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Is Binocular Visual Space Constantly Curved?

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Summary. This is a contribution to the theory of binocular vision due to Luneburg and Blank. We give a short introduction to the theory and consider the various assumptions that have been made. Then we discuss several methods in order to test Luneburg's basic assumption of constant curvature empirically.

Key words: Binocular visual metric—Constant curvature—Alley experiments—Double Vieth-Müller circles.

Richard Luneburg [10] tried to describe the geometry of binocular vision by a Riemannian metric. A central point of this theory (which was completed by Albert A. Blank [2]) is that this so called visual metric has constant Gaussian curvature K ('Constant curvature condition', CCC) which mostly will be negative. CCC is assumed in the evaluation of all experiments which try to determine empirically the coefficients of the visual metric. There are, however, no cogent theoretical arguments in favour of CCC, and neither do we have a systematic empirical verification for it. The aim of this paper is to discuss mathematically some of the empirical tests which have been proposed.

In Section 1, we give a short introduction to the theory and discuss the various assumptions which have been made. Some confusion has arisen from the fact that Luneburg and his successors changed their assumptions several times. In Section 2, we discuss the experiments mentioned above, this is the central part of the paper. Section 3 collects some useful facts on the geometry of constant curvature (we avoid any specification of K) and gives the proofs of the theorems. These are stated more or less precisely in the references, but without proof. The differential geometric methods which are used can be found in [4, 5, 10].

1. *The Theory of Binocular Vision According to Luneburg and Blank*

1.1. Visual Versus Physical Geometry

Geometry is not only a branch of mathematics, but also, in the original meaning of the word, a kind of experience of our environment. In this sense of the word, geometry consists of the metrical relations of space. We come to know these

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relations by length measuring with solid rods. A sufficiently precise mathematical description of this geometry is given by the euclidean 3-space $\mathbb{R}^3 = \{x = (x_1, x_2, x_3); x_1, x_2, x_3 \in \mathbb{R}\}$, equipped with the euclidan metric

$$ds^2_{\text{phys}} = dx_1^2 + dx_2^2 + dx_3^2.$$

(Strictly speaking, the coordinates (x_1, x_2, x_3) are determined by the geometry only up to a euclidean transformation; one has to make a suitable choice.) Let us call the pair $(\mathbb{R}^3, ds^2_{\text{phys}})$ the *physical space*, and its geometry the *physical* one, since length measuring is a physical act.

But there is also a quite different type of geometry. By means of our vision, we receive geometrical information about the surrounding space. Objects appear to be near or far, large or small, lines seem to be straight or curved. Let us use the term *visual geometry* for this kind of information. The visual geometry cannot be determined by length measuring like the physical one, but only by systematically questioning a test person. This is a task not of physics, but of the psychology of perception.

Common experience already shows that the two geometries do not agree; this is a source of optical illusions. Far away objects appear to be small, and depth perception is not so good that one could distinguish very great (e.g. astronomical) distances.

Experiments show the differences between the two geometries more exactly. E.g. it has been pointed out already by H. v. Helmholtz, that apparently straight lines can be physically curved. One especially surprising example was given by A. Ames [8, 12]: the so called *distorted rooms*. These are oblique-angled rooms with curved walls which by means of binocular vision from a certain point of view look like a rectangular room with plane walls ('ordinary room').

R. Luneburg proposed to describe the visual geometry by some Riemannian metric ds^2_{vis}, called *visual metric*, which is characterized by the following properties [12]:

VM_1. The geodesics of the metric ds^2_{vis} appear to be the straight lines.

VM_2. Any two pairs of points appear to be equidistant if and only if their distances with respect to ds^2_{vis} are the same.

Luneburg showed that any two metrics fulfilling VM_2 differ only by a constant positive factor (see 3.1). Call two such metrics *equivalent*. Given any equivalence class, one selects a metric by means of a certain normalization.

Call *visual field* the set of all points in the physical space which a fixed observer in a given situation can see without moving his head. This is some open subset $V \subset \mathbb{R}^3$. According to Luneburg, we assume the following fundamental hypothesis:

H_1. For any observer and any visual field V, there exists a metric ds^2_{vis} on V which fulfills VM_1 and VM_2.

The pair (V, ds^2_{vis}) will be called the *visual space* of the observer. The visual metric will depend on the observer and the visual field. The aim of Luneburg's theory is to work out the general features of this metric that are common to the binocular vision of all normal-sighted persons.

1.2. Bipolar Coordinates

For a more precise description of this metric we have to introduce suitable coordinates on the visual field V. Fix euclidean coordinates (x_1, x_2, x_3) in the physical space such that the x_3-axis is vertical and the observer's eyes are fixed at the points $L = (0, 1, 0)$ (left eye) and $R = (0, -1, 0)$ (right eye). So the unit of length is half the eye distance (about 3 cm). We may assume that V is a subset of the half space $H = \{x \in \mathbb{R}^3; x_1 > 0\}$ such that $0 = (0, 0, 0)$ lies in the closure $\bar{V}$ of V. 0 is called the *egocenter* of the observer.

In the following, we restrict our attention essentially to the horizontal half plane $E = \{x \in H; x_3 = 0\}$ which most of the investigations deal with. Call $U := V \cap E$ the restricted visual field. Binocular depth perception arises from the parallax between the right and the left eye. This is described mathematically by the angle $\tilde{\gamma}(x) = \sphericalangle(L, x, R)$ ($x \in E$ arbitrary) called *bipolar parallax*. For the second coordinate choose the *bipolar latitude* $\varphi(x) = (\alpha(x) - \beta(x))/2$, where α and β are the angles $\alpha(x) = \sphericalangle(x, L, R)$, $\beta(x) = \sphericalangle(x, R, L)$. The mapping $(\tilde{\gamma}, \varphi)\colon E \to (0, \pi)$ $(-\pi/2, \pi/2)$ is a diffeomorphism, and one has the transformation formulas

($\tilde{1}$) $\tan \tilde{\gamma} = 2x_1/u$ where $u = x_1^2 + x_2^2 - 1$

(2) $\tan 2\varphi = 2x_1x_2/(x_1^2 - x_2^2 + 1)$

(3) $x_1 = (\cos 2\varphi + \cos \tilde{\gamma})/\sin \tilde{\gamma}$

(4) $x_2 = \sin 2\varphi/\sin \tilde{\gamma}$.

These so called *bipolar coordinates* have been introduced by Luneburg. However, it is more convenient to use the coordinate

(5) $\gamma := 2 \tan (\tilde{\gamma}/2)$,

$\gamma \in (0, \infty)$, instead of $\tilde{\gamma}$. Then we have to replace ($\tilde{1}$) by

$$\gamma = ((4x_1^2 + u^2)^{1/2} - u)/x_1. \tag{1}$$

We then have the simple formula $\gamma((x_1, 0)) = 2/x_1$ for all $x_1 > 0$. On the other hand, for any object being farther than 15 cm, the relative error $(\gamma - \tilde{\gamma})/\tilde{\gamma}$ will be less than 2%. Hence the change from $\tilde{\gamma}$ to γ is purely technical and without empirical significance. We will use the notation $x = (\gamma(x), \varphi(x))$ for any $x \in U$.

