for additional properties in order to make convergence nice enough. One important tool is the C^2-differentiability of the horospheres which has been proved by Eberlein (unpublished) and Heintze, Im Hof ([13]) in the case of nonpositive curvature. We get this result by replacing the curvature restriction with certain convergence properties which are fulfilled on manifold of bounded asymptote. This is a large class of manifolds without conjugate points containing the Anosov manifolds and the manifolds without focal points. Of course, one may not expect the stable and unstable foliation to be transversal to each other since for instance in the flat case these two foliations agree. So it is a natural problem to also investigate the intersection of the foliations, corresponding to the contact points of two different horospheres (Theorem 1). In Section 5, we apply our results to manifolds without focal points: If two geodesics are asymptotic to each other in both the negative and positive directions, then they bound a flat, totally geodesic strip of surface. This generalizes the "flat strip theorem" of O'Sullivan ([15]) since no curvature restriction is needed. In Section 7, we give an application to Anosov manifolds and show that there are no nontrivial isometries of compact Anosov manifolds which are diffeotopic to the identity.

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1. Lagrange Tensors

Let M be a riemannian manifold of dimension $n+1$. A geodesic c in our context, by assumption, will always be parametrized by arc length, i.e. the values of the tangent field c' belong to the unit sphere bundle SM of M. For each $v \in SM$, we denote the geodesic with initial velocity v by c_v. Let

$$Nc := \{x \in T_{c(t)}M;\ x \perp c'(t),\ t \in I\}$$

be the normal bundle of an arbitrary curve $c: I \to M$, I some real interval. A *normal* $\binom{1}{1}$ *tensor field* along c is a smooth bundle endomorphism of Nc. An important example we get from the riemannian curvature tensor, namely the linear mapping $x \mapsto R(x, c'(t))\, c'(t)$, $x \in N_t c$, $t \in I$. If c is a geodesic c_v, we call this tensor field $R_v(t)$ or $R(t)$, if there is no danger of confusion.

By a *Jacobi tensor* along c_v we mean a normal $\binom{1}{1}$ tensor field Y along c with transversal derivative (i.e. $\ker Y(t) \cap \ker Y'(t) = 0$ for some $t \in \mathbb{R}$, where Y' denotes the covariant derivative with respect to c_v'), such that the following differential equation holds:

$$Y'' + R_v \cdot Y = 0. \tag{1}$$

Each Jacobi tensor, applied to all parallel normal vector fields along c_v, gives rise to an n-dimensional space of Jacobi fields along c_v.

For two Jacobi tensors Y and Z along c_v we define a new tensor field $W(Y, Z) = Y'^* Z - Y^* Z'$ called the *Wronskian* of Y and Z, where $*$ denotes the adjoint with respect to the riemannian metric. By curvature identities, W turns out to be covariantly constant (see [7]). This fact leads to an important subset of Jacobi tensors, called *Lagrange tensors*, namely those Jacobi tensors A whose Wronskian $W(A, A)$ vanishes. If A is nowhere singular, this is equivalent to saying that the tensors $A'A^{-1}$ and A^*A' are symmetric. The importance of the Lagrange tensors consists in the following: If we have got a nowhere singular Lagrange tensor A, we can compute each Jacobi tensor Z along the same geodesic from its initial or boundary values: There exist constant tensors C_1, C_2 and some t_0 in the closure of the parameter domain of the geodesic such that

$$Z(t) = A(t)\left[C_1 \int_{t_0}^{t} (A^*A)^{-1}(u)\, du + C_2\right] \tag{2}$$

where the integration is taken with respect to the identification of the spaces $N_t c$ by parallel transport along the geodesic c. Moreover, for each Lagrange tensor the singularity points are isolated (see [6, 7]).

Lagrange tensors can be described geometrically as follows: Let H be an oriented C^2-hypersurface in M. The normal bundle has a canonical trivialization using the oriented unit normal vectors: $NH = H \times \mathbb{R}$. There exists a neighbourhood U of the 0-section such that the mapping $k = \exp|_U: U \to M$ is a diffeomorphism. Let $V := k_* \left(\frac{d}{dt}\right)$ be the velocity field of the geodesics $t \mapsto k(h, t)$, $h \in H$. Consider the V-invariant vector fields J along a fixed geodesic $c(t) = k(h_0, t)$. They are solutions of the equation $L_V J = 0$, hence

$$J' = (\nabla V) J, \tag{3'}$$

where ∇V is the $\binom{1}{1}$ field $x \mapsto \nabla_x V$, $x \in Nc$. Of course, J is a Jacobi field, as one sees by differentiating once more: $J'' = \nabla_V \nabla_J V = -R(J, V)V$ since $\nabla_V V$ and $[J, V]$ vanish. As before, we can take Equation (3′) as a tensor differential equation

$$Y' = (\nabla V) Y \tag{3}$$

for some normal $\binom{1}{1}$ tensor field Y along c which is uniquely determined by its initial value $Y(0)$. Moreover, Y is nonsingular everywhere in the interval $I := N_{h_0} H \cap U$. Recall that $-(\nabla V)|_H$ is the shape operator S_H of the oriented hypersurface H since $V|_H$ is an oriented unit normal field on H. In particular, by Equation (3), $Y'Y^{-1}(0) = \nabla V(h_0)$ is a symmetric tensor and hence the Jacobi tensor field Y is in fact Lagrange. On the other hand, each Lagrange tensor Y along any geodesic c arises in this way: If $Y(t)$ is nonsingular at some point $t \in \mathbb{R}$, and H some hypersurface with oriented normal vector $c'(t)$ and shape operator

$$S_H(c(t)) = Y'Y^{-1}(t), \tag{4}$$

then the Jacobi fields Yx, x constant normal field along c, are variational fields of geodesics normal to H, so called *H-Jacobi fields*. We will say that Y is *related to H* if Equation (4) holds for some t. Of course, if Y is related to some hypersurface H, it is related also to all parallel hypersurfaces $H_r = k((H \times \{r\}) \cap U)$ for $r \in \mathbb{R}$, where U is again the regularity domain of $\exp|_{NH}$.

