A NOTE ON SYMMETRIC AND HARMONIC SPACES

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Using the relationship between Jacobi fields and the second fundamental tensor of hypersurfaces, we give a rather explicit formula for the mean curvature of distance spheres on irreducible symmetric spaces. We use this in order to give a new proof of the theorem of A. J. Ledger which states that irreducible symmetric harmonic spaces are of rank one. Our proof does not refer to the theory of harmonic spaces at all. We only use some well known facts about symmetric spaces (see [4]).

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Let M be a complete connected $(n+1)$-dimensional Riemannian manifold. Call TM the tangent bundle with footpoint map $f\colon TM \to M$, and SM the unit sphere bundle $\{x \in TM : \|x\| = 1\}$. Let $\exp : TM \to M$ be the Riemannian exponential map. Fix some $v \in TM$ and call $g : \mathbb{R} \to M$ the geodesic $g(t) = \exp tv$ with tangent vector $v(t) = g'(t)$. Consider the normal bundle of g with fibres

$$N_t = \{x \in T_{g(t)}M : \langle x, v(t)\rangle = 0\} .$$

It is convenient to identify all fibres by means of the parallel transport along g; the identification space will be called N_v. This is a real vector space with a well defined inner product $\langle\, ,\rangle$. We define a family of self-adjoint endomorphisms $R_v(t)$ on N_v by $R_v(t)x = R(x, v(t))v(t)$, where

$$R : TM \times TM \times TM \to TM$$

is the Riemannian curvature tensor. Let $A_v(t) \in \operatorname{End} N_v$ be the solution of the following initial value problem:

$$A_v'' + R_v \cdot A_v = 0\,, \quad A_v(0) = 0\,, \quad A_v'(0) = I \tag{1}$$

where $'$ denotes the derivative with respect to t and I the identity on N_v. For any $r > 0$, we have $A_{rv}(t) = A_v(rt)/r$, since $R_{rv}(t) = r^2 R_v(rt)$. If v is the zero vector at some point $p \in M$, then $N_v = T_pM$ and $A_v(t) = tI$.

For any $v \in TM$, denote by $\exp_*(v) : T_vTM \to T_{\exp(v)}M$ the derivative of exp at v. Via the canonical identification of $T_{f(v)}M$ with $T_vT_{f(v)}M \subset T_vTM$, we may consider $N_v \subset T_{f(v)}M$ as a subspace of T_vTM. Up to this identification, exp is related to A by

$$\exp_*(tv)x = t^{-1}A_v(t)x$$

for any $x \in N_v$.

Let $|p, q|$ denote the Riemannian distance between any two points $p, q \in M$. Outside the conjugate locus of p, the Riemannian spheres $S_t(p) =$

$\{q \in M : |p, q| = t\}$, $t > 0$, are regularly immersed hypersurfaces. Choose the orientation with respect to the inner normal vector. The corresponding shape operator (second fundamental tensor) of $S_t(p)$ at the point $\exp tv$, $v \in S_pM$, is given by the self-adjoint endomorphism $U_v(t) := A_v'(t)A_v(t)^{-1}$ of N_v [2, 3]. Due to (1), U obeys the Riccati equation

$$U' + U^2 + R = 0. \tag{2}$$

Denote the mean curvature of $S_t(p)$ at $\exp tv$ by $h(v, t) = \operatorname{trace} U_v(t)$. This is known to be the logarithmic derivative of $a(v, t) := \det A_v(t)$, and hence $a' = h \cdot a$ [2, 3].

We call M *harmonic* if $a(v, t)$ and hence $h(v, t)$ do not depend on $v \in SM$ [1, 7]. For the concept of a *symmetric space* and related topics, we refer to [4]. Now we are ready to state the theorems.

THEOREM 1. (A.J. Ledger [5].) *Let M be an irreducible Riemannian globally symmetric space. Then M is harmonic if and only if it is a symmetric space of rank one.*

THEOREM 2. *Let M be a locally symmetric Riemannian manifold. Then M is harmonic if and only if it is either flat or its universal covering is isometric to a rank one symmetric space.*

Proof of Theorem 2. If the universal covering of M is isometric to $\mathbb{R}^n$ or a rank one symmetric space, then M is locally two-point-homogeneous, hence harmonic. On the other hand, M is harmonic if and only if its universal covering $\hat{M}$ is. This is a Riemannian globally symmetric space, hence $\hat{M} = \hat{M}_0 \times \hat{M}_1 \times \ldots \times \hat{M}_k$, where $\hat{M}_0$ is flat and $\hat{M}_i$ are irreducible symmetric spaces (see [4; p. 208 and p. 310]). By a theorem of Lichnerowicz [6], a harmonic space cannot be a product unless it is flat. Hence M is flat or irreducible. By Theorem 1, it has to be of rank one.

Proof of Theorem 1. We may assume $M = G/K$, where G is some real analytic semisimple Lie group and K a compact subgroup. Call $\mathbf{G}, \mathbf{K}$ the corresponding Lie algebras, $\mathbf{P}$ the orthogonal complement of $\mathbf{K}$ with respect to the Killing form B on $\mathbf{G}$. Via the projection $\pi : G \to G/K = M$, we may identify $\mathbf{P}$ with T_eM, where $e = \pi(1)$, 1 being the unit element in G. Since M is irreducible, we may assume that M is of compact (respectively noncompact) type and the Riemannian metric on T_eM is given by $-B|_{\mathbf{P}}$ (respectively $B|_{\mathbf{P}}$).