Luneburg and his successors agreed that these coordinates are related to perception as follows [2, 7, 11, 12]:

H_2. The perceived distance $r(x)$ of some point $x \in U$ from the egocenter of the observer (located in 0) depends only on the parallax $\gamma(x)$. In other words: all points of the excentric circle $\{x \in U; \gamma(x) = \text{const}\}$ (so called *Vieth-Muller circle*, *VMC*) seem to be equidistant from the observer.

H_3. The coordinate $\varphi(x)$ is the apparent angle between x and the x_1-axis. The hyperbolas $\{x \in U;\ \varphi(x) = \text{const}\}$ (so called *Hillebrand-hyperbolas*) appear to be radial rays, and the transformations $(\gamma, \varphi) \mapsto (\gamma, \varphi + \text{const})$, called *$\varphi$-shifts*, are visual isometries ('*iseiconic transformations*').

More precisely, we may sum up H_1, H_2, H_3 by the following

Luneburg-Hypothesis (*LH*)
For any observer and any visual field $U \subset E$, there exist smooth (i.e. C^∞) functions $\rho\colon (0, \infty) \to (0, \infty)$ and $s\colon [0, \infty) \to \mathbb{R}$ with the following properties:

1) $\rho' > 0$, $\lim_{\gamma \to \infty} \rho(\gamma) = 0$

2) $s(0) = 0$, $s'(0) = 1$

3) The metric

$$(M0) \quad ds^2 = \rho'(\gamma)^2\, d\gamma^2 + s(\rho(\gamma))^2\, d\varphi^2$$

obeys the rules VM_1 and VM_2.

The functions ρ and s have to be determined by experiment. They have the following geometric meaning: $\rho(\gamma)$ denotes the perceived distance between the observer and some point $x \in U$ with parallax $\gamma(x) = \gamma$. So ρ relates physical and visual geometry. On the other hand, the function s is defined in terms of visual geometry alone: $2\pi s(r)$ would be the circumference of the circle of radius r centered at 0 in the visual space. One may replace s by the curvature function $K = -s''/s$ (assume $s > 0$); the functions K and s determine each other.

1.3. Extension to 3-Space

The bipolar coordinates defined in 1.2 can be extended to H as follows: If $x \in H$, define $\varphi(x)$, $\tilde{\gamma}(x)$, $\gamma(x)$ as before, but now all the angles have to be taken in the plane defined by the three points L, R, x. This is called the *plane of elevation* E_x, and the angle $\vartheta(x)$ between the two half planes E and E_x is called *elevation*. $(\gamma, \varphi, \vartheta)$ is a coordinate system on H with transformation formulas (Set $d := (x_1^2 + x_3^2)^{1/2}$):

(1′) $\gamma = ((4d^2 + u^2)^{1/2} - u)/d, \quad u = d^2 + x_2^2 - 1$

(2′) $\tan 2\varphi = 2d \cdot x_2/(d^2 - x_2^2 + 1)$

(6) $\tan \vartheta = x_3/x_1$

and so on (see e.g. [7]). Now Luneburg assumes the following:

LH′ If $V \subset H$, then the visual metric is of the form

$$(M0') \quad ds^2 = \rho'(\gamma)^2\, d\gamma^2 + s(\rho(\gamma))(d\varphi^2 + \cos^2\varphi\, d\vartheta^2),$$

where ρ and s are the functions defined in *LH*.

It follows from *LH′* that the surfaces $\gamma = \text{const}$ (so called *Vieth-Muller-Tori*) are perceived as parts of euclidean spheres of radius $s(\rho(\gamma))$. Luneburg notes that this can be only a first order approximation [12, p. 633]. The sky ($\gamma = 0$) looks more like a flattened dome rather than like a sphere (moon phenomenon). Little work has been done in order to test LH' empirically. This is one reason why we will restrict our attention to the horizontal plane E. On the other hand, if LH' is a reasonable hypothesis, it suffices to determine the functions ρ and s by considering the visual geometry of U. In particular, if the curvature K is constant on U and if LH' is valid, then K will be constant on the whole of V.

1.4. Various Approaches to ρ and s

It is very difficult to determine the two functions ρ and s at once by means of experiments. So one has looked for further reasonable restrictions on the metric $(M0)$. The most restrictive hypotheses were made by Luneburg:

CCC (*Constant Curvature Condition*) [12, p. 631]

$K = \text{const}$, i.e. $s = \sin_K$ (see 3.2)

APC (*Angle Preservation Condition*) [11, p. 44, 45]

The *γ-shifts* $(\gamma, \varphi) \to (\gamma + \text{const}, \varphi)$ are angle-preserving, i.e. there is some constant $\sigma > 0$ and a function $m: (0, \infty) \to (0, \infty)$ such that

$$(M1)\quad ds^2 = m(\gamma)^2(\sigma^2\, d\gamma^2 + d\varphi^2).$$

CCC will be discussed later. APC was introduced by Luneburg in order to explain the distorted rooms of Ames (see 1.1). One constructs these by shifting the γ-coordinates of the walls of an ordinary room by a constant value. Since these rooms are visually indistinguishable, he first claimed that the γ-shifts should be isometries (iseiconic transformations) [11, p. 19]. But this turned out to be inconsistent with CCC [11, p. 47]. The best that one could do was to require the γ-shifts to be conformal mappings; this is APC.

Theorem 1 (Luneburg). *If one assumes* CCC *and* APC, *the metric* $(M0)$ *is uniquely determined by the constants* K *and* σ *up to equivalence. Any such metric is equivalent to one of the following:*

$$(M2)\quad ds^2 = 2(e^{\sigma\gamma} + K\, e^{-\sigma\gamma})^{-1}(\sigma^2\, d\gamma^2 + d\varphi^2),$$

where $\sigma > 0$ *and* $K \in [-1, \infty)$. *Any two of these metrics are not equivalent to each other. We have*

$$\rho(\gamma) = (\tan_{K/4})^{-1}(2\, e^{-\sigma\gamma}),$$

where $(\tan_k)^{-1}$ *denotes the inverse function of* $\tan_k$ (*see* 3.2). *One has*

$$(\tan_k)^{-1}(t) = \begin{cases} k^{-1/2}\tan^{-1}(k^{1/2}t) & \text{for } k > 0 \\ t & \text{for } k = 0 \\ (-k)^{-1/2}\tanh^{-1}((-k)^{1/2}t) & \text{for } k < 0 \end{cases}$$

and $|t| < (-k)^{-1/2}$.

(Proof: 3.3)

A weak version of this theorem was stated in [11, pp. 48, 56, 103, 104]. The metric ($M2$) appears again in [12, p. 636] without theoretical justification. Although the arguments in favour of APC are not cogent (see below), many authors use the metric ($M2$) [1, 9, 14,15]. Only the two constants K and σ are individual characteristics which have to be specified by experiments, according to this assumption.