There is another characterization of Lagrange tensors which we will need. Recall that each tangent vector of the tangent bundle, $w \in T_v TM$ with $v \in T_p M$, can be represented in a unique way by a pair of vectors $(w_H, w_V) \in T_p M \times T_p M$, called the *horizontal* and the *vertical component*; we will identify w with (w_H, u_V) (see [7]). This splitting leads to a metric

$$\langle w, u \rangle := \langle w_H, u_H \rangle + \langle w_V, u_V \rangle$$

on TM and hence on SM, the so called *Sasaki metric*. Recall further that the *geodesic flow*, by definition, is the one-parameter group of diffeomorphisms ϕ_t: $SM \to SM$, $\phi_t(v) = c_v'(t)$. Its differential ϕ_{t*} can be described in terms of Jacobi fields as follows: If $w \in T_v SM$ is the initial tangent vector of some curve $U: [0, \varepsilon) \to SM$ and $w(t) := \phi_{t*} w$, then $w_H(t)$ is a Jacobi field along $c_{U(0)}$ with covariant derivative $w_H'(t) = w_V(t)$, namely the variational vector field $\left.\frac{\partial}{\partial s}\right|_{s=0}$ of the geodesic family $c_{U(s)}$. In particular, if $H \subset M$ is any oriented C^k-hypersurface with oriented unit normal field V, then $V(H)$ is a C^{k-1}-submanifold of SM, and a Jacobi tensor Y along any geodesic $c_{V(p)}$, $p \in H$ is a Lagrange tensor related to H if and only if

$$\{(Y(0)x, Y'(0)x);\ x \perp V(p)\} = T_{V(p)} V(H).$$

As an example consider a geodesic $c\colon \mathbb{R} \to M$ without conjugate points. The Lagrange tensor A given by the initial values

$$A(0)=0, \quad A'(0)=1$$

(compute $W(A,A)(0)$!) is nonsingular for all real $t \neq 0$. Clearly A is related to all riemannian spheres centered in $c(0)$. Furthermore the Lagrange tensors D_s which are defined for all $s \neq 0$ by the boundary values

$$D_s(0)=1, \quad D_s(s)=0$$

(compute $W(D_s, D_s)(s)$!) are nonsingular for all $t \neq s$ and related to the riemannian spheres centered in $c(s)$. Using Equation (2) and evaluating the equation $W(A, D_s) = 1$ at $t=s$, we can compute D_s in terms of A as follows for all t between 0 and s:

$$D_s(t) = A(t) \int_t^s (A^*A)^{-1}(u)\,du.$$

It is well known that the fields D_s converge to some Lagrange tensor D as $s \to \infty$, called the *stable Jacobi tensor* (see e.g. [7]); consequently for $t \geqq 0$

$$D(t) = A(t) \int_t^\infty (A^*A)^{-1}(u)\,du.$$

Since $(A^*A)^{-1}$ is a positive definite, symmetric tensor, D is nonsingular for all $t \geqq 0$. Also consider the so called antistable Jacobi tensor $E := \lim_{s \to -\infty} D_s$. Jacobi fields J which are both stable and antistable are called *central*, that means $J = Dx = Ex$ for some covariantly constant field x in Nc. In the following sections we will consider hypersurfaces to which D and E are related.

Remark. We will use the same symbol for any vector $x \in N_t c$ and the covariantly constant vector field along c with value x at $c(t)$.

2. Busemann Function and Horospheres

Let M be a complete, simply connected riemannian manifold without conjugate points. For every $p, q \in M$ call $d(p,q) =: |p,q|$ the *distance function* between p and q. For each unit vector $v \in SM$ and each $s \geqq 0$ define the function $b_{vs}(q) := s - |c_v(s), q|$, $q \in M$, further the ball $B_{vs} := b_{vs}^{-1}((0,s])$. These functions are smooth except at $c_v(s)$ and, by triangle inequality, increasing with s and absolutely bounded by $|c_v(0), q|$. So the function $b_v := \lim_{s \to \infty} b_{vs}$ is defined everywhere on M. Call $H_v := b_v^{-1}(0)$ the *horosphere* and $B_v := b_v^{-1}((0,\infty))$ the *horodisc* of v. If $v(t) := c_v'(t)$, then clearly $b_{v(t)} = b_v - t$, $H_{v(t)} = b_v^{-1}(t)$. A vector $w \in S_q M$, $q \in M$ arbitrary, is called *asymptotic to* v if $\nabla b_{vs_i}(q) \to w$ for some sequence $s_i \to \infty$. b_v is called the *Busemann function* of v.

Lemma. *Suppose $q \in H_v$, $w \in S_q M$ asymptotic to v. Then for each $s \geqq 0$*

$$-b_{-w,s} \geqq b_v \geqq b_{w,s}.$$

Proof. Let $s_i \to \infty$ be a real sequence such that the gradient vectors $w_i = \nabla b_{vs_i}(q)$ converge to w. For an arbitrary $\varepsilon > 0$ and each $x \in M$ there is a number $i \in \mathbb{N}$ such that

(a) $b_v(x) \geqq b_{vs_i}(x) - \varepsilon/3$,

(b) $0 = b_v(q) \geqq b_{vs_i}(q) - \varepsilon/3$, so $s_i \geqq |c_v(s_i), q| - \varepsilon/3$,

(c) $|c_{w_i}(s), c_w(s)| \leqq \varepsilon/3$, so $|c_v(s_i), q| \geqq s + |c_v(s_i), c_w(s)| - \varepsilon/3$.