Now choose some maximal abelian subalgebra $\mathbf{A}$ of $\mathbf{P}$; call $\mathbf{A}^\perp$ the orthogonal complement of $\mathbf{A}$ in $\mathbf{P}$. Hence $\mathbf{A}^\perp \subset N_v$ for all $v \in \mathbf{A}$. Let $r := \dim \mathbf{A}$ be the rank of M. The mappings $\operatorname{ad}(v)$, $v \in \mathbf{A}$, are commuting semisimple endomorphisms of $\mathbf{G}$ with purely imaginary or zero (respectively purely real) eigenvalues, so they are simultaneously diagonizable on the complexification $\mathbf{G}_{\mathbb{C}}$. A nonzero linear form $\alpha : \mathbf{A} \to \mathbb{R}$ is called a *root* of M if there exists a non-zero $x \in \mathbf{G}_{\mathbb{C}}$ such that $\operatorname{ad}(v)x = i\alpha(v)x$ (respectively $\operatorname{ad}(v)x = \alpha(v)x$) for any $v \in \mathbf{A}$. Denote the set of roots by Rt.

The Riemannian curvature tensor at e of the symmetric space M is given by $R(x, y)z = -[[x, y], z]$ for any $x, y, z \in \mathbf{P}$ [4; p. 180]. In particular, $R_v(0) = -\operatorname{ad}(v)^2|_{N_v}$. Recall that R is covariantly constant since M is symmetric. Hence $R_v(t) =: R_v = -\operatorname{ad}(v)^2$ for all $t \in \mathbb{R}$. By the foregoing, $\{R_v|_{\mathbf{A}^\perp} : v \in \mathbf{A}\}$ is a set of commuting self-adjoint endomorphisms of $\mathbf{A}^\perp$. The eigenvalues are among the $\pm\, \alpha(v)^2$.

Let $Rt' \subset Rt$ be the set of those roots α such that there exists some non-zero $x \in \mathbf{A}^\perp$ with $R_v x = \alpha(v)^2 x$ (respectively $R_v x = -\alpha(v)^2 x$) for all $v \in \mathbf{A}$. It can be shown but is not used here that $Rt' = Rt$. Now R_v, $v \in \mathbf{A}$, is zero on $N_v \cap \mathbf{A}$ and diagonizable on $\mathbf{A}^\perp$ with eigenvalues $\alpha(v)^2$ (respectively $-\alpha(v)^2$), $\alpha \in Rt'$.

Denote the cotangent (respectively hyperbolic cotangent) function by ct. Equation (2) shows that the set of eigenvalues of $U_v(t)$ is $\{\alpha(v)\cdot ct(\alpha(v)t) : \alpha \in Rt'\}$ on $\mathbf{A}^\perp$ and $\{t^{-1}\}$ on $\mathbf{A} \cap N_v$. Hence we get for the trace

$$h(v,t) = (r-1)t^{-1} + \sum_{\alpha \in Rt'} k_\alpha \alpha(v) ct(\alpha(v)t)$$

with suitable nonnegative integers k_α.

1st case: M is of compact type. For any $v \in \mathbf{A}$ let $\lambda(v)$ be the maximum of all $\alpha(v)$ for $\alpha \in Rt'$. Then $h(v,t)$ depends smoothly on t for $0 < t < \pi/\lambda(v)$, and $h(v,t) \to -\infty$ if $t \to \pi/\lambda(v)$. Therefore, harmonicity implies that λ is constant on $S\mathbf{A} := \mathbf{A} \cap S_e M$. Assume that $r-1 = \dim S\mathbf{A} \geqslant 1$. For any $\alpha \in Rt'$ let $h_\alpha \in \mathbf{A}$ be the vector with $\langle h_\alpha, x\rangle = \alpha(x)$ for all $x \in \mathbf{A}$. Choose $\mu \in Rt'$ such that $\|h_\mu\| \geqslant \|h_\alpha\|$ for all $\alpha \in Rt'$. Call $w := h_\mu/\|h_\mu\| \in S\mathbf{A}$. Then $\mu(w) = \langle h_\mu, h_\mu\rangle/\|h_\mu\| = \|h_\mu\|$, and for all $\alpha \in Rt'$ we have

$$\alpha(w) = \langle h_\alpha, w\rangle \leqslant \|h_\alpha\| \leqslant \|h_\mu\| = \mu(w).$$

If $\alpha \notin \mathbb{R}\mu$, then $\langle h_\alpha, w\rangle < \|h_\alpha\|$. If $\alpha = k\mu, k > 0$, then $\|h_\alpha\| = k\|h_\mu\|$, hence $k \leqslant 1$ by the choice of μ. If $\alpha = -k\mu$, $k > 0$, then $\langle h_\alpha, w\rangle < 0$. It follows that $\alpha(w) < \mu(w)$ for any $\alpha \neq \mu$. By continuity, there is a neighbourhood U of w in $S\mathbf{A}$ such that $\alpha(v) < \mu(v)$ for all $v \in U$. Hence $\mu|_U = \lambda$. But the linear form μ is nowhere constant on a sphere of positive dimension. Hence we have $r = 1$.

2nd case: M is of noncompact type. Then $h(v,t) \to \sum_{\alpha \in Rt'} k_\alpha \alpha(v)$ for $t \to \infty$. But the linear form $\sum_{\alpha \in Rt'} k_\alpha \alpha$ cannot be constant on $S\mathbf{A}$ if $r > 1$.

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