After R. Luneburg's sudden death in 1949, his work was continued by his colleagues at Knapp Memorial Laboratories. They retained CCC, but replaced APC by the following hypothesis [2, 7]. For a given visual field $U \subset E$ call $\gamma_0 := \inf\{\gamma(x); x \in U\}$, and $\tilde{\rho}(\Gamma) := \rho(\Gamma + \gamma_0)$ for any $\Gamma > 0$.

VFC (Visual Field Condition)

The functions $\tilde{\rho}$ and s are the same for any visual field $U \subset E$.

This assumption perfectly explains the distorted rooms of Ames: If U_1 is the interior of some distorted room which looks like the interior of an ordinary room U_2, then the two configurations differ only by a γ-shift; by VFC, this is an isometry from U_1 onto U_2.

There are two new ideas in this theory. (i) The determination of the entire function ρ is left to the experiment (for one visual field). (ii) The visual metric depends explicitly on the visual field U (since γ_0 depends on U). However, it is not clear whether this dependence is always as simple as VFC states. The more extensive experimental tests of A. Zajaczkowska partially contradict this assumption: 'Neither Luneburg's mapping function nor that of Hardy and Blank can be considered as final' [15, p. 527]. A more recent investigation of Battro et al. [1] states that in large open fields even the sign of the curvature can vary if the visual field is changed. We do not discuss the point further in this paper.

Hardy, Blank et al. [7] also tried to justify CCC. They claimed that there are 2-dimensional totally geodesic submanifolds ('*visual planes*') in V in any given position and orientation (see also [2]). But this is difficult to verify by experiments. We believe that one should find a test of CCC which can be carried out in the horizontal plane E. We want to show that this is possible by means of some 'classical' experiments. The method has been indicated by Luneburg [12, p. 639] and Blank [2, p. 918]. We want to specify these ideas.

We assume in the following nothing but LH; hence the visual metric has the form ($M0$) for a fixed observer and a fixed visual field $U = E \cap V$, $V \subset H$.

2. *Empirical tests of CCC*

2.1. The Alley Experiments

These experiments are described in [7, pp. 20, 21] as follows: 'Two lights are fixed at the points $Q_1^+ = (\gamma_1, \varphi_1)$ and $Q_1^- = (\gamma_1, \varphi_1)$, equidistant from the observer and symmetric to the median. Other lights are then introduced successively in pairs $Q_n^\pm$ at predesignated stations approaching the observer. The observer is asked to adjust the pair $Q_n^\pm$ according to two different sets of instructions: (a) Adjust the lights $Q_2^\pm, Q_3^\pm, \ldots, Q_n^\pm$ until the two rows of lights appear to be straight, parallel to each other and parallel to the median.' The word 'parallel' is explained by 'neither converging nor diverging in distance'. '(b) With only the two lights $Q_1^{\pm}$ left on, set the pair $Q_u^{\pm}$ to appear symmetric to the median and to have the same apparent separation as the two fixed lights. The result of experiment (a) is called a *parallel alley*: of experiment (b), a *distance alley*.' (See also [2, 3, 12, 13, 15].)

In the visual space, the distance alley may be characterized as a locus of constant distance from the median ray (x_1-axis). The parallel alleys are not so easy to interpret. Call *parallel lines* the 'walls' of these alleys. According to Luneburg, these are 'the visual geodesics which are sensed as being perpendicular to the subjective frontal plane.' [7, p. 21] The subjective frontal plane is not the x_2x_3-plane, since the metric ($M0$) cannot be extended to the x_2-axis. It is a visual plane orthogonal to the median which contains the egocenter 0. Hence it is not a subset of V. In order to make the characterization of Luneburg precise (see PAA below), we have to enlarge the visual space (U, ds^2), as Luneburg does for the special case of constant curvature. Call $\rho_0 := \sup\{\rho(\gamma(x));\, x \in U\}$. Then we define the *enlarged visual space* (M, ds^2) to be the open disk $M = \{x \in \mathbb{R}^2;\, x_1^2 + x_2^2 < \rho_0^2\}$ together with the metric

$$(M0_e)\quad ds^2 = dr^2 + s(r)^2\, d\varphi^2$$

where (r, φ) are the ordinary polar coordinates on M (see 3.2). The visual space (U, ds^2) is isometrically imbedded in M by the mapping $f\colon U \to M, f(x) = \rho(\gamma(x), \varphi(x))$. (We are writing $x = (r(x), \varphi(x))$ for any $x \in M$, $x \neq 0$.) U is a part of the physical space, but the embedding of U into the physical space cannot be extended to M.

Now we can interpret the parallel alleys. Call $g\colon (0, \rho_0) \to M$ the median normal geodesic ray; hence $\varphi(g(u)) = 0$ for all $u \in (0, \rho_0)$. (*Normal* means parametrization with respect to arc length.) Let $h\colon (-\rho_0, \rho_0) \to M$ be the normal geodesic with $h(0) = 0$ and $(h(u)) = \pi/2$ for all $u \in (0, \rho_0)$. By identifying U and $f(U)$, we consider U as a subset of M. The assumptions of Luneburg can now be stated as follows.

UPA (*Unique Perpendicular Assumption*)

There is a neighborhood G of $g([0, \rho_0))$ in M such that any $x \in G$ can be joined to h by a unique geodesic (up to parametrization) p_x in G which is perpendicular to h.

PAA (*Parallel Alley Assumption*)

For any $x \in U \cap G$, $p_x \cap U$ is the parallel line through the point x.

The first condition UPA says that the ray g has no focal points; this is equivalent to saying that $s' > 0$ on $(0, \rho_0)$ which is automatically true if $K \leqq 0$. Otherwise it says that the visual space is not too large, e.g. isometric to a part of a half sphere, if $K = \text{const} > 0$. This is purely technical in nature. The second condition PAA is crucial.

If CCC holds and UPA and PAA are valid, the results of the alley experiments can be predicted as follows (see [7] for more details). If $(r, \varphi) \in U \subset M$ is a point of distance d from g, we have by the sine law for rectangular triangles (Eq. (5) in 3.2): $\sin_K d = \sin\varphi \cdot \sin_K r$. Hence, if d_x is the *distance line* (wall of the distance alley) through x, then for all points (r, φ_{dr}) on d_x we have

$$\sin\varphi_{dr}\, \sin_K r = \sin\varphi(x)\, \sin_K r(x). \tag{1}$$

On the other hand, if (r, φ) is a point on a parallel line p_x which meets h at $h(u)$, we have by the cosine law for rectangular triangles (Eq. (6) in 3.2): $\sin\varphi = \cos[(\pi/2) - \varphi] = \tan_K u/\tan_K r$. Hence it follows for all points (r, φ_{pr}) on p_x

$$\sin\varphi_{pr}\tan_K r = \sin\varphi(x)\tan_K r(x). \tag{2}$$

Using the equations (1) and (2) in 3.2, one can eliminate $\sin_K r$ and $\tan_K r$ from the preceding equations (1) and (2), and the result is

$$\frac{\sin^2\varphi_{dr} - \sin^2\varphi(x)}{\sin^2\varphi_{pr} - \sin^2\varphi(x)} = \cos_K r(x). \tag{3}$$