Therefore

$$\begin{aligned} b_v(x) &\geqq s_i - |c_v(s_i), x| - \varepsilon/3 \\ &\geqq |c_v(s_i), q| - |c_v(s_i), x| - 2\varepsilon/3 \\ &\geqq s + |c_v(s_i), c_w(s)| - |c_v(s_i), x| - \varepsilon \\ &\geqq s - |c_w(s), x| - \varepsilon \\ &= b_{ws}(x) - \varepsilon, \end{aligned}$$

which proves the second inequality of the assertion.

In order to prove the first one, choose another $i \in \mathbb{N}$ fulfilling the following properties:

(a) $b_v(x) \leqq b_{vs_i}(x) + \varepsilon/3$,

(b) $0 = b_v(q) \leqq b_{vs_i}(q) + \varepsilon/3$, hence $s_i \leqq |c_v(s_i), q| + \varepsilon/3$,

(c) $|c_{w_i}(-s), c_w(-s)| \leqq \varepsilon/3$, so $|c_v(s_i), q| \leqq |c_v(s_i), c_w(-s)| - s + \varepsilon/3$.

Therefore

$$\begin{aligned} b_v(x) &\leqq s_i - |c_v(s_i), x| + \varepsilon/3 \\ &\leqq |c_v(s_i), q| - |c_v(s_i), x| + 2\varepsilon/3 \\ &\leqq -s + |c_v(s_i), c_w(-s)| - |c_v(s_i), x| + \varepsilon \\ &\leqq -s + |c_w(-s), x| + \varepsilon \\ &= -b_{-w,s}(x) + \varepsilon, \end{aligned}$$

which proves the first part of the assertion.

Proposition 1. *Let M be a complete, simply connected riemannian manifold without conjugate points. Then the Busemann function b_v is C^1-differentiable with gradient*

$$\nabla b_v = \lim_{s \to \infty} \nabla b_{vs}$$

(pointwise convergence) for each unit vector $v \in SM$.

Proof. Let $v \in SM$, $q \in M$ be arbitrary; without restriction of generality we can assume $q \in H_v$ (see above). There exists a vector $w \in S_q M$ asymptotic to v. Since the difference of the upper and the lower bound of b_v in the lemma, $b_{ws} + b_{-ws}$, has vanishing gradient at the point q, it is some $o(|x, q|^2)$, and hence b_{ws} and b_{-ws} are first order approximations of b_v around q. Therefore b_v is differentiable at q and hence C^1. Moreover, $w = \nabla b_v(q)$, so each unit vector which is asymptotic to v is a gradient vector of b_v. This proves the assertion.

3. Continuous Asymptote

Let T be any linear endomorphism of an euclidean vector space E. Recall the definition of the *norm* $\|T\| = \max\{\|Tx\|;\ \|x\| = 1\}$ and the so called *lower norm* $((T)) = \min\{\|Tx\|;\ \|x\| = 1\}$. If T is invertible, we have $((T)) = \|T^{-1}\|^{-1}$, otherwise $((T)) = 0$. The following properties are easy to show: If T_1, T_2 are endomorphisms, then $((T_1 T_2)) \geqq ((T_1))((T_2))$. If T is symmetric, then

$$((T)) = \min\{\langle Tx, x\rangle;\ \|x\| = 1\} = \min\{|\lambda|;\ \lambda \text{ eigenvalue of } T\}.$$

Let $T(t)$ be an integrable family of positive definite symmetric endomorphisms, then $((\int T(t)\,dt)) \geqq \int ((T(t)))\,dt$. Recall further that for symmetric endomorphisms there is a partial order relation: $T_1 < T_2$ ($T_1 \leqq T_2$) if and only if $T_2 - T_1$ is positive (semi-) definite. One has $\|T\| \leqq r$ for some positive number r if and only if $-k \cdot 1 \leqq T \leqq k \cdot 1$. (See [7] for more details.)

As in the previous section, let M be a complete, simply connected manifold without conjugate points. For each $v \in SM$ let A_v, D_{sv}, D_v the Jacobi tensors A, D_s, D along the geodesic c_v, as defined in Section 1, in order to emphasize the underlying geodesic. A Jacobi tensor Y defined for each $v \in SM$ is called *continuous* if the initial values $Y_v(0)$, $Y_v'(0)$ are continuous as $\binom{1}{1}$ tensors of the vector bundle

$$\nu := \{(x, v) \in TM \times SM;\ x \perp v\}$$

over SM.

Proposition 2. *Let M be complete, simply connected without conjugate points. If the stable Jacobi tensor D is continuous, then for each $v \in SM$ the convergence of the gradients ∇b_{vs} to ∇b_v is uniform within each compact subset of M. In particular, ∇b_v is a continuous vector field on M; in fact it is C^{1-}.*

Proof. Recall that for each $t > r > 0$, the tensors

$$D_t'(0) - D_r'(0) = \int_r^t (A^* A)^{-1}(u)\,du$$

and

$$D'(0) - D_t'(0) = \int_t^\infty (A^* A)^{-1}(u)\,du$$

are symmetric and positive definite, since so is $A^* A$. So we have

$$D_r'(0) < D_t'(0) < D'(0).$$

Now for fixed $v \in SM$ call $V_s := \nabla b_{vs}$, $V := \nabla b_v$. For each $q \in M$, the Lagrange tensor $D_{t, V_s(q)}$ with $t := |q, c_v(s)|$ is related to the spheres centered in $c_v(s)$. So Equation (4) in Section 1 implies that

$$\nabla V_s(q) = D'_{t, V_s(q)}(0).$$

Let K be a compact subset of M, r some positive number. There is some $s_0 \in \mathbb{R}$ such that the distance $d(c_v(s), K)$ exceeds r for all $s \geqq s_0$. Thus for all $q \in K$ and all $s \geqq s_0$

$$D'_{r, V_s(q)}(0) < \nabla V_s(q) < D'_{V_s(q)}(0).$$

These bounds are depending continuously on the vector $V_s(q)$, hence on q. Therefore they are uniformly bounded for all $q \in K$, and hence there exists some constant number $L > 0$ such that $\|\nabla V_s(q)\| \leqq L$ for all $q \in K$ and all $s \geqq s_0$. This implies that the vector fields V_s are equicontinuous and so, by Ascoli's theorem, uniformly convergent (see [3], Prop. 7.5.6). Moreover, L is a Lipschitz constant for V, so V is C^{1-}.