The left-hand side of Eq. (3) can be determined by the alley experiments for any $r = \rho(\gamma)$. If CCC holds, this has to be the same, whatever γ and $\varphi(x)$ are, if $\gamma(x)$ is fixed. Also the converse is true, as Blank states [2, p. 918]:

Theorem 2. *Assume* UPA, PAA. *For $x \in U \cap G$ and any small enough $\gamma \geqq \gamma(x)$, let $(\gamma, \varphi_{d\gamma}(x))$ be on the distance line d_x and $(\gamma, \varphi_{p\gamma}(x))$ on the parallel line p_x. Assume that the quotient*

$$q(x, \gamma) \coloneqq \frac{\sin^2\varphi_{d\gamma}(x) - \sin^2\varphi(x)}{\sin^2\varphi_{p\gamma}(x) - \sin^2\varphi(x)}$$

is constant with respect to γ and $\varphi(x)$ and depends only on $\gamma(x)$. Then the metric $(M0)$ has constant curvature on U.

(Proof: 3.4).

Hence CCC can be tested by the alley experiments. If CCC is valid, the metric $(M0)$ can be determined completely. The sign of the curvature K is given by (3), since $K \gtreqless 0$ iff $\cos_K r \lesseqgtr 1$ for arbitrary $r \in (0, \rho_0)$, hence iff $\varphi_d \lesseqgtr \varphi$. If $K \neq 0$, one can use equivalence to fix $K = \pm 1$, hence $s = \sin_K$ equals sin resp. sinh. $\rho(\gamma) = r$ then can be computed from (3) and (1). If $K = 0$, one has to fix $r(x)$ arbitrarily for some x and to compute $\rho(\gamma) = r$ by (1).

2.2. The Double Vieth-Müller circle Experiments

(a) The Three-Point Experiment

Consider the two Vieth-Müller circles associated with two given values $\gamma_1 < \gamma_2$ of the bipolar parallax. Let $Q_0 = (\gamma_1, \varphi_0)$ and $Q_1 = (\gamma_1, \varphi_1)$ be two points moving on the outer circle $\gamma = \gamma_1$ and let $Q_2 = (\gamma_1, \varphi_2)$ be a freely adjustable point on the circle $\gamma = \gamma_2$. The observer is asked to leave Q_0 and Q_1 fixed and to adjust the point Q_2 so that the sensed distance from Q_1 to Q_0 equals the sensed distance from Q_0 to Q_2. (See [7], p. 25, also [2, 12].)

Which relation holds between $\psi_1 \coloneqq \varphi_1 - \varphi_0$ and $\psi_2 \coloneqq \varphi_2 - \varphi_0$ if CCC is valid? By the cosine law (see Eq. (4a), (4b) in 3.2) there are certain functions F, G, H such that for any geodesic triangle with side lengths a, b, c and angle C opposite to c

$$F(c) = G(a, b) + H(a, b)\cos C.$$

Hence with $a := \rho(\gamma_1)$, $b := \rho(\gamma_2)$, $c = |Q_1, Q_0| = |Q_2, Q_0|$ (distance with respect to the visual metric ($M0$)) we get for the triangles $(0, Q_0, Q_1)$ and $(0, Q_0, Q_2)$

$$F(c) = G(a, a) + H(a, a) \cos \psi_1,$$

$$F(c) = G(a, b) + H(a, b) \cos \psi_2.$$

Hence the two cosines are affinely related:

$$\cos \psi_1 = M \cos \psi_2 + N \tag{1}$$

where

$$M = H(a, b)/H(a, a) = \sin_K b/\sin_K a, \tag{2}$$

$$N = \frac{G(a, b) - G(a, a)}{H(a, a)} = \begin{cases} \dfrac{\cos_K a(\cos_K b - \cos_K a)}{K \sin_K^2 a} & \text{if } K \neq 0 \\ (1 - b^2/a^2)/2 & \text{if } K = 0. \end{cases} \tag{3}$$

M and N can be determined by the experiment. If $K = 0$, we have $2N = 1 - M^2$, hence

$$N^2/((N - 1)^2 - M^2) = 1. \tag{4a}$$

If $K \neq 0$, we get from (2) using $K \sin_K^2 = 1 - \cos_K^2$ (see 3.2)

$$\cos_K^2 b = 1 - M^2 K \sin_K^2 a$$

and from (3)

$$\cos_K^2 b = N^2 K^2 \sin_K^4 a/\cos_K^2 a + 2NK \sin_K^2 a + 1 - K_K^2 \sin a.$$

Combining these two equations, we end up with

$$N^2/((N - 1)^2 - M^2) = \cos_K^2 a. \tag{4b}$$

(See [7] for more details)

If CCC holds, we can decide also by this experiment which sign K has. Call P the left-hand side of (4a) and (4b): we have $K \gtreqless 0$ iff $P \lesseqgtr 1$. For $K \neq 0$, we can fix $K = \pm 1$ again and compute $\rho(\gamma_1) = a$ and $\rho(\gamma_2) = b$ by means of (4b) and (2). If $K = 0$, we may fix $\rho(\gamma_1) = a$ arbitrarily and compute $\rho(\gamma_2) = b$ by (2).

It is not known whether conversely CCC can be derived from (1) alone, since this gives us no information about the isosceles triangles. We can get the latter by another experiment which also gives a more exact method for the determination of the slope M in Equation (1).

(*b*) *The Four-Point Experiment*

Let $Q_1 = (\gamma_1, \varphi_1)$ and $Q_2 = (\gamma_1, \varphi_2)$ be two points on the circle $\gamma = \gamma_1$ and let $Q_3 = (\gamma_2, \varphi_3)$ and $Q_4 = (\gamma_2, \varphi_4)$ be two other points which slide on the circle $\gamma = \gamma_2$ (γ_1, γ_2 as above). The observer is asked to equate the sensed distance from Q_3 to Q_4 to the sensed distance from Q_1 to Q_2. [7, p. 27; 12, 2].

Again we may predict what happens if CCC holds. Set $a = \rho(\gamma_1)$, $b = \rho(\gamma_2)$ as before, and call $\psi_3 = \varphi_1 - \varphi_2$, $\psi_4 = \varphi_3 - \varphi_4$. Call $d := |Q_1, Q_2| = |Q_3, Q_4|$. Then by the cosine law (see (a)) we have

$$G(a, a) + H(a, a) \cos \psi_3 = F(d) = G(b, b) + H(b, b) \cos \psi_4.$$

Moreover, if $d = 0$ we have $\psi_3 = \psi_4 = 0$, hence

$$G(a, a) + H(a, a) = F(0) = G(b, b) + H(b, b).$$

So we derive the relation

$$\frac{1 - \cos \psi_3}{1 - \cos \psi_4} = \frac{H(b, b)}{H(a, a)} = \frac{\sin_K^2 b}{\sin_K^2 a} = M^2 \tag{5}$$

for arbitrary $K \in \mathbb{R}$.

(*c*) *Application to the CCC-Problem*

One can use a combination of both experiments for a test of CCC. For this, one has to be sure that there are enough isosceles triangles:

ITA (*Isosceles Triangle Assumption*)

For all values a, c with $c \leqslant a < \rho_0$, there exists an isosceles triangle $(0, Q_1, Q_2)$ in U with side lengths $a = |0, Q_1| = |0, Q_2|$, $c = |Q_1, Q_2|$.

Theorem 3. *Assume* ITA. *Assume that for all* γ_1, γ_2 *with* $\gamma_2 > \gamma_1 > \gamma_0 := \inf\{\gamma(x); x \in U\}$, *there exist real constants* L, M, N *such that in the three-point and the four-point experiments for any choice of* ψ_1 *and* ψ_3 (*notation as above*) *the following relations hold*:

(a) $\cos \psi_1 = M \cos \psi_2 + N$

(b) $1 - \cos \psi_3 = L(1 - \cos \psi_4)$.

Then the metric ($M0$) *on* U *fulfills* CCC.

(Proof: 3.5)

Unfortunately, the assumption ITA cannot be verified directly since the distances are not measurable. A necessary condition is that the perceived φ-domain is larger than 60 angular degrees. By the Rauch comparison theorem (see [4]), this is also sufficient if $K \leqslant 0$. In the general case, one may replace ITA in Theorem 3 by the following two assumptions which are easy to verify:

(i) The perceived φ-domain is larger than 90 degrees.

(ii) UPA holds.

The argument goes as follows: If $e \in (0, \rho_0)$ is small enough, ITA holds for all $a, c < e$, by (i). This is because small triangles in M look nearly like euclidean triangles. Call e_0 the supremum of all such e. We have to show $e_0 = \rho_0$. Assume that this is not true, i.e. $e_0 < \rho_0$, then it follows by (ii) that $s'(e_0) > 0$. Now by Theorem 3, CCC holds on the subdomain $M' := \{x \in M; r(x) < e_0\}$ of the enlarged visual space M. (Replace ρ_0 by e_0!) Claim: ITA is fulfilled on M' even for all a, c with

$c < a + \varepsilon$ for some positive ε. If $K \leqslant 0$ on M', this is clear by (i); if $K > 0$, then it follows because M' is isometric to a small disk on a sphere of radius $1/K^{1/2}$, and this disk lies on one half sphere since $s'(e_0) = \sin'_K(e_0) > 0$. Therefore, by continuity ITA is fulfilled also for all a, c with $c \leqslant a < e_0 + \delta$ for some positive δ. This is a contradiction to the choice of e_0.

A result similar to Theorem 3 has been conjectured by Luneburg [12, p. 639], but the argument he gave is only a transformation of the problem.

2.3. Discussion

We do not see any good reason why CCC *a priori* should be valid. Free movability of rigid bodies cannot justify this assumption since the same body will have apparently different size in different positions. Also the lack of absolute localization proved by the distorted rooms of Ames (compare [12, p. 631]) does not give an argument for CCC, as we saw in 1.4[1].

On the other hand, this assumption is crucial for the known empirical methods in order to determine the functions ρ and s of the metric ($M0$). Hence we have to look for an empirical test of CCC. For simplicity, this should be performable in the horizontal plane E.

There are several experiments which are *negative* tests of CCC. By this we mean the following. Under the assumption of CCC certain relations between measurable data have to hold; these can be examined by experiments. E.g. the experiments of the equipartitioned geodesics [7, 2] belong to this group. As far as we know, however, the alley and double VMC experiments described above are the only performed *positive* tests of CCC: they examine a set of relations which are valid if and *only if* CCC is true.

The alley experiments have been reported by many authors [3, 7, 12, 13, 15], but only Shipley [13] specifies the variance of the quotient $q(x, \gamma)$ in Theorem 2 with respect to γ (unfortunately not with respect to $\varphi(x)$, too). Most of his results support CCC. However, one has to suppose always the condition PAA which is somewhat arbitrary and 'conspicuously absent in the instructions to the observers', as Blank states [2, p. 919]. On the other hand, the term 'parallel' together with the explanation 'walls that appear neither to converge nor to diverge' [7, p. 49] is difficult to understand and not unequivocal. E.g. if the visual geometry is supposed to be hyperbolic (constant $K < 0$), *any* two geodesics either converge or diverge. There are also several observers for which in Shipley's experiments the quotient $q(x, \gamma)$ does depend on γ in a systematic (not random) way. This irregularity could arise

[1] We wish to mention Luneburg's interesting effort to justify CCC with $K < 0$ on planes of constant physical height h below the horizontal plane E by what he called Cyclopean projection [11, p. 51–56]. However, according to his theory developed in 1.3, these planes are not totally geodesic with respect to the visual metric. Moreover, he overlooked the following: If he were right, the curvature value would tend to $-\infty$ for $h \to 0$. Hence this metric cannot be extended to E.

from the difficulties mentioned above. See [2, p. 919, 920] for a more explicit discussion of the point.

Therefore, the second quite independent CCC-test by means of the double VMC experiments is very important. One does not need additional assumptions, and the instructions to the observers are simple and clear. Moreover, one can also test by these experiments whether the φ-shifts are really isometries. The reported results [2, 7, 9, 14] impressively agree with the predictions following from CCC. However, none of these performed experiments are sufficient for a *positive* CCC-test. Since Zajaczkowska [14] and Indow et al. [9] worked with the assumption of the Luneburg metric ($M2$), only the two constants K and σ had to be determined. For this it was sufficient to consider two fixed γ-values γ_1 and γ_2. Hardy et al. [7] only assumed CCC, therefore the function ρ had to be determined entirely by experiment. Hence they fixed the γ_1-value and only varied γ_2. For a positive CCC test, we need the values $L(\gamma_1, \gamma_2)$, $M(\gamma_1, \gamma_2)$, $N(\gamma_1, \gamma_2)$ for arbitrary $\gamma_1 < \gamma_2$, hence both VMCs have to be varied.

Comparing the methods 2.1 and 2.2, we note a remarkable difference. The parallel alley experiment uses the immediate impression of straightness, for the double VMC experiments, the perception of distance is needed. If the results of both experiments agree, this is a hint that the two defining axioms VM_1 and VM_2 are compatible. This was not clear at the beginning, since we saw that VM_2 alone characterizes the visual metric. The compatibility of both axioms means that apparently straight lines are shortest with respect to our intuitive distance perception.

It was not our purpose to vote in favour or against CCC. We only believe that this assumption is basic for Luneburg's theory of binocular vision, hence we want to propose a systematic empirical test of CCC along the lines developed above.