Proposition 3. *Let M be complete, simply connected without conjugate points. Assume that the stable Jacobi tensor D is continuous. Then "asymptotic" is an equivalence relation on SM. Moreover, if $w \in SM$ is asymptotic to $v \in SM$, then the Busemann functions b_v and b_w agree up to some constant.*

Proof. Clearly "asymptotic" is reflexive. In order to prove transitivity assume that $u \in S_o M$ and $w \in S_q M$ are asymptotic to $v \in S_p M$. Call $u_t := \nabla b_{vt}(0)$, $w_t := \nabla b_{vt}(q)$, then $u_t \to u$, $w_t \to w$ as $t \to \infty$. Since c_{u_t} and c_{w_t} meet each other at the point $c_v(t)$, we have also $w_t = \nabla b_{u_t, s_t}(q)$, where $s_t = |q, c_v(t)|$. Since s_t becomes arbitrarily big for big t, we get

$$w = \lim_{t \to \infty} w_t = \nabla b_u(q)$$

due to the uniform convergence of the gradient fields (Prop. 2). Therefore, w is asymptotic to u. The proof of the symmetry is given by the particular case $w = v$. We proved also that $\nabla b_w = \nabla b_v$ if w is asymptotic to v; that gives the second part of the assertion.

Next we consider horospheres and Busemann functions of opposite directions. By G_v we denote for each $v \in SM$ the horosphere H_{-v} together with the opposite orientation, in other words v is, by definition, an oriented normal vector of G_v. A vector $w \in SM$ asymptotic to $v \in SM$ is called *bi-asymptotic* if also $-w$ is asymptotic to $-v$. Also call the geodesics c_v and c_w asymptotic or bi-asymptotic if so are v and w.

Proposition 4. *Let M be as in Proposition 3. Then for each $v \in SM$*

(i) *$H_v \cap G_v = \bar{B}_v \cap \bar{B}_{-v}$,*

(ii) *∇b_v and $-\nabla b_{-v}$ agree at the points of $H_v \cap G_v$,*

(iii) *$H_v \cap G_v$ is a connected set.*

(iv) *Exactly the geodesics intersecting H_v perpendicularly at points of $H_v \cap G_v$ are bi-asymptotic to c_v.*

Proof. B_v and B_{-v} cannot have common points since intersection of some B_{vs} and B_{-vs} would contradict to the triangle inequality. Therefore, all intersection points of H_v and G_v are contact points, which proves (i) and (ii). In order to prove (iii), assume p and q are sitting in two different connected components of $H_v \cap G_v$. p and q are minima of the function $b_v|_{G_v}$, since b_v is nonpositive outside B_v. Now assume that all critical points of $b_v|_{G_v}$ are of this type, that means $\mathrm{Crit}(b_v|_{G_v}) = H_v \cap G_v$. According to Proposition 2, ∇b_v is a C^{1-} vector field, hence also its TG_v-projection $\nabla(b_v|_{G_v})$ is C^{1-}. So we can push down the whole of G_v along the integral curves of $\nabla(b_v|_{G_v})$ and so the set of critical points, $H_v \cap G_v$, is a deformation retract of G_v. But this is impossible since G_v is connected and $G_v \cap H_v$ not, according to our general assumption.

Hence, there exists a critical point $r \in G_v$ with $b_v(r) =: -s < 0$. Call $\nabla b_v(r) =: w$; w is a normal vector of G_v since r is critical. It follows from Proposition 3 that $b_w = b_v + s$ and $b_{-w} = b_{-v}$. So for instance at the point p we have $b_w(p) = s > 0$, $b_{-w}(p) = 0$, and therefore $p \in \bar{B}_w \cap \bar{B}_{-w}$. But it has already been proved in (i) that then $p \in H_w \cap G_w$, in particular $b_w(p) = 0$, which is a contradiction.

Proof of (iv): Unit vectors which are orthogonal to both H_v and G_v and well oriented, are gradient vectors of both b_v and $-b_{-v}$ and hence bi-asymptotic. On the other hand, a bi-asymptotic geodesic cuts both H_v and G_v perpendicularly. By the same argument as in the proof of (iii) these two intersection points must coincide.

It is an open question whether the stable Jacobi tensor is continuous on each manifold without conjugate points. Necessary and sufficient conditions for this property are given in [7]. In the next section, we are going to discuss one sufficient condition and give examples.

4. Uniform Convergence and Bounded Asymptote

Theorem 1. *Let M be a complete, simply connected, m-dimensional riemannian manifold without conjugate points. Assume that the convergence $D'_{vs}(0) \to D'_v(0)$ is uniform for all v in an arbitrary compact set $L \subset SM$. Then*

(i) *Busemann functions and horospheres are of differentiability class C^2. The shape operator of the horosphere H_v is given by $-\nabla^2 b_v$. If $w \in SM$ is asymptotic to v, then D_w is related to H_v.*

(ii) *The classes of asymptotic vectors form a continuous, m-dimensional foliation X of SM. The leaves X_v, $v \in SM$, are C^1 vector fields on M which are invariant under the geodesic flow ϕ.*

(iii) *If Y is the foliation of SM with leafs $Y_v := -X_{-v}$, then the leaves of X and Y have connected, ϕ-invariant intersection sets, namely the classes of biasymptotic vectors.*

(iv) *If w is biasymptotic to v and $w \neq \phi_t v$ for all $t \in \mathbb{R}$, then there exists a central Jacobi field along c_w.*

Proof. For fixed $v \in SM$ let $V_s = \nabla b_{vs}$, $V = \nabla b_v$ as above. The uniform convergence of the D_{vs} implies the continuity of the limit D_v with respect to v. So, by Proposition 2, $V_s \to V$ uniformly in each compact set $K \subset M$. Hence $V_s(q) = D'_{t, V_s(q)}(0)$ with $t := |q, c_v(s)|$ converges uniformly to $D'_{V(q)}(0)$ for all $q \in K$. Therefore the vector field V is C^1 with derivative $\nabla V(q) = D'_{V(q)}(0)$. Thus, the Busemann function b_v is C^2 and leads to a C^2 structure of the hypersurface H_v with shape operator $S = -\nabla V|_{H_v} = -\nabla^2 b_v|_{H_v}$. This proves (i). The equivalence classes of asymptotic vectors are given by the C^1 vector fields $V = \nabla b_v$. These are solutions of the differential equation

$$\nabla V(p) = D'_{V(p)}(0)$$

with continuous coefficients given by $D'(0)$, so they depend continuously on their initial values v, and (ii) is proved. (iii) is clear from Proposition 4.

In order to prove (iv) assume (without restriction of generality) that w is oriented normal vector of H_v and also of G_v (see Prop. 4). Hence the corresponding C^1-submanifolds of SM, $\tilde{H}_v := \nabla b_v(H_v)$ and $\tilde{G}_v := \nabla b_v(G_v)$ intersect each other at w. If this intersection was transversal, the intersection point w would be isolated. But this is impossible since $H_v \cap G_v$ and hence $\nabla b_v(H_v \cap G_v)$ are connected and contain at least two points. So there exists a common tangent vector

$$0 \neq u \in T_w H_v \cap T_w G_v$$

which gives rise to a central Jacobi field along c_w.

Call X the *stable* and Y the *unstable* foliation of SM.

We want to describe next a rather large class of riemannian manifolds where all previous assumptions are satisfied. M is called *manifold with bounded asymptote* if it is complete, connected, without conjugate points, and if there exists a uniform bound $\rho \geqq 1$ for the stable Jacobi tensor D such that

$$\|D_v(t)\| \leqq \rho \qquad \text{for all } v \in SM,\ t \geqq 0.$$

For example all manifolds without focal points are of this type since then $\|D_v(t)\|$ is monotonely decreasing for each $v \in SM$, so $\rho = 1$ (Section 5). Another important class of examples is given by the manifolds with geodesic flow of Anosov type as we will see in Section 7. Gulliver ([11]) showed that these manifolds may have focal points.

Proposition 5. *Let M be a manifold with bounded asymptote $\|D\|\,|_{SM \times \mathbb{R}_+} \leqq \rho$. Then the convergence $D_s'(0) \to D'(0)$ is uniform in SM:*

$$\|D_v'(0) - D_{sv}'(0)\| \leqq \rho^2/s \qquad \textit{for all } v \in SM,\ s > 0.$$

Proof. Using Equation (2) in Section 1 we get on the interval $(0, \infty)$

$$D = A \cdot \int^{\infty} (A^* A)^{-1}, \qquad A = D \cdot \int_0 (D^* D)^{-1}.$$

Call $X := A^{-1} D$, then

$$X(t) = \int_t^{\infty} (A^* A)^{-1} = D'(0) - D_t'(0),$$

$$X^{-1}(t) = (D^{-1} A)(t) = \int_0^t (D^* D)^{-1}.$$

Compute the lower norm of X^{-1}:

$$\|X(t)\|^{-1} = ((X^{-1}(t))) \geqq \int_0^t (((D^* D)^{-1})) = \int_0^t \|D^* D\|^{-1} \geqq t/\rho^2,$$

since $\|D^* D\| = \|D\|^2 \leqq \rho^2$. So $\|D'(0) - D_s'(0)\| = \|X(s)\| \leqq \rho^2/s$.

Clearly asymptotic rays have bounded distance on manifolds with bounded asymptote as one sees by integration along the horospheres orthogonal to both rays. The opposite statement is not necessarily true without curvature assumptions (see Section 6).

5. No Focal Points

By definition, a complete riemannian manifold M has *no focal points* if each Jacobi field $J(t)$ with $J(0)=0$ has monotonely increasing length $\|J(t)\|$ for all $t \geqq 0$. Clearly such manifold cannot have conjugate points, so the Jacobi tensors A, D_s, D are defined everywhere on SM. $\|A(t)\|$ is monotonely increasing for $t \geqq 0$, $\|D_s(t)\|$ monotonely decreasing for $t \leqq s$ and hence also $\|D(t)\|$ is monotonely decreasing for all $t \in \mathbb{R}$. In particular $\|D(t)\| \leqq 1$ for $t \geqq 0$, so M has bounded asymptote. Since for all $v \in SM$, all parallel vector fields x normal to c_v (notation: $x \perp v$) and $t \leqq s > 0$

$$2 \cdot \langle D'_{sv}(t)\,x,\ D_{sv}(t)\,x \rangle = (\|D_{sv}(t)\,x\|^2)' \leqq 0,$$

the symmetric tensors $D^*_{sv}(t)\,D'_{sv}(t)$ are negative semi-definite for all $t \leqq s > 0$, hence passing to the limit

$$(D^*_v D'_v)(t) \leqq 0 \qquad \text{for all } v \in SM,\ t \in \mathbb{R}.$$

Clearly then for the antistable Jacobi tensor E we have $E^* E' \geqq 0$ everywhere on $SM \times \mathbb{R}$, since, by definition, $E_v(t) = D_{-v}(-t)$.