3. *Mathematical Appendix*

3.1. Uniqueness of a Visual Metric

Let ds_1^2 and ds_2^2 be two Riemannian metrics on V which both obey to VM_2. Calling two pairs of points equivalent if they have the same distance, this yields the same equivalence relation with respect to either metric. Call $TV = V \times \mathbb{R}^3$ the set of all tangent vectors of V. Then for any two vectors $v, w \in TV$ we have $\|v\|_1 = \|w\|_1$ if and only if $\|v\|_2 = \|w\|_2$, where $\|x\|_i$ for any $x \in TV$ denotes the length of the vector x with respect to ds_i^2, $i = 1, 2$. In particular, if $\|v\|_1 = 1$ and $\|v\|_2 = r$ for some $r > 0$, we have $\|w\|_2 = r$ for all $w \in TV$ such that $\|w\|_1 = 1$. Hence, by homogeneity, we have $\|w\|_2/\|w\|_1 = r$ for arbitrary $w \in TV$, therefore $ds_2^2 = r^2\, ds_1^2$.

3.2. Some Geometry of Constant Curvature

Throughout this paper, we are using the following model space (M, ds^2) for the 2-dimensional geometry of constant curvature K. Let M be the euclidean disk

$$M = \{x = (x_1, x_2) \in \mathbb{R}^2;\ x_1^2 + x_2^2 < r_0^2\}$$

for a suitable number $r_0 \in (0, \infty]$. Introduce polar coordinates (r, φ) on M:

$$r(x) = (x_1^2 + x_2^2)^{1/2} \quad \text{for all } x \in M,$$
$$\cos \varphi(x) = x_1/r(x), \qquad \sin \varphi(x) = x_2/r(x), \quad \varphi(x) \in (-\pi, \pi]$$

for $x \neq 0$. The metric is defined as follows:

$$ds^2 = dr^2 + \sin_K^2 r \, d\varphi^2$$

where $\sin_K$ is the solution of the initial value problem

$$\sin_K'' + K \sin_K = 0, \qquad \sin_K(0) = 0, \qquad \sin_K'(0) = 1.$$

We have

$$\sin_K r = \begin{cases} K^{-1/2} \sin (K^{1/2} r) & \text{if } K > 0 \\ r & \text{if } K = 0 \\ (-K)^{-1/2} \sinh ((-K)^{1/2} r) & \text{if } K < 0 \end{cases}$$

We also introduce the functions $\cos_K \coloneqq \sin_K'$ and

$$\tan_K \coloneqq \sin_K/\cos_K. \tag{1}$$

If $K > 0$, $\tan_K$ is defined on the interval $(-\pi/2K^{1/2}, \pi/2K^{1/2})$. In this case, we require $r_0 \leqslant \pi/2K^{1/2}$ which follows from UPA (see 2.1). Moreover, we have the useful relation

$$K \sin_K^2 + \cos_K^2 = 1, \tag{2}$$

since $(K \sin_K^2 + \sin_K'^2)' = 2 \sin_K' (K \sin_K + \sin_K'') = 0$.

The trigonometry of constant curvature is ruled by the sine and the cosine law (see [5]): suppose that we have a geodesic triangle in M with side lengths a, b, c and angles A, B, C opposite to the corresponding sides. We have the sine law

$$\frac{\sin A}{\sin_K a} = \frac{\sin B}{\sin_K b} = \frac{\sin C}{\sin_K c} \tag{3}$$

and the cosine law for the angle C

$$\cos_K c = \cos_K a \cos_K b + K \sin_K a \sin_K b \cos C \tag{4a}$$

if $K \neq 0$, and

$$c^2 = a^2 + b^2 - 2ab \cos C \tag{4b}$$

if $K = 0$.
For rectangular triangles ($C = \pi/2$) we have in particular

$$\sin A = \sin_K a/\sin_K c \tag{5}$$

by (3). Moreover, from the cosine laws for the angles A and C we can eliminate a and get for $K \neq 0$

$$\cos_K c/\cos_K b = \cos_K b \cos_K c + K \sin_K b \sin_K c \cos A,$$

hence

$$\cos A = \cos_K c(1 - \cos_K^2 c)/K \sin_K b \sin_K c \cos_K b,$$

and using (2) and (1), we get

$$\cos A = \tan_K b/\tan_K c \tag{6}$$

which is also valid if $K = 0$.

3.3. Proof of Theorem 1

A comparison of $(M1)$ and $(M0)$ (with $s = \sin_K$, CCC) shows that one has to solve the ordinary system

$$\rho'^2 = \sigma^2 m^2; \qquad (\sin_K \circ \rho)^2 = m^2. \tag{1}$$

This is equivalent to solving the differential equation

$$\rho'^2 = \sigma^2(\sin_K \circ \rho)^2 \tag{2}$$

and setting $m^2 = \rho'^2/\sigma^2$. According to LH, the function ρ in $(M0)$ has to be positive, its derivative negative, so we get from (2)

$$\rho' = -\sigma \sin_K \circ \rho. \tag{3}$$

We are looking for all positive solutions ρ of (3) such that $\lim_{\lambda \to \infty} \rho(\gamma) = 0$ as it was demanded in LH.

Assertion. These are the following functions $\rho_{K,c}\colon (0, \infty) \to (0, \infty)$

$$\rho_{K,c}(\gamma) = (\tan_{K/4})^{-1}(c\, e^{-\sigma\gamma}) \tag{4}$$

where $c > 0$ is an arbitrary constant with $c \leqslant 2/(-K)^{1/2}$ if $K < 0$.

Proof. One has

$$((\tan_{K/4})^{-1})'(t) = 1\Big/\left(1 + \frac{K}{4} t^2\right).$$

Call $R(\gamma) := c\, e^{-\sigma\gamma}$, then $R' = -\sigma R$. Hence

$$(\rho_{K,c})' = -\sigma R\Big/\left(1 + \frac{K}{4} R^2\right). \tag{5}$$

On the other hand we have for all $z \in \mathbb{R}$ the formula $\sin 2z = 2 \sin z \cos z = 2 \tan z/(\tan^2 z + 1)$. This formula extends also to all $z \in \mathbb{C}$ where the right-hand side is defined. From this we get for all $K \in \mathbb{R}$, $t \in \mathbb{R}$:

$$\sin_K t = \tan_{K/4} t\Big/\left(1 + \frac{K}{4} \tan_{K/4}^2 t\right).$$

Using (5), one gets that $\rho_{K,c}$ is a solution:

$$\sin_K \circ \rho_{K,c} = R\Big/\left(1 + \frac{K}{4} R^2\right) = -(\rho_{K,c})'/\sigma.$$

The solutions of (3) are in 1-1-correspondence to the initial values. For $K \leqslant 0$, $\rho_{K,c}(0) = (\tan_{K/4})^{-1}(c)$ can be an arbitrary positive number. Hence we found all positive solutions in this case, and they all fulfill $\lim_{\gamma\to\infty} \rho(\gamma) = (\tan_{K/4})^{-1}(0) = 0$. If $K > 0$, then the set of initial values of the solutions (4), $\rho_{K,c}(0) = 2K^{-1/2} \tan^{-1} (\frac{1}{2}K^{1/2}c)$, is the interval $(0, \pi/K^{1/2})$. The solutions with initial values $n\pi/K^{1/2}$, $n \in \mathbb{N}$, are constants, and the remaining positive solutions are of the type $\rho_{K,c} + n\pi/K^{1/2}$. But these two types of solutions do not fulfill $\lim_{\gamma\to\infty} \rho(\gamma) = 0$, but only those of type (4), so the assertion is proved.

If ρ is a solution of (3) with curvature value K, then $\tilde{\rho} \coloneqq k\rho$ for arbitrary $k > 0$ solves (3) with K replaced by $\tilde{K} = K/k^2$. This is because $\sin_{\tilde{K}} (k\rho) = k \sin_K \rho$. The two corresponding metrics, $ds^2 = \rho'(\gamma)^2\, d\gamma^2 + \sin_K^2 \rho\, d\varphi^2$ and $d\tilde{s}^2 = \tilde{\gamma}'(\gamma)^2\, d\gamma^2 + \sin_{\tilde{K}}^2 \tilde{\rho}\, d\varphi^2$ are related by $d\tilde{s}^2 = k^2\, ds^2$, hence equivalent. Now, if $\rho = \rho_{K,c}$, we have $\tilde{\rho} = \rho_{\tilde{K},kc}$. Therefore, one can normalize the metric by choosing $c = 2$ (say). Each two distinct solutions $\rho_K \coloneqq \rho_{K,2}$ are not equivalent. However, by the restriction $c \leqslant 2/(-K)^{1/2}$ for $K < 0$, one has to assume $K \geqslant -1$. This is no restriction: if $K < -1$ and $\rho_{K,c}$ is a solution, then $c \leq 2/(-K)^{1/2}$. Now $\rho_{K,c}$ is equivalent to $\rho_{\tilde{K}}$ with $\tilde{K} = 4K/c^2 \geqslant -1$.

The solution ρ_{-1} plays a particular role: it is the unique one with $\rho(0) = \infty$. This would mean for a visual metric that an impression of infinite distance is possible.

3.4. Proof of Theorem 2

1. We are working in the extended visual space M. U is considered as a subset of M. Let us write $x = (r(x), \varphi(x))$ for all $x \in M$, where (r, φ) are the polar coordinates introduced in 3.2. Let $K\colon M \to \mathbb{R}$ be the Gaussian curvature function and $k = K \circ g$, where $g\colon [0, \rho_0) \to M$ is the median ray ($\varphi = 0$), parametrized with respect to arc length. It is sufficient to prove that k is constant since the curvature function of the metric $(M0_e)$ is the same along any geodesic ray emanating from 0. Let $a \in (0, \rho_0)$ be fixed arbitrarily. We will show that $k|_{[0,a]}$ is constant.

2. Call $x(t) \coloneqq (a, t)$ for $t \in I = (-\varepsilon, \varepsilon)$. If $\varepsilon > 0$ is small enough, then $x(t) \in G$ for all such t (compare UPA). Call d_t the distance line and p_t the parallel line through $x(t)$. Parametrize both curves with respect to their r-coordinate, i.e. a parameter u is chosen such that $r(d_t(u)) = r(p_t(u)) = u$. Fix $b \in (0, a]$ arbitrarily and call

$$\varphi_d(t) \coloneqq \varphi(d_t(b)),$$
$$\varphi_p(t) \coloneqq \varphi(p_t(b)).$$

By assumption of the theorem, there exists a constant q (only dependent on $a = r(x(t))$) such that

$$\sin^2 \varphi_d(t) - \sin^2 t = q \quad (\sin^2 \varphi_p(t) - \sin^2 t) \tag{1}$$

for all $t \in I$. Differentiating twice and taking the value at $t = 0$, one gets

$$\varphi_d'(0)^2 - 1 = q(\varphi_p'(0)^2 - 1), \tag{1$''_0$}$$

because $\varphi_d(0) = \varphi_p(0) = 0$. We will now compute the values of $\varphi_d'(0)$ and $\varphi_p'(0)$ in terms of the fundamental solutions of the Jacobi differential equation

$$(J)\ j'' + kj = 0.$$

Call $s, c\colon [0, a] \to \mathbb{R}$ the solutions of (J) with

$$s(0) = 0, \qquad s'(0) = 1; \qquad c(0) = 1, \qquad c'(0) = 0.$$

3. For this, consider the curves $e(t) := p_t(b) = (b, \varphi_p(t))$ and $f(t) := d_t(b) = (b, \varphi_d(t))$ for $t \in I$. We have

$$e'(0) = (d/dt)_{t=0} \quad (b, \varphi_p(t)) = \varphi_p'(0)F(b, 0)$$

where $F(r, \varphi)$ is the vector field $(\partial/\partial\varphi)(r, \varphi)$. Similarly,

$$f'(0) = \varphi_d'(0)F(b, 0).$$

Now $A(u) := F(u, 0)$ is a Jacobi field along g with initial values $A(0) = 0$, $A'(0) = h'(0)$; $h\colon (-\rho_0, \rho_0) \to M$ was the geodesic perpendicular to g as defined in 2.1. Call $E(u)$ the parallel vector field along g with $E(0) = h'(0)$. Then we have $A(u) = s(u)E(u)$ for $0 \leqslant u \leqslant a$, since both sides of this equation are Jacobi fields with the same initial values. Hence it follows

$$e'(0) = \varphi_p'(0)s(b)E(b), \tag{2}$$

$$f'(0) = \varphi_d'(0)s(b)E(b). \tag{3}$$

4. There is a quite different description of the left-hand sides of Eq. (2) and (3). Reparametrize d_t and p_t as follows: Let $\tilde{d}_t(u)$ be the nearest point to $g(u)$ on the curve d_t for $0 \leqslant u \leqslant a$, and let $\tilde{p}_t\colon [0, a] \to M$ be the curve p_t parametrized with constant velocity ($\|\tilde{p}_t'\| = \text{const}$) such that $\tilde{p}_t(0)$ lies on h and $\tilde{p}_t(a) = x(t)$. Call $v(t, u)$, $w(t, u)$ the parameter transformations: $\tilde{p}_t(u) = p_t(v(t, u))$, $\tilde{d}_t(u) = d_t(w(t, u))$. Then we have $v(0, u) = u$ for all $u \in [0, a]$ and

$$(\partial/\partial t)_{t=0}\tilde{p}_t(u) = (\partial/\partial t)_{t=0}p_t(u) + \frac{\partial v}{\partial t}(0, u)g'(u).$$

The second term of the right-hand side has to vanish: Since the boundary curves of the variation $t \mapsto \tilde{p}_t(u)$ for $u = 0$ and $u = a$ intersect the geodesic g perpendicularly and the u-parameter is proportional to arc length, all curves $t \mapsto \tilde{p}_t(u)$ for $0 \leqslant u \leqslant a$ have to be orthogonal to g, hence the g'-component of $(\partial/\partial t)_{t=0}\tilde{p}_t(u)$ has to vanish. Using the abbreviation $B(u) = (\partial/\partial t)_{t=0}\tilde{p}_t(u)$, one gets

$$B(u) = (\partial/\partial t)_{t=0}p_t(u), \tag{4}$$

in particular

$$B(b) = e'(0). \tag{4a}$$

If we define the vector field D along g by $D(u) := (\partial/\partial t)_{t=0}\, \tilde{d}_t(u)$, we get by a similar consideration

$$D(u) = (\partial/\partial t)_{t=0}\, d_t(u), \tag{5}$$

in particular

$$D(b) = f'(0). \tag{5a}$$

This is because the curves $t \mapsto \tilde{d}_t(u)$ intersect g perpendicularly by the very definition of the parameter u of $\tilde{d}_t$.