Lemma (Eberlein [4]). *Each central Jacobi field on a manifold without conjugate points is parallel.*

Proof. Let J be a central Jacobi field along c_v, $v \in SM$, that means $Dx = J = Ex$ for some $x \perp v$. From $0 \leqq \langle E'x, Ex \rangle = \langle D'x, Dx \rangle \leqq 0$ it follows that $\langle D^* D'x, x \rangle = 0$, hence $D^* D'x = 0$ due to $D^* D' \leqq 0$. Since for $t \geqq 0$ the tensor $D^*(t)$ is an isomorphism, one has $J'(t) = D'(t)\,x = 0$, and the same is valid for negative t-values using E instead of D.

Call a subset $S \subset M$ *convex* if for all $p, q \in S$ the geodesic segment from p to q lies completely in S. A C^2-function $f\colon M \to \mathbb{R}$ is called *concave* if its Hessean $\nabla^2 f$ is negative semi-definite. The sets $f^{-1}((t, \infty))$ are convex for each concave function f, $t \in \mathbb{R}$.

Theorem 2. *Let M be a simply connected manifold without focal points. Then*

(i) *The Busemann functions b_v are C^2-differentiable and concave for all $v \in SM$, and the horodiscs are convex.*

(ii) *The sets $H_v \cap G_v$ are convex.*

(iii) *If some geodesic is bi-asymptotic to some other geodesic $c_0 \neq c$, then c and c_0 have constant distance $a > 0$ and there is a totally geodesic, isometric imbedding F: $[0, a] \times \mathbb{R} \to M$ with $c = F|_{\{a\} \times \mathbb{R}}$, $c_0 = F|_{\{0\} \times \mathbb{R}}$.*

Proof. According to Theorem 1, $\nabla^2 b_v(q) = D'_{\nabla b_v(q)}(0) \leqq 0$, so b_v is concave. Hence B_v and also $H_v \cap G_v = \bar{B}_v \cap \bar{B}_{-v}$ are convex sets. This proves (i) and (ii). Now suppose that c is biasymptotic to c_0. Let v be the initial velocity $c'_0(0)$ and reparametrize c so that $q := c(0) \in H_v$. By Proposition 4, (iv) we know that $q \in H_v \cap G_v$. This set is convex, so the geodesic segment d_0 connecting $p := c_0(0)$ with q lies in $H_v \cap G_v$. Let a be the length of d_0, and for each $0 \leqq s \leqq a$ call c_s the geodesic with initial vector $v_s := \nabla b_v(d_0(s))$. Then the mapping $F\colon [0, a] \times \mathbb{R} \to M$, $F(s, t) := c_s(t)$ is an isometric imbedding since the Jacobi fields

$$\frac{\partial F}{\partial s}(s, t) = D_{v_s}(t)\, d'(s) = E_{v_s}(t)\, d'(s)$$

are orthogonal to the geodesic c_s and, due to the lemma, parallel, hence of unit length. From this it follows that the curve $d_t := F|_{[0,a]\times\{t\}}$ has length a for all $t\in\mathbb{R}$. If d_t was not the shortest curve connecting $c_0(t)$ with $c(t)$, then $|c(t), c_0(t)|<a$, and repeating the construction starting with the points $c_0(t)$ and $c(t)$ instead of p and q we would get a curve $\tilde{d}_0$ of length $|c(t), c_0(t)|<a$ between p and q which is impossible. So also d_t is a geodesic segment. Hence all covariant derivatives $\frac{D}{ds}\frac{\partial F}{\partial s}$, $\frac{D}{dt}\frac{\partial F}{\partial s}$ and of course $\frac{D}{dt}\frac{\partial F}{\partial t}$ vanish. Therefore F is totally geodesic.

6. Bounded Curvature

Let M be an arbitrary riemannian manifold and $c\colon I\to M$ some geodesic, Y a nowhere singular Jacobi tensor along c. Then the tensor field $U=Y'Y^{-1}$ which is symmetric in the Lagrange case is a solution of the Riccati equation

$$U'+U^2+R=0. \tag{5}$$

If U is symmetric and defined on $I=(0,\infty)$ and the sectional curvatures of all 2-planes containing c' are bounded from below by some constant $-r^2$, in other words $R\geqq -r^2\cdot 1$, then due to Eberlein [4] the following estimate is known for all $t>0$:

$$-r\cdot 1\leqq U(t)\leqq r\cdot\coth(r\,t)\cdot 1.$$

In particular $\|U(t)\|\leqq 2r$ for all $t\geqq T:=(1/r)\cdot\operatorname{arcoth}(2/r)$, and if $U(t)$ is defined for all $t\in\mathbb{R}$, then we get $\|U(t)\|\leqq r$ for all $t\in\mathbb{R}$ since we can shift the parameter of c arbitrarily. For the corresponding Jacobi fields it follows that the derivative cannot differ too much from the value of the Jacobi field:

$$\|Y'x\|=\|Y'Y^{-1}Yx\|\leqq\|U\|\,\|Yx\|$$

for all parallel fields $x\perp c'$.

Assume that $c\colon\mathbb{R}\to M$ is a complete geodesic without conjugate points with the curvature restriction made above: $R\geqq -r^2\cdot 1$. Call $X:=A^{-1}D=(A^*A)^{-1}$, apply the above estimates of solutions of the Riccati equation (5) to the tensors $A'A^{-1}$ and $D'D^{-1}$ and use the fact that $W(A,D)=1$, then the result will give a lower bound for the Jacobi tensor A: For all $t\geqq T$

$$((A(t)))\geqq(4r\,\|X(t)\|)^{-1/2}.$$

(see [4, 6, 7] for details). In particular, each Ax, $x\perp c'$, is unbounded, since $X(t)\to 0$ for $t\to\infty$.

Using Proposition 4, we get immediately

Proposition 6. *Let M be a manifold with ρ-bounded asymptote and all sectional curvatures bounded from below by $-r^2$. Then*

$$((A_v(t)))\geqq(\rho\cdot\sqrt{4r})^{-1}\sqrt{t}$$

for all $v\in SM$, $t\geqq T=(1/r)\operatorname{arcoth}(2/r)$.

Two geodesic rays c_1, c_2: $[0, \infty) \to M$, by definition, are of the same *type* if $|c_1(t), c_2(t)|$ is uniformly bounded for all $t \geqq 0$. If M is like in the proposition, the uniform estimate of A implies that no two geodesic rays starting at the same point can be of the same type (see [7]). Hence geodesic rays are asymptotic if and only if they are of the same type. From this Remark and Theorem 2 one gets immediately the "flat strip theorem" of O'Sullivan [15].

More important in our context is that the curvature bound leads to an estimate of a part of the geodesic flow:

Theorem 3. *Let M be a complete, connected riemannian manifold of dimension $n+1$ without conjugate points and sectional curvature bounded from below. Then the following statements are equivalent:*

(i) *M is a manifold of bounded asymptote.*

(ii) *There are two n-dimensional subbundles $\mathscr{X}$ and $\mathscr{Y}$ of TSM which are invariant under the geodesic flow ϕ and orthogonal to its tangent vectors such that for some constant $\beta > 0$*

$$\|\phi_{t*} x\| \leqq \beta \|x\| \quad \textit{for all } t \geqq 0, \ x \in \mathscr{X},$$

$$\|\phi_{t*} y\| \leqq \beta \|y\| \quad \textit{for all } t \leqq 0, \ y \in \mathscr{Y}.$$

(iii) *There is a continuous foliation X of SM whole leaves are $(n+1)$-dimensional C^1-manifolds which are invariant and stable under the geodesic flow in the following sense: There is a constant $\beta > 0$ such that for all $t \geqq 0$*

$$\|\phi_{t*}|_{TX}\| \leqq \beta.$$

Proof. (i) $\Rightarrow$ (iii). Suppose $\|D|_{SM \times [0,\infty)}\| \leqq \rho$. First assume that M is simply connected. Then we have the foliation X of Theorem 1 on M, with leaves $X_v = \nabla b_v(M)$ for all $v \in SM$. D_v is related to H_v and X_v contains the oriented normal vectors of H_v, so for each $x \in T_v X_v$ we have

$$\phi_{t*} x = (D_v(t) x_H, D_v'(t) x_V) \quad \text{for all } t \in \mathbb{R}.$$

Due to the curvature bound we know that $\|D'(t) y\| \leqq r \|D(t) y\|$, so

$$\|\phi_{t*} x\|^2 = \|D_v(t) x_H\|^2 + \|D_v'(t) x_V\|^2 \leqq \rho^2 \|x_H\|^2 + \rho^2 r^2 \|x_H\|^2,$$

so

$$\|\phi_{t*} x\| = \beta \|x\| \quad \text{with} \quad \beta := \rho \sqrt{1+r^2} \quad \text{for all } t \geqq 0.$$

If M is not simply connected, we do the same business on the universal covering and project back to M. The projections of the leaves are regularly imbedded submanifolds of SM with empty intersection, since the tangent space at each $v \in SM$ is uniquely prescribed by $\{(x, D_v'(0) x) \in T_v SM;\ x \perp v\}$.

(iii) $\Rightarrow$ (ii) with $\mathscr{X} = TX$, $\mathscr{Y} = TY$, where $Y_v := \{y \in SM; -y \in X_{-v}\}$.

(ii) $\Rightarrow$ (i). We claim that each Jacobi field of the form $J(t) = (\phi_{t*} x)_H$, $x \in \mathscr{X}$, has to be stable: $J(t) = D(t) x_H$. The reason is its bounded length. Consider the difference Jacobi fields $J_s(t) := J(t) - D_s(t) x_H$ which converge to $J(t) - D(t) x_H$. Suppose $\|(J - D x_H)'(0)\| > 2\delta$ for some $\delta > 0$, then $\|J_s'(0)\| > \delta$ for sufficiently big s. Since J_s

vanishes initially, it can be expressed by $A \cdot J_s'(0)$, and so we get on one hand for all $t>0$

$$\|J_s(t)\| = \|A(t)\, J_s'(0)\| \geqq ((A(t)))\, \|J_s'(0)\| \geqq ((A(t))) \cdot \delta.$$

On the other hand for all $s>0$

$$\|J_s(s)\| = \|J(s)\| \leqq \beta$$

which is a contradiction, since $\{((A(s)));\ s>0\}$ is unbounded. So $J(t)=D(t)\,x_H$.

A dimension argument now shows that all double tangent vectors of the form $(y, D'(0)\,y)$ *ly in* $\mathscr{X}$. Therefore, for $t \geqq 0$,

$$\begin{aligned}\|D(t)\,y\| &\leqq \|(D(t)\,y, D'(t)\,y)\| = \|\phi_{t*}(y, D'(0)\,y)\| \leqq \beta\, \|(y, D'(0)\,y)\| \\ &\leqq \beta \cdot \sqrt{1+r^2}\, \|y\|.\end{aligned}$$

So M has ρ-bounded asymptote with $\rho := \beta \cdot \sqrt{1+r^2}$.