5. It is easy to compute the vector fields B and D. B is a Jacobi field since the curves $\tilde{p}_t$ are geodesics. We have the boundary conditions (using (4))

$$B(a) = x'(0) = (d/dt)_{t=0}(a, t) = F(a, 0) = s(a)E(a)$$

and

$$B'(0) = \frac{D}{\partial u}\frac{\partial}{\partial t}\tilde{p}_t(u)|_{t=0} = \frac{D}{\partial t}\frac{\partial}{\partial u}\tilde{p}_t(u)|_{t=0} = \frac{D}{dt}l(t)G(t)|_{t=0}$$

where $l(t)$ denotes the length of p_t and $G(t)$ is the parallel vector field along h with $G(0) = g'(0)$; $D/\partial t$ is the covariant derivative of a vector field with respect to t. We have $l'(0) = 0$ since both boundary curves of the variation p_t are perpendicular to g as we noticed in *4*. So we end up with

$$B'(0) = 0$$

since G is parallel. There is exactly one Jacobi field with these boundary conditions (because of UPA), namely

$$B(u) = \frac{s(a)}{c(a)}c(u)E(u).$$

So we get by (2) and (4a)

$$\varphi_p'(0) = s(a)c(b)/c(a)s(b). \tag{6}$$

The vector field D along g has constant length and is orthogonal to g', by the definition of $\tilde{d}_t$. Hence D is parallel. It suffices to compute $D(a)$. By (5), we have

$$D(a) = (d/dt)_{t=0}\, d_t(a) = x'(0) = s(a)E(a).$$

It follows for all $u \in [0, a]$

$$D(u) = s(a)E(u),$$

hence by (3)

$$\varphi_d'(0) = s(a)/s(b). \tag{7}$$

6. Now we insert (6) and (7) into $(1)_0''$ and get for all $b \in (0, a]$

$$(q - 1)s(b)^2 = q(s(a)^2/c(a)^2)c(b)^2 - s(a)^2.$$

So there are constant numbers R, S, T such that for all $b \in (0, a]$

$$Rs(b)^2 = Sc(b)^2 + T. \tag{8}$$

We differentiate this equation with respect to b.

$$0 = Rss' - Scc' \tag{8)'$$

$$\begin{aligned} 0 &= R(s'^2 - ks^2) - S(c'^2 - kc^2) \qquad (8)'' \\ &= Rs'^2 - Sc'^2 + kT \end{aligned}$$

using the equations (J) and (8). Hence with (J) again and (8)$'$

$$\begin{aligned} Tk' &= 2Rkss' - 2Skcc' \\ &= 2k(Rss' - Scc') = 0. \qquad (8)''' \end{aligned}$$

Since $T = s(a)^2 \neq 0$, we get $k' = 0$, so k is constant.

3.5. Proof of Theorem 3

We want to show a cosine law for triangles in $U \subset M$ with vertex in 0. Then CCC follows from [6].

Fix some arbitrary $e \in (0, \rho_0)$. We will show CCC on $U_e \coloneqq U \cap \{x \in M; r(x) < e\}$. Let $(0, P, Q)$ be a given triangle in U_e with $a = r(P)$, $b = r(Q)$, $c = |P, Q|$, $C = |\varphi(P) - \varphi(Q)|$. We may assume $b \leqslant a < e$ and $\varphi(P) > \varphi(Q)$. We consider only those triangles with $c < a$. We have to construct certain functions F, G, H independent of the given triangle such that

$$F(c) = G(a, b) + H(a, b) \cos C. \tag{1}$$

By ITA, there are points $Q_1, Q_2, Q_3, Q_4 \in U$ such that $r(Q_1) = r(Q_2) = e$, $r(Q_3) = r(Q_4) = a$, and $|Q_1, Q_2| = |Q_3, Q_4| = c$. We may assume $\varphi(Q_1) > \varphi(Q_2)$ and $\varphi(Q_3) > \varphi(Q_4)$. Call $D = \varphi(Q_1) - \varphi(Q_2)$ and $E = \varphi(Q_3) - \varphi(Q_4)$. Since the φ-shifts are isometries, we do not restrict generality by choosing $P = Q_4$: it suffices to prove (1) for arbitrarily narrow triangles $(0, P, Q)$ in U_e. Now we apply the three-point experiment to Q_3, $Q_4 = P$, Q and the four-point experiment to Q_1, Q_2, Q_3, Q_4. By the assumptions (a) and (b) of the theorem we have

$$\cos E = M(a, b) \cos C + N(a, b), \tag{2}$$

$$-\cos E = (1 - \cos D(c, e))/L(a, e) - 1. \tag{3}$$

(2) + (3) yields (1) if we set

$$F(c) = \cos D(c, e),$$

$$G(a, b) = L(a, e)(N(a, b) - 1) + 1,$$

$$H(a, b) = L(a, e)M(a, b),$$

recall that e was fixed.

It remains to prove the regularity of F, G, H which is assumed in [6]. Consider the triangles $(0, P_0, P_t)$ with $P_t = (e, t)$ for all $t \in [0, \pi)$. These are probably not in U but that does not matter for the definition of F. Let $c(t) := |P_0, P_t|$; this is a smooth function with $c' > 0$. Now $F(c(t)) = \cos t$, hence F is a smooth function with $F'(c) < 0$ for all $c \in (0, 2\rho_0)$.

By similar arguments, the angles $\psi_1, \psi_2, \psi_3, \psi_4$ in the equations (*a*) and (*b*) of the theorem are depending smoothly on the distances of the triangle vertices, hence L, M, N depend smoothly on these distances, by (*a*) and (*b*). This is clear for L. Now consider the case $\psi_2 = 0$, i.e. $|Q_0, Q_1| = \rho(\gamma_1) - \rho(\gamma_2)$ in the three-point experiment. Call ψ_1^0 the corresponding ψ_1-value, this depends smoothly on $\rho(\gamma_1)$ and $\rho(\gamma_2)$. By (*a*), we have

$$\cos \psi_1^0 = M + N.$$

Inserting this in Equation (a) one gets

$$M = (\cos \psi_1^0 - \cos \psi_1)/(1 - \cos \psi_2).$$

So M depends smoothly on the triangle data, hence also N, by (*a*).

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