7. Anosov Manifolds

A closed riemannian manifold M of dimension $n+1$ is called *Anosov* if its geodesic flow ϕ is of Anosov type (in other notation a C- or U-system, see [1, 2]). That means that the bundle TSM splits into three subbundle $\mathscr{X}$, $\mathscr{Y}$ and $\mathscr{Z}$; $\mathscr{X}$ and $\mathscr{Y}$ have fibre dimension n and are orthogonal to the geodesic flow ϕ, and $\mathscr{Z}$ is a line bundle which is tangent to ϕ, and there are constants β, $k>0$ such that for all $t \geqq 0$

$$\begin{array}{lll}\|\phi_{t*}\,x\| \leqq \beta\,\|x\| \cdot e^{-kt}, & \phi_{-t*}\,x \geqq \beta^{-1} \cdot \|x\|\, e^{kt} & \text{for } x \in \mathscr{X}, \\ \|\phi_{t*}\,y\| \geqq \beta^{-1}\,\|y\| \cdot e^{kt}, & \phi'_{t*}\,y \leqq \beta \cdot \|y\|\, e^{-kt} & \text{for } y \in \mathscr{Y}.\end{array}$$

For this it is sufficient that there exists a ϕ-invariant subbundle $\mathscr{X}$ of fibre dimension n such that there are constants $\beta \geqq 1$, $k>0$ such that for all $t \geqq 0$, $x \in \mathscr{X}$

$$\|\phi_{t*}\,x\| \leqq \beta\, \|x\|\, e^{-kt}.$$

The relation for negative t we get from the ϕ-invariance, and the bundle $\mathscr{Y}$ is given by "looking into the opposite direction", that means $\mathscr{Y}_v := \mathscr{X}_{-v}$ for each $v \in SM$. Here we use the canonical identification of $T_v SM$ and $T_{-v} SM$. Klingenberg [14] showed that closed Anosov manifolds don't admit conjugate points, so it follows from Theorem 3 that they have bounded asymptote. The Anosov estimates show that the bundles $\mathscr{X}$ and $\mathscr{Y}$ are transversal to each other. Therefore there are no central Jacobi fields, and the intersection of the foliations X and Y are given exactly by the integral curves of the geodesic flow on SM. Eberlein [4] proved also the inverse statement: A closed riemannian manifold without conjugate points and without any central Jacobi field is Anosov.

It is well known (see [1]) that compact manifolds of negative sectional curvature are Anosov. That can be easily seen as follows: If $-k^2$ is the upper curvature bound, it follows from the Rauch theorem (see [8], p 178) that for $s>0$, $0 \leqq t \leqq s$

$$\|D_s(t)\| \leqq \frac{\sinh(k(s-t))}{\sinh(ks)}.$$

Passing to the limit $s \to \infty$, we get $\|D(t)\| \leqq e^{-kt}$ for $t \geqq 0$. If there is a lower curvature bound $-r^2$, too (which clearly exists for compact manifolds), we can estimate also the derivative $\|D'(t)\| \leqq r \cdot e^{-kt}$, so M is Anosov with $\mathscr{X}$ as in Theorem 3 and $\beta = \sqrt{1+r^2}$.

The following theorem has been shown by Grove [10] for manifolds with negative curvature using completely different methods.

Theorem 4. *Let M be a compact Anosov manifold, $D(M)$ the diffeomorphism group of M with identity component $D(M)^0$, and $I(M)$ the isometry group. Then*

$$D(M)^0 \cap I(M) = \{1\},$$

in other words, there exists no nontrivial isometry of M which is diffeotopic to the identity.

Proof. Let $\{f_t; 0 \leqq t \leqq 1\}$ be a differentiable family of diffeomorphisms such that $f_0 = \mathrm{id}$, $f_1 \in I(M)$. The length of the path $w_p: [0,1] \to M$, $w_p(t) := f_t(p)$ which leads from p to $f_1(p)$, depends continuously on p. Therefore it is bounded above on the compact manifold M, say by some constant $L>0$.

Now call $\hat{M}$ the universal covering of M with projection map $\pi: \hat{M} \to M$. For each point $\hat{p}$ in the fibre $\pi^{-1}(p) \subset \hat{M}$, $p \in M$, let $w_{\hat{p}}: [0,1] \to \hat{M}$ be the unique lift of w_p starting at $\hat{p}$. The mapping $\hat{f}_1: \hat{M} \to \hat{M}$, $\hat{f}_1(\hat{p}) := w_{\hat{p}}(1)$ is an isometry of $\hat{M}$, since lifting preserves all local properties. Since $\hat{w}_p$ has the same length as w_p, the displacement function $|\hat{p}, \hat{f}_1(\hat{p})|$ is bounded above by L for all $\hat{p} \in \hat{M}$.

In particular, if $c: \mathbb{R} \to \hat{M}$ is any geodesic, then the geodesic $\hat{f}_1 \circ c$ has bounded distance from c and is thus biasymptotic to c (see Section 6). If c is different from $\hat{f}_1 \circ c$ up to reparametrization, then Theorem 1, (iv), shows the existence of a central Jacobi field J along c. Hence also $\pi_* J$ is central on M which is impossible as we mentioned above. So $\hat{f}_1$ can only translate each geodesic, hence $\hat{f}_1 = \mathrm{id}_{\hat{M}}$ and therefore $f_1 = \mathrm{id}_M$.

It is well known that the Anosov flows are structural stable, so the Anosov metrics on a closed differentiable manifold form an open subset of the space $\mathscr{M}$ of all riemannian metrics on M (see [1, 2]). A similar statement is not true in general for metrics with bounded asymptote as the following proposition shows:

Proposition 7. *Let M be a closed differentiable manifold, g_0 a metric on M with vanishing Ricci curvature. Then in the space $\mathscr{M}$ of all riemannian metrics on M each neighbourhood of g_0 contains metrics with conjugate points.*

Proof. For each $g \in \mathscr{M}$ let r_g be the scalar curvature and $A(g) := \int_M r_g \, d_g M$ (integration w.r.t. the volume element of g) the total scalar curvature. Ehrlich proved in [5] that in each neighbourhood U of g_0 there exist metrics $g \in U$ with $A(g) > 0$. This implies the existence of conjugate points for g as Green showed in [9].

Therefore at least on Ricci-flat manifolds there are no open conditions that imply "no conjugate points".

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